The pro-p-Iwahori Hecke algebra of a reductive p-adic group I

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Abstract

Let R be a commutative ring, let F be a locally compact non-archimedean field of finite residual field k of characteristic p, and let $\mathbf{G}$ be a connected reductive F-group. We show that the pro-p-Iwahori Hecke R-algebra of $G = \mathbf{G}(F)$ admits a presentation similar to the Iwahori-Matsumoto presentation of the Iwahori Hecke algebra of a Chevalley group, and alcove walk bases satisfying Bernstein relations.

Contents

1 Introduction

The study of congruences between classical modular forms naturally leads to representations over arbitrary commutative rings R, rather than to complex representations. In our local setting, that means studying R-modules with a smooth action of $G = \mathbf{G}(F)$ where F is a locally compact non-archimedean field of finite residue field k, and $\mathbf{G}$ is a connected reductive group over F.

For **any** $(R, F, \mathbf{G})$, we describe the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ of G, where p is the characteristic of k, generalizing the Iwahori and Matsumoto presentation of the Iwahori Hecke R-algebra of a Chevalley group. The proof of the quadratic relations is done by reduction to the analog Hecke R-algebra of a finite reductive group. The pro-p-Iwahori subgroups of G are the analogues of the p-Sylow subgroups of a finite group and the study of the smooth representations of G over an algebraically closed field C of characteristic p involves naturally $\mathcal{H}_C(1)$. The Iwahori Hecke R-algebra $\mathcal{H}_R$ of G is a quotient of $\mathcal{H}_R(1)$ and all our results transfer to analogous and simpler results for the Iwahori Hecke R-algebra.

The Iwahori-Matsumoto presentation of the pro-p-Iwahori Hecke R-algebra of G leads naturally to the definition of R-algebras $\mathcal{H}_R(q_s, c_s)$ associated to a group $W(1)$ and parameters (q_s, c_s) satisfying simple conditions. The group $W(1)$ is an extension by a commutative group Z_k of an extended affine Weyl group W attached to a reduced root system Σ, more general than the group W appearing in the Lusztig affine Hecke algebras $\mathcal{H}_R(q_s, q_s - 1)$. The R-algebra $\mathcal{H}_R(q_s, c_s)$ is a free R-module of basis indexed by the elements of $W(1)$ satisfying the braid relations and quadratic relations with coefficients (q_s, c_s). We show that the algebra $\mathcal{H}_R(q_s, c_s)$ admits an alcove walk basis indexed by the elements of $W(1)$ for any Weyl chamber, a product formula involving alcove walk bases associated to different Weyl chambers, and Bernstein relations. When the q_s are invertible in R we obtain a presentation of the algebra $\mathcal{H}_R(q_s, c_s)$ generalizing the Bernstein-Lusztig presentation for the Iwahori Hecke algebra of a split group. Our proofs proceed by reduction to the case $q_s = 1$.

The main point is that we have no restriction on the triple $(R, F, \mathbf{G})$, the reductive group $\mathbf{G}$ may be not split, the local field may have characteristic p, the commutative ring R may be the ring of integers $\mathbb{Z}$ or a field of characteristic p.

When $\mathbf{G}$ is split, the complex Iwahori Hecke algebra $\mathcal{H}_{\mathbb{C}}$ of G was well understood. It is the affine Hecke algebra attached to a based root datum of G and to the cardinal q of k (the first proof is due to Iwahori and Matsumoto for a Chevalley group). Starting from the Iwahori-Matsumoto presentation of $\mathcal{H}_{\mathbb{C}}$, Bernstein and Lusztig gave another presentation of $\mathcal{H}_{\mathbb{C}}$, from which one can recover the center of $\mathcal{H}_{\mathbb{C}}$ and which is an essential step for the classification of its simple modules. The classification was done for $\mathbf{G} = GL(n)$ by Zelevinski and Rogawski, and for $\mathbf{G}$ simple with a connected center by Kazhdan-Lusztig, and Ginzburg, using equivariant K-theory of the variety of Steinberg triples. Görtz realized that the Bernstein basis could be understood using Ram's alcove walks, and gave a simpler proof of the Bernstein presentation of an affine Hecke algebra of a based root datum with unequal invertible parameters. When $\mathbf{G}$ is split, I had shown that the pro-p-Iwahori Hecke algebra of G admits an Iwahori-Matsumoto presentation and an integral Bernstein basis, using Haines minimal expressions. A student Nicolas Schmidt of Grosse-Klonne, in his unpublished diplomarbeit, defined the alcove walk basis, proved the product formula, and studied the Bernstein relations for algebras $\mathcal{H}_R(q_s, c_s)$ containing the algebras arising from a split $\mathbf{G}$, but not all those arising from a general $\mathbf{G}$.

Our motivation for studying the pro-p-Iwahori Hecke algebra of G comes from the theory of smooth representations of G over an algebraically closed field C of characteristic p. In the complex case, the relation between the $\mathcal{H}_{\mathbb{C}}(1)$-modules and the smooth $\mathbb{C}$-representations of G is well understood. This is not the case when $\mathbb{C}$ is replaced by C, but the pro-p-Iwahori Hecke algebra $\mathcal{H}_C(1)$ plays an important role in the theory of smooth C-representations of G. In a forthcoming work with Abe, Henniart and Herzig, we classify the irreducible admissible representations of G over an algebraically closed field C of characteristic p which are not supercuspidal, in term of the irreducible admissible supercuspidal (= supersingular) representations of the Levi subgroups and of parabolic induction. The Bernstein relations in the pro-p-Iwahori Hecke C-algebra $\mathcal{H}_C(1)$ of G, which is isomorphic to a C-algebra $\mathcal{H}_C(0, c_s)$ with parameters $q_s = 0$, are an ingredient of the classification.

In a sequence to this paper, we will give application of the Bernstein relations to the center of $\mathcal{H}_R(1)$ and to the classification of the simple $\mathcal{H}_C(1)$-modules.

Rachel Ollivier was the first to understand the importance of the different integral Bernstein basis for the pro-p-Iwahori Hecke algebra of G, in her work on the inverse Satake isomorphism and on the classification of the supersingular modules (for $\mathbf{G}$ split). I had many conversations with Noriyuki Abe on the Bernstein relations during his stay at the Institute of mathematiques of Jussieu in 2013, and their relations with the change of weight in irreducible smooth admissible C-representations of G. The unpublished diplomarbeit of Nicolas Schmidt explaining in details the application by Görtz of the alcove walks of Ram to the Bernstein presentation of the Lusztig affine Hecke algebras, and his own work on the generic algebras generalizing the pro-p-Iwahori Hecke algebras when $\mathbf{G}$ is split, were extremely helpful for writing this article.

2 Main Results

2.1 Iwahori-Masumoto presentation.

Let $\mathbf{G}$ be a connected reductive group over a local non-archimedean field F of finite residue field k of characteristic p with q elements, and let R be a commutative ring. We fix an Iwahori subgroup I of G. Its pro-p-radical $I(1)$ is called a pro-p-Iwahori group. The

Iwahori Hecke ring

$$\mathcal{H} = \mathbb{Z}[I\backslash G/I]$$

with the convolution product, is isomorphic to the ring of intertwiners $\mathrm{End}_{\mathbb{Z}[G]}\,\mathbb{Z}[I\backslash G]$ of the regular right representation $\mathbb{Z}[I\backslash G]$ of G associated to I. The Iwahori Hecke R-algebra obtained by base change

$$\mathcal{H}_R = R \otimes_{\mathbb{Z}} \mathcal{H} = R[I\backslash G/I]. \tag{1}$$

is isomorphic to the R-algebra of intertwiners $\mathrm{End}_{R[G]}\,R[I\backslash G]$. We replace I by $I(1)$ and define in the same way the pro-p-Iwahori Hecke ring $\mathcal{H}(1) = \mathbb{Z}[I(1)\backslash G/I(1)]$ and the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$.

The sets $I\backslash G/I$ and $I(1)\backslash G/I(1)$ have a natural group structure, isomorphic to the Iwahori Weyl group W and the pro-p-Iwahori Weyl group $W(1)$ defined as follows.

The Iwahori group I is the parahoric subgroup of G stabilizing an alcove $\mathfrak{C}$ in the building of the adjoint group of G. We choose an apartment $\mathfrak{A}$ containing $\mathfrak{C}$. The apartment $\mathfrak{A}$ is associated to a maximal F-split subtorus $\mathbf{S}$ of $\mathbf{G}$. The apartment $\mathfrak{A}$ is a finite dimensional affine euclidean real space with a locally finite set $\mathcal{H}$ of hyperplanes, such that the orthogonal reflections with respect to $H \in \mathcal{H}$ generate an affine Weyl group $W(\mathfrak{H})$, and $\mathfrak{C}$ is a connected component of $\mathfrak{A} - \cup_{H\in\mathcal{H}} H$. We denote by $\mathbf{Z}$ and $\mathbf{N}$ the centralizer and the normalizer of $\mathbf{S}$ in $\mathbf{G}$. The group $N = \mathbf{N}(F)$ acts on $\mathfrak{A}$ by affine automorphisms respecting $\mathcal{H}$ such that its subgroup $Z = \mathbf{Z}(F)$ acts by translations. The parahoric subgroups of G generate a subgroup G^{aff}, and G is generated by $Z \cup G^{aff}$. The unique parahoric subgroup Z_0 of Z is contained in the unique maximal compact subgroup $\tilde{Z}_0$, acts trivially on $\mathfrak{A}$, and

$$Z \cap G^{aff} = Z_0, \tag{2}$$

$$Z \cap I = Z_0, \quad Z \cap I(1) = Z_0(1), \quad I = I(1)Z_0. \tag{3}$$

The action of $N^{aff} = N \cap G^{aff}$ on the apartment $\mathfrak{A}$ induces an isomorphism

$$W^{aff} = N^{aff}/Z_0 \to W(\mathcal{H}). \tag{4}$$

The groups $Z_0(1) \subset Z_0$ are normalized by N and the maps $n \mapsto InI,\ n \mapsto I(1)nI(1)$ induce bijections

$$W = N/Z_0 \to I\backslash G/I, \quad W(1) = N/Z_0(1) \to I(1)\backslash G/I(1). \tag{5}$$

The *pro-p-Iwahori-Weyl group* $W(1)$ is an extension of the *Iwahori-Weyl group* W by $Z_0/Z_0(1) \simeq I/I(1)$

$$1 \to Z_0/Z_0(1) \to W(1) \to W \to 1. \tag{6}$$

The extension does not split in general (see [**?**]).

For $n \in N$, the double coset InI depends only on the image $w \in W$ of n and the corresponding intertwiner in the Iwahori Hecke ring $\mathcal{H}$ is denoted by T_w. Thus $(T_w)_{w\in W}$ is a natural basis of $\mathcal{H}$. We do the same for $\mathcal{H}(1)$ and $W(1)$. The relations satisfied by the products of these elements follow from the fact that W is a semi-direct product of the affine Weyl group W^{aff} by the image Ω in W of the N-normalizer of $\mathfrak{C}$,

$$W = W^{aff} \rtimes \Omega. \tag{7}$$

Let $S^{aff} \subset W^{aff}$ be the set of orthogonal reflections with respect to the walls of $\mathfrak{C}$ (using the isomorphism (**??**)). The length ℓ of the Coxeter group (W^{aff}, S^{aff}) inflates to a length of W constant on the double cosets modulo Ω, and to a length of $W(1)$ constant on the double cosets modulo the inverse image $\Omega(1)$ of Ω. The Bruhat order of W^{aff} inflates to $W(1)$ and to W [?].

For $n \in N$ of image w in W or in $W(1)$, the sets

$$InI/I \simeq I(1)nI(1)/I(1) \tag{8}$$

have the same number of elements by (??). This number, denoted by q_w, is a power of q and $q_s = q_{s'}$ when $s, s' \in S^{aff}$ are conjugate in W (denoted by $s \sim s'$).

Theorem 2.1. Iwahori-Matsumoto presentation of $\mathcal{H}$ *The Iwahori Hecke ring $\mathcal{H}$ is the free $\mathbb{Z}$-module with basis $(T_w)_{w\in W}$ endowed with the unique ring structure satisfying*

- *The braid relations: $T_w * T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$.*
- *The quadratic relations: $T_s * T_s = q_s + (q_s - 1)T_s$ if $s \in S^{aff}$.*

The Iwahori Hecke R-algebra $\mathcal{H}_R$ has the same presentation over R by base change (??). By a general property of Hecke algebras ([?] I.3.5) :

Corollary 2.2. *The linear map $\mathcal{H} \to \mathbb{Z}$:*

$$T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_\omega \mapsto q_w = q_{s_1} \dots q_{s_{\ell(w)}} \quad \text{if} \quad w = s_1 \dots s_{\ell(w)} u \quad (s_i \in S^{aff}, u \in \Omega),$$

is a ring homomorphism.

The elements in the basis $(T_w)_{w\in W(1)}$ of $\mathcal{H}(1)$ verify the braid relations, and similar quadratic relations but more complicated. The integers $q_s - 1$ are replaced by elements of $\mathbb{Z}[I/I(1)]$, that we define now.

We denote by $S^{aff}(1) \subset W(1)$ the subset above S^{aff}. Let $s \in S^{aff}(1)$ and let $K_{\mathfrak{F}_s}$ be the parahoric subgroup of G fixing the face of $\mathfrak{C}$ supported on the wall fixed by s. The square s^2 is an element of $Z_k = Z_0/Z_0(1)$. The quotient of $K_{\mathfrak{F}_s}$ by its pro-p-radical is the group $K_{\mathfrak{F}_s,k}$ of k-points of a finite reductive connected group over k of semi-simple rank 1. The group Z_k is a maximal split torus of $K_{\mathfrak{F}_s,k}$ and the root system $\Phi_{\mathfrak{F}_s}$ of $K_{\mathfrak{F}_s,k}$ with respect to Z_k is contained in the root system Φ of G with respect to S. We denote by $U_{\alpha_s,k}$ the root subgroup associated to a a reduced root $\alpha_s \in \Phi_{\mathfrak{F}_s}$ (we have $U_{2\alpha_s,k} \subset U_{\alpha_s,k}$ if $2\alpha_s \in \Phi_{\mathfrak{F}_s,k}$) and by N_k the normalizer of Z_k in $K_{\mathfrak{F}_s,k}$. For $u \in U^*_{\alpha_s,k} = U_{\alpha_s,k} - \{1\}$, the intersection

$$U_{-\alpha_s,k} u U_{-\alpha_s,k} \cap N_k = \{m_{\alpha_s}(u)\}$$

consists of a single element. For $u, u' \in U^*_{\alpha,k}$, the product $m(u')^{-1}m(u)$ is an element of $Z'_{k,s} = Z_k \cap K'_{\mathfrak{F}_s,k}$ where $K'_{\mathfrak{F}_s,k}$ is the group generated by $U_{\alpha_s,k}$ and $U_{-\alpha_s,k}$. Let

$$c_s = \sum_{u' \in U^*_{\alpha_s,k}} m(u')^{-1} m(u) = \sum_{t \in Z'_{k,s}} c_s(t) t. \tag{9}$$

We have

$$\sum_{t \in Z'_{k,s}} c_s(t) = q_s - 1, \quad c_s(t) |\{ts(t^{-1}) \mid t \in Z_k\}| \equiv -1 \bmod p.$$

Theorem 2.3. Iwahori-Matsumoto presentation of $\mathcal{H}(1)$ *The pro-p-Iwahori Hecke ring $\mathcal{H}(1)$ is the free $\mathbb{Z}$-module with basis $(T_w)_{w\in W(1)}$ endowed with the unique ring structure satisfying*

- *The braid relations: $T_w * T_{w'} = T_{ww'}$ if $w, w' \in W(1), \ell(w) + \ell(w') = \ell(ww')$.*
- *The quadratic relations:*

$$T_s * T_s = q_s s^2 + c_s T_s \quad \text{for } s \in S^{aff}(1).$$

The quadratic relation of T_s in $\mathcal{H}(1)$ is the same than the quadratic relation of T_s in the finite Hecke algebra $\mathcal{H}(K_{\mathfrak{F}_s,k}, U_{\alpha_s,k})$ ([**?**] proof of Prop. 6.8). The braid relations allow to identify the group algebra $\mathbb{Z}[\Omega(1)]$ to a subalgebra of $\mathcal{H}(1)$. The $\mathbb{Z}$-module of basis (T_w) for w in the inverse image $W^{aff}(1)$ of W^{aff} in $W(1)$ is a subalgebra $\mathcal{H}^{aff}(1)$ and $\mathcal{H}(1)$ is a twisted product

$$\mathcal{H}(1) \simeq \mathcal{H}^{aff}(1)\hat{\otimes}_{Z[Z_k]}\mathbb{Z}[\Omega(1)].$$

The pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ has the same presentation by base change.

Theorems **??** and **??** imply:

Corollary 2.4. *The surjective R-linear map $\mathcal{H}_R(1) \to \mathcal{H}_R$:*

$$T_{w(1)} \mapsto T_w \quad \text{for } w(1) \in W(1) \text{ of image } w \in W,$$

is a R-algebra homomorphism.

The properties of the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ are transported to the Iwahori Hecke R-algebra $\mathcal{H}_R$ via this surjective R-algebra homomorphism.

Conversely, we describe the conditions on the elements $(q_s, c_s) \in R \times R[Z_k]$ for $s \in S^{aff}(1)$, equivalent to the existence of an R-algebra $\mathcal{H}_R(q_s, c_s)$ of basis $(T_w)_{w\in W(1)}$ satisfying the braid and quadratic relations of Thm. **??**.

Proposition 2.5. *The algebra $\mathcal{H}_R(q_s, c_s)$ exists if and only if, for all $t \in Z_k, w \in W(1)$, $s \sim s'$ in $S^{aff}(1)$, $ws'w^{-1}s^{-1} \in Z_k$, we have:*

1) $q_s = q_{st} = q_{s'}$.
2) $c_{st} = c_s t$, $\sum_{t\in Z_k} c_{s'}(t)wtw^{-1} = ws'w^{-1}s^{-1}c_s$.

Let $(q_s, c_s) \in R \times R[Z_k]$ for $s \in S^{aff}(1)$ satisfying the properties 1) and 2) of this proposition. We introduce indeterminates $\mathfrak{q}_s$ for $s \in S^{aff}(1)$ satisfying 1). Let $\mathbf{q}_s = \mathfrak{q}_s^2$. The $R[(\mathbf{q}_s)]$-algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ is called the generic algebra.

Proposition 2.6. *The algebra $\mathcal{H}_R(q_s, c_s)$ is the specialisation by $\mathbf{q}_s \mapsto q_s$ of the generic algebra.*

The generic algebra is a $R[(\mathbf{q}_s)]$-subalgebra of $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$.

The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$- algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$ is isomorphic to $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1}c_s)$.

This proposition allows to prove by reduction to the simpler case $q_s = 1$ for all $s \in S^{aff}$, some relations between the elements of the algebra $\mathcal{H}_R(q_s, c_s)$.

Remark 2.7. The presentation of $\mathcal{H}$ (Thm. **??**) generalizing the Iwahori-Matsumoto presentation for a Chevalley group [**?**], is not in the litterature but can be extracted by quoting different results from Bruhat-Tits ([**?**] 5.2.12 Prop. (i) and (ii)) and exercises in Bourbaki ([**?**] Ch. IV, §2, Ex. 8, 22, 23, 24, 25). See Borel [**?**] when $\mathbf{G}$ is semi-simple, for the "non-connected" Iwahori subgroup $\tilde{I} = I\tilde{Z}_0$. When $\mathbf{G}$ is F-quasisplit, Z is a torus, $I = \tilde{I}$, then $\mathcal{H}$ is a Lusztig affine Hecke algebra attached to a based root datum of G and to a system of unequal parameters (q_s) [**?**]. But for a general $\mathbf{G}$, $\mathcal{H}$ is not a Lusztig affine Hecke algebra. When $\mathbf{G}$ is F-split, the quadratic relations in Theorem **??** are proved in [**?**] without reduction to finite reductive groups.

2.2 Alcove walk bases and Bernstein relations

The quadratic relations in the generic algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ can be written as

$$T_s * T_s^* = T_s^* * T_s = \mathbf{q}_s s^2, \text{ where } T_s^* = T_s - c_s.$$

The choice of a special vertex x_0 of $\mathfrak{C}$ identifies the apartment $\mathfrak{A}$ with an euclidean real vector space V. The affine hyperplanes in $\mathfrak{H}$ are $\mathrm{Ker}(\alpha + k)$ for $k \in \mathbb{Z}$ and α in a reduced root system Σ in the dual of V. Let Δ be the basis of Σ such that $\alpha \in \Delta$ takes positive values on the alcove $\mathfrak{C}$ and $\operatorname{Ker}\alpha$ is a wall of $\mathfrak{C}$.

Let Δ' be any basis of Σ.

The complement of an affine hyperplane $H \in \mathfrak{H}$ in V consists of two half-spaces. The Δ'-negative half-space is the set of $v \in V - \mathfrak{H}$ where $\alpha(v) + k < 0$ if $H = \operatorname{Ker}(\alpha + k)$ for $k \in \mathbb{Z}$ and α positive for Δ'. For $w \in W(1)$ and for $s \in S^{aff}(1)$ acting on V by the orthogonal reflection with respect to $H_s \in \mathfrak{H}$, we set

$$\epsilon_{\Delta'}(w,s) = 1 \quad \text{if and only if } w(\mathfrak{C}) \text{ is contained in the } \Delta'\text{-negative side of } w(H_s).$$

Otherwise we set $\epsilon_{\Delta'}(w,s) = -1$. When we walk from $w(\mathfrak{C})$ to $ws(\mathfrak{C})$ we cross the hyperplane H_s in the positive direction if $\epsilon_{\Delta'}(w,\tau) = 1$ and in the negative direction if $\epsilon_{\Delta'}(w,\tau) = -1$. We set in the generic algebra

$$T_s^{\epsilon_{\Delta'}(w,s)} = T_s \text{ if } \epsilon_{\Delta'}(w,\tau) = 1 \text{ and } T_s^{\epsilon_{\Delta'}(w,s)} = T_s^* \text{ if } \epsilon_{\Delta'}(w,\tau) = -1.$$

Theorem 2.8. *For $w \in W(1)$ of decomposition $w = s_1 \dots s_{\ell(w)} u$ with $u \in \Omega(1)$ and $s_i \in S^{aff}(1)$ for $1 \leq i \leq \ell(w)$, the element of the generic algebra*

$$E_{\Delta'}(w) = T_{s_1}^{\epsilon_{\Delta'}(1,s_1)} T_{s_2}^{\epsilon_{\Delta'}(s_1,s_2)} \dots T_{s_{\ell(w)}}^{\epsilon_{\Delta'}(s_1 \dots s_{\ell(w)-1}, s_{\ell(w)})} T_u \tag{10}$$

does not depend on the decomposition of w. It satisfies

$$E_{\Delta'}(w) - T_w \in \oplus_{w' < w} \mathbb{Z} T_{w'}. \tag{11}$$

For $w, w' \in W(1)$, we have the product formula:

$$E_{\Delta'}(w) E_{w^{-1}(\Delta')}(w') = \mathbf{q}_{w,w'} E_{\Delta'}(ww'), \quad \mathbf{q}_{w,w'} = (\mathbf{q}_w \mathbf{q}_{w'} \mathbf{q}_{ww'}^{-1})^{1/2}. \tag{12}$$

The theorem implies that $(E_{\Delta'}(w))_{w \in W(1)}$ is a basis of $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ as a $\mathbb{Z}$-module. The theorem is proved by reduction to $q_s = 1$.

The *finite Weyl group* $W_0 = N/Z$ identifies with the subgroup of W^{aff} generated by the orthogonal reflections s_β with respect to $\operatorname{Ker}\beta$ for $\beta \in \Delta'$. The Iwahori-Weyl group W is the semi-direct product

$$W = \Lambda \rtimes W_0, \quad \text{where } \Lambda = Z/Z_0,$$

the kernel of the map $W(1) \to W_0$ is $\Lambda(1) = Z/Z_0(1)$, the simply transtive action of W_0 on the bases of Σ inflates to an action of W and of $W(1)$, and

$$W(1) = \Lambda(1) W_0(1),$$

where $W_0(1)$ is the image inverse of W_0 in $W(1)$.

Corollary 2.9. *The $\mathbb{Z}$-module of basis $(E_{\Delta'}(\lambda))_{\lambda \in \Lambda(1)}$ is a subring $\mathfrak{A}_{\Delta'}$ of the generic algebra with product*

$$E_{\Delta'}(\lambda) E_{\Delta'}(\lambda') = q_{\lambda,\lambda'} E_{\Delta'}(\lambda\lambda').$$

The existence of these subrings $\mathfrak{A}_{\Delta'}$ have deep consequences on the structure of the generic algebra (as in the case $\mathbf{G}$ split where $\mathfrak{A}_{\Delta'}$ is commutative) which will be given in a sequence to this work.

Let $s \in W_0(1)$ above the reflection $s_\beta \in W_0$ with respect to $\operatorname{Ker}\beta$ for a root β in $\Delta' \cup -\Delta'$ which takes positive values on $\mathfrak{C}$. Let $\lambda \in \Lambda(1)$. The Bernstein relation in the Iwahori Hecke ring $\mathcal{H}(1)$ provides the expansion of

$$E_{\Delta'}(s)(E_{s(\Delta')}(\lambda) - E_{\Delta'}(\lambda)) = E_{\Delta'}(s(\lambda)) E_{\Delta'}(s) - E_{\Delta'}(s) E_{\Delta'}(\lambda) \tag{13}$$

in the basis $(E_{\Delta'}(w))_{w \in W(1)}$.

Let $\nu : \Lambda(1) \to V$ be the homomorphism such that λ act on V by translation by $\nu(\lambda)$. We have $\beta \circ \nu(\lambda) = 0$ if and only if $\nu(\lambda)$ is fixed by s.

Lemma 2.10. *When $\beta\circ\nu(\lambda)=0$, we have $E_{s(\Delta')}(\lambda)=E_{\Delta'}(\lambda)$.*

As the translation by $\nu(\lambda)$ stabilizes $\mathfrak{H}$, we have $\beta\circ\nu(\lambda)\in\mathbb{Z}$. When it is not 0, we denote its sign by $\epsilon_\beta(\lambda)$.

Theorem 2.11. (Bernstein relation in the generic algebra) *We suppose $\beta\circ\nu(\lambda)\neq 0$. Then,*

$$E_{\Delta'}(s)(E_{s(\Delta')}(\lambda)-E_{\Delta'}(\lambda))=\epsilon_\beta(\lambda)\epsilon_{\Delta'}(1,s)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)c(k)E_{\Delta'}(\mu(k)),$$

where $c(k)\in\mathbb{Z}[Z_k]$, $\mu(k)\in\Lambda(1)$, $\beta\circ\nu(\mu(k))=2k-|\beta\circ\nu(\lambda)|$, and

$$\mathbf{q}(k)=\prod_{s\in S^{aff}/\sim}\mathbf{q}_s^{m_k(s)},\quad m_k(s)\in\mathbb{N},\quad \sum_s m_k(s)=\ell(\lambda)-\ell(\mu(k)).$$

The theorem is proved by reduction to $q_s=1$. The values of $\mathbf{q}(k),c(k)$ and $\mu(k)$ are explicit (Cor. **??**) and depend on (s,λ) but not on Δ'. They are simpler when the image of $\beta\circ\nu$ is $\mathbb{Z}$ (the other possibility is $2\mathbb{Z}$). We have $m_k(s)\geq 1$ when $\ell(s\lambda)<\ell(\lambda)$. When $\beta\circ\nu(\lambda)\neq 0$, moving the term indexed by $k=0$ from the right hand side to the left hand side, the Bernstein relation becomes:

$$E_{\Delta'}(s\lambda)-E_{s(\Delta')}(s\lambda)=\epsilon_{s(\Delta')}(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)\mathbf{q}_s^{-1}c(k)E_{\Delta'}(\mu(k))\quad\text{if } \ell(s\lambda)<\ell(\lambda),$$

$$E_{\Delta'}(s\lambda)-E_{\Delta'}(s)E_{\Delta'}(\lambda)=\epsilon_{\Delta'}(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)c(k)E_{\Delta'}(\mu(k))\quad\text{if } \ell(s\lambda)>\ell(\lambda).$$

This form is well adapted to the specialisation $\mathbf{q}_s\to 0$ for all $s\in S^{aff}$ because almost of the terms in the sum will vanish.

Lemma 2.12. *We suppose $\beta\circ\nu(\lambda)>1$ and $0<k<\beta\circ\nu(\lambda)$. Then, $\mathbf{q}(k)\neq 1$.*

When $\ell(s\lambda)<\ell(\lambda)$, then $\mathbf{q}(k)\mathbf{q}_s^{-1}\neq 1$ for $1<k<|\beta\circ\nu(\lambda)|-1$. Moreover, $\mathbf{q}(1)\mathbf{q}_s^{-1}=1$ if $w(\beta)\in\Delta'$ and $\nu(w(\lambda))$ is Δ'-dominant for some $w\in W_0$.

