

The Satake isomorphism modulo p with weight

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Abstract

Let F be a local non archimedean field with a finite residue field k of characteristic p , let G be a connected reductive group over F , let K be a special parahoric subgroup of $G(F)$, and let V be an irreducible smooth representation of K over a large extension k' of k . We prove a Satake isomorphism for the Hecke algebra $\mathcal{H}(G, K, V)$, generalizing the work of Herzig who assumed F of characteristic 0, G unramified over F and K hyperspecial. We show that $\mathcal{H}(G, K, V)$ is commutative for all (K, V) when G splits over an unramified extension of F and we give an example of a non commutative Hecke algebra $\mathcal{H}(G, K, V)$.

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Introduction

Let G be a connected reductive group over a local non archimedean field F with a finite residue field k of characteristic p . Suppose that K is a special parahoric subgroup of $G(F)$ with pro- p -radical K_+ of K . Then $\overline{G}(k) = K/K_+$ is the group of k -points of a connected reductive group $\overline{G}$ defined over k . Suppose that V is an irreducible representation of $\overline{G}(k)$ over an extension k' of k containing a root of 1 of order the prime to p part of the exponent of $\overline{G}(k)$. We view V as a representation of K . The Hecke algebra $\mathcal{H}_{k'}(G, K, V)$ is the algebra of intertwining operators of the compactly induced representation $\text{c-Ind}_K^{G(F)} V$. We forget V when V is trivial. Let $B = Z.U$ be a minimal parabolic F -subgroup of G with unipotent radical U and Levi subgroup Z . The natural image in $\overline{G}(k)$ of the intersection $K \cap B(F)$ is the group of points of a minimal parabolic k -subgroup $\overline{B} = \overline{Z}.\overline{U}$ of $\overline{G}$. The space $V^{\overline{U}(k)}$ is one dimensional with a natural action of $\overline{Z}(k)$ by a character $\chi_V : \overline{Z}(k) \rightarrow (k')^*$. The group Z is a connected reductive F -group which is anisotropic modulo its center (it is not the connected center of G). The intersection $K \cap Z(F)$ is the unique parahoric subgroup Z_0 of Z , the intersection $K_+ \cap Z(F)$ is the pro-unipotent radical Z_{0+} of Z_0 and $\overline{Z}(k) = Z_0/Z_{0+}$. The group $Z(F)/Z_0$ is commutative and finitely generated, isomorphic to the image of the Kottwitz morphism for $Z(F)$. We view $V^{\overline{U}(k)}$ as V^{U_0} where $U_0 = U(F) \cap K$ and χ_V as a k' -character of Z_0 . There exists a canonical morphism of Hecke algebras

$$\mathcal{S} : \mathcal{H}_{k'}(G, K, V) \rightarrow \mathcal{H}_{k'}(Z, Z_0, V^{U_0})$$

called the Satake morphism. Let Z_V^- be the monoid of antidominant elements of $Z(F)$ normalizing the character χ_V . We will prove:

Theorem 0.1. *The Satake morphism $\mathcal{S}$ is injective of image the subalgebra of functions with support contained in Z_V^- .*

We denote by $\mathcal{H}_{k'}(Z_V^-, Z_0, V^{U_0})$ the subalgebra of $\mathcal{H}_{k'}(Z, Z_0, V^{U_0})$ with support in Z_V^- .

Definition 0.2. *The Satake isomorphism modulo p with weight V is the isomorphism*

$$\mathcal{H}_{k'}(G, K, V) \rightarrow \mathcal{H}_{k'}(Z_V^-, Z_0, V^{U_0})$$

induced by $\mathcal{S}$.

When G is F -anisotropic modulo its center, then K is the unique parahoric subgroup of $G(F)$, and V is a character $\chi : K \rightarrow (k')^*$. It is tempting to denote $\mathcal{H}_{k'}(G, \chi) := \mathcal{H}_{k'}(G, K, V)$. We give an example of (G, χ) with $\mathcal{H}_{k'}(G, \chi)$ non commutative. We prove that $\mathcal{H}_{k'}(G, \chi)$ is commutative for all χ when G is a F -torus, or is semi-simple simply connected, or splits over an unramified extension of F .