Applying Thm. **??**, **??** with $\Delta'=-\Delta$, we obtain a presentation of the generic algebra.

We denote by $S(1)$ the inverse image in $W(1)$ of the set S of s_α for $\alpha\in\Delta$, and we write $E_{-\Delta}(w)=E(w)$ for $w\in W(1)$. We have $E(s)=T_s$ for $s\in S(1)$.

Theorem 2.13. (Bernstein presentation of the generic algebra)

The $R[\mathbf{q}_s]$-algebra $\mathcal{H}_{R[\mathbf{q}_s]}(\mathbf{q}_s,c_s)$ is the free $R[\mathbf{q}_s]$-module of basis $(E(w))_{w\in W(1)}$ endowed with the unique $R[\mathbf{q}_s]$-algebra structure satisfying:

- *Braid relations for $w,w'\in W_0(1)$, $E(w)E(w')=E(ww')$ if $\ell(w)+\ell(w')=\ell(ww')$.*
- *Quadratic relations for $s\in S(1)$, $E(s)^2=\mathbf{q}_s+c_sE(s)$.*
- *Product for $\lambda,\lambda'\in\Lambda(1)$, $E(\lambda)E(\lambda')=\mathbf{q}_{\lambda,\lambda'}E(\lambda\lambda')$.*
- *Bernstein relations for $s\in S(1),\lambda\in\Lambda(1)$,*

$E(s(\lambda))E(s)=E(s)E(\lambda)$ when $\nu(\lambda)$ is fixed by s,

$E(s(\lambda))E(s)-E(s)E(\lambda)=\epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)c(k)E_o(\mu(k))$, when $\nu(\lambda)$ is not fixed by s.

3 Review of Bruhat-Tit's theory

The aim of this chapter is to give precise references for the properties extracted from Bruhat-Tit's theory which will be used in the proofs of our results. The reader familiar with this theory should skip this chapter and proceed directly to chapter **??**.

We keep the notations given in chapter **??**.

For an algebraic group $\mathbf{H}$ defined over F, we denote $H = \mathbf{H}(F)$. Let $X^*(H)$ and $X_*(H)$ the group of F-characters and F-cocharacters of $\mathbf{H}$.

We insist on the fact that the characteristic of F may be 0 or p, and that the root system $\Phi \subset X^*(S)$ of $\mathbf{G}$ may be not reduced, Φ is the union of its irreducible components ([**?**] VI. §1.2)

$$\Phi = \sqcup_{j=1}^{r} \Phi_j. \tag{14}$$

A basis Δ of Φ is the union of basis of Φ_j, $\Delta = \sqcup_{j=1}^{r} \Delta_j$. The set of coroots $\Phi^\vee \subset X_*(S)$ is the union of the sets of coroots of Φ_j, $\Phi^\vee = \sqcup_j \Phi_j^\vee$. The real vector space generated by $\Phi^\vee$ is a direct sum of the vector spaces V_j generated by $\Phi_j^\vee$,

$$V = \oplus_j V_j. \tag{15}$$

The Weyl group W_0 of Φ is the direct product of the Weyl groups of Φ_j, $W_0 = \prod_j W_{0,j}$. The actions of $W_{0,j}$ on V_j is irreducible, and the decomposition of V is orthogonal for a fixed positive definite bilinear form $(\ ,\ )$ on V invariant by the action of W_0 ([**?**] VI §1.2, V §3.7). For $\alpha \in \Phi$, we have the root group $\mathbf{U}_\alpha$ (containing $\mathbf{U}_{2\alpha}$ if 2α is a root). We denote by ω the valuation of F normalized by $\omega(F - \{0\}) = \mathbb{Z}$.

3.1 Main Theorem

The results contained in the 316 pages of [**?**] and [**?**] are valid for the group G, by the fundamental theorem ([**?**] 5.1.20, 5.1.23) :

Theorem 3.1. *$(Z, U_\alpha)_{\alpha\in\Phi}$ is a root datum generating G, and admitting a discrete valuation $(\varphi_\alpha)_{\alpha\in\Phi}$ compatible with the valuation ω of F.*

The definition of "a root datum generating G" and of "a discrete valuation compatible with ω" is given in ([**?**] 6.1.1 and 6.1.2 (8)) and in ([**?**] 6.2.1, 6.2.21).

Let G', resp. G'_i, be the subgroup of G generated by the root groups U_α for $\alpha \in \Phi$, resp. Φ_i, and let $Z' = Z \cap G', Z'_i = Z \cap G'_i$, for $1 \le i \le r$, be the intersections of Z with these groups.

Then, $(Z', U_\alpha)_{\alpha\in\Phi}$, resp. $(Z'_i, U_\alpha)_{\alpha\in\Phi_i}$, is a root datum generating G', resp. G'_i, admitting a discrete valuation compatible with ω. The subgroups G', G'_i are normal in G, each element of G'_i commutes with each element of G'_j if $i \neq j$, and $G'_i \cap G'_j$ is contained in the center of G'; this center is contained in Z and $G = ZG'$ ([**?**] 6.1.5, 6.2.12).

3.2 The element $m_\alpha(u)$ for $u \in U_\alpha^*$

For $\alpha \in \Phi$ and $u \in U_\alpha^* = U_\alpha - \{1\}$, let $(v'_\alpha(u), m_\alpha(u), v''_\alpha(u))$ in $U_{-\alpha} \times G \times U_{-\alpha}$ be the unique triple such that ([**?**] 6.1.2 (2)):

$$u = v'_\alpha(u) m_\alpha(u) v''_\alpha(u). \tag{16}$$

Remark 3.2. *If $2\alpha \in \Phi$ and $u \in U_{2\alpha}^*$, we have $U_{2\alpha} \subset U_\alpha$ and $m_{2\alpha}(u) = m_\alpha(u)$ by unicity.*

If $\alpha \in \Phi, u \in U_\alpha^*$, then $m_\alpha(u^{-1}) = m_\alpha(u)^{-1}$.

The group $W_0 = N/Z$ identifies with the Weyl group of Φ ([**?**] §5). The image s_α of $m_\alpha(u)$ in W_0 is the reflection defined by α.

Lemma 3.3. *The group N is generated by Z and $\cup_{\alpha\in\Phi} m_\alpha(U_\alpha^*)$.*

Proof. ([**?**] 6.1.2 (10), 6.1.3 c)) where can replace M_α by $m_\alpha(U_\alpha^*)$. □

Proposition 3.4. *Let Δ be a basis of Φ. We can choose $u_\alpha \in U_\alpha^*$ for all $\alpha \in \Delta$ such that, when $\alpha \neq \beta$,*

$$m_\alpha(u_\alpha)m_\beta(u_\beta)\ldots = m_\beta(u_\beta)m_\alpha(u_\alpha)\ldots, \tag{17}$$

where the number of factors is the order $n(\alpha,\beta)$ of $s_\alpha s_\beta \in W_0$.

Proof. a) When $\mathbf{G}$ is F-split, semisimple, and simply connected (we recall that $\mathbf{G}$ is connected), we choose a Chevalley system $x_\alpha : \mathbf{G_a} \to \mathbf{U}_\alpha$ for $\alpha \in \Phi$. The elements $u_\alpha = x_\alpha(1)$ for $\alpha \in \Delta$ satisfy the proposition ([?] Lemma 56). We reduce to this case in two steps.

b) From a) to the split case using a z-extension. We suppose that $\mathbf{G}$ is F-split. There exists a reductive connected F-group $\mathbf{H}$ with a simply-connected derived group $\mathbf{H}^{der}$ which is a central extension of $\mathbf{G}$ by a split F-torus ([?] Prop. 3.1, Remark 3.3 when the characteristic of F is 0; their proof is valid in positive characteristic). There exists a maximal F-split subtorus S_H of H of image S in G, and $S_H \cap H^{der}$ is a maximal F-subtorus of H^{der}. The root groups of H^{der} are equal to the root groups in H and identify with the root groups in G by the map $H \to G$ ([?] Thm. 22.6). The group H^{der} satisfies the condition of a). The image by the map $H \to G$ of a set of elements in H^{der} satisfying the proposition is a set of elements in G satisfying the proposition.

c) From the split case to the general case. By ([?] Prop. 7.2 (11)) G contains a split connected subgroup G_{nm} with the same maximal split torus S and the following properties: the system of roots of S in G_{nm} is the subset $\Phi_{nm} \subset \Phi$ of non multipliable roots, the root group in G_{nm} of $\gamma \in \Phi_{nm}$ is $U_{\gamma/2} \cap G_{nm}$ if $\gamma/2 \in \Phi$ and $U_\gamma \cap G_{nm}$ otherwise. A basis Δ of Φ gives a basis $\Delta_{nm} = \{\alpha_{nm} \mid \alpha \in \Delta\}$ of Φ_{nm}, where $\alpha_{nm} = \alpha$ if α is not multipliable and $\alpha_{nm} = 2\alpha$ otherwise. The root subgroup in G_{nm} of α_{nm} is contained in U_α for $\pm\alpha \in \Delta$. The proposition is true for G_{nm} by b). A set of elements in G_{nm} satisfying the proposition is contained in G. Applying Remark **??** the proposition is true for G.

□

3.3 The apartment

The existence of the apartment is a consequence of the existence of the discrete valuation $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ compatible with ω on the root datum $(Z, U_\alpha)_{\alpha\in\Phi}$ generating G.

Remark 3.5. A valuation is constructed for the classical groups ([?] Chapter 10), or using a Chevalley-Steinberg system when $\mathbf{G}$ is F-quasi-split ([?] 4.1.3, 4.2.2, 4.2.3). In general, $\mathbf{G}$ is quasi-split over an unramified finite Galois extension F'/F. A valuation for $\mathbf{G}(F')$ descends to G ([?] 9.1.11, 9.2.10) but not necessarily the Chevalley-Steinberg valuation ([?] 5.1.15).

We consider the homomorphism $v : Z \mapsto V$ defined by

$$\alpha(v(z)) = (\omega \circ \alpha)(z) \quad (z \in S, \alpha \in \Phi). \tag{18}$$

The maximal compact subgroup $\tilde{Z}_0$ of Z and the center of G are contained in the kernel of v. The index of the subgroup $S\tilde{Z}_0 \subset Z$ is finite.

For $\alpha \in \Phi$, φ_α is a function from $U_\alpha^* = U_\alpha - \{1\}$ to $\mathbb{R}$, satisfying properties described in ([?]. 6.2.1 (V0) to (V5)), which is compatible with ω :

$$\varphi_\alpha(z^{-1}uz) = \varphi_\alpha(u) - v(z) \quad \text{for all } \alpha \in \Phi, u \in U_\alpha^*, z \in Z, \tag{19}$$

and discrete : $\Gamma_\alpha = \varphi_\alpha(U_\alpha^*)$ is a discrete subset in $\mathbb{R}$ ([?] 6.2.21). If Δ is a basis of Φ, φ is determined by $(\varphi_\alpha)_{\alpha\in\Delta}$ ([?] 6.2.8). We have ([?] 6.2.2):

$$\Gamma_{-\alpha} = \Gamma_\alpha \quad \text{if } \alpha \in \Phi, \text{ and } \varphi_{2\alpha} = 2\varphi_\alpha|_{U_{2\alpha}^*},\ \Gamma_{2\alpha} \subset 2\Gamma_\alpha \quad \text{if } \alpha, 2\alpha \in \Phi. \tag{20}$$

For $u \in U_\alpha^*$, the elements $v'_\alpha(u), v''_\alpha(u) \in U_{-\alpha}^*$ defined in (**??**) satisfy ([**?**] 6.2.1 (V5)):

$$\varphi_{-\alpha}(v'_\alpha(u)) = \varphi_{-\alpha}(v''_\alpha(u)) = -\varphi_\alpha(u). \tag{21}$$

For $x \in V$, the family $\varphi + x = ((\varphi + x)_\alpha)_{\alpha\in\Phi}$ defined by

$$(\varphi + x)_\alpha(u) := \varphi_\alpha(u) + \alpha(x) \quad \text{for all } \alpha \in \Phi, u \in U_\alpha, \tag{22}$$

is also a discrete valuation compatible with ω ([**?**] 6.2.5). The set of discrete valuations compatible with ω on $(Z, U_\alpha)_{\alpha\in\Phi}$ is ([**?**] 5.1.23):

$$\mathfrak{A} = \{\varphi + x, \quad \text{for} \quad x \in V\}. \tag{23}$$

This is an affine euclidean real space with an action ν of N by affine automorphisms. Recalling Lemma **??**, the action ν of N is defined by: $\nu(z)$ is the translation by $-v(z)$ for $z \in Z$, and $\nu(m_\beta(v))$ is the orthogonal reflection $s_{\beta+\varphi_\beta(v)}$ with respect to the affine hyperplane :

$$H_{\beta+\varphi_\beta(v)} = \varphi + \mathrm{Ker}(\beta + \varphi_\beta(v)),$$

for $\beta \in \Phi, v \in U_\beta^*$. We have ([**?**] 6.2.7 (1)) :

$$m_\beta(v).\psi = \psi - (\beta(v) + \varphi_\beta(v))\beta^\vee \Leftrightarrow (m_\beta(v).\psi)_\alpha = \psi_{s_\beta(\alpha)} - \varphi_\beta(v)\alpha(\beta^\vee) \ (\alpha \in \Phi). \tag{24}$$

The equivalence follows from (**??**). The action ν of N on $\mathfrak{A}$

$$(\nu(n)(\psi))_\alpha(u) = \psi_{w^{-1}(\alpha)}(n^{-1}un) \quad (\alpha \in \Phi, u \in U_\alpha^*, n \in N \mapsto w \in W_0), \tag{25}$$

determines the valuation φ, and conversely. The set of hyperplanes

$$\mathfrak{H} = \{H_{\alpha+r} = \varphi + \mathrm{Ker}(\alpha + r) \mid \alpha \in \Phi_{red}, r \in \Gamma_\alpha\} \tag{26}$$

is stable under the action ν of N ([**?**] 6.2.10). By (**??**), when $\alpha, 2\alpha \in \Phi, r \in \Gamma_{2\alpha}$, we have $r/2 \in \Gamma_\alpha$ and $H_{2\alpha+r} = H_{\alpha+r/2} \in \mathfrak{H}$.

The affine space $\mathfrak{A}$ contains a valuation ψ such that $0 \in \psi(U_\alpha^*)$ for all α in the set Φ_{nm} of non-multipliable roots ([**?**] 6.2.15). We suppose, as we may, that $0 \in \Gamma_\alpha$ for all $\alpha \in \Phi_{nm}$. By (**??**),

$$0 \in \Gamma_\alpha \quad \text{for all } \alpha \in \Phi. \tag{27}$$

In particular, φ is special ([**?**] 6.2.13).

For $1 \le j \le r$, $\varphi_j = (\varphi_\alpha)_{\alpha\in\Phi_j}$ is a discrete valuation of the root datum $(Z'_j, (U_\alpha)_{\alpha\in\Phi_j})$ compatible with ω, $\mathfrak{A}$ is a product of affine euclidean real spaces $\mathfrak{A}_j = \varphi_j + V_j$, the set $\mathfrak{H}$ is the union of the sets $\mathfrak{H}_j = \{\varphi_j + \mathrm{Ker}(\alpha + r) \mid \alpha \in \Phi_{j,red}, r \in \Gamma_\alpha\}$ of affine hyperplanes in $\mathfrak{A}_j$ embedded in $\mathfrak{A}$ by

$$H_j \mapsto \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_{j-1} \times H_j \times \mathfrak{A}_{j+1} \times \ldots \times \mathfrak{A}_r \quad (H_j \in \mathfrak{H}_j, 1 \le j \le r-1). \tag{28}$$

The action ν of N on $\mathfrak{A}$ factorizes through an action ν_j of N on $\mathfrak{A}_j$ such that $\nu(n)(\psi_1, \ldots, \psi_r) = (\nu_1(n)\psi_1, \ldots, \nu_r(n)(\psi_r))$ for $(\psi_1, \ldots, \psi_r) \in \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_r$.

3.4 The affine Weyl group

Let $\mathfrak{T}$ be the set of orthogonal reflections s_H with respect to the hyperplanes $H \in \mathcal{H}$ (??) and let $W(\mathfrak{H}) \subset \nu(N)$ be the group generated by $\mathfrak{T}$. The group $W(\mathfrak{H})$ is normal in $\nu(N)$.

The group $W(\mathfrak{H})$ is an affine Weyl group associated to a reduced root system Σ of V^* ([?] VI §2.1 Prop.1 and Prop. 2, §2.5 Prop. 8), and [?] 6.2.22), and

(29)
$$\mathfrak{H} = \{H_{\beta+n} = \varphi + \mathrm{Ker}(\beta+n) \ |\beta \in \Sigma, n \in \mathbb{Z}\}.$$

The normal subgroup $\Lambda(\mathfrak{H}) \subset W(\mathfrak{H})$ of translations identifies with the $\mathbb{Z}$-module $Q(\Sigma^\vee)$ generated by the set of coroots $\Sigma^\vee$ ([?] VI.2.1 Prop. 1). We have $H_{\beta+n+1} = H_{\beta+n} - (1/2)\beta^\vee$ where $\beta^\vee$ is the coroot of β. The translation by $\beta^\vee$ is $s_{\beta+n}s_{\beta+n+1}$ where $s_{\beta+n} = s_{H_{\beta+n}}$([?] V §2.4 Prop. 5).

Two points $x, y \in \mathfrak{A}$ are called $\mathfrak{H}$ equivalent if : for all $H \in \mathfrak{H}$, either $x, y \in H$ or they are in the same connnected component of $\mathfrak{A} - H$ ([?] V §1.2, [?] 1.3). A facet $\mathfrak{F} \subset \mathfrak{A}$ is an equivalence class. A facet of $\mathfrak{F}$ is a facet contained in the closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$. A vertex is a point which is a facet. A chamber of $\mathfrak{A}$ (a connected component of $\mathfrak{A} - \cup_{H\in\mathfrak{H}} H$) is called an alcove ([?] V.1.3 Déf.2) to avoid a confusion with the chambers relatively to $\mathfrak{H}_\varphi = \{H \in \mathfrak{H} \mid \varphi \in H\}$ that we call Weyl chambers. The group $W(\mathfrak{H})$ acts simply transitively on the alcoves of $\mathfrak{A}$ ([?] VI.2.1).

We choose an alcove $\mathfrak{C} \subset \mathfrak{A}$ of vertex the special point φ. A face of $\mathfrak{C}$ is a facet of $\mathfrak{C}$ contained in a single $H \in \mathfrak{H}$, called its support. A wall of $\mathfrak{C}$ is an hyperplane $H \in \mathfrak{H}$ containing a face of $\mathfrak{C}$ ([?] V. §1.4, Déf. 3). The set

$$S(\mathfrak{C}) = \{s_H \mid H \in \mathfrak{H} \text{ wall of } \mathfrak{C}\}$$

of orthogonal reflections s_H with respect to the walls H of $\mathfrak{C}$, generates $W(\mathfrak{H})$ and $(W(\mathfrak{H}), S(\mathfrak{C}))$ is a Coxeter system.

The type of a facet $\mathfrak{F}$ of $\mathfrak{C}$ is the set

(30)
$$S_{\mathfrak{F}} = \{s_H \mid H \in \mathfrak{H} \text{ wall of } \mathfrak{C},\ \mathfrak{F} \subset H\}.$$

We have $S_{\mathfrak{C}} = \emptyset$. A facet $\mathfrak{F}$ of $\mathfrak{C}$ is determined by its type because

$$\mathfrak{F} = \{x \in \overline{\mathfrak{C}} \mid x \in H \Leftrightarrow \mathfrak{F} \subset H \quad \text{for any wall } H \text{ of } \mathfrak{C}\}.$$

The bijection between the facets of $\mathfrak{C}$ and their types reverses the inclusion:

$$\mathfrak{F}' \text{ is a facet of } \mathfrak{F} \ \Rightarrow\ S_{\mathfrak{F}} \subset S_{\mathfrak{F}'}.$$

The types of the facets of $\mathfrak{C}$ are the subsets of $S(\mathfrak{C})$ generating a finite subgroup. A facet of $\mathfrak{A}$ is the image by an element of $W(\mathfrak{H})$ of a unique facet of $\mathfrak{C}$ and we can define the type of any facet ([?] 1.3.5).

Let W_φ be the group generated by S_φ. Then (W_φ, S_φ) is a Coxeter system. As φ is a special point ([?] V §3.10), $W(\mathfrak{H})$ is a semi-direct product ([?] V.3.10 Prop. 9 and Def. 1 and [?] 1.3):

(31)
$$W(\mathfrak{H}) \simeq \Lambda(\mathfrak{H}) \rtimes W_\varphi,$$

the groups W_φ, the Weyl group of Σ, the Weyl group of Φ, and the group $W_0 = N/Z$ are isomorphic. The group W_φ, acts simply transitively on the chambers of $\mathfrak{A}$.

The set of affine roots is the subset of automorphisms of $\mathfrak{A}$

$$\Sigma^{aff} = \{\beta + n \mid \beta \in \Sigma, n \in \mathbb{Z}\}.$$

The action of the group $W(\mathfrak{H})$ on $\mathfrak{A}$ induces an action on Σ^{aff}. We have $\mathrm{Ker}(\beta+n) = \mathrm{Ker}(\beta'+n')$ if and only if $\beta'+n' \in \{\beta+n, -\beta-n\}$. We choose $x \in \mathfrak{C}$. The sign of $(\beta+n)(x)$ does not depend on the choice of x. When the sign is positive we say that the affine root $\beta+n$ is $\mathfrak{C}$-positive and we write $\beta+n>0$. The set of $\mathfrak{C}$-affine positive roots is denoted by $\Sigma^{aff,+}$. Let

$$\Delta_\Sigma^{aff}(\mathfrak{C}) = \{\beta+n \in \Sigma^{aff} \mid \mathrm{Ker}(\beta+n) \text{ is a wall of } \mathfrak{C},\ (\beta+n)(x)>0\} \tag{32}$$

The set $\Delta_\Sigma^{aff}(\mathfrak{C})$ is in bijection with $S(\mathfrak{C})$ by the map $\beta+n \mapsto s_{\beta+n}$ and with a subset $\Delta_\Sigma(\mathfrak{C})$ of Σ by the gradient map $\beta+n \mapsto \beta$. For $s \in S(\mathfrak{C})$ we denote by $\beta_s+n_s \in \Delta(\mathfrak{C})$ its antecedent. We have:

1. An affine root is $\mathfrak{C}$-positive if and only if it belongs to $\sum_{\beta+n\in\Delta(\mathfrak{C})}\mathbb{Q}_+(\beta+n)$.
2. $s(\beta_s+n_s)<0$ and $s(\beta+n)>0$ for $s\in S, \beta+n\in\Delta(\mathfrak{C}), \beta+n\neq\beta_s+n_s$,
3. $w(\beta_s+n_s) = (\beta_{s'}+n_{s'}) \Leftrightarrow wsw^{-1}=s'$ for $s,s'\in S(\mathfrak{C}), w\in W(\mathfrak{H})$.

For a facet $\mathfrak{F}$ of $\mathfrak{C}$, the set

$$\Delta_{\Sigma,\mathfrak{F}}^{aff} = \{\beta+n \in \Delta(\mathfrak{C}) \mid \mathfrak{F}\subset \mathrm{Ker}(\beta+n)\}. \tag{33}$$

is in bijection with the type $S_\mathfrak{F}$ of $\mathfrak{F}$ by the map $\beta+n\mapsto s_{\beta+n}$ and with a subset $\Delta_{\Sigma,\mathfrak{F}}$ of Σ by the gradient map. We have $\Delta_{\Sigma,\mathfrak{C}} = \Delta_{\Sigma,\mathfrak{C}}^{aff} = \emptyset$ and $\Delta_{\Sigma,\varphi} = \Delta_{\Sigma,\varphi}^{aff}$ is a basis of Σ.

The affine group $W(\mathfrak{H})$ is the direct product of the affine Weyl groups $W(\mathfrak{H}_j)$ for $1\leq j\leq r$, the reduced root systems Σ_j associated to $W(\mathfrak{H}_j)$ are the irreducible components of Σ, the facets of $\mathfrak{A}$ are products of facets of $\mathfrak{A}_j$ ([?] 6.2.12, [?] V §3.8 Prop. 6), the sets $S(\mathfrak{C})$, S_φ, $S_\mathfrak{F}$, Σ^{aff}, $\Sigma^{aff,+}$, $\Delta_\Sigma^{aff}(\mathfrak{C})$, $\Delta_\Sigma(\mathfrak{C})$, $\Delta_{\Sigma,\mathfrak{F}}^{aff}$, $\Delta_{\Sigma,\mathfrak{F}}$, are the disjoint unions of the similar sets, for $1\leq j\leq r$. This allows to reduce to the case where the root system Σ is irreducible.

By ([?] VI.1.8 Prop. 25, VI.2.3 Prop.5), the alcove $\mathfrak{C}_j$ is the set of φ_j+x for $x\in V$ satisfying

$$\gamma(x)>0 \text{ for all } \gamma\in\Delta_{\Sigma_j,\varphi_j} \text{ and } \tilde\beta_j(x)<1 \ \Leftrightarrow\ 0<\gamma(x)<1 \text{ for all } \gamma\in\Sigma_j^+. \tag{34}$$

where $\tilde\beta_j = \sum_{\gamma\in\Delta_{\Sigma_j}} n_\gamma\gamma$ is the highest root of Σ_j^+ given explicitely in the tables of Bourbaki ([?] pp. 250-275). We have

$$\Delta_{\Sigma_j}^{aff}(\mathfrak{C}_j) = \Delta_{\Sigma_j,\varphi_j}\cup\{-\tilde\beta_j+1\}.$$

Returning to Σ, we have $\Delta_\Sigma^{aff}(\mathfrak{C}) = \Delta_{\Sigma,\varphi}\cup\{-\tilde\beta_1+1,\ldots,-\tilde\beta_r+1\}$.

The vertices of $\mathfrak{C}_j$ are $\{\varphi_j, \varphi_j+n_\beta^{-1}\omega_\beta\ (\beta\in\Delta_{\Sigma_j,\varphi_j})\}$ where $\omega_{\beta^\vee}$ is a fundamental coweight ([?] VI.2.3 Cor. to Prop. 5). The vertex $\varphi+n_\beta^{-1}\omega_{\beta^\vee}$ is special if and only if $n_\beta=1$ ([?] VI §1.10, 2.2 Cor., §2.2 Prop. 3). Any set of vertices of $\mathfrak{C}_j$ is the set of vertices of a facet of $\mathcal{C}$. For $Y\subset\Delta_{\Sigma_j,\varphi_j}$, the facet $\mathfrak{F}_{\varphi_j,Y}$ of vertices $\varphi_j, (\varphi_j+n_\beta^{-1}\omega_{\beta^\vee})_{\beta\in Y}$ is the set of φ_j+x such that $\gamma(x)=0$ for $\gamma\in\Delta_{\Sigma_j}-Y$, and $0<\beta(x)<1$ for $\beta\in Y$. The facet $S_{\mathfrak{F}_Y}$ of vertices $(\varphi_j+n_\beta^{-1}\omega_{\beta^\vee})_{\beta\in Y}$ is the set of φ_j+x such that $\gamma(x)=0$ for $\gamma\in\Delta_{\Sigma_j}-Y$, $\tilde\beta_j(x)=1$ and $0<\beta(x)<1$ for $\beta\in Y$. We have

$$\Delta_{\Sigma_j,\mathfrak{F}_{\varphi_j,Y}}^{aff} = \Delta_{\Sigma_j,\varphi_j}-Y, \quad \Delta_{\Sigma_j,\mathfrak{F}_Y}^{aff} = (\Delta_{\Sigma_j,\varphi_j}-Y)\cup\{-\tilde\beta_j+1\}.$$

Lemma 3.6. *The translation by $v_j\in V_j$ stabilises $\mathfrak{H}_j$ if and only if $\gamma(v_j)\in\mathbb{Z}$ for all $\gamma\in\Sigma_j$. The translation by v_j normalizes $\mathfrak{C}_j$ if and only if $v_j=0$.*

Proof. $\gamma(x)+k=0$ is equivalent to $\gamma(x+v_j)+k-\gamma(v_j)=0$ and for $r\in\mathbb{R}$, $\mathrm{Ker}\,\gamma+r\in\mathfrak{H}_j$ if and only if $r\in\mathbb{Z}$. The image of $\mathfrak{C}_j$ by $\gamma\in\Delta_{\Sigma_j}$ is an interval $]a,b[$. The image of $\mathfrak{C}_j+v_j$ by γ is the interval $]a+\gamma(v_j), b+\gamma(v_j)[$. If $\mathfrak{C}_j+v_j=\mathfrak{C}_j$, we have $\gamma(v_j)=0$ for all $\gamma\in\Delta_{\Sigma_j}$, hence $v_j=0$. □

3.5 The filtration of U_α

The properties ([?] 6.2.1) of the valuation φ imply that, for $\alpha \in \Phi$ and $r \in \mathbb{R}$, the set

$$U_{\alpha,r} = \{u \in U_\alpha \mid \varphi_\alpha(u) \geq r\} \tag{35}$$

is a compact open subgroup of U_α, and $(U_{\alpha,r})_{r\in\Gamma_\alpha}$ is a strictly decreasing filtration of union U_α and trivial intersection. For $\alpha \in \Phi, r \in \Gamma_\alpha$, let U_{α,r_+} be the group $U_{\alpha,r'}$ for $r' \in \Gamma_\alpha, r' > r$, and r' minimal for these properties, and let

$$U_{\alpha,r,k} = U_{\alpha,r}/U_{\alpha,r_+}.$$

When the root system is not reduced, we make the following observation:

Lemma 3.7. *For $\alpha, 2\alpha \in \Phi, r \in (1/2)\Gamma_{2\alpha} \subset \Gamma_\alpha$, the sequence*

$$1 \to U_{2\alpha,2r,k} \to U_{\alpha,r,k} \to U_{\alpha,r}/U_{\alpha,r_+}U_{2\alpha,2r} \to 1$$

is exact.

Proof. We have to show that $U_{2\alpha,2r} \cap U_{\alpha,r_+} = U_{2\alpha,(2r)_+}$. By (**??**), the left hand side is $U_{2\alpha,2r'}$ where r' is the smallest element of Γ_α with $r' > r$, and for $r'' \in \Gamma_{2\alpha}$, the strict inequality $2r < r''$ is equivalent to the inequality $2r' \leq r''$. The right hand side is $U_{2\alpha,r''}$ and $U_{2\alpha,2r'} = U_{2\alpha,r''} = U_{2\alpha,(2r)_+}$. □

For $\alpha, 2\alpha \in \Phi$, we introduce the set

$$\Gamma'_\alpha = \Gamma_\alpha - \{r \in (1/2)\Gamma_{2\alpha} \mid U_{\alpha,r} = U_{\alpha,r_+}U_{2\alpha,2r}\} = \{r \in \Gamma_\alpha \mid U_{2\alpha,2r,k} \neq U_{\alpha,r,k}\}. \tag{36}$$

This set is never empty ([?] 4.2.21). When $\alpha \in \Phi, 2\alpha \notin \Phi$ we put $\Gamma'_\alpha = \Gamma_\alpha$. We have a bijection

$$\cup_{\alpha\in\Phi_{red}}\alpha + \Gamma_\alpha \to \Phi^{aff} = \cup_{\alpha\in\Phi}\alpha + \Gamma'_\alpha \tag{37}$$

sending $\alpha + r$ to $2\alpha + 2r$ if $r \notin \Gamma'_\alpha$, and to $\alpha + r$ otherwise.

For $\alpha \in \Phi, r \in \Gamma_\alpha$, we say that $\alpha + r$ is $\mathfrak{C}$-positive when $\alpha(x) + r > 0$ for $x \in \mathfrak{C}$. The bijection (**??**) respects $\mathfrak{C}$-positivity.

For $\alpha \in \Phi$, there exists a unique positive number $e_\alpha > 0$ such that the map

$$\alpha + r \mapsto e_\alpha(\alpha + r) : \cup_{\alpha\in\Phi}\alpha + \Gamma_\alpha \to \Sigma^{aff} \tag{38}$$

is surjective, and restricts to a bijection from the $\mathfrak{C}$-positive elements of $\cup_{\alpha\in\Phi_{red}}\alpha + \Gamma_\alpha$ onto $\Sigma^{aff,+}$. Let

$$\Delta^{aff}_\Phi(\mathfrak{C}),\ \Delta_\Phi(\mathfrak{C}),\ \Delta^{aff}_{\Phi,\mathfrak{F}},\ \Delta_{\Phi,\mathfrak{F}} \quad \text{and} \quad \Delta'^{aff}_\Phi(\mathfrak{C}),\ \Delta'_\Phi(\mathfrak{C}),\ \Delta'^{aff}_{\Phi,\mathfrak{F}},\ \Delta'_{\Phi,\mathfrak{F}} \tag{39}$$

be the sets of $\mathfrak{C}$-positive elements of $\cup_{\alpha\in\Phi_{red}}\alpha + \Gamma'_\alpha$ and of Φ^{aff} in bijection with (**??**), (**??**) via the bijections induced by (**??**) and (**??**). The set $\Delta_{\Phi,\varphi} = \Delta^{aff}_{\Phi,\varphi}$ is a basis of Φ.

It is obvious that $e_\alpha = e_{-\alpha}$. With (**??**),

$$\Gamma_\alpha = \gamma_\alpha\mathbb{Z} \quad \gamma_\alpha = e_\alpha^{-1}, \quad \text{if } \alpha \in \Phi_{red}. \tag{40}$$

When $\alpha, 2\alpha \in \Phi$, we have $e_{2\alpha} = (1/2)e_\alpha$, $\Gamma_{2\alpha}$ is a group because $0 \in \Gamma_0$ (**??**) ([?] Cor. 6.2.16), there exists a unique positive integer $f_\alpha \in \mathbb{N}_{>0}$ such that

$$\Gamma_{2\alpha} = \gamma_{2\alpha}\mathbb{Z} \quad \gamma_{2\alpha} = 2f_\alpha e_\alpha^{-1}, \quad \text{if } \alpha, 2\alpha \in \Phi. \tag{41}$$

Lemma 3.8. *e_α is a positive integer for all $\alpha \in \Phi$, which is divisible by $2f_\alpha$ if $2\alpha \in \Phi$.*

Proof. By the proof of [?] Lemma **I**.2.10, Γ_α contains $n_\alpha^{-1}\mathbb{Z}$ where $n_\alpha \in \mathbb{N}_{>0}$ for any $\alpha \in \Phi$. □

3.6 The adjoint building

For $x \in V$ and $\alpha \in \Phi$,

$$\text{the smallest element } r_x(\alpha) \in \Gamma_\alpha \text{ such that } \alpha(x) + r_x(\alpha) \geq 0, \tag{42}$$

depends only on the facet $\mathfrak{F}$ containing $\varphi + x$, and is also denoted by $r_{\mathfrak{F}}(\alpha)$. Let $\beta = e_\alpha \alpha \in \Sigma$. When $\alpha \in \Phi_{red}$, the smallest integer $r_x(\beta) \in \mathbb{Z}$ such that $\beta(x) + r_x(\beta) \geq 0$ is $e_\alpha r_{\mathfrak{F}}(\alpha)$.

Example 3.9. *$r_\varphi(\alpha) = 0$ for all $\alpha \in \Phi$.*
$r_{\mathfrak{C}}(\alpha) = 0$ if $\alpha \in \Phi$ is $\mathfrak{C}$-positive,
$r_{\mathfrak{C}}(\alpha) = e_\alpha^{-1}$ if $\alpha \in \Phi_{red}$ is $\mathfrak{C}$-negative,
$r_{\mathfrak{C}}(2\alpha) = 2f_\alpha e_\alpha^{-1}$ if $\alpha, 2\alpha \in \Phi$ are $\mathfrak{C}$-negative.

Example 3.10. *Let $\mathfrak{F}$ be a facet of $\mathfrak{C}$ contained in the wall $H_{\alpha+r}, r \in \Gamma_\alpha$, of $\mathfrak{C}$. Then $r_{\mathfrak{F}}(\alpha) = -r_{\mathfrak{F}}(-\alpha) = r$.*

Let U_x be the group generated by $\cup_{\alpha \in \Phi} U_{\alpha, r_x(\alpha)}$ and let N_x be the fixator of $\varphi + x$ in N. The group N_x normalizes U_x, and

$$P_x = N_x U_x \tag{43}$$

is a group (denoted by $\hat{P}_x$ iin [?] 7.1.8). These groups depending only on the facet $\mathfrak{F}$ containing $\varphi + x$, are also denoted by $U_{\mathfrak{F}}, N_{\mathfrak{F}}, P_{\mathfrak{F}}$. For $\alpha \in \Phi$ we have

$$P_{\mathfrak{F}} \cap U_\alpha = U_{\alpha, r_{\mathfrak{F}}(\alpha)}. \tag{44}$$

This is clear if α is not multipliable. If $\alpha, 2\alpha \in \Phi$ this is true because $U_{2\alpha, r_{\mathfrak{F}}(2\alpha)} = U_{\alpha, r_{\mathfrak{F}}(2\alpha)/2} \cap U_{2\alpha}$ is contained in $U_{\alpha, r_{\mathfrak{F}}(\alpha)}$ as $r_{\mathfrak{F}}(2\alpha)$ is the smallest element of $\Gamma_{2\alpha}$ satisfying $r_{\mathfrak{F}}(2\alpha) \geq 2r_{\mathfrak{F}}(\alpha)$ by (**??**) ([?] 7.4.1).

Definition 3.11. *The adjoint building is*

$$\mathfrak{B}(G_{ad}) := G \times \mathfrak{A} / \sim \tag{45}$$

where $\sim$ is the equivalence relation on $G \times \mathfrak{A}$ defined by

$$(g, \varphi + x) \sim (h, \varphi + y) \Leftrightarrow \textit{there exists } n \in N \mid \varphi + y = n.(\varphi + x) \textit{ and } g^{-1}hn \in P_x,$$

with the natural action of G, induced by $(g, (h, \psi)) \mapsto (gh, \psi)$ for $g, h \in G, \psi \in \mathfrak{A}$.

The map $\psi \mapsto (1, \psi) : \mathfrak{A} \to \mathfrak{B}(G_{ad})$ is an N-equivariant embedding. The facets, resp. alcoves, of $\mathfrak{B}(G_{ad})$ are the images by G of the facets, resp. alcoves, of $\mathfrak{A}$. The G-orbit of a facet contains a unique facet of the alcove $\mathfrak{C}$.

The group P_x is obviously the G-fixator of $(1, \varphi + x)$. The G-fixator $P_{\mathfrak{F}}$ of a facet $\mathfrak{F}$ is the intersection of the G-fixators of its vertices. We denote by U^+, resp. $U^+_{\mathfrak{F}}$ the subgroup of G generated by U_α, resp. $U_{\alpha, r_{\mathfrak{F}}(\alpha)}$, for $\alpha \in \Phi^+_{red}$. The product maps

$$\prod_{\alpha \in \Phi^+_{red}} U_\alpha \to U^+, \qquad \prod_{\alpha \in \Phi^+_{red}} U_{\alpha, r_{\mathfrak{F}}(\alpha)} \to U^+_{\mathfrak{F}} \tag{46}$$

are homeomorphisms ([?] 6.1.6, 6.4.9), whatever ordering we choose on Φ^+_{red}. We have a similar result for $\Phi^-_{red} = -\Phi^+_{red}, U^-_{\mathfrak{F}}, U^-$.

3.7 Parahoric subgroups

We denote by F^s a maximal separable extension of F, by F^{unr} the maximal unramified extension of F contained in F^s, by $\mathcal{I} = \mathrm{Gal}(F^s/F^{unr})$ the inertia group and by $\sigma \in \mathrm{Gal}(F^{unr}/F)$ the Frobenius automorphism. Let $Z(\hat{G})$ be the center of the Langlands dual group of $\hat{G}$ with the natural action of $\mathrm{Gal}(F^s/F)$, also called the Borovoi algebraic fundamental group of $\mathbf{G}$.

Kottwitz ([**?**] 7.1 to 7.4) defined a functorial surjection

$$\kappa_G : G \to X^*(Z(\hat{G}))^{\sigma}_{\mathcal{I}}. \tag{47}$$

vanishing on the unipotent subgroups U_α for $\alpha \in \Phi$.

Definition 3.12. *A parahoric subgroup of G is the fixator $K_{\mathfrak{F}} = \ker \kappa_G \cap P_{\mathfrak{F}}$ in the kernel of κ_G of a facet $\mathfrak{F}$ of the building $\mathcal{B}(G_{ad})$.*

A pro-p-parahoric subgroup $K_{\mathfrak{F}}(1)$ of G is the pro-p-radical of a parahoric subgroup $K_{\mathfrak{F}}$ of G.

An Iwahori, resp. pro-p-Iwahori, subgroup of G is the parahoric, resp. pro-p-parahoric, subgroup fixing an alcove.

This definition of parahoric subgroup by Haines and Rapoport [**?**], coincides with the definition by Bruhat and Tits, denoted by $\mathfrak{G}^0_{\mathfrak{F}}(\mathcal{O}^\natural)$ in [**?**].

The pro-p-radical $K_{\mathfrak{F}}(1)$ of a parahoric group $K_{\mathfrak{F}}$ is the largest open normal pro-p-subgroup ([**?**] 3.6). The quotient $K_{\mathfrak{F},k} = K_{\mathfrak{F}}/K_{\mathfrak{F}}(1)$ is the group of k-points of a connected reductive group over the residue field k of F.

A parahoric subgroup of G is G-conjugate to a parahoric subgroup fixing a facet of the alcove $\mathfrak{C}$ of $\mathfrak{A}$. *The Iwahori, resp. pro-p-Iwahori, sugroups of G are conjugate.*

From now on, $\mathfrak{F}$ is a facet of $\mathfrak{C}$, I is the Iwahori subgroup fixing $\mathfrak{C}$, and positive means $\mathfrak{C}$-positive.

The group Z admits a unique parahoric subgroup Z_0, which is the kernel of the Kottwitz morphism κ_Z ([**?**] 4.1.1). The group Z_0 is a subgroup of finite index of the maximal compact subgroup $\tilde{Z}_0$ of Z. The group N normalizes $Z, Z_0, Z_0(1)$, and the subgroup $Z_0^{(p)}$ of elements of Z_0 of finite order prime to p. The quotient $Z_k = Z_{0,k} = Z_0/Z_0(1)$ is the group of points over k of a torus (non necessarily split). The quotient map $Z_0 \to Z_k$ restricted to $Z_0^{(p)}$ is an isomorphism, Z_0 is a semi-direct product

$$Z_0 = Z_0(1) \rtimes Z_0^{(p)} \simeq Z_0(1) \rtimes Z_k. \tag{48}$$

The group $\Lambda = Z/Z_0$ is finitely generated and commutative, of torsion subgroup $\tilde{Z}_0/Z_0$. We have $Z_0 = \tilde{Z}_0$ when $Z = S$ is a split torus or when G is unramified, or semi-simple and simply connected ([**?**] Section 11). The group $\Lambda(1) = Z/Z_0(1)$ is finitely generated, of torsion subgroup $\tilde{Z}_0/Z_0(1)$ and may be not commutative.

The same considerations apply to the split torus S.

Proposition 3.13. $Z \cap K_{\mathfrak{F}} = Z_0$.

Proof. ([**?**] Lemma 4.2.1). □

The group $U_{\mathfrak{F}}$ generated by $U^+_{\mathfrak{F}} \cup U^-_{\mathfrak{F}}$ (**??**) is normalized by $\tilde{Z}_0$. The unipotent groups $U_\alpha, \alpha \in \Phi$, beeing contained in $\mathrm{Ker}\,\kappa_G$, we deduce from (**??**)

$$K_{\mathfrak{F}} \cap U_\alpha = U_{\alpha, r_{\mathfrak{F}}(\alpha)} \text{ for } \alpha \in \Phi, \quad K_{\mathfrak{F}} \cap U^+ = U^+_{\mathfrak{F}}, \quad K_{\mathfrak{F}} \cap U^- = U^-_{\mathfrak{F}}. \tag{49}$$

Proposition 3.14. ([?] 5.2.4) $K_{\mathfrak{F}} = Z_0 U_{\mathfrak{F}} = U_{\mathfrak{F}}^- U_{\mathfrak{F}}^+ U_{\mathfrak{F}}^- Z_0 = U_{\mathfrak{F}}^- U_{\mathfrak{F}}^+ (N \cap K_{\mathfrak{F}})$.

For $\alpha \in \Phi$, we denote $r_{\mathfrak{F}}^*(\alpha) = r_{\mathfrak{F}}(\alpha)_+$ is α is constant on $\mathfrak{F}$, $\alpha(\mathfrak{F}) = -r_{\mathfrak{F}}(\alpha) \in \Gamma_\alpha$, and $r_{\mathfrak{F}}^*(\alpha) = r_{\mathfrak{F}}(\alpha)$ otherwise. We have ([?] Lemma I.2.1):

(50) $$K_{\mathfrak{F}}(1) \cap U_\alpha = U_{\alpha, r_{\mathfrak{F}}^*(\alpha)}.$$

Remark 3.15. *When $\alpha, 2\alpha \in \Phi$ and $2r_{\mathfrak{F}}(\alpha) \in \Gamma_{2\alpha}$ we have $2r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(2\alpha)$, and $r_{\mathfrak{F}}^*(2\alpha) = r_{\mathfrak{F}}(2\alpha)_+$.*

We denote
$$K_{\mathfrak{F}}(1) \cap U^+ = U_{\mathfrak{F}}^+(1), \quad K_{\mathfrak{F}}(1) \cap U^- = U_{\mathfrak{F}}^-(1).$$
As in (**??**), the product map

(51) $$\prod_{\alpha \in \Phi_{red}^+} U_{\alpha, r_{\mathfrak{F}}^*(\alpha)} \to U_{\mathfrak{F}}^+(1),$$

is an homeomorphism whatever ordering we choose on Φ_{red}^+ ([?] 5.2.3). We have a similar result for $U_{\mathfrak{F}}^-(1)$.

Proposition 3.16. *(Iwahori decomposition) $K_{\mathfrak{F}}(1) = U_{\mathfrak{F}}^+(1) Z_0(1) U_{\mathfrak{F}}^-(1)$ and the factors commute.*

Proof. ([?] Prop. I.2.2). □

Corollary 3.17. *(Iwahori decomposition) The Iwahori group $I = I(1)Z_0$ admits the Iwahori decomposition $I = I^- Z_0 I^+ = I^+ Z_0 I^-$, the factors commute, and the product maps*
$$\prod_{\alpha \in \Phi_{red}^+} U_{\alpha,0} \to I^+, \quad \prod_{\alpha \in \Phi_{red}^+} U_{\alpha, e_\alpha^{-1}} \to I^-$$
are homeomorphisms.

Proof. Example **??**. □

Corollary 3.18. *The map $\mathfrak{F} \mapsto K_{\mathfrak{F}}$ is decreasing and the map $\mathfrak{F} \mapsto K_{\mathfrak{F}}(1)$ is increasing:*
$$K_{\mathfrak{F}}(1) \subset K_{\mathfrak{F}'}(1) \subset K_{\mathfrak{F}'} \subset K_{\mathfrak{F}},$$
if $\mathfrak{F}$ is a facet of a facet $\mathfrak{F}'$.

Proof. If $\mathfrak{F}$ is a facet of $\mathfrak{F}'$, the inclusions $K_{\mathfrak{F}'}(1) \subset K_{\mathfrak{F}'} \subset K_{\mathfrak{F}}$ are clear. The inclusion $K_{\mathfrak{F}}(1) \subset K_{\mathfrak{F}'}(1)$ follows from (**??**). □

For $\alpha \in \Phi$ and $r \in \Gamma_\alpha$, let $U_{\alpha,r}^* = U_{\alpha,r} - U_{\alpha,r_+}$.

Lemma 3.19. *Let $\alpha \in \Phi$ constant on $\mathfrak{F}$, $-\alpha(\mathfrak{F}) \in \Gamma_\alpha$. We have*
$$m_\alpha(U_{\alpha, r_{\mathfrak{F}}(\alpha)}^*) \subset K_{\mathfrak{F}} - K_{\mathfrak{F}}(1).$$

Proof. Let $u \in U_{\alpha, r_{\mathfrak{F}}(\alpha)}^*$. Then $m_\alpha(u) = v'_\alpha(u)^{-1} u v''_\alpha(u)^{-1}$ with $v'_\alpha(u), v''_\alpha(u) \in U_{-\alpha, -r_{\mathfrak{F}}(\alpha)}$ by (**??**) and (**??**). By (**??**), $m_\alpha(u) \in K_{\mathfrak{F}}$ because $r_{\mathfrak{F}}(-\alpha) = -r_{\mathfrak{F}}(\alpha)$ by Example **??**. The image of $m_\alpha(u)$ in $K_{\mathfrak{F},k}$ is not trivial because $r_{\mathfrak{F}}^*(\alpha) = r_{\mathfrak{F}}(\alpha)_+ \neq r_{\mathfrak{F}}(\alpha)$. □

3.8 Finite quotients of parahoric groups

For $H \in \mathfrak{H}$, the subset $\Phi'_H \subset \Phi$ of α such that $H = \mathrm{Ker}(\alpha + r)$ for $r \in \Gamma'_\alpha$ is never empty. Let

$$\Phi'_{\mathfrak{F}} = \cup_{\mathfrak{F} \subset H \in \mathfrak{H}} \Phi'_H.$$

We recall the set $\Delta'_{\Phi,\mathfrak{F}}$ defined in (**??**).

Proposition 3.20. *The torus S_k is a maximal k-split torus of $K_{\mathfrak{F},k}$, the root system of $K_{\mathfrak{F},k}$ with respect to S_k is $\Phi'_{\mathfrak{F}}$. The set $\Delta'_{\Phi,\mathfrak{F}}$ is a basis of $\Phi'_{\mathfrak{F}}$. The root subgroup associated to $\alpha \in \Phi'_{\mathfrak{F}}$ is*

$$U_{\mathfrak{F},k,\alpha} = U_{\alpha,r_{\mathfrak{F}}}/U_{\alpha,r^*_{\mathfrak{F}}(\alpha)} = U_{\alpha,r_{\mathfrak{F}}(\alpha)}/U_{\alpha,r_{\mathfrak{F}}(\alpha)+}.$$

Proof. ([**?**] 5.1.31). □

Remark 3.21. *When $\alpha, 2\alpha \in \Phi$, if 2α belongs to $\Phi'_{\mathfrak{F}}$ but not α, we have*

$$U_{\mathfrak{F},k,2\alpha} = U_{\alpha,r_{\mathfrak{F}}(\alpha)}/U_{\alpha,r_{\mathfrak{F}}(\alpha)+},$$

because $2r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(2\alpha)$ and $U_{\alpha,r_{\mathfrak{F}}(\alpha)} = U_{\alpha,r_{\mathfrak{F}}(\alpha)+}U_{2\alpha,r_{\mathfrak{F}}(2\alpha)}$, by (**??**) *and* (**??**).

A minimal parabolic subgroup of $K_{\mathfrak{F},k}$ is $B_{\mathfrak{F},k} = Z_k U^+_{\mathfrak{F},k}$ of unipotent radical $U^+_{\mathfrak{F},k} = \prod_{\alpha \in \Phi'^+_{\mathfrak{F}}} U_{\alpha,\mathfrak{F},k}$ whatever ordering we choose on the set $\Phi'^+_{\mathfrak{F}}$ of positive roots of $\Phi'_{\mathfrak{F}}$. Let $N_{\mathfrak{F},k}$ be the subgroup of $K_{\mathfrak{F},k}$ generated by Z_k and $m_\alpha(u_k)$ for $\alpha \in \Phi'_{\mathfrak{F}}, u_k \in U^*_{\alpha,\mathfrak{F},k}$, and let $s_{\alpha,k} \in N_{\mathfrak{F},k}/Z_k$ and $s_\alpha(u_k) \in N_{\mathfrak{F},k}$ be the images of $m_\alpha(u_k)$.

Proposition 3.22. ([**?**] 5.2 Lemma, [**?**] 2.20) *The finite group $K_{\mathfrak{F},k}$ is a strongly split BN-pair of characteristic p with $B = B_{\mathfrak{F},k},\ N = N_{\mathfrak{F},k},\ S = \{s_{\alpha,k} \mid \alpha \in \Delta'_{\Phi,\mathfrak{F}}\}$.*

A parabolic subgroup of $K_{\mathfrak{F},k}$ containing $B_{\mathfrak{F},k}$ is called standard. If $\mathfrak{F}$ is a facet of a facet $\mathfrak{F}'$, we have $\Phi'_{\mathfrak{F}'} \subset \Phi'_{\mathfrak{F}}$.

Proposition 3.23. *If $\mathfrak{F}$ is a facet of a facet $\mathfrak{F}'$, the subgroup $M_{\mathfrak{F},k,\mathfrak{F}'}$ of $K_{\mathfrak{F},k}$ generated by Z_k and $U_{\alpha,,k}$ for $\alpha \in \Phi'_{\mathfrak{F}'}$ is the Levi subgroup of a standard parabolic subgroup $Q_{\mathfrak{F},k,\mathfrak{F}'} = M_{\mathfrak{F},k,\mathfrak{F}'}B_{\mathfrak{F},k}$.*

The parahoric subgroup $K_{\mathfrak{F}'}$ is the inverse image of $Q_{\mathfrak{F},k,\mathfrak{F}'}$ in $K_{\mathfrak{F}}$, the pro-p-parahoric subgroup $K_{\mathfrak{F}'}(1)$ is the inverse image of the unipotent radical of $Q_{\mathfrak{F},k,\mathfrak{F}'}$, and $K_{\mathfrak{F}',k} \simeq M_{\mathfrak{F},k,\mathfrak{F}'}$.

Proof. ([**?**] 4.6.33 and 5.1.32). □

Corollary 3.24. *The reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ induces isomorphisms*

$$K_{\mathfrak{F}'}\backslash K_{\mathfrak{F}}/K_{\mathfrak{F}'} \simeq Q_{\mathfrak{F},k,\mathfrak{F}'}\backslash K_{\mathfrak{F}}/Q_{\mathfrak{F},k,\mathfrak{F}'}, \quad K_{\mathfrak{F}'}(1)\backslash K_{\mathfrak{F}}/K_{\mathfrak{F}'}(1) \simeq U^+_{\mathfrak{F},k,\mathfrak{F}'}\backslash K_{\mathfrak{F}}/U^+_{\mathfrak{F},k,\mathfrak{F}'} \tag{52}$$

Corollary 3.25. *The pro-p-Iwahori subgroup $I(1)$ and the Iwahori subgroup I are the inverse images in $K_{\mathfrak{F}}$ of $U^+_{\mathfrak{F},k}$ and of $B_{\mathfrak{F},k}$.*

For $\alpha \in \Phi'_{\mathfrak{F}}$, the parahoric subgroup $K_{\mathfrak{F}}$ contains $U_{\alpha,r_{\mathfrak{F}}(\alpha)}$ (**??**). By Lemma **??**, the reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ induces an isomorphism

$$u \mapsto u_k : U^*_{\alpha,r_{\mathfrak{F}}(\alpha)} \to U^*_{\alpha,\mathfrak{F},k} = U_{\alpha,\mathfrak{F},k} - \{1\}.$$

and sends $m_\alpha(u) \in K_{\mathfrak{F}}$ to $m_\alpha(u_k) \in K_{\mathfrak{F},k}$. We denote by $s_\alpha \in N/Z = W_0$, $s_{\alpha+r_{\mathfrak{F}}(\alpha)} \in N/Z_0, s_\alpha(u) \in N/Z_0(1)$ the images of $m_\alpha(u)$. For $u, u' \in U^*_{\alpha,r_{\mathfrak{F}}(\alpha)}$ we have $m_\alpha(u')^{-1}m_\alpha(u) \in Z \cap K_{\mathfrak{F}}$ and $Z \cap K_{\mathfrak{F}} = Z \cap I = Z_0$ by Prop. **??**. We have

$$B_{\mathfrak{F},k}m_\alpha(u_k)B_{\mathfrak{F},k}/B_{\mathfrak{F},k} \simeq U^+_{\mathfrak{F},k}m_\alpha(u_k)U^+_{\mathfrak{F},k}/U^+_{\mathfrak{F},k} \simeq U_{\mathfrak{F},k,\alpha}.$$

Corollary 3.26. *For a facet $\mathfrak{F}$, $\alpha \in \Phi'_{\mathfrak{F}}$ and $u \in U^*_{\alpha,\mathfrak{F}}$, we have*

$$Im_\alpha(u)I/I \simeq U_{\mathfrak{F},k,\alpha} \simeq I(1)m_\alpha(u)I(1)/I(1).$$

For $s \in S(\mathfrak{C})$, and $\mathfrak{F}_s$ the face contained in the wall H_s fixed by s, $\Phi'_{\mathfrak{F}_s}$ has a single element α_s. The number of elements of $Im_{\alpha_s}(u)I/I$, resp. $U_{\mathfrak{F}_s,k,\alpha_s}$, is denoted by q_s, resp. q_{s_k}. Cor. **??** implies:

Corollary 3.27. *For $s \in S(\mathfrak{C})$, $q_s = q_{s_k} = |I(1)m_{\alpha_s}(u)I(1)/I(1)|$.*

3.9 The Iwahori Weyl groups

Let $G^{aff} \subset G$ be the subgroup generated by the parahoric subgroups. The group G^{aff} is generated by $Z_0 \cup U^+ \cup U^-$ (Prop. **??**), and is a normal subgroup of G. We have $G = G^{aff}Z$. Set $N^{aff} = N \cap G^{aff}, Z^{aff} = Z \cap G^{aff}$.