Returning to a general G , the Satake isomorphism implies :

Proposition 0.3. *When G is F -quasi-split, or semi-simple and Z is simply connected, or splits over an unramified extension of F , then the Hecke algebra $\mathcal{H}_{k'}(G, K, V)$ is commutative for all (K, V) .*

We describe now the center of $\mathcal{H}_{k'}(G, K, V)$ via the Satake isomorphism. Let $Z_V'^-$ be the monoid of antidominant elements $z \in Z(F)$ such that $zxz^{-1}x^{-1}$ acts trivially on V^{U_0} for any $x \in Z_V$. The center of the algebra $\mathcal{H}_{k'}(Z_V^-, Z_0, V^{U_0})$ is the subalgebra $\mathcal{H}(Z_V'^-, Z_0, V^{U_0})$ with support in $Z_V'^-$. This algebra is naturally isomorphic to the group algebra $k'[Z_V'^-/Z_0]$ of the commutative finitely generated monoid $Z_V'^-/Z_0$.

Proposition 0.4. *The Hecke algebra $\mathcal{H}(G, K, V)$ is a finitely generated module over its center and the center is a finitely generated algebra isomorphic to $k'[Z_V'^-/Z_0]$.*

History Originally, Satake [Sa] (1963) formulated “the Satake isomorphism” for the complex Hecke algebra $\mathcal{H}_{\mathbb{C}}(G', K')$ of a pair (G', K') consisting of a p -adic group G' and a maximal compact subgroup K' satisfying assumptions inspired by the real case and that he verified for classical simple p -adic groups. The development of the theory of the maximal compact subgroups of the points of the connected reductive F -group G , mainly by Bruhat-Tits, showed that the assumptions of Satake are satisfied by the pair $(G(F), \tilde{K})$ where $\tilde{K}$ is a special maximal compact subgroup (a special parahoric subgroup K is contained in a unique special maximal compact subgroup $\tilde{K}$). The classical and excellent reference is the article of Cartier [Cart] (1979) in the Corvallis proceedings. Let $\tilde{Z}_0$ be the unique maximal compact subgroup of $Z(F)$ and let W_f be the finite relative Weyl group of G . The classical Satake isomorphism says that the $\mathbb{C}$ -algebra $\mathcal{H}_{\mathbb{C}}(G, \tilde{K})$ is canonically isomorphic to the $\mathbb{C}$ -algebra of all W_f -invariant elements of $\mathbb{C}[Z(F)/\tilde{Z}_0]$:

$$\mathcal{H}_{\mathbb{C}}(G, \tilde{K}) \simeq \mathbb{C}[Z(F)/\tilde{Z}_0]^{W_f}$$

via the morphism

$$f \mapsto (z \mapsto \delta_B^{1/2}(z) \int_{U(F)} f(zu) du)$$

where du gives $U_0 = U(F) \cap K$ measure 1 and δ_B is the modular function on B . For $z \in Z(F)$ we have $\delta_B(z) := |\det(\text{Ad}(z); \text{Lie}(U(F)))|_F$. The image of f is essentially the normalized orbital integral of f on $Z(F)$. Let $D(z) := |\det(\text{Ad}(z) - 1; \text{Lie}(G(F)/Z(F)))|_F^{1/2}$ and let S be the maximal F -split subgroup of Z . We have

$$\delta_B^{1/2}(z) \int_{U(F)} f(zu) du = D(z) \int_{G(F)/S(F)} f(gzg^{-1}) \frac{dg}{ds} \quad \text{when } D(z) \neq 0$$

where dg, ds give $K, S(F) \cap K$ the measure 1. The normalization of the Satake isomorphism by $\delta_B^{1/2}$ implies its independence of the choice of B and the symmetry of its image by W_f . For arithmetic applications, the ring $\mathcal{H}_{\mathbb{Z}}(G, \tilde{K})$ is more important and Satake shows that it is a polynomial ring with m variables over $\mathbb{Z}$ when G is simple without center and m is the rank of its maximal F -split torus. The importance of the Satake isomorphism became extreme when Langlands [Lang] (1970) introduced a new complex group ${}^L G(\mathbb{C})$ and viewed the Satake isomorphism when G is quasi-split and split over an unramified extension (we say now that G is F -unramified) as a parametrization of the characters of $\mathcal{H}_{\mathbb{C}}(G, \tilde{K})$ by certain semi-simple conjugacy classes of ${}^L G(\mathbb{C})$. Langlands used this parametrization to define local L -functions of automorphic forms.

In the theory of types the parahoric subgroups are more useful than the maximal compact subgroups. The Hecke algebras $\mathcal{H}_{\mathbb{Z}}(G, K', V)$ with weight a complex irreducible supercuspidal representation V have been described by Morris [Mo] (1993), for any parahoric subgroup K' of $G(F)$. The simple modules of the complex weighted Hecke algebras associated to the parahoric subgroups of $G(F)$ are used to classify the irreducible complex representations of $G(F)$ of level 0. Over a field of characteristic p all the smooth irreducible representations of $G(F)$ have level 0, but they are not known.

Recently (2009), Rapoport Haines and Rostami [HRa], [HR] with a view of applications to Shimura varieties, showed that the assumptions of Satake are satisfied by $(G(F), K)$ when K is a special parahoric subgroup, and they related $Z(F)/Z_0$ with the L -group of Z . The classical complex Satake isomorphism suitably normalized for $(G(F), K)$ is of the form

$$\mathcal{H}_{\mathbb{C}}(G, K) \simeq \mathbb{C}[Z(F)/Z_0]^{W_f}.$$

Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Over a field of characteristic p the normalization is impossible. Barthel and Livné [BL1], [BL2] (1994) described the (unnormalized) Satake isomorphism modulo p with weight when $G = GL(2)$, and used it to obtain the classification

of the irreducible admissible $\overline{\mathbb{F}}_p$ -representations of $GL(2, F)$ with a central character in term of supercuspidal representations.

For an Iwahori subgroup of $G(F)$, i.e. for a minimal parahoric subgroup of $G(F)$, the weighted Hecke $\overline{\mathbb{F}}_p$ -algebras $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, I, V)$ have been described in 2006 [V]. They are finitely generated modules over their center and the center is a finitely generated $\overline{\mathbb{F}}_p$ -algebra; the simple $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, I, V)$ -modules have a central character [Vig]; the supersingular modules have been described when $G = GL(n)$ [V2] [O1].

Recently (2009), Herzig [H], [H2], generalized to $G = GL(n)$ for all $n \geq 2$ the results of Barthel and Livne. Herzig describes the Satake isomorphism of weighted Hecke algebra

$$\mathcal{H}_{\overline{\mathbb{F}}_p}(G, \tilde{K}, V) \simeq \mathcal{H}_{\overline{\mathbb{F}}_p}(Z_V^-, Z_0, V^{U_0}) \quad ,$$

that Herzig proved under the assumptions that G is F -unramified and that $\tilde{K}$ hyperspecial. In this case $K = \tilde{K}$.

Plan of the paper The paper is divided in five parts. In the first part, we recall the definition of the weighted Hecke algebra $\mathcal{H}_{k'}(G, K, V)$ and we notice that the representation of $G(F)$ induced from (K, V) is a $G(F) \times \mathcal{H}_{k'}(G, K, V)$ -module, a fact which will be used later with the Iwasawa decomposition for the special parahoric subgroup K to define the Satake morphism.

In the second part, we suppose that the reductive connected F -group G is F -anisotropic modulo the center. The group $G(F)$ has a unique parahoric subgroup K and the quotient of K by its pro- p -radical is the group of points of a k -torus. The representation V of K is one dimensional and can be replaced by a k' -character χ of K . We denote $\mathcal{H}_{k'}(G, K, V) = \mathcal{H}_{k'}(G, \chi)$. We consider the problem of the commutativity of the Hecke algebra $\mathcal{H}_{k'}(G, \chi)$. We show that the commutativity is equivalent to the vanishing of a certain element in $H^2(G_\chi/K, E^*)$ where G_χ the normalizer of χ in $G(F)$ and E is any algebraically closed field of characteristic 0 or large enough different from p . We recall for a general G the functorial properties of the parahoric subgroups of $G(F)$ described as the intersection of the kernel of the Kottwitz morphism with the fixator of a facet of the Bruhat-Tits building of the adjoint group G_{ad} , by Haines and Rapoport. In the last two subsections we give an example of a non commutative Hecke algebra $\mathcal{H}_{k'}(G, \chi)$, and we prove that $\mathcal{H}_{k'}(G, \chi)$ is commutative for any character χ of K when G is split over an unramified extension of F .

In the third part, using the results of Haines, Rapoport and Rostami for a special parahoric subgroup of $G(L)$, we describe the Cartan and Iwasawa decomposition for a special parahoric subgroup of $G(F)$ and the intersections between the Cartan double classes and the Iwasawa double classes; this will allow later to prove the injectivity of the Satake isomorphism and that the support of the image of the Satake morphism is antidominant - via the Herzig's trick. We study the action of the reduction from K to its reductive quotient $\overline{G}(k)$, to the intersections $P_{z,0} = K \cap P_z(F)$ of K with the parabolic subgroups of $G(F)$ associated to the elements z of $Z(F)$. We prove the property allowing later to describe a k' -basis of the Hecke algebra $\mathcal{H}_{k'}(G, K, V)$.