Definition 3.28. *We call*

$$W_0 = N/Z,\ W^{aff} = N^{aff}/Z_0,\ W = N/Z_0,\ W^{aff}(1) = N^{aff}/Z_0(1),\ W(1) = N/Z_0(1)$$

the finite, affine, Iwahori, pro-p-affine, pro-p-Iwahori, Weyl groups of G.

The action of N^{aff} on $\mathfrak{A}$ identifies $W(\mathfrak{H})$ with W^{aff} and $S(\mathfrak{C})$ with a set denoted by S^{aff}. One of the most important application from the theory of Bruhat-Tits is ([**?**] 5.2.12) and ([**?**] 6.5, 5.1.3):

Theorem 3.29. *(G^{aff}, I, N^{aff}) is a double Tits system and the inclusion $G^{aff} \subset G$ is $I - N^{aff}$-adapted of connected type.*

To be a double Tits system means ([**?**] 1.2.6, 5.1.1) that (G^{aff}, I, N^{aff}) and (G^{aff}, B^{aff}, N) are Tits systems with affine Weyl group W^{aff} and finite Weyl group W_0 respectively.

Proposition 3.30. Bruhat Decompositions for G^{aff} ([**?**] 1.2.7). *We have*

$$G^{aff} = B^{aff}N^{aff}B^{aff} = IN^{aff}I = I(1)N^{aff}I(1).$$

The maps $n \mapsto B^{aff}nB^{aff}, n \mapsto InI, n \mapsto I(1)nI(1)$ induce bijections

$$W_0 \simeq B^{aff}\backslash G^{aff}/B^{aff}, \quad W^{aff} \simeq I\backslash G^{aff}/I, \quad W^{aff}(1) \simeq I(1)\backslash G^{aff}/I(1).$$

We have a similar Bruhat decomposition for G. Let $B = ZU^+$.

Proposition 3.31. Bruhat Decompositions for G. *We have*

$$G = BNB = INI = I(1)NI(1).$$

The maps $n \mapsto BnB, n \mapsto InI, n \mapsto I(1)nI(1)$ induce bijections

$$W_0 \simeq B\backslash G/B,\ W \simeq I\backslash G/I,\ W(1) \simeq I(1)\backslash G/I(1),\ InI/I \simeq I(1)nI(1)/I(1).$$

Proof. For B and I, the equalities and the isomorphism with W_0 follow from $G = G^{aff}Z, N = N^{aff}Z, B = B^{aff}Z$ and from Prop. **??**. For the isomorphism with W_0 see also ([**?**] 5.1.32). The isomorphism with W follows from [**?**] 4.2.2 (iii) where W, Z_0 are denoted by $\hat{W}, H$.

We deduce $G = I(1)NI(1)$ from $G = INI$, $I = Z_0I(1)$ and $Z_0 \subset N$. We have $InI/I \simeq I(1)nI(1)/I(1)$ because

$$I/(I \cap nIn^{-1}) = I(1)Z_0/(I(1) \cap nI(1)n^{-1})Z_0 \simeq I(1)/(I(1) \cap nI(1)n^{-1}).$$

If $I(1)nI(1) = I(1)n'I(1)$ we have $InI = In'I$ and the images w, w' of n, n' in W are equal. As $I = Z_0I(1)$, the double coset $InI = I(1)nZ_0I(1)$ is a disjoint union of $I(1)nzI(1)$ for $z \in Z_0/Z_0(1) = Z_k$. This implies that the images of n, n' in $W(1)$ are equal. □

The group G is the disjoint union

$$G = \sqcup_{w \in W_0} U^- Z n_w U^+, \quad n_w \in N^{aff} \text{ above } w,$$

by the Bruhat decomposition and the fact that U^- is conjugate to U^+ by the longest element of W_0. The Iwahori decomposition of I implies that

$$N \cap I = Z_0. \tag{53}$$

Let $N_{\mathfrak{C}}$ be the N-normalizer of the alcove $\mathfrak{C}$ and Ω, resp. $\Omega(1)$, be its the image in W, resp. $W(1)$. We have

$$N_{\mathfrak{C}} \cap N^{aff} = Z_0, \quad N = N^{aff} N_{\mathfrak{C}}, \quad N_{\mathfrak{C}} \cap Z = \operatorname{Ker} v, \tag{54}$$

because W^{aff} acts simply transitively on the set of alcoves, a translation normalizing $\mathfrak{C}$ is trivial (Lemma **??**), and $z \in Z$ acts on $\mathfrak{A}$ by translation by $-v(z)$ (**??**). The groups

$$G/G^{aff} \simeq Z/Z^{aff} \simeq N/N^{aff} \simeq W/W^{aff} \simeq N_{\mathfrak{C}}/Z_0 \tag{55}$$

are commutative because Z/Z_0 is commutative ([**?**]). The G-normalizer of $\mathfrak{C}$ is $G_{\mathfrak{C}} = N_{\mathfrak{C}} I$ ([**?**] page 105-106), and the G^{aff}-normalizer $G_{\mathfrak{C}} \cap G^{aff}$ of $\mathfrak{C}$ is equal to his fixator by (**??**):

$$G_{\mathfrak{C}} \cap G^{aff} = I. \tag{56}$$

Remark 3.32. ([**?**] Lemma 17, [**?**] Lemma 4.3) *The kernel of the Kottwitz morphism κ_G is G^{aff}.*

The subgroup $W^{aff} \subset W$ is normal. From (**??**), the morphism $W \to W/W^{aff}$ splits, and we have

$$W = W^{aff} \rtimes \Omega, \quad W(1) = W^{aff}(1)\Omega(1), \quad W^{aff}(1) \cap \Omega(1) = Z_k. \tag{57}$$

The extension $W(1) \to W$ of kernel Z_k does not split in general [**?**].

The set S^{aff} is invariant by conjugation by Ω, hence the length ℓ of the Coxeter group (W^{aff}, S^{aff}) is invariant by conjugation by Ω. The length extends to a map on W and $W(1)$, still called a length and denoted by ℓ :

$$\ell(w(1)) = \ell(w) = \ell(w'), \tag{58}$$

for $w(1) \in W(1)$ of image $w \in W$ and $w' \in W^{aff}$ such that $w = w'u, u \in \Omega$. The set of elements of length 0 in W, resp. $W(1)$, is Ω, resp. $\Omega(1)$.

The cardinal q_w of $|InI/I| = |I(1)nI(1)/I(1)|$ for $n \in N$ of image $w \in W$ or $W(1)$ (Prop. **??**), can be explicitely computed from its values q_s for $s \in S^{aff}$ (Cor. **??**) :

Proposition 3.33. *1) For $w_1, w_2 \in W$, $q_{w_1 w_2} = q_{w_1} q_{w_2}$ is equivalent to $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$.*

2) For $s, s' \in S^{aff}$ conjugate in W, we have $q_s = q_{s'}$.

Proof. The map $n \mapsto [I : I \cap nIn^{-1}]$ is invariant by conjugation by $N_{\mathfrak{C}}$ and $\ell(w), q_w$ are invariant by congugation by Ω. It suffices to prove the braid relations 1) for W^{aff}. For W^{aff}, the braid relations follow from the properties of the affine Tits system (G^{aff}, I, N^{aff}). This is well known but the only reference that I am aware of is ([**?**] IV §2 Ex. 3,8,23). ☐

4 Iwahori-Masumoto presentations

4.1 Generalities on Hecke rings

In this preliminary subsection, G is an arbitrary locally profinite group containing a compact open subgroup I, and R is a commutative ring.

The Hecke R-algebra $\mathcal{H}_R(G,I)$ of I in G is the ring of I-bi-invariant compactly supported functions from G to R, with the convolution product $*$. The value at 1 is an isomorphism from the intertwining algebra $\mathrm{End}_{RG} R[I\backslash G]$ to $\mathcal{H}_R(G,I)$. We have

$$\mathcal{H}_R(G,I) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G,I).$$

We call $\mathcal{H}_{\mathbb{Z}}(G,I)$ the Hecke ring of I in G.

For $g \in G$, the characteristic function of IgI is denoted by T_g. The Hecke R-algebra $\mathcal{H}_R(G,I)$ is a free R-module of basis $(T_g)_{g\in I\backslash G/I}$.

When I is contained in a subgroup G' of G, the free R-submodule of basis $(T_g)_{g\in I\backslash G'/I}$ in $\mathcal{H}_R(G,I)$ is a subalgebra identified with the Hecke R-algebra $\mathcal{H}_R(G',I)$.

For $g,h \in G$, the convolution product $T_g * T_h$ is

(59) $$T_g * T_h = \sum_{x\in I\backslash IgIhI/I} (T_g * T_h)(x) T_x,$$

where $(T_g * T_h)(x)$ is the cardinal of $(IgI \cap xIh^{-1}I)/I$ ([?] I.3.4 (3)), or equivalently,

(60) $$(T_g * T_h)(x) \text{ is the cardinal of } \{u \in Y_g \mid u^{-1}x \in gIhI\},$$

where Y_g is a system of representatives of the cosets $I/(gIg^{-1}\cap I)$. A system of representatives of the coset IgI/I is $Y_g g$. The number of elements of IgI/I is denoted by q_g. The linear map

(61) $$d : \mathcal{H}_R(G,I) \to R, \quad T_g \mapsto q_g \quad (g \in G)$$

respects the product ([?] I.3.5). For $g,h \in G$, the formula (**??**) implies

$$q_g q_h = \sum_{x\in I\backslash IgIhI/I} (T_g \circ T_h)(x) q_x$$

For $x \in IgIhI$, $(T_g * T_h)(x)$ is a positive integer $\leq \min(q_g, q_h)$ and we have $(T_g * T_h)(gh) \geq 1$. Therefore $q_g q_h \geq q_{gh}$ and

(62) $$q_g q_h = q_{gh} \quad \text{is equivalent to } T_g * T_h = T_{gh}.$$

We have

(63) $$T_g * T_h = T_{gh} \quad \text{if } g \text{ or } h \text{ normalizes } I.$$

When I is normal in G, the algebra $\mathcal{H}_R(G,I)$ identifies with the group ring $R[G/I]$ and the map d corresponds to the augmentation map $R[G/I] \to R$.

4.2 Iwahori-Matsumoto presentation

We return to the reductive group G. The Hecke ring of the Iwahori subgroup I, resp. pro-p-Iwahori subgroup $I(1)$, in G is called the Iwahori Hecke ring $\mathcal{H}$, resp. the pro-p-Iwahori Hecke ring $\mathcal{H}(1)$, of G. For $n \in N$ of image w in W, resp. $W(1)$, we write $T_n = T_w$ in $\mathcal{H}$, resp. $\mathcal{H}(1)$.

Proposition 4.1. *The Iwahori Hecke ring $\mathcal{H}$, resp. pro-p-Iwahori Hecke ring $\mathcal{H}(1)$, is a free $\mathbb{Z}$-module with basis $(T_w)_{w\in W}$, resp. $(T_w)_{w\in W(1)}$, satisfying the braid relations*

$$T_w * T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w').$$

Proof. For the basis, Prop. **??**. For the braid relations, (**??**), Prop. **??** and **??**. □

Let $S^{aff}(1) \subset W(1)$ be the inverse image of S^{aff}. The elements T_s in $\mathcal{H}$, resp. $\mathcal{H}(1)$, for $s \in S^{aff}$, resp. $s \in S^{aff}(1)$, satisfy quadratic relations. It is possible to prove them using Bruhat-Tits theory as in [**?**] when $\mathbf{G}$ is split. But we will obtain them from the quadratic relations in the Hecke rings of finite groups with a strongly split BN-pair [**?**] using Prop. **??**.

Let $s \in S^{aff}$, H_s the wall of the alcove $\mathfrak{C}$ fixed by s and let $\mathfrak{F}$ be a facet of $\mathfrak{C}$ contained in H_s (for instance the face $\mathfrak{F}_s$). We have $H_s = \mathrm{Ker}(\alpha + r_{\mathfrak{F}}(\alpha))$ where $\alpha \in \Delta'_{\Phi,\mathfrak{F}}$ (defined in (**??**)). For $u \in U^*_{\alpha,r_{\mathfrak{F}}}$, the image of $m_\alpha(u)$ in W is s. Let $s_\alpha(u) \in S^{aff}(1)$ be the image of $m_\alpha(u)$ in $W(1)$. Then $T_s \in \mathcal{H}$ belongs to the Hecke subring $\mathcal{H}(K_{\mathfrak{F}}, I)$ by Lemma **??**. Similarly, $T_{s_\alpha(u)} \in \mathcal{H}(1)$ belongs to the Hecke subring $\mathcal{H}(K_{\mathfrak{F}}, I(1))$.

The Iwahori subgroup I is the inverse image by the reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ of a minimal Borel subgroup $B_{\mathfrak{F},k}$ of $G_{\mathfrak{F},k}$ and $I(1)$ is the inverse image of the unipotent radical $U^+_{\mathfrak{F},k}$ of $B_{\mathfrak{F},k}$ (Cor. **??**). The Hecke rings $\mathcal{H}(K_{\mathfrak{F}}, I)$ and $\mathcal{H}_{\mathfrak{F},k} = \mathcal{H}(K_{\mathfrak{F},k}, B_{\mathfrak{F},k})$ are isomorphic. The Hecke rings $\mathcal{H}(K_{\mathfrak{F}}, I(1))$ and $\mathcal{H}_{\mathfrak{F},k}(1) = \mathcal{H}(K_{\mathfrak{F},k}, U^+_{\mathfrak{F},k})$ are isomorphic.

The finite group $K_{\mathfrak{F},k}$ is a strongly split BN-pair of characteristic p with $B = B_{\mathfrak{F},k}$ (Prop. **??**). The reduction u_k of $u \in U^*_{\alpha,r_{\mathfrak{F}}}$ in $K_{\mathfrak{F},k}$ belongs to $U^*_{\alpha,k}$, $s_\alpha(u_k), s_k$ are the images of $m_\alpha(u_k)$ in $N_{\mathfrak{F},k}, W_{\mathfrak{F},k} = N_{\mathfrak{F},k}/Z_k$ and $q_{s_k} = |U_{\alpha,\mathfrak{F},k}|$ (Cor. **??**).

The quadratic relations satisfied by $T_s \in \mathfrak{H}, T_s \in \mathcal{H}(K_{\mathfrak{F}}, I), T_{s_k} \in \mathcal{H}_{\mathfrak{F},k}$ are the same. The quadratic relations satisfied by $T_{s_\alpha(u)} \in \mathfrak{H}(1), T_{s_\alpha(u)} \in \mathcal{H}(K_{\mathfrak{F}}, I(1)), T_{s_\alpha(u_k)} \in \mathcal{H}_{\mathfrak{F},k}(1)$ are the same.

Proposition 4.2. (Quadratic relations in $\mathfrak{H}$)

$$T_{s_k} * T_{s_k} = q_{s_k} + (q_{s_k} - 1)T_{s_k} \quad \text{in } \mathcal{H}_{\mathfrak{F},k},$$
$$T_s * T_s = q_s + (q_s - 1)T_s \quad \text{in } \mathfrak{H}.$$

where $q_s = q_{s_k}$.

Proof. In $\mathbb{Z}[1/p] \otimes \mathcal{H}_{\mathfrak{F},k}$ ([**?**] Thm. 3.3). The quadratic relation holds true in $\mathcal{H}_{\mathfrak{F},k}$ because all the elements belong to $\mathcal{H}_{\mathfrak{F},k}$. □

We need more notation for the quadratic relation satisfied by $T_{s_\alpha(u_k)} \in \mathcal{H}_{\mathfrak{F},k}(1)$. Let $G'_{\alpha,k}$ be the group generated by $U_{\alpha,k}$ and $U_{-\alpha,k}$ and let $Z'_k = Z_k \cap G'_{\alpha,k}$ in $K_{\mathfrak{F},k}$. We have $m_\alpha(u_k) \in G'_{\alpha,k}$ and $m^2_{\alpha(u_k)} = s_\alpha(u_k)^2 \in Z'_k$. The braid relations identify $\mathbb{Z}[Z'_k]$ with a subring of $\mathcal{H}_{\mathfrak{F},k}(1)$. Let $c_{s_\alpha(u_k)} = \sum_{t\in Z'_k} c_{s_\alpha(u_k)}(t)t \in \mathbb{Z}[Z'_k]$ where

$$c_{s_\alpha(u_k)}(t) = |m_\alpha(u_k)U_{\alpha,k}m_\alpha(u_k) \ \cap \ U_{\alpha,k}m_\alpha(u_k)tU_{\alpha,k}|. \tag{64}$$

Proposition 4.3. (Quadratic relations in $\mathfrak{H}(1)$)

$$T_{s_\alpha(u_k)} * T_{s_\alpha(u_k)} = q_{s_k} s_\alpha(u_k)^2 + c_{u_k} T_{s_\alpha(u_k)} \quad \text{in } \mathcal{H}_{\mathfrak{F},k}(1),$$
$$T_{s(1)} * T_{s(1)} = q_{s(1)} s(1)^2 + c_{s(1)} T_{s(1)} \quad \text{in } \mathcal{H}(1),$$

where $s(1) = s_\alpha(u_k)t, t \in Z_k, c_{s(1)} = c_{s_\alpha(u_k)}t$, $q_{s(1)} = q_{s_k}$.

Proof. The first relation follows from the proof of Prop. 6.8 (iii) in [**?**].

$T_{s_\alpha(u_k)t} * T_{s_\alpha(u_k)t} = T_{s_\alpha(u_k)} * T_{s_\alpha(u_k)} s(t)t = q_{s_k} s_\alpha(u_k)^2 s(t)t + c_{s_\alpha(u_k)} T_{s_\alpha(u_k)} s(t)t = q_s(s_\alpha(u_k)t)^2 + c_{s_\alpha(u_k)} t T_{s_\alpha(u_k)t}$. □

The Iwahori Hecke ring $\mathcal{H}$ and the pro-p-Iwahori Hecke ring $\mathcal{H}(1)$ are uniquely determined by Prop. **??** and the quadratic relations Prop. **??** and Prop. **??**.

Proposition 4.4. *The coefficient $c_{s_\alpha(u_k)}$ of the quadratic relation satisfies:*

1) $c_{s_\alpha(u_k)}(t)|\{ts_k t^{-1} s_k^{-1}) \mid t \in Z_k\}| \equiv -1 \ mod\ p$, *for* $t \in Z'_k$.

2) $\sum_{t \in Z'_k} c_{s_\alpha(u_k)}(t) = q_s - 1$.

3) $c_{s_\alpha(u_k)} = \sum_{u'_k \in U^*_{\alpha,k}} m_\alpha(u'_k)^{-1} m_\alpha(u_k)$.

Proof. 1) ([**?**] 6.10 (i)).

2) ([**?**] beginning of Proof of 6.10).

3) We have $c_{s_\alpha(u_k)} = \sum_{u' \in U^*_{\alpha,k}} t(u')$ where $t : U^*_{\alpha,k} \to Z'_k$ is the map defined by

$$mu'm = xmt(u')y \quad x, y \in U^*_{\alpha,k}, m = m_\alpha(u_k),$$

deduced from the Bruhat decomposition of $G'_{\alpha,k}$ in a disjoint union of $U_{\alpha,k} n U_{\alpha,k}$ for $n \in Z'_k \cup mZ'_k$. Multiplying this relation on the left by $m^{-1}x^{-1}$ and on the right by $y^{-1}m^{-1}$, and remembering the definition of $m_\alpha(u')$ (**??**), we obtain $m_\alpha(u') = t(u')m^{-1}$. As $m_\alpha(u') = m_\alpha(u'^{-1})^{-1}$ (Remark **??**), we have also $m_\alpha(u'^{-1})^{-1} m_\alpha(u_k) = t(u')$. Hence $c_{s_\alpha(u_k)} = \sum_{u' \in U^*_{\alpha,k}} m_\alpha(u')^{-1} m_\alpha(u_k)$. □

4.3 Generic algebra

Let R be a commutative ring, let

(65) $$W^{aff}, S^{aff}, \Omega, W, Z_k, W(1),$$

satisfying:

- a1 (W^{aff}, S^{aff}) is a Coxeter system.
- a2 Ω is a group acting on S^{aff}, hence on W^{aff}.
- a3 W is the semi-direct product $W^{aff} \rtimes \Omega$.
- a3 Z_k is a commutative group.
- a4 $1 \to Z_k \to W(1) \to W \to 1$ is an extension of W by Z_k.

For a subset X of W, we denote by $X(1)$ the inverse image of X in $W(1)$.

The length ℓ of (W^{aff}, S^{aff}) beeing invariant by conjugation by Ω, extends to a length ℓ of W constant on the double cosets of Ω, and inflates to a length on $W(1)$, still denoted by ℓ. The subgroup of elements of length 0 in $W(1)$ is $\Omega(1)$. The inverse image of W^{aff} in $W(1)$ is a normal subgroup $W^{aff}(1)$ such that $Z_k = W^{aff}(1) \cap \Omega(1)$ and $W(1) = W^{aff}(1)\Omega(1)$ as in (**??**).

For $w \in W(1)$ and $t \in Z_k$, $w(t) = wtw^{-1}$ depends only on the image of w in W because Z_k is commutative. By linearity the conjugation defines an action

$$(w, c) \mapsto w \bullet c : W(1) \times R[Z_k] \to R[Z_k]$$

of $W(1)$ on $R[Z_k]$ factorizing through the map $W(1) \to W$. Let $S^{aff}(1)$ be the inverse image of S^{aff} in $W(1)$. For $s, s' \in S^{aff}$ we write $s \sim s'$ if $s, s' \in S^{aff}$ are conjugate in W and $s(1) \sim s'(1)$ for $s(1), s'(1) \in S^{aff}(1)$ above s, s' if $s \sim s'$.

Theorem 4.5. *Let $(q_s, c_s) \in R \times R[Z_k]$ for all $s \in S^{aff}(1)$. We have, for all $s \sim s'$ in $S^{aff}(1)$, $w \in W(1)$, $swz = ws'$, $z, t \in Z_k$,*

a5 $q_s = q_{st} = q_{s'}$,

a6 $c_{st} = c_s t$ *and* $w \bullet c_{s'} = ws'w^{-1}s^{-1}c_s$ *if* $\ell(sw) > \ell(w)$, $q_s w \bullet c_{s'} = q_{s'} ws'w^{-1}s^{-1}c_s$ *if* $\ell(sw) < \ell(w)$,

if and only if the R-free module of basis $(T_w)_{w\in W(1)}$ admits an R-algebra structure satisfying

$$\text{the braid relations:} \quad T_w T_{w'} = T_{ww'} \quad \text{for } w, w' \in W(1),\ \ell(w) + \ell(w') = \ell(ww'),$$
$$\text{the quadratic relations:} \quad T_s T_s = q_s T_{s^2} + c_s T_s \quad \text{for } s \in S^{aff}(1).$$

This algebra is denoted by $\mathcal{H}_R(q_s, c_s)$ and called the R-algebra of $W(1)$ with parameters (q_s, c_s).

When R has no zero divisors and $q_s \neq 0$ for all $s \in S^{aff}(1)$, then a6 is simply:

$$c_{st} = c_s t, \quad w \bullet c_{s'} = ws'w^{-1}s^{-1}c_s. \tag{66}$$

Proof. 1) We show that the conditions a5 on (q_s) and a6 on (c_s) are necessary. The braid relations identify $R[Z_k]$ with a subalgebra of $\mathcal{H}_R(q_s, c_s)$ and $T_w t = T_{wt} = wtw^{-1}T_w$ for $w \in W(1), t \in Z_k$ hence

$$T_w c = (w \bullet c) T_w \quad (c \in R[Z_k], w \in W(1)). \tag{67}$$

The equalities $q_s = q_{st}$ and $c_{st} = c_s t$ follow from

$$q_s (st)^2 + c_s t T_{st} = (T_s T_s) s^{-1} tst = T_s t T_s t = T_{st} T_{st} = q_{st}(st)^2 + c_{st} T_{st}.$$

The equalities $q_{s'} = q_s$ and $w \bullet c_{s'} = ws'w^{-1}s^{-1}c_s$ for $s, s' \in S^{aff}(1), w \in W(1), swz = ws'$ for some $z \in Z_k$, follow from the associativity of the product

$$T_s(T_w T_{s'}) = (T_s T_w) T_{s'}. \tag{68}$$

a) Case $\ell(sw) = \ell(ws') = \ell(w) + 1$. By the braid and quadratic relations,

$$\begin{aligned}
&T_s(T_w T_{s'}) = T_s T_{ws'} = T_s T_{swz} = T_s T_s T_{wz} = q_s s^2 T_{wz} + c_s T_{swz}.\\
&(T_s T_w) T_{s'} = T_{sw} T_{s'} = T_{ws'z^{-1}} T_{s'} = ws'z^{-1}(ws')^{-1} T_{ws'} T_{s'}\\
&= ws'z^{-1}(ws')^{-1} T_w T_{s'} T_{s'} = ws'z^{-1}(ws')^{-1} T_w (q_{s'} {s'}^2 + c_{s'} T_{s'}).
\end{aligned}$$

We compute $ws'z^{-1}(ws')^{-1} T_w {s'}^2 = ws'z^{-1}(ws')^{-1} w{s'}^2 w^{-1} T_w = s^2 wzw^{-1} T_w = s^2 T_{wz}$. This implies

$$(T_s T_w) T_{s'} = q_{s'} s^2 T_{wz} + ws'z^{-1}(ws')^{-1}(w \bullet c_{s'}) T_{ws'}$$

We compare and deduce $q_{s'} = q_s$, $w \bullet c_{s'} = ws'z(ws')^{-1} c_s = ws'w^{-1}s^{-1}c_s$.

b) Case $\ell(sw) = \ell(ws') = \ell(w) - 1$. We expand first $T_w T_{s'}$ and $T_s T_w$ using $T_w = T_{ws'^{-1}} T_{s'} = T_s T_{s^{-1}w}$ by the braid relations. By the quadratic relations,

$$\begin{aligned}
&T_w T_{s'} = T_{ws'^{-1}}(q_{s'} {s'}^2 + c_{s'} T_{s'}) = q_{s'} T_{ws'} + (w{s'}^{-1} \bullet c_{s'}) T_{ws'^{-1}} T_{s'} = q_{s'} T_{ws'} + T_w ({s'}^{-1} \bullet c_{s'}),\\
&T_s T_w = (q_s s^2 + c_s T_s) T_{s^{-1}w} = q_s T_{sw} + c_s T_w.
\end{aligned}$$

Recalling $sw = ws'z^{-1}$ we have $\ell(sws') = \ell(ws') + 1$, we compute

$$\begin{aligned}
&T_s(T_w T_{s'}) = q_{s'} T_{sws'} + T_s T_w ({s'}^{-1} \bullet c_{s'}) = q_{s'} T_{sws'} + (q_s T_{sw} + c_s T_w)({s'}^{-1} \bullet c_{s'}),\\
&(T_s T_w) T_{s'} = q_s T_{sws'} + c_s T_w T_{s'} = q_s T_{sws'} + q_{s'} c_s T_{ws'} + c_s T_w ({s'}^{-1} \bullet c_{s'}).
\end{aligned}$$

We compare to get $q_{s'} = q_s,\ q_{s'} c_s T_{ws'} = q_s T_{sw}({s'}^{-1} \bullet c_{s'})$. Writing

$$T_{sw}({s'}^{-1} \bullet c_{s'}) = sw{s'}^{-1}w^{-1}T_{ws'}({s'}^{-1} \bullet c_{s'}) = sw{s'}^{-1}w^{-1}(w \bullet c_{s'})T_{ws'},$$

we obtain $q_s(w \bullet c_{s'}) = q_{s'} ws'w^{-1}s^{-1}c_s$.

2) The existence and unicity is proved in ([?] Thm. 3.1.5). $\square$

Remark 4.6. a) When (q_s) satisfies a5, $(c_s = q_s - 1)$ satisfies a6.

b) Suppose that R has no zero divisors and $q_s \neq 0$ for all $s \in S^{aff}(1)$. Then a6 implies: $s \bullet c_s = c_s$ (take $w = s \in S^{aff}(1)$ in a6). This means that T_s commutes with c_s.