In the fourth part, we recall the classification of the irreducible representations of a finite group of Lie type (the points of a reductive connected group over a finite field) in the natural characteristic, following Cabanes and Curtis. We check that an automorphism stabilizing a maximal torus and the root subgroups fixes the highest weight χ_V if and only if it fixes the isomorphism class of the representation V . This is an ingredient to show that both the Hecke algebras $\mathcal{H}_{k'}(G, K, V)$ and $\mathcal{H}_{k'}(G, \chi_V)$ have a k' -basis indexed by a system of representatives of Z_V^-/Z_0 .

In the fifth and last part, we prove the Satake isomorphism following Herzig. It is remarkable that the method of Herzig generalizes to non commutative weighed Hecke algebras. The Satake morphism is a consequence of the Iwasawa decomposition for K ,

which implies that the induced representation from (Z_0, V^{U_0}) to $Z(F)$ is isomorphic to the $U(F)$ -invariant elements of the induced representation of (K, V) to $G(F)$. The Satake isomorphism implies finiteness results for the weighted Hecke algebra $\mathcal{H}_{k'}(G, K, V)$. The weighted Hecke algebra is finitely generated over its center, the center is the k' -algebra of a finitely generated monoid Z_V'/Z_0 , the simple $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, K, V)$ -modules are finitely generated and have a character given by a $\overline{\mathbb{F}}_p$ -character of Z_V'/Z_0 ; if this character is supersingular the simple $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, K, V)$ -module can reasonably be called supersingular.

1 Definition of the Hecke algebra $\mathcal{H}(G, K, V)$

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See ([?] 1.8.6). The Hecke algebra $\mathcal{H}(G, K, V) = \mathcal{H}_{k'}(G, K, V)$ is the k' -algebra of compactly supported functions $f : G(F) \rightarrow \text{End}_{k'} V$ satisfying $f(k_1 g k_2) = k_1 f(g) k_2$ for all $k_1, k_2 \in K, g \in G(F)$, and the multiplication of $f_1, f_2 \in \mathcal{H}(G, K, V)$ is given by convolution:

$$(f_1 * f_2)(g) = \sum_{y \in G(F)/K} f_1(y) \circ f_2(y^{-1}g)$$

for $g \in G(F)$. We will forget V and we can replace k' by the ring $\mathbb{Z}$ when V is the trivial representation. Hence $\mathcal{H}_{\mathbb{Z}}(G, K)$ is the convolution ring of $\mathbb{Z}$ -basis the double classes of G modulo K .

By Frobenius reciprocity,

$$\mathcal{H}(G, K, V) \simeq \text{End}_{k'G(F)} \text{c-Ind}_K^{G(F)} V \quad ,$$

where $\text{c-Ind}_K^{G(F)} V$ is the compact induced representation of V to $G(F)$ (the space of compactly supported functions $\varphi : G(F) \rightarrow V$ satisfying $\varphi(kg) = k\varphi(g)$ for all $k \in K, g \in G(F)$, with the action of $G(F)$ by right translation). The subspace $[1, V]$ of functions $[1, v]$ with support K and value $v \in V$ at 1 is K -stable and isomorphic to V by the value at 1. We have

$$\text{c-Ind}_K^{G(F)} V = \bigoplus_{g \in G/K} g \cdot [1, V] \quad .$$

The action of $f \in \mathcal{H}(G, K, V)$ on $g \cdot [1, v]_K$ is the function $f \cdot g \cdot [1, v]_K = g \cdot f \cdot [1, v]_K : G(F) \rightarrow V$ defined by

for

$$(1) \quad h \mapsto \sum_{KhK = \cup_x Kx} g x^{-1} \cdot [1, f(x)v]_K = \sum_{Kh^{-1}K = \cup_x xK} g x \cdot [1, f(x^{-1})v]_K.$$

Hence $\text{Ind}_K^{G(F)} V$ is a $G(F) \times \mathcal{H}(G, K, V)$ -module.

The non compact induced representation $\text{Ind}_K^{G(F)} V$ of (K, V) to $G(F)$ is defined as $\text{c-ind}_K^{G(F)} V$ but without restriction of the support.

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Lemma 1.1. $\text{Ind}_K^{G(F)} V$ is a $G(F) \times \mathcal{H}(G, K, V)$ -module.