$c_s sts^{-1} = c_s t$ for $t \in Z_k$ (use $st \bullet c_{st} = c_{st},\ s \bullet c_s = c_s, c_{st} = c_s t$ and the commutativity of Z_k). Hence, if $c_s \neq 0$, the group $\{tst^{-1}s^{-1} \mid t \in Z_k\}$ is finite.

c) Given $q_s \in R$ for all $s \in S^{aff}/\sim$ one defines $q_{s(1)} = q_s$ for $s \in S^{aff}(1)$ above $s' \in S^{aff}, s' \sim s$. Then $(q_s)_{s \in S^{aff}(1)}$ satisfies a5, and any set $(q_s)_{s \in S^{aff}(1)}$ satisfying a5 arises for a set $(q_s)_{s \in S^{aff}/\sim}$.

Example 4.7. *1. The Iwahori Hecke ring is $\mathcal{H} = \mathcal{H}_{\mathbb{Z}}(q_s, q_s - 1)$ with q_s given by Cor.* **??** *and $Z_k = \{1\}, W = W(1)$.*

2. The pro-p-Iwahori Hecke ring $\mathcal{H}(1) = \mathcal{H}_{\mathbb{Z}}(q_s, c_s)$ with q_s given by Cor. **??***, c_s as in Prop.* **??***, for $s \in S^{aff}(1)$.*

3. The group algebra $R[W(1)] = \mathcal{H}_R(1, 0)$ with $q_s = 1, c_s = 0$ for all $s \in S^{aff}(1)$.

4. The Lusztig affine Hecke R-algebras with parameters (q_s) with q_s an invertible square in R [?] are examples of R-algebras $\mathcal{H}_R(q_s, q_s - 1)$ with $Z_k = \{1\}, W = W(1)$.

The R-algebra $\mathcal{H}_R^{aff}(q_s, c_s)$ of $W^{aff}(1)$ is a subalgebra of the R-algebra $\mathcal{H}_R(q_s, c_s)$ of W. By the braid relations, the R-linear map such that $u \mapsto T_u$ for $u \in \Omega(1)$, embeds the group R-algebra $R[\Omega(1)]$ of $\Omega(1)$ in $\mathcal{H}_R(q_s, c_s)$. The intersection $R[\Omega(1)] \cap \mathcal{H}_R^{aff}(q_s, c_s)$ is the group R-algebra $R[Z_k]$ of Z_k.

Proposition 4.8. *The R-algebra $\mathcal{H}_R(q_s, c_s)$ is isomorphic to the twisted tensor product*

$$R[\Omega(1)] \hat{\otimes}_{R[Z_k]} \mathcal{H}_R^{aff}(q_s, c_s)$$

with the product $(T_u \hat{\otimes} T_w)(T_{u'} \hat{\otimes} T_{w'}) = T_{uu'} \hat{\otimes} T_{u'^{-1}wu'} T_{w'}$ for $u, u' \in \Omega(1), w, w' \in W^{aff}(1)$.

Proof. Clear. $\square$

Remark 4.9. *Let $T_s^* = T_s - c_s$ for $s \in S^{aff}(1)$. The quadratic relation in $\mathcal{H}_R(q_s, c_s)$ is*

$$T_s^* T_s = T_s T_s^* = q_s s^2 \quad \text{or} \quad T_{s^{-1}}^* T_s = T_s T_{s^{-1}}^* = q_s.$$

For $u \in \Omega(1)$, we have $c_{u^{-1}su} = T_u^{-1} c_s T_u, \quad T_{u^{-1}su}^* = T_u^{-1} T_s^* T_u.$

Proof. We have $T_s^* T_s = T_s T_s - c_s T_s = q_s s^2$ and $T_s s^{-2} = s^{-2} T_s = T_{s^{-1}},\ c_{s^{-1}} = c_s s^{-2}$, because $s^2 \in Z_k$. The product $T_s^* T_s$ commutes because c_s and s^2 commute with T_s. Comparing the quadratic relation for $T_{usu^{-1}} = T_u T_s T_u^{-1}$ with the quadratic relation for T_s multiplied on the left by T_u and on the right by T_u^{-1} we obtain $c_{usu^{-1}} = T_u c_s T_u^{-1}$. $\square$

Let $w = s_1 \ldots s_{\ell(w)} u$ with $u \in \Omega(1)$ and $s_i \in S^{aff}(1)$ for $1 \leq i \leq \ell(w)$, and let

$$T_{w^{-1}}^* := T_u^{-1} T_{s_{\ell(w)}^{-1}}^* \ldots T_{s_1^{-1}}^*.$$

Proposition 4.10. *We suppose that q_s is invertible in R for all $s \in S^{aff}/\sim$.*

1) T_w is invertible in $\mathcal{H}_R(q_s, c_s)$ of inverse $q_w^{-1} T_{w^{-1}}^*$.

2) $T_w^* = T_{s_1}^* \ldots T_{s_{\ell(w)}}^* T_u$ and does not depend on the decomposition of w.

3) $T^*_{w^{-1}}T_w = T_w T^*_{w^{-1}} = q_w$.

4) $T^*_w\ c = (w \bullet c)\ T^*_w$ *for* $c \in R[Z_k]$.

5) $T^*_w\ T_u = T^*_{wu} = T_u T^*_{u^{-1}wu}$ *for* $w \in W(1), u \in \Omega(1)$.

Proof. By Remark **??**, $T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_u$ (by the braid relations, it is independent on the decomposition) is invertible of inverse

$$T_w^{-1} = q_w^{-1} T_{u^{-1}} T^*_{s^{-1}_{\ell(w)}} \dots T^*_{s_1^{-1}} = q_w^{-1} T^*_{w^{-1}},$$

with $w^{-1} = u^{-1} s^{-1}_{\ell(w)} \dots s_1^{-1}$. Replacing w^{-1} by $w = u(u^{-1}s_1 u)\dots(u^{-1}s_n u)$ with $n = \ell(w) = \ell(w^{-1})$ and $u^{-1}s_i u \in S^{aff}$, we obtain $T^*_w = T_u T^*_{u^{-1}s_1 u} \dots T_{u^{-1}s_n u}$. By Remark **??**, $T^*_w = T^*_{s_1} \dots T^*_{s_n} T_u$. As $T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_u$ was independent of the decomposition, the same is true for $T^*_w = T^*_{s_1} \dots T^*_{s_{\ell(w)}} T_u$.

From $T^*_w = T^*_{s_1} \dots T^*_{s_{\ell(w)}} T_u$ and

a) $T_s T^*_s = T^*_s T_s = q_s$ (Remark **??**), we deduce $T_w T^*_w = T^*_w T_w = q_w$.

b) $(T_s - c_s)t = T_{st} - c_s sts^{-1} = sts^{-1}(T_s - c_s)$ for $t \in Z_k$ (use that Z_k is commutative and Remark **??**), we deduce $T^*_w\ c = (w \bullet c) T^*_w$ for $c \in R[Z_k]$,

c) $T^*_s T_u = T^*_{su} = T_u T^*_{u^{-1}su}$ (Remark **??**), we deduce $T^*_w\ T_u = T^*_{wu} = T_u T^*_{u^{-1}wu}$ for $u \in \Omega(1)$. □

4.4 $\mathbf{q}_w \mathbf{q}_{w'} = \mathbf{q}_{ww'} \mathbf{q}^2_{w,w'}$

Let $\mathbf{q}_s$ be indeterminates for $s \in S^{aff}/\sim$. The set $\mathfrak{T}$ of conjugates of S^{aff} in W does not depend on S^{aff}. Let $\mathfrak{T}(1)$ be the inverse image of $\mathfrak{T}$ in $W(1)$. For $\tau(1) \in \mathfrak{T}(1)$ above $\tau \in \mathfrak{T}$ conjugate in W to $s \in S^{aff}$, i.e. $\tau \sim s$, let $\mathbf{q}_{\tau(1)} = \mathbf{q}_\tau = \mathbf{q}_s$. For a sequence $(s_1, \dots, s_n)$ in S^{aff}, let

(69) $\mathfrak{T}(s_1, \dots, s_n) = (t_1 = s_1, t_2 = s_1 s_2 s_1^{-1}, \dots, t_n = s_1 \dots s_{n-1} s_n (s_1 \dots s_{n-1})^{-1})$ in $\mathfrak{T}$.

Definition 4.11. *For* $w(1) \in W(1)$ *above* $w = s_1 \dots s_n u$ *with* $s_i \in S^{aff}$ *for* $1 \le i \le n$ *and* $u \in \Omega$, *let* $\mathfrak{T}_{w(1)} = \mathfrak{T}_w = \mathfrak{T}_{s_1 \dots s_n}$ *be the set of elements of odd multiplicity in* $\mathfrak{T}(s_1, \dots, s_n)$ *and let*

$$\mathbf{q}_{w(1)} = \mathbf{q}_w = \mathbf{q}_{s_1 \dots s_n} = \prod_{\tau \in \mathfrak{T}_w} \mathbf{q}_\tau.$$

For $w(1), w'(1) \in W(1)$ *above* $w, w' \in W$, *let*

$$\mathbf{q}_{w(1),w'(1)} = \mathbf{q}_{w,w'} = (\mathbf{q}_w \mathbf{q}_{w'} \mathbf{q}^{-1}_{ww'})^{1/2}.$$

When $w \in W^{aff}$, $\mathfrak{T}_w$ consists of the elements $w' s w'^{-1}$ for all triples $(w', w'', s) \in W^{aff} \times W^{aff} \times S^{aff}$ such that $w = w' s w''$ and $\ell(w) = \ell(w') + \ell(w'') + 1$.

Remark 4.12. *When* $w \in W^{aff}$, $\mathfrak{T}_w$ *contains* $s \in S^{aff}$ *if and only if* $\ell(sw) < \ell(w)$.

When the decomposition $w = s_1 \dots s_{\ell(w)}$ is reduced, $\mathfrak{T}(s_1, \dots, s_{\ell(w)}) = \mathfrak{T}_w$ ([**?**] IV.1.4 Lemme 2 and Remarque; see also [**?**] 1.3.14), and

$$\mathbf{q}_w = \mathbf{q}_{s_1} \dots \mathbf{q}_{s_{\ell(w)}},$$

$\mathfrak{T}_w, q_w$ depend only on w.

Lemma 4.13. *Let* $w, w' \in W$. *Then* $\mathbf{q}_{w,w'} = 1$ *if and only if* $\ell(w) + \ell(w') = \ell(ww')$.

Proof. By the braid relations $\mathbf{q}_w \mathbf{q}_{w'} = \mathbf{q}_{ww'}$ if and only if $\ell(w) + \ell(w') = \ell(ww')$. □

Lemma 4.14. *Let $w, w' \in W^{aff}$. We have*

$$\mathfrak{T}_{ww'} = (\mathfrak{T}_w \cup \mathfrak{T}_{ww'w^{-1}}) - (\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}).$$

Proof. If $w = s_1 \dots s_n, w' = s'_1 \dots s'_m$ are reduced decomposition of w, w' then the multiset $\mathfrak{T}(s_1, \dots, s_n, s'_1 \dots s'_m)$ is a union of $\mathfrak{T}_w$ and of $w\mathfrak{T}_{w'}w^{-1} = \mathfrak{T}_{ww'w^{-1}}$. The elements of $\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}$ have multiplicity 2, the other ones have multiplicity 1. This implies the formula for $\mathfrak{T}_{ww'}$. □

Remark 4.15. *Let $w, w' \in W^{aff}$. Then, $\ell(ww') = \ell(w) + \ell(w') - 2\,\mathrm{Card}(\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}})$.*

The computation of $\mathbf{q}_{w,w'}$ can be done using the following lemma.

Lemma 4.16. *Let $w, w' \in W^{aff}$ and $u, u' \in \Omega$. We have*

$$\mathbf{q}_{w,w'} = \prod_{\tau \in \mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}} \mathbf{q}_\tau, \quad \mathbf{q}_{wu,w'u'} = \mathbf{q}_{w,uw'u^{-1}}.$$

Proof. The formula for $\mathbf{q}_{w,w'}$ follows from Lemma **??**. The group Ω normalizes S^{aff} and $w, uw'u^{-1}, wuw'u' = wuw'u^{-1}uu'$ belong to W^{aff}. We compute:
$\mathbf{q}^2_{wu,w'u'} = \mathbf{q}_{wu}\mathbf{q}_{w'u'}\mathbf{q}^{-1}_{wuw'u'} = \mathbf{q}_w\mathbf{q}_{w'}\mathbf{q}^{-1}_{wuw'u^{-1}uu'} = \mathbf{q}_w\mathbf{q}_{uw'u^{-1}}\mathbf{q}^{-1}_{wuw'u^{-1}} = \mathbf{q}^2_{w,uw'u^{-1}}$. □

Example 4.17. Let w, w' in W.
$\mathbf{q}_w = \mathbf{q}_{w^{-1}}$, $\mathbf{q}_{w,w^{-1}} = \mathbf{q}_w$.
$\mathbf{q}_{w^{-1},ww'w^{-1}} = \prod_{\tau \in \mathfrak{T}_w \cap \mathfrak{T}_{w'}} \mathbf{q}_\tau$ is equal to $\mathbf{q}_w = \prod_{\tau \in \mathfrak{T}_w} \mathbf{q}_\tau$ if and only if $\mathfrak{T}_w \subset \mathfrak{T}_{w'}$ if and only if $\ell(w'w) = \ell(w') - \ell(w)$.

4.5 Reduction to $q_s = 1$

We explain a method to reduce the proof of a property of the R-algebra $\mathcal{H}_R(q_s, c_s)$ (Thm. **??**) to the simpler case where $q_s = 1$ for all $s \in S^{aff}/\sim$.

Let $(\mathfrak{q}_s)_{s \in S^{aff}/\sim}$ be indeterminates, let $\mathbf{q}_s = \mathfrak{q}_s^2$ be the square of $\mathfrak{q}_s$ for $s \in S^{aff}/\sim$, and let $\mathbf{c}_s$ be elements in $R[(\mathbf{q}_s)][Z_k]$ for $s \in S^{aff}(1)$ satisfying a6 (**??**), where $R[(\mathbf{q}_s)]$ is the R-algebra generated by $\mathbf{q}_s$ for all $s \in S^{aff}/\sim$.

The "generic" algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q_s}, \mathbf{c}_s)$ is a $R[(\mathbf{q}_s)]$-subalgebra of the $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q_s}, \mathbf{c}_s)$,

$$\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q_s}, \mathbf{c}_s) \subset \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q_s}, \mathbf{c}_s). \tag{70}$$

For $w \in W(1)$, let $\mathfrak{q}_w = \prod_{\tau \in \mathfrak{T}_w} \mathfrak{q}_\tau$ where $\mathfrak{q}_\tau = \mathfrak{q}_s$ if $\tau \sim s$. In $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q_s}, \mathbf{c}_s)$, the elements

$$\tilde{T}_w := \mathfrak{q}_w^{-1} T_w \quad (w \in W(1)). \tag{71}$$

form a $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-basis satisfying the braid relations and the quadratic relations with parameters $(1, \mathfrak{q}_s^{-1}\mathbf{c}_s)$:

$$\tilde{T}_s\tilde{T}_s = \tilde{T}_{s^2} + \mathfrak{q}_s^{-1}\mathbf{c}_s\,\tilde{T}_s \quad (s \in S^{aff}(1)). \tag{72}$$

Applying Thm. **??**, we obtain :

Proposition 4.18. *The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-linear map*

$$T_w \mapsto \tilde{T}_w : \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1}\mathbf{c}_s) \to H_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, \mathbf{c}_s) \tag{73}$$

is an algebra isomorphism.

We can often reduce to the case $q_s = 1$ by considering:

1) The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$.
2) The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra isomorphism (**??**).
3) The generic $R[(\mathbf{q}_s)]$-subalgebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s) \subset \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$.
4) The specialisation $\mathcal{H}_R(q_s, c_s) = R \otimes_{R[(\mathbf{q}_s)]} \mathcal{H}_{R[(\mathbf{q}_s)}(\mathbf{q}_s, c_s)$ sending $\mathbf{q}_s$ to q_s for all $s \in S^{aff}/\sim$.

We give an example:

Proposition 4.19. *The properties 2) to 5) of Prop.* **??** *are valid in $\mathcal{H}_R(q_s, c_s)$ for any choice of $q_s \in R, s \in S^{aff}/\sim$.*

Proof. By Prop. **??**, the properties 2) to 5) of the proposition are true in the algebra $\mathcal{H}_{R[(\mathbf{q}_s,\mathbf{q}_s^{-1})]}(\mathbf{q}_s, c_s)$. They are relations between elements of the generic $R[(\mathbf{q}_s)]$-subalgebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$. They remain true in the algebra $\mathcal{H}_R(q_s, c_s) = R \otimes_{R[(\mathbf{q}_s)]} \mathcal{H}_{R[(\mathbf{q}_s)}(\mathbf{q}_s, c_s)$ obtained by the specialisation sending $\mathbf{q}_s$ to q_s for $s \in S^{aff}/\sim$. □

5 Alcove walk bases and Bernstein relations

Let $R, W^{aff}, S^{aff}, \Omega, W, Z_k, W(1), (q_s, c_s)$ as in subsection **??**, satisfying a1 to a6, and the following hypotheses:

b1 W^{aff} is the affine Weyl group of a reduced root system Σ, generated by the orthogonal reflections with respect to a set of affine hyperplanes

$$\mathfrak{H} = \{\mathrm{Ker}(\beta + k) \mid \beta \in \Sigma, k \in \mathbb{Z}\}$$

in an euclidean real vector space V, and S^{aff} is the set of orthogonal reflections with respect to the walls of an alcove $\mathfrak{C}$ of vertex 0 in V.

b2 The action of W^{aff} on V extends to an action of W such that for any $w \in W$, an element $w_0^{-1} w$ acts by a translation respecting $\mathfrak{H}$, for some w_0 in the fixator W_0 of 0 in W^{aff}.

b3 For $s, s' \in S^{aff}$ such that ss' has finite order $n(s, s')$, there exist $s(1), s'(1) \in S^{aff}(1)$ above s, s' such that $s(1)s'(1)s(1)\ldots = s'(1)s(1)s'(1)\ldots$ where the two products have $n(s, s')$ factors.

We use the notations of section **??**, without Σ in the index in (**??**) and (**??**) because Σ is now the unique root system (there is no Φ). The group Ω which normalizes S^{aff} is the stabilizer of $\mathfrak{C}$ in $W = W^{aff} \rtimes \Omega$.

We denote by Λ, resp. Λ^{aff}, the subgroup of W, resp. W^{aff}, acting by translations on V and by

$$\nu : \Lambda \to V$$

the homomorphism such that $\lambda \in \Lambda$ acts by translations by $\nu(\lambda)$. The group Λ is normalized by $w_0 \in W_0$: $w_0 \lambda w_0^{-1}$ acts by translation by $w_0.\nu(\lambda)$, the homomorphism ν is W_0-equivariant : $\nu(w_0 \lambda w_0^{-1}) = w_0.\nu(\lambda)$, and

$$W = \Lambda \rtimes W_0. \tag{74}$$

The lattice $Q(\Sigma^\vee)$ of V generated by the set $\Sigma^\vee$ of coroots of Σ is equal to $\nu(\Lambda^{aff})$ and $\nu(\Lambda)$ is contained in the lattice $P(\Sigma^\vee)$ of weights of $\Sigma^\vee$, that is, the elements $v \in V$ such that $\alpha(v) \in \mathbb{Z}$ for all $\alpha \in \Sigma$,

$$Q(\Sigma^\vee) \subset \nu(\Lambda) \subset P(\Sigma^\vee).$$

The action of W on V inflates to an action of $W(1)$ trivial on Z_k and the homomorphism ν inflates to an homomorphism $\nu : \Lambda(1) \to V$ vanishing on Z_k, where $\Lambda(1)$ is the inverse image of Λ in $W(1)$. We have

$$W(1) = \Lambda(1) W_0(1), \tag{75}$$

where $W_0(1)$ is the inverse image of W_0 in $W(1)$, $\Lambda(1) \cap W_0(1) = Z_k$ and $\Lambda(1)$ is normal in $W(1)$.

Remark 5.1. *The data arising from (R, F, G) satisfies b_j for $j = 1, 2, 3$.*

Proof. The property b3 follows from Prop. **??** applied to the root datum generating the finite quotients of the parahoric sugroups of G. See the subsection **??** b1 and b2. We have $\Lambda = Z/Z_0,\ \Lambda(1) = Z/Z_0(1)$. The extension $\Lambda(1) \to \Lambda$ of kernel Z_k does not split in general [**?**]. □

5.1 Hyperplanes between $\mathfrak{C}$ and $w(\mathfrak{C})$

Let $\Sigma^{aff} = \{\beta + k \ \mid \beta \in \Sigma, k \in \mathbb{Z}\}$. The set $\mathfrak{T}$ of conjugates of S^{aff} in W considered in Subsection **??** is equal to $\mathfrak{T} = \{s_{\beta+k} \mid \beta + k \in \Sigma^{aff}\}$.

Lemma 5.2. *For any $\beta + k \in \Sigma^{aff}, \lambda \in \Lambda$, $s_{\beta+k-\beta\circ\nu(\lambda)} = \lambda s_{\beta+k} \lambda^{-1}$.*

Proof. Let $x \in V$. The action of W^{aff} on V is faithful. We have $s_{\beta+k}.x = x - (\beta(x)+k)\beta^\vee$ and

$$\lambda s_{\beta+k} \lambda^{-1}.x = s_{\beta+k}.(x - \nu(\lambda) + \nu(\lambda) = x - (\beta(x - \nu(\lambda)) + k)\beta^\vee.$$

□

Remark 5.3. *We deduce: $s_\beta \sim s_{\beta+2k}, s_{\beta-1} \sim s_{\beta-1+2k}$ for all $\beta \in \Sigma, k \in \mathbb{Z}$. If $\beta \circ \nu(\lambda) = \mathbb{Z}$, then $s_\beta \sim s_{\beta-1}$.*

For $\tau \in \mathfrak{T}$, let H_τ be the affine hyperplane fixed pointwise by τ. The map $\tau \mapsto H_\tau : \mathfrak{T} \to \mathfrak{H}$ is bijective. When two facets of V are not contained in the connected component of $V - H_\tau$, we say that H_τ is separates them. By ([**?**] IV. 1 Ex. 16 h)), *for $w \in W^{aff}$, the set of hyperplanes of $\mathfrak{H}$ separating $\mathfrak{C}$ and $w(\mathfrak{C})$ is*

$$\mathfrak{H}_w = \{H_\tau \mid \tau \in \mathfrak{T}_w\}, \tag{76}$$

where the finite set $\mathfrak{T}_w$ of cardinal $\ell(w)$ was defined in Subsection **??**.

Example 5.4. *Let $s \in S^{aff}, w \in W^{aff}$. Then $\ell(sw) = \ell(w) + 1$ means that $w(\mathfrak{C})$ and $\mathfrak{C}$ are on the same side of the wall H_s of $\mathfrak{C}$ fixed by s.*

Definition 5.5. *Let $x \in \mathfrak{C},\ \beta \in \Sigma,\ w \in W(1)$ above $w \in W$. We define $\ell_\beta(w(1)) = \ell_\beta(w) \in \mathbb{Z}$ as the integer such that*

$$\ell_\beta(w) < \beta(w(x)) < \ell_\beta(w) + 1 \tag{77}$$

The integer $\ell_\beta(w)$ does not depend on the choice of $x \in \mathfrak{C}$, and depends only on the action of w on V.

Let Σ^+, Σ^- be the set of positive, negative, roots of Σ (we say positive instead of $\mathfrak{C}$-positive). When $\beta \in \Sigma^+$, $0 < \beta(x) < 1$ by (**??**).

Lemma 5.6. *For $w \in W$, $\beta \in \Sigma^+$ and $k \in \mathbb{Z}$, the hyperplane $\mathrm{Ker}(\beta + k)$ separates the alcoves $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if*

$$k \in [0, -\ell_\beta(w) - 1] \text{ and } \ell_\beta(w) \leq -1 \text{ , or } k \in [-\ell_\beta(w), -1] \text{ and } \ell_\beta(w) \geq 1.$$

Proof. Then $\mathrm{Ker}(\beta+k) \in \mathfrak{H}$ separates $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if $\beta(x)+k$ and $\beta(w(x))+k$ have a different sign.

Let $\beta \in \Sigma^+$. Then $k < \beta(x)+k < 1+k$ and $\ell_\beta(w)+k < \beta(w(x))+k < \ell_\beta(w)+k+1$. Hence $\beta(x)+k$ is positive if and only if $k \geq 0$ and $\beta(w(x))+k$ is negative if and only if $\ell_\beta(w)+1+k \leq 0$. This holds if and only if $\ell_\beta(w) \leq -1$ and $k \in [0, -\ell_\beta(w)-1]$. Similarly $\beta(x)+k$ negative and $\beta(w(x))+k$ positive is equivalent to $k \in [-\ell_\beta(w), -1]$ and $\ell_\beta(w) \geq 1$. □

Proposition 5.7. *The length of $w \in W$ or $W(1)$ is $\ell(w) = \sum_{\beta \in \Sigma^+} |\ell_\beta(w)|$.*

Proof. Let $w \in W^{aff}$. The length of w is the cardinal of $\mathfrak{T}_w$. Use (**??**) and Lemma **??**. The number of $\mathrm{Ker}(\beta+k) \in \mathfrak{H}$ with $\beta \in \Sigma^+, k \in \mathbb{Z}$ separating $\mathfrak{C}$ and $w(\mathfrak{C})$ is $|\ell_\beta(w)|$. This remains valid for $w \in W$ because Ω normalizes $\mathfrak{C}$, and for $w \in W(1)$ because Z_k acts trivially. □

Example 5.8. 1) When w acts trivially, $\ell_\beta(w) = 0$ if $\beta \in \Sigma^+$ and $\ell_\beta(w) = -1$ if $\beta \in \Sigma^-$.

2) Let $w \in W_0$ and $\beta \in \Sigma$. We have $\beta(w(x)) = w^{-1}(\beta)(x)$. Hence $\ell_\beta(w) = 0$ if $w^{-1}(\beta) \in \Sigma^+$, and $\ell_\beta(w) = -1$ if $w^{-1}(\beta) \in \Sigma^-$. The hyperplane $\mathrm{Ker}\,\beta$ separates $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if $w^{-1}(\beta) \in \Sigma^-$. The length $\ell(w)$ of $w \in W_0$ is the number of $\beta \in \Sigma^+$ such that $w^{-1}(\beta) \in \Sigma^-$.

Proposition 5.9. *Let $\beta \in \Sigma, \lambda \in \Lambda, w \in W_0$. We have*

1) *$\ell_\beta(\lambda) = \beta \circ \nu(\lambda)$ if β is positive and $\beta \circ \nu(\lambda) - 1$ if β is negative.*
2) *$\ell_\beta(\lambda w) = \beta \circ \nu(\lambda)$ if $\beta \in w(\Sigma^+)$ and $\beta \circ \nu(\lambda) - 1$ if $\beta \in w(\Sigma^-)$.*
3) *$\ell_\beta(w\lambda) = w^{-1}(\beta) \circ \nu(\lambda)$ if $\beta \in w(\Sigma^+)$ and $w^{-1}(\beta) \circ \nu(\lambda) - 1$ if $\beta \in w(\Sigma^-)$.*

Proof. Let $x \in \mathfrak{C}$. We recall that $\beta \circ \nu(\lambda)$ is an integer.

When β is positive we have $0 < \beta(x) < 1$ and $\beta \circ \nu(\lambda) < \beta(x + \nu(\lambda) < 1 + \beta \circ \nu(\lambda)$.

When β is negative, $-1 < \beta(x) < 0$ and $-1 + \beta \circ \nu(\lambda) < \beta(x + \nu(\lambda) < \beta \circ \nu(\lambda)$.

We have $\lambda w(x) = w(x) + \nu(\lambda)$, $w\lambda(x) = w(x + \nu(\lambda))$,

$\beta(\lambda w(x)) = \beta(w(x)) + \beta \circ \nu(\lambda) = w^{-1}(\beta)(x) + \beta \circ \nu(\lambda)$,

$\beta(w\lambda(x)) = w^{-1}(\beta)(x + \nu(\lambda)) = w^{-1}(\beta)(x) + w^{-1}(\beta) \circ \nu(\lambda)$.