Proof. We check that the action of $f \in \mathcal{H}(G, K, V)$ on $\text{c-Ind}_K^{G(F)} V$ extends to $\text{Ind}_K^{G(F)} V$. For any $x \in G(F)$ and for any $\phi \in \text{Ind}_K^{G(F)} V$,

$$(f \cdot \phi)(x) = (f \cdot (\phi|_{x^{-1} \cdot \text{supp}(f)}))(x) = (f \cdot \phi|_C)(x)$$

for any large compact subset C containing $x^{-1} \cdot \text{supp}(f)$ is well defined, because the support of $f \cdot g \cdot [1, v]_K$ is contained in $\text{supp}(f)g^{-1}$, and the value at x of $f \cdot g \cdot [1, v]_K$ is 0 when g does not belong to $x^{-1} \cdot \text{supp}(f)$.

We check that $f.\phi \in \text{c-Ind}_K^{G(F)} V$. For $k \in K$ and $x \in G(F)$ we have

$$(f.\phi)(kx) = (f.\phi|_C)(kx) = k.((f.\phi|_C)(x)) = k.((f.\phi)(x)) \quad .$$

for any compact $C \subset G(F)$ containing $(kx)^{-1}\text{supp}(f)$ and $x^{-1}\text{supp}(f)$.

We check that the linear map $\phi \mapsto f.\phi$ is $G(F)$ equivariant. For any $g, x \in G(F)$ we have $(g.f.\phi)(x) = (f.g.\phi)(x)$ because

$$(f.\phi)(xg) = (f.\phi|_C)(xg) = (g.f.\phi|_C)(x) = (f.g.\phi_C)(x) = (f.(g.\phi)|_{Cg^{-1}})(x) \quad .$$

for any compact $C \subset G(F)$ containing $(xg)^{-1}\text{supp}(f)$ and $x^{-1}\text{supp}(f)g$.

□

2 G is F -anisotropic modulo the center

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We suppose that G is a connected reductive group which is F -anisotropic modulo the center (a maximal F -split subtorus of G is central). We have the following properties:

1. The group G is of type A ([BTIII] 4.3).
2. The group $G(F)$ admits a unique parahoric subgroup K and a unique maximal compact subgroup $\tilde{K}$ ([BTII] 5.1.27, 5.2.7). By unicity, K and $\tilde{K}$ are normal in $G(F)$.
3. The group K/K_+ is the group of k -rational point of a k -torus ([BTII] 5.1.32). Hence K/K_+ is a commutative finite group of order prime to p .
4. The group

$$\Lambda := G(F)/K$$

is commutative, finitely generated, of torsion $\Lambda_{\text{tor}} = \tilde{K}/K$ ([HR] 1.0.1). In particular $\tilde{K}/K$ is commutative and the commutator subgroup $(G(F), G(F))$ of $G(F)$ is contained in K .

5. Let S be the maximal F -split subtorus of G . Then $\tilde{K} \cap S(F) = K \cap S(F)$ is the maximal compact subgroup S_0 of $S(F)$ which is the unique parahoric subgroup of $S(F)$. The subgroup $S(F)/S_0$ of Λ has finite index and identifies with the cocharacter group S_* of S .
6. We have $S(F)/S_0 = \Lambda$ hence $K = \tilde{K}$ when G is F -unramified, i.e. F -quasi-split and splits over an unramified extension ([HR] 11.1.2).

2.1 Hecke algebra

An irreducible k' -representation V of K is given by a character $\chi : K \rightarrow k'^*$. The kernel H of χ is a normal subgroup of K and the quotient K/H is cyclic of finite order prime to p . Any such subgroup of K is the kernel of a homomorphism $K \rightarrow k'^*$. The normalizer $G_\chi = G_V$ of χ in $G(F)$ is the group

$$\{ g \in G(F) \mid g x g^{-1} x^{-1} \in H \text{ for all } x \in K \} \quad .$$

It is obvious that it contains K and normalizes H . Let

$$G'_\chi = G'_V := \{ g \in G(F) \mid g x g^{-1} x^{-1} \in H \text{ for all } x \in G_V \} \quad .$$

It is obvious that $K \subset G'_V \subset G_V$. We put

$$\Lambda'_V := G'_V/K \quad , \quad \Lambda_V := G_V/K \quad .$$

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Lemma 2.1. G'_V is a normal subgroup of G_V of finite index.