□

Corollary 5.10. *We have for $(\lambda, w) \in \Lambda \times W_0$,*

$$\ell(\lambda w) = \sum_{\beta \in \Sigma^+ \cap w(\Sigma^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Sigma^+ \cap w(\Sigma^-)} |\beta \circ \nu(\lambda) - 1|,$$

$$\ell(w\lambda) = \sum_{\beta \in \Sigma^+ \cap w^{-1}((\Sigma^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Sigma^+ \cap w^{-1}((\Sigma^-)} |\beta \circ \nu(\lambda) + 1|.$$

Proof. Prop. **??**, **??** imply the above equality of $\ell(\lambda w)$ and

$$\ell(w\lambda) = \sum_{\beta \in \Sigma^+ \cap w(\Sigma^+)} |w^{-1}(\beta) \circ \nu(\lambda)| + \sum_{\beta \in \Sigma^+ \cap w(\Sigma^-)} |w^{-1}(\beta) \circ \nu(\lambda) - 1|.$$

Replace $w^{-1}(\beta)$ by β in the first sum and by $-\beta$ in the second sum. □

Corollary 5.11. *For $\lambda \in \Lambda, w \in W_0$,*

$$\ell(\lambda) = \sum_{\beta \in \Sigma^+} |\beta \circ \nu(\lambda)|,$$
$$\ell(w) = \ell(w^{-1}) = |\Sigma^+ \cap w(\Sigma^-)|,$$
$$\ell(\lambda w) = \ell(\lambda) - \ell(w) \text{ if and only if } \beta \circ \nu(\lambda) > 0 \text{ for } \beta \in \Sigma^+ \cap w(\Sigma^-),$$
$$\ell(\lambda w) = 0 \text{ if and only if } \beta \circ \nu(\lambda) = 0 \text{ for } \beta \in \Sigma^+ \cap w(\Sigma^+) \text{ and } \beta \circ \nu(\lambda) = 1 \text{ for } \beta \in \Sigma^+ \cap w(\Sigma^-).$$

Compare with [**?**] Appendice.

A Weyl chamber of V is a chamber relatively to $\mathfrak{H}_0 = \{H \in \mathfrak{H} \mid 0 \in H\}$. There exists a basis Δ' of Σ such that the chamber is the set of $x \in V$ with $\beta(x) > 0$ for all $\beta \in \Delta'$. A closed Weyl chamber is the closure of a Weyl chamber.

Example 5.12. *For $\lambda, \lambda' \in \Lambda$, $\ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda')$ if $\nu(\lambda), \nu(\lambda')$ belong to the same closed Weyl chamber.*

Proof. If $x, x' \in V$ belong to a closed Weyl chamber, then $\beta(x)\beta(x') \geq 0$ for all $\beta \in \Sigma$. Then $|\beta(x + x')| = |\beta(x)| + |\beta(x')|$. Apply $\sum_{\beta \in \Sigma} |\beta \circ \nu(\lambda)| = 2\ell(\lambda)$. ☐

Proposition 5.13. *$\mathbf{q}_w$ is constant on a conjugacy class of $W(1)$ contained in $\Lambda(1)$. In particular, the length is constant on such a class.*

The two assertions are equivalent if and only if $S^{aff}/\sim$ has a single element.

Proof. It is equivalent to prove that $\mathbf{q}_w$ is constant on the conjugacy classes of W contained in Λ. We have $W = \Lambda \rtimes W_0$. As Λ is commutative, we have to prove that $\mathbf{q}_{w\lambda w^{-1}} = \mathbf{q}_w$ for $\lambda \in \Lambda, w \in W_0$.

We will not use it, but we mention that it is easy to prove that the equality of the lengths. For $\beta \in \Sigma, w \in W_0$, the equalities

$$\beta \circ \nu(w\lambda w^{-1}) = \beta \circ w.\nu(\lambda) = w^{-1}(\beta) \circ \nu(\lambda) \tag{78}$$

imply $2\ell(w\lambda w^{-1}) = \sum_{\beta \in \Sigma} |w^{-1}(\beta) \circ \nu(\lambda)| = \sum_{\beta \in \Sigma} |\beta \circ \nu(\lambda)| = 2\ell(\lambda)$.

The group W_0 is generated by s_α for $\alpha \in \Sigma^+$. It suffices to prove that $\mathbf{q}_{s_\alpha \lambda s_\alpha} = \mathbf{q}_\lambda$ for $\alpha \in \Sigma, \lambda \in \Lambda$. We compare $\mathfrak{T}_\lambda$ with $\mathfrak{T}_{s_\alpha \lambda s_\alpha}$. We claim that :

$$s_\alpha \mathfrak{T}_\lambda s_\alpha = \mathfrak{T}_{s_\alpha \lambda s_\alpha} \quad \text{when } \alpha \circ \nu(\lambda) = 0,$$

$$s_\alpha \mathfrak{T}'_\lambda s_\alpha = \mathfrak{T}'_{s_\alpha \lambda s_\alpha} \quad \text{when } \alpha \circ \nu(\lambda) \neq 0,$$

and $\mathfrak{T}_\lambda = \mathfrak{T}'_\lambda \sqcup \{s\}$, $\mathfrak{T}_{s_\alpha \lambda s_\alpha} = \mathfrak{T}'_{s_\alpha \lambda s_\alpha} \sqcup \{s'\}$ where $(s, s') = (s_\alpha, s_{\alpha + \alpha\circ\nu(\lambda)})$ if $\alpha \circ \nu(\lambda) > 0$, and $(s, s') = (s_{\alpha + \alpha\circ\nu(\lambda)}, s_\alpha)$ if $\alpha \circ \nu(\lambda) < 0$.

Let us admit the claim. Definition **??** implies that $\mathbf{q}_{s_\alpha \lambda s_\alpha} = \mathbf{q}_\lambda$ when $\alpha \circ \nu(\lambda) = 0$.

It remains the case $\alpha \circ \nu(\lambda) \neq 0$. Definition **??** and Remark **??** imply that

If $\alpha \circ \nu(\lambda) > 0$, $\mathbf{q}_\lambda = \mathbf{q}_{s_\alpha} \mathbf{q}_{s_\alpha \lambda}, \mathbf{q}_{s_\alpha \lambda s_\alpha} = \mathbf{q}_{s_{\alpha + \alpha\circ\nu(\lambda)}} \mathbf{q}_{s_\alpha \lambda}$.

If $\alpha \circ \nu(\lambda) < 0$, $\mathbf{q}_{s_\alpha \lambda s_\alpha} = \mathbf{q}_{s_\alpha} \mathbf{q}_{\lambda - \alpha}, \mathbf{q}_\lambda = \mathbf{q}_{s_{\alpha + \alpha\circ\nu(\lambda)}} \mathbf{q}_{\lambda s_\alpha}$.

Hence $\mathbf{q}_{s_\alpha \lambda s_\alpha} = \mathbf{q}_\lambda \Leftrightarrow \mathbf{q}_{s_\alpha} = \mathbf{q}_{s_{\alpha + \alpha\circ\nu(\lambda)}} \Leftrightarrow s_\alpha \sim s_{\alpha + \alpha\circ\nu(\lambda)}$. Apply Lemma **??**.

This ends the proof of the proposition, if we admit the claim.

The claim is a consequence of Lemma **??**, noting that by (**??**), $\ell_\beta(s_\alpha \lambda s_\alpha) = \ell_{s_\alpha(\beta)}(\lambda)$ for $\beta \in \Sigma$.

Let $\beta \in \Sigma^+, k \in \mathbb{Z}$. If $\beta \neq \alpha$, then $s_\alpha(\beta) \in \Sigma^+$ and $\mathrm{Ker}(\beta + k)$ separates $\mathfrak{C}$ from $\lambda.\mathfrak{C}$ if and only if $\mathrm{Ker}(s_\alpha(\beta) + k)$ separates $\mathfrak{C}$ from $s_\alpha \lambda s_\alpha.\mathfrak{C}$. Hence $s_{\beta + k} \in \mathfrak{T}_\lambda$ if and only if $s_\alpha(\beta) + k \in \mathfrak{T}_{s_\alpha \lambda s_\alpha}$. We have $s_{s_\alpha(\beta)+k} = s_\alpha s_{\beta+k} s_\alpha$.

We have $\ell_\alpha(\lambda) = \alpha \circ \nu(\lambda)$. If $\alpha \circ \nu(\lambda) = 0$, no hyperplane $\mathrm{Ker}(\alpha + k)$ separates $\mathfrak{C}$ from $\lambda.\mathfrak{C}$ or from $s_\alpha \lambda s_\alpha.\mathfrak{C}$.

We suppose now $\ell_\alpha(\lambda) = \alpha \circ \nu(\lambda) \neq 0$. We have $\ell_\alpha(s_\alpha \lambda s_\alpha) = \ell_{-\alpha}(\lambda) = -\ell_\alpha(\lambda)$.

$\mathrm{Ker}(\alpha + k)$ separates $\mathfrak{C}$ from $\lambda.\mathfrak{C}$ if and only if k in $[0, -\ell_\alpha(\lambda) - 1]$ or in $[-\ell_\alpha(\lambda), -1]$, depending on the sign of $\ell_\alpha(\lambda)$.

$\mathrm{Ker}(\alpha + k')$ separates $\mathfrak{C}$ from $s_\alpha \lambda s_\alpha.\mathfrak{C}$ if and only if k' in $[0, \ell_\alpha(\lambda) - 1]$ or in $[\ell_\alpha(\lambda), -1]$, depending on the sign of $\ell_\alpha(\lambda)$.

We have $s_\alpha s_{\alpha+k} s_\alpha = s_{\alpha-k}$ and $\mathrm{Ker}(\alpha + k)$ separates $\mathfrak{C}$ from $\lambda.\mathfrak{C}$ if and only if $-k$ in $[1, \ell_\alpha(\lambda)]$ or in $[\ell_\alpha(\lambda) + 1, 0]$, depending on the sign of $\ell_\alpha(\lambda)$.

We deduce $\mathfrak{T}_\lambda = \mathfrak{T}'_\lambda \sqcup \{s\}$ and $\mathfrak{T}_{s_\alpha \lambda s_\alpha} = \mathfrak{T}'_{s_\alpha \lambda s_\alpha} \sqcup \{s'\}$ where $s_\alpha \mathfrak{T}'_\lambda s_\alpha = \mathfrak{T}'_{s_\alpha \lambda s_\alpha}$ and $(s, s') = (s_\alpha, s_{\alpha + \ell_\alpha(\lambda)})$ if $\ell_\alpha(\lambda) > 0$, $(s, s') = (s_{\alpha+\ell_\alpha(\lambda)}, s_\alpha)$ if $\ell_\alpha(\lambda) > 0$.

This ends the proof of the claim. ☐

Let $\tilde\beta$ be the highest root of the irreducible component Σ_j of Σ containing β and let $\tilde s \in S^{aff}$ such that $\tilde s = s_{-\tilde\beta+1}$.

Proposition 5.14. *Let $\beta \in \Sigma_j$.*

$(\beta \circ \nu)(\Lambda^{aff}) = 2\mathbb{Z}$ when Σ_j has rank 1.

$(\beta \circ \nu)(\Lambda^{aff}) = 2\mathbb{Z}$ when Σ_j has type $C_n, n \geq 2$, and β is a long root of Σ_j.

Otherwise $(\beta \circ \nu)(\Lambda^{aff}) = \mathbb{Z}$.

Proof. The translation subgroup Λ^{aff} of W^{aff} is generated by $s_\gamma s_{\gamma+1}$ for $\gamma \in \Sigma^+$, and $\beta \circ \nu(s_\gamma s_{\gamma+1}) = \beta(\gamma^\vee) = n(\beta, \gamma)$ is a Cartan integer. The group $\beta \circ \nu(\Lambda^{aff})$ is generated by the Cartan integers $n(\beta, \gamma)$ for $\gamma \in \Sigma$ and contains $2 = n(\beta, \beta)$. When γ does not belong to the irreducible component Σ_j we have $0 = n(\beta, \gamma)$.

On the Cartan matrix ([**?**]VI.Planches), we see that $n(\beta, \gamma) \in 2\mathbb{Z}$ for all $\gamma \in \Sigma_j$ if and only if Σ_j has a single element, or Σ_j is of type C_n and β is a long root. ☐

Lemma 5.15. *Let $\beta \in \Sigma_j$. There exists $s' \in S^{aff}, w \in W^{aff}$ such that*

$$s_{\beta+1} s_\beta = s_\beta w s' w^{-1}, \quad \ell(s_{\beta+1} s_\beta) = 2\ell(w) + 2.$$

If Σ_j has rank 1, then $s_{\beta+1} s_\beta = s_\beta \tilde s$.
If Σ_j has type $C_n, n \geq 2$, and β is a long root of Σ_j, then $s' = \tilde s$.

Proof. With the formula $s_{\beta+k}(x) = x - (\beta(x) + k)\beta^\vee$ one checks that $s_{\beta-1} = s_\beta s_{\beta+1} s_\beta$. We choose a reduced decomposition $s_{\beta-1} = s_1 \ldots s_n$ with $s_j \in S^{aff}$ for $1 \leq j \leq n = \ell(s_{\beta-1})$. By the strong exchange condition ([**?**] Thm. 1.3.11 c)), there exists a unique integer i such that $1 = s_1 \ldots \hat{s_i} \ldots s_n$. Set $s' = s_i, w = s_1 \ldots s_{i-1} = (s_{i+1} \ldots s_n)^{-1}$. Then

$$s_{\beta-1} = w s' w^{-1} \text{ with } \ell(s_{\beta-1}) = 2\ell(w) + 1, \text{ and } s_{\beta+1} s_\beta = s_\beta w s' w^{-1}.$$

By Lemma **??**, the hyperplanes $\mathrm{Ker}(\beta + k)$, for $k \in \mathbb{Z}$, separating $\mathfrak{C}$ and $s_\beta s_{\beta-1}(\mathfrak{C})$ are $\mathrm{Ker}(\beta)$ and $\mathrm{Ker}(\beta+1)$ because $\nu(s_\beta s_{\beta-1}) = -\beta^\vee$. As $\mathfrak{C}$ and $s_\beta s_{\beta-1}(\mathfrak{C})$ are not on the same side of $\mathrm{Ker}\,\beta$, we have $\ell(s_{\beta-1}) < \ell(s_\beta s_{\beta-1})$ [**?**] (V.3.2 Thm.1) hence $\ell(s_{\beta+1} s_\beta) = 2\ell(w) + 2$.

We have $s_{\beta+1} = s_\beta s_{\beta-1} s_\beta$ because $s_{-\beta+1} = s_{\beta-1} = s_{-\beta} s_{-\beta-1} s_{-\beta}$ for all $\beta \in \Sigma_j$. Replace $-\beta$ by β.

If Σ_j has rank 1 and $\beta > 0$, then $\tilde s = s_{\beta-1} = s_\beta s_{\beta+1} s_\beta$ hence $s_{\beta+1} s_\beta = s_\beta \tilde s$. If $\beta < 0$ we have $\tilde s = s_{\beta+1} = s_\beta s_{\beta-1} s_\beta$ hence $s_{\beta+1} s_\beta = s_\beta \tilde s$.

If Σ_j has type $C_n, n \geq 2$, the long roots of Σ_j are W_0-conjugate and the highest root $\tilde\beta_j$ is a long root. Let $\beta \in \Sigma_j$ be a long root of Σ_j, and $w \in W_0$ such that $w(\beta) = \tilde\beta_j$. The affine hyperplanes $w(\mathrm{Ker}\,\beta - 1) = \mathrm{Ker}(\tilde\beta_j - 1)$ are equal because $\beta(x) = \tilde\beta_j(w(x))$ for $x \in V$. Therefore $\tilde s = s_{\tilde\beta_j - 1}$ is conjugate to $s_{\beta-1}$ by w. No element of $S^{aff} - \{\tilde s\}$ is conjugate to $\tilde s$ in W^{aff} ([**?**] 3.3 and [**?**] VI PLanche III). We deduce $s' = \tilde s$. ☐

5.2 Alcove walk

By ([?] Definition 2.3.1), an orientation o of $(V,\mathfrak{H})$ is given by distinguishing for each affine hyperplane $H\in\mathfrak{H}$, a positive half-space among the two half-spaces which form the complement of H in V (the non-positive half-space is called negative) such that for all $H\in\mathfrak{H}$, either 1) or 2) holds:

1) For any finite subset of $\mathfrak{H}$, the intersection of the negative half-spaces is non-empty.
2) For any finite subset of $\mathfrak{H}$, the intersection of the positive half-spaces is non-empty.

The group $W(1)$ acts on the orientations of $(V,\mathfrak{H})$. The image by $w\in W(1)$ of an orientation o is the orientation $o\bullet w$ is such that the $o\bullet w$-positive side of $H\in\mathfrak{H}$ is the image by w^{-1} of the o-positive side of $w(H)$. The action of $W(1)$ factorizes through an action of W.

Definition 5.16. *The spherical orientation $o_{\Delta'}$ associated to a basis Δ' of Σ. For $H\in\mathfrak{H}$ there exists a unique pair $(\beta,k)\in\Sigma^{aff}$ with*

$$H=\operatorname{Ker}(\beta+k)\quad \textit{and } \beta \textit{ is } \Delta'\textit{-positive.}$$

The $o_{\Delta'}$-positive side of H is the set of $x\in V$ where $\beta(x)+k>0$.

The set of spherical orientations is stable by the action of $W=\Lambda\rtimes W_0$:

Proposition 5.17. *A spherical orientation $o_{\Delta'}$ is fixed by Λ and $o_{\Delta'}\bullet w=o_{w^{-1}(\Delta')}$ for $w\in W_0$.*

Proof. Let $\lambda\in\Lambda, x\in V,\beta\in\Sigma,k\in\mathbb{Z}$.

We suppose that β is Δ'-positive. Then x belongs to the $o_{\Delta'}\bullet\lambda$-positive side of $\operatorname{Ker}(\beta+k)$ if $x+\nu(\lambda)$ belongs to the $o_{\Delta'}$-positive side of $\operatorname{Ker}(\beta+k)+\nu(\lambda)$. We have $\operatorname{Ker}(\beta+k)+\nu(\lambda)=\operatorname{Ker}(\beta+k-(\beta\circ\nu)(\lambda))$ and $\beta(x+\nu(\lambda))+k-(\beta\circ\nu)(\lambda)=\beta(x)+k$. Therefore $o_{\Delta'}\bullet\lambda=o_{\Delta'}$.

We suppose that β is $w^{-1}(\Delta')$-positive, that is, $w(\beta)$ is Δ'-positive, and that x belongs to the $o_{w^{-1}(\Delta')}$-positive side of $\operatorname{Ker}(\beta+k)$. We have $\beta(x)+k>0$ and $\beta(x)=w(\beta)(w.x)$. Hence $w.x$ belongs to the $o_{\Delta'}$-positive side of $\operatorname{Ker}(w(\beta)+k)$. We have $\operatorname{Ker}(w(\beta)+k)=w.\operatorname{Ker}(\beta+k)$. Hence x belongs to the $o_{\Delta'}\bullet w$-positive side of $\operatorname{Ker}(\beta+k)$. ☐

Let o be an orientation of $(V,\mathfrak{H})$. We say that we cross $H\in\mathfrak{H}$ in the o-positive direction if we go from the o-negative side to the o-positive side (in the o-negative direction otherwise). Let $(w,s)\in W\times S^{aff}$. When we walk from the alcove $w.\mathfrak{C}$ to the alcove $ws.\mathfrak{C}$, we cross the affine hyperplane $H_{wsw^{-1}}\in\mathfrak{H}$ fixed by wsw^{-1}.

Definition 5.18. *Let o be an orientation of $(V,\mathfrak{H})$ and let $(w,s)\in W\times S^{aff}$. Let*

- $\epsilon_o(w,s)=1$ *if $w.\mathfrak{C}$ belongs to the o-negative side of $H_{wsw^{-1}}$,*
- $\epsilon_o(w,s)=-1$ *if $w.\mathfrak{C}$ belongs to the o-positive side of $H_{wsw^{-1}}$.*

Let $\epsilon_o(w(1),s(1))=\epsilon_o(w,s)$ for $w(1),s(1)$ in $W(1)$ above w,s.

When we walk from $w.\mathfrak{C}$ to $ws.\mathfrak{C}$, we cross $H_{wsw^{-1}}$ in the o-positive, resp. o-negative, direction if $\epsilon_o(w,s)=1$, resp. -1. We say that we cross $H_{wsw^{-1}}$ in the $\epsilon_o(w,s)$ direction with respect to o.

Let $s_1,\dots,s_n$ in S^{aff}. The walk from $\mathfrak{C}$ to $s_1\dots s_n(\mathfrak{C})$ following the gallery $\mathfrak{C}$, $s_1.\mathfrak{C}$, $s_1s_2.\mathfrak{C},\dots,s_1\dots s_n.\mathfrak{C}$, crosses the hyperplanes

$$H_{s_1}=H_{\tau_1},\ s_1.H_{s_2}=H_{\tau_2},\ \dots,\ s_1s_2\dots s_{n-1}.H_{s_n}=H_{\tau_n},$$

where $\mathfrak{T}(s_1,\dots,s_n)=(\tau_1,\dots,\tau_n)$ (**??**), in the

$$\epsilon_o(1,s_1),\epsilon_o(s_1,s_2),\dots,\epsilon_o(s_1\dots s_{i-1},s_i),\dots\epsilon_o(s_1\dots s_{n-1},s_n) \tag{79}$$

directions with respect to o.

Example 5.19. *For $J \subset \Delta$, let $S_J = \{s_\beta \mid \beta \in J\}$, let W_J be the subgroup of W_0 generated by S_J and let w_J be the element of maximal length in W_J. We have $w_J^2 = 1$, $S_\Delta = S$, $W_\Delta = W_0$ and $w_\Delta(\Delta) = -\Delta$.*

1) For $w \in W_J, s \in S_J$, we have $\epsilon_{o_{w_J(\Delta)}}(w,s) = 1$ if and only if $\ell(ws) > \ell(w)$.

2) For $w \in W_0, s \in S$, we have $\epsilon_{o_{-\Delta}}(w,s) = 1$ if and only if $\ell(ws) > \ell(w)$.

Proof. We prove 1). Let $\Sigma_J \subset \Sigma$ be the root system generated by J. Let $\beta \in J$ such that $s = s_\beta$. We have $H_{sws^{-1}} = \mathrm{Ker}\, w(\beta)$. Let $x \in \mathfrak{C}$. The alcove $w.\mathfrak{C}$ is contained in the $o_{w_J(\Delta)}$-negative side of $H_{sws^{-1}}$ if and only if $w(\beta)$ is $w_J(\Delta)$-negative because $w(\beta)(w.x) = \beta(x)$ is positive. The root $w(\beta)$ belongs to Σ_J; hence $w(\beta)$ is $w_J(\Delta)$-negative, if and only if $w_J w(\beta)$ is Δ-negative, if and only if $w(\beta)$ is Δ-positive, if and only if $\ell(ws_\beta) > \ell(w)$. □

Lemma 5.20. *Let $s, s' \in S^{aff}$ with ss' of finite order $n(s,s')$. Then the sequences with $n(s,s')$ terms*

$$(\epsilon_o(1,s), \epsilon_o(s,s'), \epsilon_o(ss',s), \ldots) \text{ and } (\epsilon_o(1,s'), \epsilon_o(s',s), \epsilon_o(s's,s'), \ldots)$$

are equal to $(1,1\ldots,1,-1,-1\ldots-1)$ and $(-1,-1\ldots,-1,1,1\ldots1)$,

or to $(-1,-1\ldots,-1,1,1\ldots1)$ *and* $(1,1\ldots,1,-1,-1\ldots-1)$,

where $(1,1\ldots,1)$ have the same length k, $0 \le k \le n(s,s')$, in both sequences.

Proof. ([**?**] Proof of Thm. 3.3.1). □

Lemma 5.21. *For (w,s) in $W \times S^{aff}$ and for $u \in \Omega$, we have $\epsilon_o(ws,s) \neq \epsilon_o(w,s)$ and*

$$\epsilon_{o\bullet w}(w',s) = \epsilon_o(ww',s), \quad \epsilon_o(wu,s) = \epsilon_o(w,usu^{-1}), \quad \epsilon_{o\bullet u}(w,s) = \epsilon_o(uwu^{-1}, u^{-1}su).$$

Proof. 1) $ws(\mathfrak{C})$ and $w(\mathfrak{C})$ are in different sides of $H_{wsw^{-1}}$.

2) $\epsilon_{o\bullet w}(w',s) = 1$ if and only if $w'.\mathfrak{C}$ is contained in the $o \bullet w$-negative side of $H_{w'sw'^{-1}}$. The $o \bullet w$-negative side of $H_{w'sw'^{-1}}$ is the image by w^{-1} of the o-negative side of $H_{ww'sw'^{-1}w^{-1}}$. Hence $\epsilon_{o\bullet w}(w',s) = 1$ if and only if $ww'.\mathfrak{C}$ is contained in the o-negative side of $H_{ww'sw'^{-1}w^{-1}}$, if and only if $\epsilon_o(ww',s) = 1$.

3) We have $u.\mathfrak{C} = \mathfrak{C}$. We have $\epsilon_o(wu,s) = 1$ if and only if $wu.\mathfrak{C} = w.\mathfrak{C}$ is contained in the o-negative side of $H_{wusu^{-1}w^{-1}}$ if and only if $\epsilon_o(w,usu^{-1}) = 1$.

4) We compute $\epsilon_{o\bullet u}(w,s) = \epsilon_o(uw,s) = \epsilon_o(uwu^{-1}, u^{-1}su)$.

□

5.3 Alcove walk bases

Notations as in Thm. **??** and Def. **??**. We will associate to any orientation of $(V,\mathfrak{H})$ a basis of the R-algebra $\mathcal{H}_R(q_s,c_s)$ of $W(1)$ with parameters $(q_s,c_s)_{s\in S^{aff}}$.

Definition 5.22. *For $(w,s) \in W(1) \times S^{aff}(1)$ and an orientation $o \in (V,\mathfrak{H})$, and for $T_s \in \mathcal{H}_R(q_s,c_s)$ we set:*

$$(80) \qquad T_s^{\epsilon_o(w,s)} = T_s \text{ if } \epsilon_o(w,s) = 1, \quad T_s^{\epsilon_o(w,s)} = T_s^* = T_s - c_s \text{ if } \epsilon_o(w,s) = -1$$

*where $\epsilon_o(w,s)$ is defined in Def. **??**. For $s_1,\ldots,s_n$ in $S^{aff}(1), u, u' \in \Omega(1)$, we set:*

$$(81) \qquad E_o(u,s_1,\ldots,s_n,u') = T_u T_{s_1}^{\epsilon_o(u,s_1)} \ldots T_{s_i}^{\epsilon_o(us_1\ldots s_{i-1},s_i)} \ldots T_{s_n}^{\epsilon_o(us_1\ldots s_{n-1},s_n)} T_{u'}.$$

We remark that $E_o(s) = T_s^{\epsilon_o(1,s)}$, $E_{o\bullet s}(s) = T_s^{\epsilon_{o\bullet s}(1,s)}$,

$$(82) \qquad E_{o\bullet s}(s) - E_o(s) = \epsilon_{o\bullet s}(1,s) c_s, \quad E_o(s) E_{o\bullet s}(s) = q_s s^2.$$

(Remark **??** and Lemma **??**).

We suppose first that $q_s = 1$ for all $s \in S^{aff}/\sim$. In this case $T_s^* = T_{s^{-1}}^{-1}$.

Proposition 5.23. *When $q_s = 1$ for all $s \in S^{aff}/\sim$, $E_o(w) = E_o(u, s_1, \dots, s_n, u')$ depends only on the product $w = us_1 \dots s_n u'$.*

Proof. ([?] Thm. 3.3.1, [?] Thm. 3.3.19).

a) Let $s, s' \in S^{aff}$ of finite order $n(s,s')$ and $s(1), s'(1) \in S^{aff}(1)$ satisfying b3 (this is the only place where b3 is needed). We show

$$E_o(s(1), s'(1), \dots) = E_o(s'(1), s(1), \dots).$$

By symetry we can suppose that the sequences in Lemma ?? are $(1, 1 \dots, 1, *, * \dots *)$ and $(*, * \dots, *, 1, 1 \dots 1)$ with k terms equal to 1. We decompose accordingly the products $s(1)s'(1) \dots = w_k w_{n(s,s')-k}$, $s'(1)s(1) \dots = w'_{n(s,s')-k} w'_k$. By the braid relations,

$$E_o(s(1), s'(1), \dots) = T_{w_k} T^{-1}_{w^{-1}_{n(s,s')-k}} \text{ and } E_o(s'(1), s(1), \dots) = T^{-1}_{w'^{-1}_{n(s,s')-k}} T_{w'_k}.$$

The element $w'^{-1}_{n(s,s')-k} w_k = w'_k w^{-1}_{n(s,s')-k}$ has length $n(s,s')$ because $w'^{-1}_{n(s,s')-k}$ ends by $s'(1)^{-1}$ while w_k begins with $s(1)$. The additivity of the lengths is satisfied and by the braid relations $T_{w'^{-1}_{n(s,s')-k}} T_{w_k} = T_{w'_k} T_{w^{-1}_{n(s,s')-k}}$. We deduce $E_o(s(1), s'(1), \dots) = E_o(s'(1), s(1), \dots)$.

b) Let $t_1 \dots t_n \in Z_k$ such that $s_1 \dots s_n = s'_1 \dots s'_n$ where $s'_i = s_i t_i$ for $1 \leq i \leq n$. The equality

$$E_o(s_1, \dots, s_n, u) = E_o(s'_1, \dots, s'_n, u).$$

is obvious by the braid relations using that the elements of Z_k have length 0, act trivially on V, and that Z_k is normal in $W(1)$.

c) We suppose that $s_{i+1} = s_i^{-1}$. We have

$$E_o(s_1, \dots, s_n, u) = E_o(s_1, \dots s_{i-1}, s_{i+2}, \dots, s_n, u),$$

because $T_s^{\epsilon_o(w,s)} T_{s^{-1}}^{\epsilon_o(ws,s^{-1})} = 1$. This follows from $\epsilon_o(w,s) \neq \epsilon_o(ws,s)$ (Lemma ??); we recall that $\epsilon_o(ws, s^{-1}) = \epsilon_o(ws, s)$ and that $q_s = 1$.

d) As the elements of the group $\Omega(1)$ normalize $S^{aff}(1)$ and have length 0, the equality

$$E_o(u, s_1, \dots, s_n, u') = E_o(us_1u^{-1}, \dots, us_nu^{-1}, uu')$$

follows from $T_u T_s^* = T_{usu^{-1}}^* T_u$ and $\epsilon_{o \bullet u}(w,s) = \epsilon_o(uwu^{-1}, usu^{-1})$ for $s \in S^{aff}(1), w \in W(1), u \in \Omega(1)$ (Prop. ?? 5)) .

e) The proposition follows from a), b), c), d) and ([?] IV.1.5 Prop. 5).