Proof. Recalling that $(G(F), G(F)) \subset K$ we have for all $g, h, x \in G_V$,

$$\begin{aligned}\chi(ghx(gh)^{-1}x^{-1}) &= \chi(g(hxh^{-1}x^{-1})g^{-1}) \chi(gxg^{-1}x^{-1}) = \chi(hxh^{-1}x^{-1})\chi(gxg^{-1}x^{-1}), \\ \chi(g^{-1}xgx^{-1})^{-1} &= \chi(g^{-1}(gxg^{-1}x^{-1})g) = \chi(gxg^{-1}x^{-1}).\end{aligned}$$

They show that G'_V is a group by choosing $g, h \in G'_V$. As Λ_H is commutative G'_V is a normal subgroup of G_V . The index of G'_V in G_V is equal to the index of Λ'_V in Λ_V . The first formula shows

$$\chi(g^d x g^{-d} x^{-1}) = \chi(gxg^{-1}x^{-1})^d$$

for all $d \in \mathbb{N}_{>0}$ and $g, x \in G_V$. Hence, if d is the exponent of the finite group K/H , we have $g^d \in G'_V$ for all $g \in G_V$. We deduce that $d\Lambda_V \subset \Lambda'_V$ and that the index of Λ'_V in Λ_V is finite because $\Lambda_H \subset \Lambda$ is commutative and finitely generated. $\square$

A function in the Hecke algebra vanishes outside G_V and for any $g \in G_V$ there exists a unique function $\tau_g \in \mathcal{H}(G, K, V) = \mathcal{H}(G, K, \chi)$ of support KgK such that $\tau_g(g) = 1$. One chooses a representative $g_\lambda \in G(F)$ of each $\lambda \in \Lambda$.

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Lemma 2.2. 1. A k' -basis of $\mathcal{H}(G, K, V)$ is $(\tau_{g_\lambda})_{\lambda \in \Lambda_V}$.

2. A k' -basis of the center $\mathcal{Z}(G, K, V)$ of $\mathcal{H}(G, K, V)$ is $(\tau_{g_\lambda})_{\lambda \in \Lambda'_V}$.

3. The center of the Hecke algebra $\mathcal{H}(G, K, V)$ is isomorphic to $k'[\Lambda'_V]$, and $\mathcal{H}(G, K, V)$ is a finitely generated module over its center.

4. The Hecke algebra $\mathcal{H}(G, K, V)$ is commutative if and only if $\chi(G_\chi, G_\chi) = 1$.

Proof. We have the product formula for $g, g' \in G_\chi$,

$$\tau_g * \tau_{g'} = \tau_{gg'} = \tau_{g'g} \chi(g^{-1}g'^{-1}gg') = \tau_{g'} * \tau_g \chi(g^{-1}g'^{-1}gg').$$

We deduce that τ_g commutes with $\tau_{g'}$ for all $g' \in G_\chi$ if and only if $g \in G'_\chi$. With the lemma 2.1, the lemma is now obvious. $\square$

2.2 First results on commutativity

It is convenient to denote sometimes G_V by G_H ; this group depends only on H . The action of G_H by conjugation on K/H is trivial, and

$$1 \rightarrow K/H \rightarrow G_H/H \rightarrow G_H/K \rightarrow 1$$

is a special kind of central extension where the quotient G_H/K is commutative and finitely generated and the kernel K/H is cyclic of order prime to p . Let $\alpha_H \in H^2(G_H/K, K/H)$ be the cohomology class of the extension. By definition, for any choice of a system of representatives $\{b_c\}$ of G_H/K in G_H/H , the class of the 2-cocycle $(c, c') \mapsto b_{cc'}^{-1}b_cb_{c'}$ in $H^2(G_H/K, K/H)$ is α_H . Let E be an algebraically closed field of characteristic 0 or not dividing the order of K/H and let $\beta_H \in H^2(G_H/K, E^*)$ be the natural image of α_H .

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Lemma 2.3. The following properties are equivalent:

1. The Hecke algebra $\mathcal{H}(G, K, V)$ is commutative,
2. G_H/H is commutative,
3. $\alpha_H = 0$,
4. $\beta_H = 0$,

5. The natural injective morphism $K/H \rightarrow E^*$ extends to a morphism $X : G_H/H \rightarrow E^*$.

Proof. It is obvious that the Hecke algebra $\mathcal{H}(G, K, V)$ is commutative if and only if

$$(G_H/H, G_H/H) \subset K/H$$

if and only if the group G_H/H is commutative if and only if the extension splits if and only if $\alpha_H = 0$, and that $\alpha_H = 0$ implies $\beta_H = 0$. By Clifford's theory ([NT] th.5.7) $\beta_H = 0$ if and only if the natural inclusion $K/H \rightarrow E^*$ extends to a morphism $X : G_H/H \rightarrow E^*$. If a morphism $X : G_H/H \rightarrow E^*$ extends the inclusion $K/H \rightarrow E^*$ then $X(G_H/H, G_H/H) = 1$. As $(G_H/H, G_H/H) \subset K/H$ and $X|_{K/H}$ is injective we deduce $(G_H/H, G_V/H) = 1$. Equivalently G_H/H is commutative. $\square$