□

Proposition 5.24. *When $q_s = 1$ for all $s \in S^{aff}/\sim$, we have the product formula*

(83) $$E_o(ww') = E_o(w) E_{o\bullet w}(w') \quad (w, w' \in W(1)).$$

Proof. Let $w = s_1 \dots s_n u, w' = s'_1 \dots s'_m u'$ with $s_i, s'_j \in S^{aff}(1), u' \in \Omega(1)$. From Prop. ?? we have

$$E_o(ww') = E_o(s_1, \dots, s_n, us'_1u^{-1}, \dots us'_mu^{-1}, uu').$$

From $\epsilon_{o\bullet w}(1,s) = \epsilon_o(w,s)$ for $(w,s) \in W(1) \times S^{aff}(1)$ (Lemma ??), the right hand side is equal to

$$E_o(s_1 \dots s_n) E_{o\bullet w}(us'_1u^{-1}, \dots us'_mu^{-1}, uu')$$

and $E_{o\bullet w}(us'_1u^{-1}, \dots us'_mu^{-1}, uu') = E_{o\bullet w}(uw') = T_u E_{o\bullet w}(w')$ from Prop. ??. Hence $E_o(ww') = E_o(s_1 \dots s_n) T_u E_{o\bullet w}(w') = E_o(w) E_{o\bullet w}(w')$. □

We recall $q_{w,w'} = (q_w q_{w'} q^{-1}_{ww'})^{1/2}$ for $w, w' \in W(1)$ (Def. ??).

Theorem 5.25. *Let o be an orientation of $(V, \mathfrak{H})$, let $w, w' \in W(1)$, and let a reduced decomposition $w = s_1 \dots s_{\ell(w)} u$ $u \in \Omega(1)$, $s_i \in S^{aff}(1)$ for $1 \leq i \leq \ell(w)$. Then,*

$$E_o(w) = E_o(s_1, \dots, s_{\ell(w)}, u) \in \mathcal{H}_R(q_s, c_s)$$

depends only on w and $E_o(w) E_{o \bullet w}(w') = q_{w,w'} E_o(ww')$.

Proof. As in the subsection (**??**), the $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$ satisfies Prop. **??** and **??**. Let $s \in S^{aff}(1)$ and $u \in \Omega(1)$. The elements $T_s, T_u, T_s^* = T_s - \mathfrak{q}_s^{-1} c_s$ and $E_o(w)$ in $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$ are sent by the isomorphism $h \mapsto \tilde{h}$ (**??**) to elements of $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$ equal to

$$T_s\tilde{} = \mathfrak{q}_s^{-1} T_s, \quad T_u\tilde{} = T_u, \quad (T_s^*)\tilde{} = \tilde{T}_s - \mathfrak{q}_s^{-1} c_s = \mathfrak{q}_s^{-1}(T_s - c_s) = \mathfrak{q}_s^{-1} T_s^*,$$
$$\tilde{E}_o(w) = (T_{s_1}^{\epsilon_o(1, s_1)})\tilde{} \dots (T_{s_i}^{\epsilon_o(s_1 \dots s_{i-1}, s_i)})\tilde{} \dots (T_{s_{\ell(w)}}^{\epsilon_o(s_1 \dots s_{n-1}, s_{\ell(w)})})\tilde{}\, \tilde{T}_u.$$

We have

$$\tilde{E}_o(w) = \mathfrak{q}_w^{-1} E_o(s_1, \dots, s_{\ell(w)}, u).$$

In $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$, the product $\mathfrak{q}_w \tilde{E}_o(w)$ depends only on w hence the same is true for $E_o(s_1, \dots, s_{\ell(w)}, u)$. The product formula in $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$ implies the product formula

(84) $$E_o(w) E_{o \bullet w}(w') = \mathbf{q}_{w,w'} E_o(ww'), \quad \mathbf{q}_{w,w'} = \mathfrak{q}_w \mathfrak{q}_{w'} \mathfrak{q}_{ww'}^{-1},$$

in $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$. The product formula holds true in the generic $R[\mathbf{q}_s]$-subalgebra $\mathcal{H}_{R[\mathbf{q}_s]}(\mathbf{q}_s, c_s)$. By specialisation, it holds true in $\mathcal{H}_R(q_s, c_s)$. □

The Bruhat partial order $<$ on (W^{aff}, S^{aff}) extends to $W = W^{aff} \rtimes \Omega : wu < w'u'$ for $w, w' \in W^{aff}, u, u' \in \Omega$ if $w < w'$ and $u = u'$. The extended Bruhat partial order $<$ on W inflates to $W(1) : w(1) < w'(1)$ for $w(1), w'(1) \in W(1)$ above $w, w' \in W$ if $w < w'$.

Corollary 5.26. *$E_o(w) = T_w + \sum_{w' < w} a_{w'} T_{w'}$ for $w' \in W(1), a_{w'} \in R$.*
Hence $(E_o(w))_{w \in W(1)}$ is an R-basis of $\mathcal{H}_R(q_s, c_s)$.

Proof. $E_o(w) - T_w$ is a finite sum of elements $T_{w'}$ where $w' = s'_1 \dots s'_r t u$ for $t \in Z_k, u \in \Omega(1)$, and $(s'_1, \dots, s'_r)$ extracted from the sequence $(s_1, \dots, s_n)$ with $r < n$. □

One does not need to change the orientation o in the product formula in $E_o(\Lambda(1))$ when $\Lambda(1)$ fixes o. By Prop. **??**:

Corollary 5.27. *Let o be a spherical orientation of $(V, \mathfrak{H})$. The R-submodule of $\mathcal{H}_R(q_s, c_s)$ of basis $(E_o(\lambda))_{\lambda \in \Lambda(1)}$ is an R-subalgebra $\mathcal{A}_o$ of $\mathcal{H}_R(q_s, c_s)$, with product*

$$E_o(\lambda) E_o(\lambda') = q_{\lambda, \lambda'} E_o(\lambda \lambda') \quad (\lambda, \lambda' \in \Lambda(1)).$$

From now on, we consider only spherical orientations o of $(V, \mathfrak{H})$. For each closed Weyl chamber $\mathfrak{D}$ of V, let $\Lambda(1)_{\mathfrak{D}}$ the monoid of elements $\lambda \in \Lambda(1)$ such that $\nu(\lambda) \in \mathfrak{D}$.

Corollary 5.28. *Let o be a spherical orientation of $(V, \mathfrak{H})$ and let $\mathfrak{D}$ be a closed Weyl chamber of V. Then, the monoid R-algebra $R[\Lambda(1)_{\mathfrak{D}}]$ embeds in $\mathcal{H}_R(q_s, c_s)$ by the linear map such that*

$$\lambda \mapsto E_o(\lambda) \quad (\lambda \in \Lambda(1)_{\mathfrak{D}}).$$

Proof. Let $\lambda, \lambda' \in \Lambda(1)$. We have $\mathbf{q}_{\lambda, \lambda'} = 1$ if and only if $\ell(\lambda \lambda') = \ell(\lambda) + \ell(\lambda')$(Lemma **??**) if and only if $\nu(\lambda), \nu(\lambda')$ belong to the same closed Weyl chamber (Example **??**). □

Example 5.29. *Let Δ' be a basis of Σ and $\lambda \in \Lambda(1)$ such that $\nu(\lambda)$ belong to the closed Δ'-dominant Weyl chamber. Then $E_{o_{\Delta'}}(\lambda) = T_\lambda$.*

Proof. When we walk from the alcove $\mathfrak{C}$ to the alcove $\mathfrak{C} + \nu(\lambda)$ we cross hyperplanes in the $o_{\Delta'}$- positive direction because we walk away from the most $o_{\Delta'}$-negative point of V which lies infinitely deep in the $-\Delta'$ dominant Weyl chamber. □

Example 5.30. *With the notations of Example **??**, let $J \subset \Delta$, let o be the spherical orientation $o_{w_J(\Delta)}$, and let $w \in W_J(1)$ of reduced decomposition $w = s_1 \dots s_n$ with $s_i \in S_J(1)$. Then*

$$E_o(w) = T_w, \quad E_{o \bullet w^{-1}}(w) = T^*_w, \quad E_o(w) E_{o\bullet w}(w^{-1}) = q_w.$$

In particular, $E_{o_{-\Delta}}(w) = T_w$, $E_{o_{-\Delta}\bullet w^{-1}}(w) = T^*_w$, $E_{o_{-\Delta}}(w)E_{o_{-\Delta}\bullet w}(w^{-1}) = q_w$ for $w \in W_0$.

Proof. 1) We have $\epsilon_o(w, s) = 1$ for $(w, s) \in W_J \times S_J$ with $\ell(ws) > \ell(w)$ (Example **??**). Applying this to $(s_1 \dots s_{i-1}, s_i)$ we get $E_o(w) = T_w$.

2) $\epsilon_{o\bullet w^{-1}}(s_1 \dots s_{i-1}, s_i) = \epsilon_o(s_n \dots s_i, s_i) \neq \epsilon_o(s_n \dots s_{i+1}, s_i) = 1$ (Lemma **??**).

3) Prop. **??**, **??**. □

Example 5.31. *For $s \in S(1), \tilde{s} \in S^{aff}(1) - S(1), w \in W_0(1)$, we have:*

$$E_{o_{-\Delta}}(\tilde{s}) = T^*_{\tilde{s}}, \quad E_{o_{-\Delta}\bullet w}(s) = T_s \text{ if and only if } \ell(ws) > \ell(w).$$

Proof. a) Let $x \in \mathfrak{C}$. We have $\epsilon_o(1, \tilde{s}) = -1$ if and only if x belongs in the $o_{-\Delta}$-positive part of $H_{\tilde{s}}$. We have $H_{\tilde{s}} = \mathrm{Ker}(-\tilde{\beta}_j + 1)$ where $\tilde{\beta}_j$ is the longest root of an irreducible component Δ_j of Δ, and $-\tilde{\beta}_i(x) + 1 > 0$. As $-\tilde{\beta}_i$ is $-\Delta$ positive, x belongs in the $o_{-\Delta}$-positive part of $\mathrm{Ker}(-\tilde{\beta}_j + 1)$.

b) $\epsilon_{o_{-\Delta}\bullet w}(1, s) = \epsilon_{o_{-\Delta}}(w, s)$ by Lemma **??**. Then, Example **??**. □

Lemma 5.32. *Let Δ' be a basis of Σ, $\beta \in \pm\Delta'$ and $w \in W(1)$ such that $\ell_\beta(w) = 0$. Then,*

$$E_{o_{\Delta'}\bullet s_\beta}(w) = E_{o_{\Delta'}}(w).$$

Proof. The $o_{\Delta'}$-positive and $o_{\Delta'} \bullet s_\beta$-positive sides of $\mathrm{Ker}(\alpha + k)$ for $\alpha + k \in \Sigma^{aff}$, are equal if $\alpha \neq \pm\beta$, because $\beta \in \pm\Delta'$ hence $s_\beta(\alpha)$ and α are simultaneously Δ'-positive or Δ'-negative.

$\ell_\beta(w) = 0$ if and only if the alcoves $\mathfrak{C}, w(\mathfrak{C})$ are on the same side of $H_{\beta+n}$ for all $n \in \mathbb{Z}$ (Lemma **??**). Hence no $H_{\beta+n}$ belongs to the set of affine hyperplanes H_τ for $\tau \in \mathfrak{T}_w$ separating $\mathfrak{C}$ and $w(\mathfrak{C})$.

We deduce that for all $\tau \in \mathfrak{T}_w$, the $o_{\Delta'}$-positive side of H_τ is equal to the $o_{\Delta'} \bullet s_\beta$-positive side of H_τ, or equivalently, $E_{o_{\Delta'}\bullet s_\beta}(w) = E_{o_{\Delta'}}(w)$. □

5.4 Bernstein relations

Let o be a spherical orientation associated to a basis Δ_o of Σ, let $\alpha \in \Delta_o$, let $s \in W(1)$ above s_α and let $\lambda \in \Lambda(1)$. We denote $s(\lambda) = s\lambda s^{-1}$.

Proposition 5.33. *$E_o(s)E_{o\bullet s}(\lambda) = E_o(s(\lambda))E_o(s)$.*

Proof. The product formula (Thm. **??**) implies $E_o(s)E_{o\bullet s}(\lambda) = q_{s,\lambda}E_o(s\lambda)$ and $E_o(s(\lambda))E_o(s) = q_{s(\lambda),s}E_o(s\lambda)$. We have $q_{s,\lambda} = q_{s(\lambda),s}$ because $q^2_{s,\lambda} = q_s q_\lambda q^{-1}_{s\lambda}$ and $q^2_{s\lambda s^{-1},s} = q_{s\lambda s^{-1}} q_s q^{-1}_{s\lambda}$ by Definition **??**, and $q_{s\lambda s^{-1}} = q_\lambda$ by Prop. **??**. □

Definition 5.34. *For $s \in S_o(1), \lambda \in \Lambda(1)$,*

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda) \tag{85}$$

is called a Bernstein element at $\lambda \in \Lambda(1)$. When it does not vanish, its expansion in the alcove walk basis $(E_o(w))_{w\in W(1)}$ is called a Bernstein relation.

We have $s_\alpha = s_{-\alpha}$. One of α or $-\alpha$ is Δ-positive, and denoted by β. We denote by $\Lambda(1)^s$ the kernel of the homomorphism

$$\beta \circ \nu : \Lambda(1) \to \mathbb{Z}.$$

$\Lambda(1)^s$ is also the group of $\lambda \in \Lambda(1)$ such that $\nu(\lambda) \in V$ is fixed by s_β. We have

$$E_{o\bullet s}(\lambda) - E_o(\lambda) = 0 \quad \text{if } \lambda \in \Lambda(1)^s,$$

by Lemma **??** because $\ell_\beta(\lambda) = (\beta\circ\nu)(\lambda)$ by Prop. **??** 1) as β is Δ-positive. The Bernstein element vanishes when $\beta \circ \nu(\lambda) = 0$.

We suppose now $(\beta \circ \nu)(\lambda) \neq 0$.

The image of $\beta \circ \nu$ is $\delta\mathbb{Z}$ with $\delta \in \{1, 2\}$. Let $\epsilon_\beta(\lambda) \in \{1, -1\}$ the sign of $\beta \circ \nu(\lambda)$ and $n_\beta(\lambda)$ the positive integer such that

$$\beta \circ \nu(\lambda) = \epsilon_\beta(\lambda)\delta n_\beta(\lambda).$$

We choose an element $\lambda_s \in \Lambda(1)$ such that $(\beta \circ \nu)(\lambda_s)$ is negative and generates $\delta\mathbb{Z}$. If $\delta = 1$ there is no other condition on λ_s. If $\delta = 2$,

$$\lambda_s = sw\tilde{s}w^{-1}, \ \ell(\lambda_s) = 2\ell(w) + 2$$

where $w \in W^{aff}(1)$, $\tilde{s} \in S^{aff}(1)$ is above $s_{-\tilde{\beta}+1}$ for the highest root $\tilde{\beta}$ of the irreducible component Σ_j of Σ containing β, and the image of λ_s in Λ is $s_{\beta+1}s_\beta$. The existence of λ_s follows from Lemma **??**.

Then we define elements B_n of $\mathcal{A}_o$ for $n \in \mathbb{N}_{>0}$ (for $\delta = 1$ or 2):

$$B_1 = (\delta - 1)(w \bullet c_{\tilde{s}})s^2 E_o(\lambda_s^{-1}) + c_s. \tag{86}$$

(the term $(w \bullet c_{\tilde{s}})$ is defined only when $\delta = 2$), and

$$B_n := \sum_{k=0}^{n-1} E_o(s(\lambda_s^k))B_1 E_o(\lambda_s^{-k}). \tag{87}$$

When $q_s = 1$ for all $s \in S^{aff}/\sim$, we will compute the expansion of the Bernstein element at λ_s. A formal computation will give the expansion of the Bernstein element at any $\lambda \in \Lambda(1)$.

Theorem 5.35. *We suppose that $q_s = 1$ for all $s \in S^{aff}/\sim$. Let $\lambda \in \Lambda(1)$ with $\beta\circ\nu(\lambda) = \epsilon_\beta(\lambda)\delta n_\beta(\lambda) \neq 0$. Then $E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ is equal to*

$$\epsilon_{o\bullet s}(1, s)B_{n_\beta(\lambda)}\ E_o(\lambda) \text{ if } \epsilon_\beta(\lambda) = -1, \quad \epsilon_o(1, s)\ E_o(s(\lambda))B_{n_\beta(\lambda)} \text{ if } \epsilon_\beta(\lambda) = 1. \tag{88}$$

For q_s general, we will deduce the expansion of the Bernstein element at any $\lambda \in \Lambda(1)$ from the case $q_s = 1$ by the method explained in subsection **??**.

We prove first Thm. **??**.

Lemma 5.36. *We suppose that $q_s = 1$ for $s \in S^{aff}/\sim$ and $(\beta\circ\nu)(\Lambda) = \mathbb{Z}$. Let $\lambda_s \in \Lambda(1)$ with $\beta \circ \nu(\lambda_s) = -1$. We have*

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1,s)c_s E_o(\lambda_s).$$

Proof. We have $\beta \in s_\beta(\Sigma^-)$ because β is Δ-positive. The hypothesis $\beta \circ \nu(\lambda_s) = -1$ implies $\ell_\beta(s\lambda_s) = 0$ by Cor. **??** and $E_o(s\lambda_s) = E_{o\bullet s}(s\lambda_s)$ by Lemma **??**. Applying the product formula,

$$\begin{aligned} E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) &= E_o(s\lambda_s) - E_o(s)E_o(\lambda_s) = E_{o\bullet s}(s\lambda_s) - E_o(s)E_o(\lambda_s) \\ &= E_{o\bullet s}(s)E_o(\lambda_s) - E_o(s)E_o(\lambda_s) = (E_{o\bullet s}(s) - E_o(s))E_o(\lambda_s) = \epsilon_{o\bullet s}(1,s)c_sE_o(\lambda_s). \end{aligned}$$

The last equality uses (**??**). ☐

Let $\tilde\beta$ be the highest root of the irreducible component Σ_j of Σ containing β. We choose $\tilde s \in S^{aff}(1)$ above $s_{-\tilde\beta+1}$.

Lemma 5.37. *We suppose that $q_s = 1$ for $s \in S^{aff}/\sim$ and $(\beta\circ\nu)(\Lambda) = 2\mathbb{Z}$. Let $\lambda_s \in \Lambda(1)$ above $s_{\beta+1}s_\beta$ satisfying $\lambda_s = sw\tilde s w^{-1}$ and $\ell(\lambda_s) = 2\ell(w)+2$ for some $w \in W^{aff}(1)$. Then*

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1,s)\ [(w\bullet c_{\tilde s})s^2 + c_sE_o(\lambda_s)].$$

Proof. Applying the product formula

$$\begin{aligned} E_o(\lambda_s) &= E_o(sw\tilde s w^{-1}) = E_o(s)E_{o\bullet s}(w)E_{o\bullet sw}(\tilde s)E_{o\bullet sw\tilde s}(w^{-1}), \\ E_{o\bullet s}(\lambda_s) &= E_{o\bullet s}(s)E_o(w)E_{o\bullet w}(\tilde s)E_{o\bullet w\tilde s}(w^{-1}). \end{aligned}$$

We note that $o\bullet sw\tilde s = o\bullet w$ because $sw\tilde s = \lambda_s w$ and the orientation o is spherical. We have $\nu(\lambda_s) = -\beta^\vee$ and $\beta\circ\nu(\lambda_s) = -2$ generates $(\beta\circ\nu)(\Lambda)$. By Lemma **??**, the two affine hyperplanes $\mathrm{Ker}(\beta+k)$ separating $\mathfrak{C}, \lambda_s(\mathfrak{C})$ are $\mathrm{Ker}\,\beta, \mathrm{Ker}(\beta+1)$. The image of $sw\tilde s w^{-1}s^{-1}$ in W is $s_{\beta+1}$. The o-positive side is equal to the $o\bullet s$-positive side for the other affine hyperplanes between $\mathfrak{C}, \lambda_s(\mathfrak{C})$. We deduce:

$$\begin{aligned} &E_{o\bullet w}(w^{-1}) = E_{o\bullet sw\tilde s}(w^{-1}) = E_{o\bullet w\tilde s}(w^{-1}),\ E_{o\bullet s}(w) = E_o(w), \\ &E_o(\lambda_s) = E_o(s)E_o(w)E_{o\bullet sw}(\tilde s)E_{o\bullet w}(w^{-1}), \end{aligned}$$

and $E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)$ is the sum of $(E_{o\bullet s}(s) - E_o(s))E_o(w)E_{o\bullet sw}(\tilde s)E_{o\bullet w}(w^{-1})$ and of $E_{o\bullet s}(s)E_o(w)(E_{o\bullet w}(\tilde s) - E_{o\bullet sw}(\tilde s))E_{o\bullet w}(w^{-1})$. By (**??**),

$$\begin{aligned} E_{o\bullet s}(\lambda_s) - E_o(\lambda_s) &= \epsilon_{o\bullet s}(1,s)c_sE_{o\bullet s}(w\tilde s w^{-1}) + \epsilon_{o\bullet w}(1,\tilde s)E_{o\bullet s}(sw)c_{\tilde s}E_{o\bullet w}(w^{-1}) \\ &= \epsilon_{o\bullet s}(1,s)(c_sE_{o\bullet s}(w\tilde s w^{-1}) + (sw\bullet c_{\tilde s})E_{o\bullet s}(sw)E_{o\bullet w}(w^{-1})) \\ &= \epsilon_{o\bullet s}(1,s)(c_sE_{o\bullet s}(w\tilde s w^{-1}) + (sw\bullet c_{\tilde s})E_{o\bullet s}(s)). \end{aligned}$$

For the second line, we note that $\epsilon_{o\bullet s}(1,s) = \epsilon_{o\bullet w}(1,\tilde s)$ because we cross the hyperplanes $\mathrm{Ker}\,\beta$ and $\mathrm{Ker}(\beta+1)$ in the same sense when we go from $\mathfrak{C}$ to $\lambda_s(\mathfrak{C}) = \mathfrak{C} + \nu(\lambda_s)$. Multiplying on the left side by $E_o(s)$ we obtain

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1,s)(c_sE_o(\lambda_s) + (w\bullet c_{\tilde s})s^2),$$

using that the product formula, $E_o(s)c = (s\bullet c)E_o(s)$ for $c \in R[Z_k]$, $s\bullet c_s = c_s$ and (**??**). ☐

We summarize: for $y \in \Lambda(1)^s, x = \lambda_s$,

$$B_1 = (\delta - 1)(w\bullet c_{\tilde s})s^2E_o(x^{-1}) + c_s. \tag{89}$$

(the term $(w \bullet c_{\tilde{s}})$ is defined only when $\delta = 2$), we proved (Lemma **??**, **??**, **??**):

$$E_{o \bullet s}(y) - E_o(y) = 0,$$
$$E_o(s)(E_{o \bullet s}(x) - E_o(x)) = \epsilon_{o \bullet s}(1, s)((\delta - 1)(w \bullet c_{\tilde{s}})s^2 + c_s E_o(x)) = \epsilon_{o \bullet s}(1, s) B_1 E_o(x).$$

By a formal computation we will deduce the expansion of the Bernstein element at any $\lambda \in \Lambda(1)$. First, we take $\lambda = x^{-1}$. We recall that $\epsilon_o(1, s) \neq \epsilon_{o \bullet s}(1, s)$.

Lemma 5.38. *We suppose that $q_s = 1$ for all $s \in S^{aff}/\sim$. Then,*

$$E_o(s)(E_{o \bullet s}(x^{-1}) - E_o(x^{-1})) = \epsilon_o(1, s) E_o(s(x^{-1})) B_1.$$

Proof. We recall that $E_o(w)$ is invertible for $w \in W(1)$ because $q_s = 1$ for $s \in S^{aff}/\sim$ and that the inverse of $E_o(\lambda)$ is $E_o(\lambda^{-1})$ because the orientation o is spherical. We multiply the equality $E_o(s)(E_{o \bullet s}(x) - E_o(x)) = \epsilon_{o \bullet s}(1, s) B_1 E_o(x)$ on the left by

$$E_o(s) E_{o \bullet s}(x)^{-1} E_o(s)^{-1} = E_o(s) E_{o \bullet s}(x^{-1}) E_{o \bullet s}(s^{-1}) = E_o(s x^{-1} s^{-1}) = E_o(s(x^{-1})),$$

and on the right by $E_o(x)^{-1} = E_o(x^{-1})$ to obtain

$$E_o(s)(E_{o \bullet s}(x^{-1}) - E_o(x^{-1})) = -\epsilon_{o \bullet s}(1, s) E_o(s(x^{-1})) B_1,$$

and we use $\epsilon_o(1, s) = -\epsilon_{o \bullet s}(1, s)$. □

Now we relate the Bernstein element at $z \in \Lambda(1)$ to the Bernstein element at z^n for $n \in \mathbb{N}_{>0}$.

Lemma 5.39. *We suppose that $q_s = 1$ for all $s \in S^{aff}/\sim$. Then,*

$$E_o(s)(E_{o \bullet s}(z^n) - E_o(z^n)) = \sum_{k=0}^{n-1} E_o(s(z^k)) E_o(s)(E_{o \bullet s}(z) - E_o(z)) E_o(z^{n-1-k}).$$

Proof. Using that the orientations o and $o \bullet s$ are fixed by $z \in \Lambda(1)$, we have
$E_{o \bullet s}(z^n) - E_o(z^n) = E_{o \bullet s}(z^{n-1})(E_{o \bullet s}(z) - E_o(z)) + (E_{o \bullet s}(z^{n-1}) - E_o(z^{n-1})) E_o(z)$.
By induction on n,

$$E_{o \bullet s}(z^n) - E_o(z^n) = \sum_{k=0}^{n-1} E_{o \bullet s}(z^k)(E_{o \bullet s}(z) - E_o(z)) E_o(z^{n-1-k}).$$

We multiply this equality on the left by $E_o(s)$, and we observe that
$E_o(s) E_{o \bullet s}(z^k) = E_o(s z^k) = E_o(s(z^k) s) = E_o(s(z^k)) E_o(s)$. □

For $n \in \mathbb{N}_{>0}$, let

$$B_n := \sum_{k=0}^{n-1} E_o(s(x^k)) B_1 E_o(x^{-k}), \tag{90}$$

with B_1 as in (**??**). For $n \in \mathbb{N}_{>0}$, by the lemma applied to $z = x$ and to $z = x^{-1}$, the Bernstein element (**??**) at x^n is equal to

$$\sum_{k=0}^{n-1} \epsilon_{o \bullet s}(1, s) E_o(s(x^k)) B_1 E_o(x) E_o(x^{n-1-k}) = \epsilon_{o \bullet s}(1, s) B_n E_o(x^n),$$

and the Bernstein element (**??**) at $x^{-n}, n \in \mathbb{N}_{>0}$, is equal to

$$\epsilon_o(1, s) \sum_{k=0}^{n-1} E_o(s(x^{-n+k+1})) E_o(s(x^{-1})) B_1 E_o(x^{-k}) = \epsilon_o(1, s) E_o(s(x^{-n})) B_n.$$

An arbitrary $\lambda \in \Lambda(1)$ is equal tor $\lambda = x^n y$ or $\lambda = yx^{-n}$ with $y \in \Lambda(1)^s$ and $n \in \mathbb{N}$. We multiply on the right by $E_o(y)$ the Bernstein element at x^n to obtain the Bernstein element at $\lambda = x^n y$

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_{o\bullet s}(1,s) B_n E_o(\lambda).$$

We have $E_o(s(y))E_o(s) = E_o(sy) = E_o(s)E_{o\bullet s}(y) = E_o(s)E_o(y),$. We multiply on the left by $E_o(s(y))$ the Bernstein element at x^{-n}, obtain the Bernstein element at $\lambda = yx^{-n}$,

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = E_o(s(y))E_o(s)(E_{o\bullet s}(x^{-n}) - E_o(x^{-n})) = \epsilon_o(1,s)E_o(s(\lambda))B_n,$$

We proved Thm. **??**.