We deduce:

Proposition 2.4. *The Hecke algebra $\mathcal{H}(G, K, V)$ is commutative in the following cases:*

1. G is quasi-split,
2. G is semi-simple and simply connected,
3. G is semi-simple and the order of G_H/K is prime to the order of K/H ,
4. G_H/K is cyclic (infinite or finite),
5. G_H/K is the product of a finite group C'' of order prime to the order of K/H and of a group C' such that the extension of C' by K/H splits.

Proof. 1. $G(F)$ is commutative because G is a torus [Spr2].
 2. $G(F)/K_+$ is commutative because $K = \tilde{K} = G(F)$ ([HR] 11.1) and K/K_+ is commutative.
 3. $H^2(G_H/K, K/H) = 0$ because $G(F) = \tilde{K}$ and an extension of finite groups of relatively prime orders splits by the Schur-Zassenhaus theorem.
 4. $H^2(G_H/K, E^*) = 0$ ([NT] Lemma 5.4.(iii)).
 5. Let B'' be a subgroup of G_H/H isomorphic to C'' and let B' be the extension of C' by K/H . We have $G_V/H = B' \times B''$ because the intersection is obviously trivial and for $b' \in B', b'' \in B''$ there exists $a \in K/H$ with $b'b''b'^{-1} = b''a$; the order of $b'b''b'^{-1}$ and the order of a are relatively prime hence $a = 1$ because K/H is commutative. The group B'' is commutative because G_H/K is commutative. Hence G_H/H is commutative if and only if B' is commutative. $\square$

2.3 Parahoric subgroups

This paragraph is valid for a general reductive connected F -group G . Let L/F be the p -adic completion of the maximal unramified extension of F , let I be the inertia Galois group of the separable closure F^s/F and let σ be a Frobenius of the unramified closure F^u/F in F^s . The group I acts continuously on L and $L^I = F$. Let $G(L)_1$ be the kernel of the Kottwitz surjective morphism ([HR] 1.0.2)

$$w_G : G(L) \rightarrow (\pi_1(G))_I \quad \text{where} \quad \pi_1(G) := X^*(Z(\hat{G})) \quad .$$

Set $G(F)_1 := (G(L)_1)^\sigma$. Let G_{der} be the derived subgroup of G , let $G_{tor} := G/G_{der}$ be the maximal toric quotient of G and let G_{ad} be the adjoint group of G . A parahoric subgroup of $G(L)$ is a subgroup of the form

$$K_{\mathcal{F}}(L) := \text{Fix}(F) \cap G(L)_1$$

for a facet $\mathcal{F}$ of the building $\mathcal{B}(G_{ad}, L)$ of the adjoint group G_{ad} of G over L and the natural action of $G(L)$ on $\mathcal{B}(G_{ad}, L)$. An Iwahori subgroup is the parahoric subgroup associated to a chamber of $\mathcal{B}(G_{ad}, L)$. A parahoric subgroup of $G(F)$ is a subgroup of the form

$$K_{\mathcal{F}}(F) := K_{\mathcal{F}}(L)^{\sigma}$$

for a σ -invariant facet $\mathcal{F}$.

1. If $G = T$ is an F -torus then $T(L)_1$ is the unique parahoric subgroup of $T(L)$.
2. If G is semisimple and simply connected then $G(L)_1 = G(L)$.
3. If G_{der} is simply connected, we have the exact sequence

$$1 \rightarrow K_{der, \mathcal{F}}(L) \rightarrow K_{\mathcal{F}}(L) \rightarrow G_{tor}(L)_1 \rightarrow 1 \quad .$$

4. If $\tilde{G} \rightarrow G$ is a z -extension of kernel R , we have the exact sequence

$$1 \rightarrow R(L)_1 \rightarrow \tilde{K}_{\mathcal{F}}(L) \rightarrow K_{\mathcal{F}}(L) \rightarrow 1 \quad .$$

5. $H^1(\sigma, K_{\mathcal{F}}(L)) = 1$. We deduce that the properties 1,2,3,4 are valid when L is replaced by F .
6. If G is an induced F -torus or a semisimple simply connected F -anisotropic group, the unique parahoric subgroup $G(F)_1$ of $G(F)$ is also the unique maximal compact subgroup of $G(F)$.