In order to pass from the case $q_s = 1$ to the general case, it is convenient to write Thm. **??** under the following form:

Corollary 5.40. *We suppose that $q_s = 1$ for all $s \in S^{aff}/\sim$. Let $\lambda \in \Lambda(1)$ with $\beta \circ \nu(\lambda) \neq 0$. Then*

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_\beta(\lambda)\epsilon_o(1,s) \sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} c(k) E_o(\mu(k)),$$

where $(c(k), \mu(k)) \in \mathbb{Z}[Z_k] \times \Lambda(1)$ does not depend on the choice of the orientation o and : When $\beta \circ \nu(\lambda) < 0$,

$$c(\delta k) = s(\lambda_s^k) \bullet c_s, \ \mu(\delta k) = s(\lambda_s^k)\lambda_s^{-k}\lambda,$$
$$\text{When } \delta = 2, \ c(2k+1) = (s(\lambda_s^k) w \bullet c_{\tilde{s}}) s^2, \ \mu(2k+1) = s(\lambda_s^k)\lambda_s^{-k-1}\lambda.$$

When $\beta \circ \nu(\lambda) > 0$,

$$c(\delta k) = s(\lambda\lambda_s^k) \bullet c_s, \ \mu(\delta k) = s(\lambda\lambda_s^k)\lambda_s^{-k},$$
$$\text{When } \delta = 2, \ c(2k+1) = (s(\lambda\lambda_s^k) w \bullet c_{\tilde{s}}) s^2, \ \mu(2k+1) = s(\lambda\lambda_s^k)\lambda_s^{-k-1}.$$

Proof. a) $\epsilon_\beta(\lambda) = -1$. Then

$$B_{n_\beta(\lambda)} \ E_o(\lambda) = E_o(s(\lambda_s^k))((\delta-1)(w \bullet c_{\tilde{s}})s^2 E_o(\lambda_s^{-1}) + c_s)E_o(\lambda_s^{-k})E_o(\lambda)$$

is the sum of $\delta = 1, 2$ terms

$$(\delta-1)(s(\lambda_s^k) w \bullet c_{\tilde{s}})s^2 E_o(s(\lambda_s^k)\lambda_s^{-k-1}\lambda) + (s(\lambda_s^k) \bullet c_s)E_o(s(\lambda_s^k)\lambda_s^{-k}\lambda)$$

For $0 \leq k < |\beta \circ \nu(\lambda)|/\delta$, let

$$\mu(\delta k) = s(\lambda_s^k)\lambda_s^{-k}\lambda, \quad \text{and when } \delta = 2, \ \mu(2k+1) = s(\lambda_s^k)\lambda_s^{-k-1}\lambda.$$

We have $\beta \circ \nu(\lambda_s) = -\delta$ and $\beta \circ \nu(s(\lambda_s)) = \delta$ hence

$$\beta \circ \nu(\mu(\delta k)) = 2\delta k + \beta \circ \nu(\lambda), \ \beta \circ \nu(\mu(2k+1)) = (2k+1)\delta + \beta \circ \nu(\lambda).$$

b) $\epsilon_\beta(\lambda) = 1$. Then

$$E_o(s(\lambda))B_{n_\beta(\lambda)} = E_o(s(\lambda))E_o(s(\lambda_s^k))((\delta-1)(w \bullet c_{\tilde{s}})s^2 E_o(\lambda_s^{-1}) + c_s)E_o(\lambda_s^{-k})$$

is the sum of δ terms

$$(\delta-1)(s(\lambda\lambda_s^k) w \bullet c_{\tilde{s}})s^2 E_o(s(\lambda\lambda_s^k)\lambda_s^{-k-1}) + (s(\lambda\lambda_s^k) \bullet c_s)E_o(s(\lambda\lambda_s^k)\lambda_s^{-k}).$$

For $0 \le k < |\beta \circ \nu(\lambda)|/\delta$, let

$$\mu(\delta k) = s(\lambda)s(\lambda_s^k)\lambda_s^{-k}, \quad \text{and when } \delta = 2,\ \mu(2k+1) = s(\lambda)s(\lambda_s^k)\lambda_s^{-k-1}.$$

We have

$$\beta \circ \nu(\mu(\delta k)) = 2\delta k - \beta \circ \nu(\lambda),\ \beta \circ \nu(\mu(2k+1)) = (2k+1)\delta - \beta \circ \nu(\lambda).$$

□

The corollary gives the expansion of $E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ in the basis $(E_o(w))_{w \in W(1)}$, because $Z_k\mu(k) \neq Z_k\mu(k')$ for $k \neq k'$, as

$$\beta \circ \nu(\mu(k)) = 2k - |\beta \circ \nu(\lambda)|,$$

and $\beta \circ \nu(\mu(k)) \neq \beta \circ \nu(\mu(k'))$ implies $Z_k\mu(k) \neq Z_k\mu(k')$ (Cor. **??**).

The Bernstein relations in the $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1}c_s)$, are given by Cor. **??** where c_s and $c_{\tilde{s}}$ are replaced by $\mathfrak{q}_s^{-1}c_s$ and $\mathfrak{q}_{\tilde{s}}^{-1}c_{\tilde{s}}$ in the formula for $c(k)$. A quick inspection shows that this means replacing $c(\delta k)$ by $\mathfrak{q}_s^{-1}c(\delta k)$ and when $\delta = 2$ replacing $c(2k+1)$ by $\mathfrak{q}_{\tilde{s}}^{-1}c(2k+1)$.

The isomorphism $h \mapsto \tilde{h} : \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1}c_s) \to \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$ (**??**) (**??**) sends $E_o(w), w \in W(1)$, to

$$\tilde{E}_o(w) = \mathfrak{q}_w^{-1}E_o(w) \in \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s),$$

and $E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ in $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1}c_s)$ to $\tilde{E}_o(s)(\tilde{E}_{o\bullet s}(\lambda) - \tilde{E}_o(\lambda))$. Multiplying by $\mathfrak{q}_s\mathfrak{q}_\lambda$ and noting that $\mathfrak{q}_{s(\lambda)} = \mathfrak{q}_\lambda$ (Prop. **??**), we obtain:

Proposition 5.41. (Bernstein relations in $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$ and in $\mathcal{H}_{R[(\mathbf{q_s})]}(\mathbf{q}_s, c_s)$)
Let $\lambda \in \Lambda(1)$ with $\beta \circ \nu(\lambda) \neq 0$. Then,

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_\beta(\lambda)\epsilon_o(1, s) \sum_{k=0}^{|\beta \circ \nu(\lambda)|-1} \mathbf{q}(k)c(k)E_o(\mu(k)),$$

$$\mathbf{q}(\delta k) = \mathfrak{q}_\lambda \mathfrak{q}_{\mu(\delta k)}^{-1} \text{ and when } \delta = 2,\ \mathbf{q}(2k+1) = \mathfrak{q}_\lambda \mathfrak{q}_{\mu(2k+1)}^{-1} \mathfrak{q}_s \mathfrak{q}_{\tilde{s}}^{-1},$$

with the notations of Thm. **??** *and Cor.* **??**.

They are also the Bernstein relations in the subalgebra $\mathcal{H}_{R[(\mathbf{q_s})]}(\mathbf{q}_s, c_s)$. We deduce:

$$\mathbf{q}(k)c(k) \in R[(\mathbf{q_s})][Z_k] \quad \text{for } 0 \le k < |\beta \circ \nu(\lambda)|.$$

This is true for any choice of R and $(c_s)_{s \in S^{aff}(1)}$ satisfying the condition a6 of subsection **??**. We may choose $R = \mathbb{Z}$ and $c_s \neq 0$ for all s. Then $c(k) \in \mathbb{Z}$ is not 0, therefore

$$\mathbf{q}(k) = \prod_{s \in S^{aff}/\sim} \mathbf{q}_s^{m_k(s)} \quad (m_k(s) \in \mathbb{N}). \tag{91}$$

The Bernstein relations in the R-algebra $\mathcal{H}_R(q_s, c_s)$ are obtained by specialisation of the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ by the map $\mathbf{q}_s \to q_s$ for $s \in S^{aff}/\sim$.

Theorem 5.42. (Bernstein relations in $\mathcal{H}_R(q_s, c_s)$)

The element $E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda)$ belongs to the subalgebra $\mathcal{A}_o$ of $\mathcal{H}_R(q_s, c_s)$ of basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$. It vanishes when $\beta\circ\nu(\lambda) = 0$, otherwise

$$E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda) = \epsilon_\beta(\lambda)\epsilon_o(1,s)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} q(k)c(k)E_o(\mu(k)),$$

with the notations of Thm. ??, Cor. ?? and Prop. ??.

For the spherical orientation $o_{-\Delta}$ associated to $-\Delta$, we have (Example ??)

$$\epsilon_{o_{-\Delta}}(1,s) = 1, \quad E_{o_{-\Delta}}(s) = T_s \quad \text{for all } s \in S(1).$$

With this orientation, writing $E(w) = E_{o_{-\Delta}}(w)$ for $w \in W(1)$, we obtain a presentation of the generic algebra $\mathcal{H}_{R[\mathbf{q}_s]}(\mathbf{q}_s, c_s)$.

Notations of Thm. ??, Cor. ?? and Prop. ??.

Corollary 5.43. (Bernstein presentation of the generic algebra)

The $R[\mathbf{q}_s]$-algebra $\mathcal{H}_{R[\mathbf{q}_s]}(\mathbf{q}_s, c_s)$ is the free $R[\mathbf{q}_s]$-module of basis $(E_o(w))_{w\in W(1)}$ endowed with the unique $R[\mathbf{q}_s]$-algebra structure satisfying:

- Braid relations for $w, w' \in W_0(1)$, $E(w)E(w') = E(ww')$ if $\ell(w)+\ell(w') = \ell(ww')$.

- Quadratic relations for $s \in S(1)$, $E(s)^2 = \mathbf{q}_s + c_sE(s)$.

- Product for $\lambda, \lambda' \in \Lambda(1)$, $E(\lambda)E(\lambda') = \mathbf{q}_{\lambda,\lambda'}E(\lambda\lambda')$.

- Bernstein relations for $s \in S(1), \lambda \in \Lambda(1)$,

$E(s(\lambda))E(s) = E(s)E(\lambda)$ when $\nu(\lambda)$ is fixed by s,

$E(s(\lambda))E(s) - E(s)E(\lambda) = \epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)c(k)E(\mu(k))$, when $\nu(\lambda)$ is not fixed by s.

5.5 Variant of the Bernstein relations

This subsection is motivated by applications to the theory of smooth representations of G over a field C of characteristic p where the algebras $\mathcal{H}_C(0, c_s)$ with $q_s = 0$ for $s \in S^{aff}/\sim$ appear naturally. Moving the term with $k = 0$ from the right hand side to the left hand side of the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ given in Prop. ??, and keeping the same notations, we obtain:

Proposition 5.44. (Variant of the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$) *When $\beta\circ\nu(\lambda) \neq 0$,*

$$E_o(s\lambda) - E_{o\bullet s}(s\lambda) = \epsilon_{o\bullet s}(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)\mathbf{q}_s^{-1}c(k)E_o(\mu(k)) \quad \text{if } \ell(s\lambda) < \ell(\lambda).$$

$$E_o(s\lambda) - E_o(s)E_o(\lambda) = \epsilon_o(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k)c(k)E_o(\mu(k)) \quad \text{if } \ell(s\lambda) > \ell(\lambda).$$

We have $m_k(s) \geq 1$ (??) when $\ell(s\lambda) < \ell(\lambda)$ and $1 \leq k < |\beta\circ\nu(\lambda)|$. The right hand side is 0 when $|\beta\circ\nu(\lambda)| = 1$.

Proof. By Cor. ??, $\beta\circ\nu(\lambda) < 0$, is equivalent to $\ell(s\lambda) < \ell(\lambda)$ and that $\beta\circ\nu(\lambda) \geq 0$ is equivalent to $\ell(s\lambda) > \ell(\lambda)$.

By Cor. ?? and Prop. ?? when $\beta\circ\nu(\lambda) < 0$, $c(0) = c_s$, $\mu(0) = \lambda$, $\mathbf{q}(0) = 1$, and when $\beta\circ\nu(\lambda) < 0$, $c(0) = s(\lambda)\bullet c_s$, $\mu(0) = s(\lambda), \mathbf{q}(0) = 1$ (by Property a5).

We will prove that

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) - \epsilon_\beta(\lambda)\epsilon_o(1,s)\mathbf{q}(0)c(0)E_o(\mu(0))$$

$$= \; q_s(E_o(s\lambda) - E_{o\bullet s}(s\lambda)) \text{ if } \epsilon_\beta(\lambda) = -1 \text{ and to } E_o(s\lambda) - E_o(s)E_o(\lambda) \text{ if } \epsilon_\beta(\lambda) = 1.$$

If $\epsilon_\beta(\lambda) = -1$, then $\epsilon_\beta(\lambda)\epsilon_o(1,s)\mathbf{q}(0)c(0)E_o(\mu(0)) = -\epsilon_o(1,s)c_sE_o(\lambda) = \epsilon_{o\bullet s}(1,s)c_sE_o(\lambda)$ by Lemma **??**. The expression to be computed is equal to (**??**)

$$E_o(s)E_{o\bullet s}(\lambda) - (E_o(s) + \epsilon_{o\bullet s}(1,s)c_s)E_o(\lambda) = E_o(s)E_{o\bullet s}(\lambda) - E_{o\bullet s}(s)E_o(\lambda).$$

This is what we want $q_s(E_o(s\lambda) - E_{o\bullet s}(s\lambda))$ by the product formula because $\ell(s\lambda) < \ell(\lambda)$ and $q_{s,\lambda} = q_s$ (proof of Prop. **??**).

If $\epsilon_\beta(\lambda) = 1$, then $-\epsilon_\beta(\lambda)\epsilon_o(1,s)\mathbf{q}(0)c(0)E_o(\mu(0)) = \epsilon_{o\bullet s}(1,s)E_o(s(\lambda))c_s$.

As $\ell(s\lambda) > \ell(\lambda)$ we have $q_{s,\lambda} = q_{s(\lambda),s} = 1$ (proof of Prop. **??**), hence

$$E_o(s)E_{o\bullet s}(\lambda) = E_o(s\lambda) = E_o(s(\lambda)s) = E_o(s(\lambda))E_o(s).$$

The expression to be computed is equal to (**??**)

$$E_o(s(\lambda))(E_o(s) + \epsilon_{o\bullet s}(1,s)c_s) - E_o(s)E_o(\lambda) = E_o(s(\lambda))E_{o\bullet s}(s) - E_o(s)E_o(\lambda).$$

Applying the product formula, this is $E_o(s\lambda) - E_o(s)E_o(\lambda)$. □

By specialisation to $\mathbf{q}_* \mapsto 0$, the k-term in the right hand side of Thm. **??** will vanish if $\mathbf{q}_k\mathbf{q}_s^{-1}$ when $\ell(s\lambda) < \ell(\lambda)$, or $\mathbf{q}_k$ when $\ell(s\lambda) > \ell(\lambda)$, is not equal to 1.

Proposition 5.45. *When $0 < k < \beta \circ \nu(\lambda)$, we have:*

1) $\mathbf{q}(k) \neq 1$.

2) When $\ell(s\lambda) < \ell(\lambda)$, $\mathbf{q}(k)\mathbf{q}_s^{-1} \neq 1$ for $1 < k < \beta \circ \nu(\lambda) - 1$.

3) When $\ell(s\lambda) < \ell(\lambda)$, $\mathbf{q}(1)\mathbf{q}_s^{-1} = 1$ if $w(\beta) \in \Delta'$ and $\nu(w(\lambda))$ is Δ'-dominant, for some $w \in W_0$.

All the objects in the proposition depend only on the images of $s, \lambda, \mu(k)$ in W. Without changing the notation we replace them by their images in W. The elements $\mu(k) \in \Lambda$ are defined for all $k \in \mathbb{Z}$ (Cor. **??**). The proof relies on different claims, proved in the next lemmas :

1. $2\sum_{s\in S^{aff}/\sim} m_k(s) = \ell(\lambda) - \ell(\mu(k))$ with $m_k(s)$ as in (**??**) and $\mu(k)$ as in Cor. **??**.
2. $\ell(\mu(k)) = \ell(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)})$ and $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\alpha^{\alpha\circ\nu(\lambda)-k}\lambda)$, with $k \in \mathbb{Z}$, $\mu_\alpha = s_{\alpha+1}s_\alpha$, $\epsilon_\beta(\lambda)$ the sign of $\beta \circ \nu(\lambda)$.
3. $\ell(\lambda) - \ell(\mu_\alpha^k\lambda) = \ell(\lambda) - \ell(\mu_\alpha^{|\alpha\circ\nu(\lambda)|-k}\lambda) \geq 2\,\mathrm{Min}(k, \alpha \circ \nu(\lambda) - k)$ for $\alpha \in \Sigma$ and $0 < k < \alpha \circ \nu(\lambda)$.
4. $\ell(\lambda) - \ell(\mu_\beta\lambda) = 2$ when $\nu(\lambda)$ is Δ'-dominant.

From claim 1, $\mathbf{q}(k) = 1$ if and only if $\ell(\lambda) = \ell(\mu(k))$ and from claims 2,3 this never occurs when $0 < k < |\beta \circ \nu(\lambda)|$. This proves part 1) of the proposition.

When $\ell(s\lambda) < \ell(\lambda)$, claim 1 implies that $\mathbf{q}(k)\mathbf{q}_s^{-1} = 1$ if and only if $\ell(\lambda) - \ell(\mu(k)) = 2$, and claims 2,3 imply that $\mathbf{q}(1)\mathbf{q}_s^{-1} = 1$ is equivalent to $\mathbf{q}(|\alpha \circ \nu(\lambda)| - 1)\mathbf{q}_s^{-1} = 1$, and implies $k = 1$ or $|\beta \circ \nu(\lambda)| - 1$. Claims 2,4 imply that the converse is true if $\nu(\lambda)$ is Δ'-anti-dominant. This proves parts 2) and 3) of the proposition. It remains only to prove the claims.

Lemma 5.46. $2\sum_{s\in S^{aff}/\sim} m_k(s) = \ell(\lambda) - \ell(\mu(k))$*, with the notation of* (**??**).

Proof. We have

$$\mathfrak{q}_\lambda \mathfrak{q}_{\mu(k)}^{-1} = \prod_{s\in S^{aff}(1)/\sim} \mathfrak{q}_s^{n_k(s)}, \quad n_s(k)\in\mathbb{Z}, \quad \sum_s n_s(k) = \ell(\lambda)-\ell(\mu(k)),$$

by choosing reduced decompositions of λ and $\mu(k)$. We have $2m_{\delta k}(s') = n_{\delta k}(s')$ for all $s'\in S^{aff}/\sim$. When $\delta=2$, we have $2m_{2k+1}(s') = n_{2k+1}(s')+1$ if $s'\sim s$, $2m_{2k+1}(s') = n_{2k+1}(s')-1$ if $s'\sim\tilde{s}$, and $2m_{2k+1}(s') = n_{2k+1}(s')$ otherwise. □

Lemma 5.47. *$\ell(\mu(k)) = \ell(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)})$ and $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\alpha^{\alpha\circ(\lambda)-k}\lambda)$ for $\alpha\in\Sigma, \lambda\in\Lambda, k\in\mathbb{Z}$.*

Proof. By Cor. **??**, the length of $x\in\Lambda$ depends only on $\nu(x)\in V$ and ν is W_0-equivariant. For $\alpha\in\Sigma$ we have $\nu(\mu_\alpha) = \nu(s_{\alpha+1}s_\alpha) = -\nu(s_\alpha s_{\alpha+1}) = -\nu(s_\alpha(\mu_\alpha)) = -\alpha^\vee$.

When $\beta\circ\nu(\Lambda)=\mathbb{Z}$, $\nu(s(\lambda_s)) = s(\nu(\lambda_s)) = \nu(\lambda_s)+\beta^\vee$ because $\beta\circ\nu(\lambda_s)=-1$. Hence

$$\nu(\mu(k)) = \nu(s(\lambda_s^k)\lambda_s^{-k}\lambda) = k\beta^\vee + \nu(\lambda) = -k\nu(\mu_\beta)+\nu(\lambda) = \nu(\mu_\beta^{-k}\lambda) \text{ if } \epsilon_\beta(\lambda)=-1,$$

$$\nu(\mu(k)) = \nu(s(\lambda)s(\lambda_s^k)\lambda_s^{-k}) = k\beta^\vee + \nu(s(\lambda)) = k\nu(s(\mu_\beta))+\nu(s(\lambda)) = \nu(s(\mu_\beta^k\lambda)),$$

if $\epsilon_\beta(\lambda)=1$. When $\beta\circ\nu(\Lambda)=2\mathbb{Z}$, $\nu(s(\lambda_s)) = \nu(\lambda_s)+2\beta^\vee$ because $\beta\circ\nu(\lambda_s)=-2$. With the same arguments $\nu(\mu(k))$ is equal to

$$\nu(\mu_\beta^{-k}\lambda) \text{ if } \epsilon_\beta(\lambda)=-1, \ \ \nu(s(\mu_\beta^k\lambda)) \text{ if } \epsilon_\beta(\lambda)=1,$$

in this case also. Hence $\ell(\mu(k)) = \ell(\mu_\beta^{-k}\lambda^{-\epsilon_\beta(\lambda)})$ if $\epsilon_\beta(\lambda)=-1$ and $\ell(\mu(k)) = \ell(s(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)}))$ if $\epsilon_\beta(\lambda)=1$. The property that the length on Λ is invariant by taking the inverse or by W_0 implies $\ell(\mu(k)) = \ell(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)})$.

We have

$$s_\alpha(\nu(\mu_\alpha^k\lambda)) = s_\alpha(-k\alpha^\vee+\nu(\lambda)) = (k-\alpha\circ\nu(\lambda))\alpha^\vee+\nu(\lambda) = \nu(\mu_\alpha^{\alpha\circ\nu(\lambda)-k}\lambda).$$

We deduce $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\alpha^{\alpha\circ\nu(\lambda)-k}\lambda)$. □

Lemma 5.48. *For $\alpha\in\Sigma, \lambda\in\Lambda$ such that $1<\alpha\circ\nu(\lambda)$ we have*

$$\ell(\lambda)-\ell(\mu_\alpha^k\lambda)\geq 2\min(k,\alpha\circ\nu(\lambda)-k) \text{ for } 0<k<\alpha\circ\nu(\lambda).$$

When $\alpha\in\Delta'$, we have $\ell(\lambda)-\ell(\mu_\alpha\lambda)=2$ if there exists $w\in W_0$ such that $w(\nu(\lambda))$ is Δ'-dominant and $w(\alpha')\in\Delta'$.

Proof. a) Reduction to $\nu(\lambda)$ Δ'-dominant and $\alpha\circ\nu(\lambda)\geq 2k$ for the inequality of the lemma.

There exists $w\in W_0$ such that $w(\nu(\lambda))$ is Δ'-dominant, that is, $\gamma\circ\nu(\lambda)\geq 0$ for all $\gamma\in\Delta'$. The homomorphism ν is W_0-equivariant, $w(\alpha)\circ\nu(w(\lambda)) = \alpha\circ\nu(\lambda)$, $w(\mu_\alpha)=\mu_{w(\alpha)}$, and the length is W_0-invariant. The pair $(w(\alpha),w(\lambda))$ satisfies the inequality of the lemma if and only if the pair (α,λ) satisfies the lemma.

From now on, $\nu(\lambda)$ Δ'-dominant. We suppose that the inequality in the lemma is true when $0<k\leq(\alpha\circ\nu(\lambda))/2$. For the other values $(\alpha\circ\nu(\lambda))/2<k<\alpha\circ\nu(\lambda)$ we have $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\beta^{\alpha\circ\nu(\lambda)-k}\lambda)$ by Lemma **??** and $0<\alpha\circ\nu(\lambda)-k<(\alpha\circ\nu(\lambda))/2$. Hence $\ell(\lambda)-\ell(\mu_\beta^{\alpha\circ\nu(\lambda)-k}\lambda)\geq 2(\alpha\circ\nu(\lambda)-k)$. Hence we can suppose $\alpha\circ\nu(\lambda)\geq 2k$ for the inequality of the lemma.

b) There exists a positive integer r such that $r\nu(\Lambda)$ is contained in $\nu(\Lambda^{aff}) = Q(\Sigma^\vee)$, the lattice generated by the coroots, because $\nu(\Lambda)\subset P(\Sigma^\vee)$, the lattice generated by the weights of $\Sigma^\vee$.

Let ρ be the half-sum of the Δ'-positive roots of Σ. If $\gamma\in\Sigma$ is Δ'-positive, $\rho(\gamma^\vee)\geq 1$ with equality if and only if $\gamma\in\Delta'$. As $\nu(\lambda)$ is Δ'-dominant, we have $\ell(\lambda) = 2\rho(\nu(\lambda))$ (Cor.

??), and for any $w \in W_0$ ([**?**] 1.3.22 Cor. 1.4.2 Prop.) $\nu(\lambda) - w(\nu(\lambda)) \in (1/r)\sum_{\gamma\in\Delta'} \mathbb{N}\gamma^\vee$, hence

$$2\rho(\nu(\lambda) - w(\nu(\lambda)) \geq 0 \quad \text{with equality if and only if} \;\; \nu(\lambda) = w(\nu(\lambda)).$$

c) There exists $w \in W_0$ such that $w(\nu(\mu_\alpha^k\lambda)) = w(-k\alpha^\vee + \nu(\lambda))$ is Δ'-dominant. As the length is W_0-invariant, we have $\ell(\mu_\alpha^k\lambda) = 2\rho(w(-k\alpha^\vee + \nu(\lambda))$, hence

$$\ell(\lambda) - \ell(\mu_\alpha^k\lambda) = 2\rho(\nu(\lambda) - w(\nu(\lambda)) + 2k\rho(w(\alpha^\vee)).$$

We deduce:

1) $\ell(\lambda)-\ell(\mu_\alpha^k\lambda) \geq 2k$ if there exists $w \in W_0$ such that $w(\alpha)$ Δ'-positive and $w(-k\alpha^\vee + \nu(\lambda))$ Δ'-dominant.

2) $\ell(\lambda)-\ell(\mu_\alpha^k\lambda) = 2k$ if and only if there exists $w \in W_0$ fixing $\nu(\lambda)$ such that $w(\alpha) \in \Delta'$ and $w(-k\alpha^\vee + \nu(\lambda))$ is Δ'-dominant.

e) Suppose $\alpha \circ \nu(\lambda) = 2k$, or equivalently $-k\alpha^\vee + \nu(\lambda)$ fixed by s_α. For $w \in W_0$, $w(-k\alpha^\vee + \nu(\lambda)) = ws_\alpha(-k\alpha^\vee + \nu(\lambda))$ and either $w(\alpha)$ or $ws_\alpha(\alpha)$ is Δ'-positive. We deduce from 1) that $\ell(\lambda) - \ell(\mu_\gamma^k\lambda) \geq 2k$.

f) Suppose $\alpha \circ \nu(\lambda) > 2k$. For $w \in W_0$ such that $w(-k\alpha^\vee + \nu(\lambda))$ is Δ'-dominant, $w(\alpha)$ is Δ'-positive because $w(\alpha)(w(-k\alpha^\vee + \nu(\lambda)) = \alpha(-k\alpha^\vee + \nu(\lambda)) = \alpha \circ \nu(\lambda) - 2k$ is positive by hypothesis. A root which takes a positive value on a point in the Δ'-dominant Weyl chamber of V is Δ'-positive. We deduce from 1) that $\ell(\lambda) - \ell(\mu_\alpha^k\lambda) \geq 2k$. This ends the proof of the inequality.

g) The equality in the lemma concerns the case $k = 1$. With the same arguments than in a), we reduce to prove that $\ell(\lambda) - \ell(\mu_\alpha\lambda) = 2$ when λ is Δ'-dominant and $\alpha \in \Delta'$.

By 2), it suffices to check that $-\alpha^\vee + \nu(\lambda)$ is Δ'-dominant. This is true because λ is Δ'-dominant and $\alpha(-\alpha^\vee + \nu(\lambda)) = -2 + \alpha \circ \nu(\lambda) \geq 0$. □

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