2.4 A non commutative Hecke algebra

Let E/F a ramified quadratic extension with ring of integers O_E , and let D/E be a division algebra of E -dimension 16, with ring of integers O_D , uniformizer p_D , residue field k_D such that the action of p_D on k_D via conjugaison is $x \mapsto x^q$. We denote by $N_{k_D/k}$ the norm of k_D/k , $y \mapsto \bar{y}$ the non trivial F -automorphism of E , G' the reductive connected E -group with $G'(E) = D^*$, and $\text{Nrd} : G' \rightarrow \mathbb{G}_{m/E}$ the reduced norm of kernel the derived group G'_{der} of G' . The group G'_{der} is an F -anisotropic inner form of $SL(n)$ with $G'_{der}(E) = D^1$ the kernel of the reduced norm, and as $SL(n)$ it is simple and simply connected. Let

$$G := \{ (d, x, y) \in (G' \times \mathbb{G}_{m/E} \times \mathbb{G}_{m/E}) \mid \text{Nrd}(d)x^2 \frac{y}{\bar{y}} = 1 \} \quad .$$

Let $R_{E/F}$ be the scalar restriction from E to F . The projection $(d, x, y) \mapsto (x, y)$ induces an exact sequence in the category of F -linear algebraic groups

EX (2) $1 \rightarrow R_{E/F}(G'_{der}) \rightarrow G \rightarrow R_{E/F}(\mathbb{G}_{m/E} \times \mathbb{G}_{m/E}) \rightarrow 1 \quad .$

The algebraic linear F -group G is an extension of two connected reductive F -groups, hence is connected and reductive. The derived group $G_{der} = R_{E/F}(G'_{der})$ of G is simple simply connected and F -anisotropic and the maximal toric quotient $G_{tor} = R_{E/F}(\mathbb{G}_{m/E} \times \mathbb{G}_{m/E})$ of G is an induced torus. Hence the unique parahoric subgroup $K = G(F)_1$ of $G(F)$ is equal to the unique maximal compact subgroup of $G(F)$. The exact sequence (2) induces exact sequences

$$\begin{aligned} 1 &\rightarrow D^1 \rightarrow G(F) \rightarrow E^* \times E^* \rightarrow 1 \quad . \\ 1 &\rightarrow D^1 \rightarrow K \rightarrow O_E^* \times O_E^* \rightarrow 1 \quad . \end{aligned}$$

We deduce that $K = G(F) \cap (O_D^* \times O_E^* \times O_E^*)$ and

$$K/K_+ = \{ (z, x, y) \in k_D^* \times k^* \times k^* \mid N_{k_D/k}(z)x^2 = 1 \} \quad .$$

For $g = (d, x, y)$, $g' = (d', x', y') \in G(F)$ we have $gg'g^{-1}g'^{-1} = (dd'd^{-1}d'^{-1}, 1, 1)$ hence $(G(F), G(F)) = (D^1, 1, 1)$. If $d \in p_D^n O_D^*$ and if $d' = \zeta$ is a root of 1 prime to p then $gg'g^{-1}g'^{-1} = (\zeta^{q^n-1}, 1, 1)$ where $d \in p_D^n O_D^*$.

Lemma 2.5. *We suppose that $k = \mathbb{F}_3$ is a field with 3 elements.*

1. *The image of $G(F)$ by the projection $(d, x, y) \mapsto d$, is the group $p_D^{2\mathbb{Z}} O_D^*$.*
2. *$K/K_+ = \{(z, x, y) \in \mathbb{F}_{81}^* \times \mathbb{F}_3^* \times \mathbb{F}_3^* \mid z \text{ is a square in } \mathbb{F}_{81}^*\}$.*
3. *The Hecke algebra $\mathcal{H}(G, K, \chi)$ is not commutative if χ is the character of K/K_+ defined by $\chi(z, x, y) = z^5$.*

Proof. 1. The image of $G(F)$ by $(d, x, y) \mapsto \text{Nrd}(d)$ is the group $\{x^2 y / \bar{y} \mid x, y \in E^*\} = p_F^{2\mathbb{Z}} O_E^*$ because each element of $p_F^{2\mathbb{Z}}(1 + p_F O_E)$ is a square and $-1 \in E$ has norm 1 in F . The inverse image of $p_F^{2\mathbb{Z}} O_E^*$ by the reduced norm Nrd is $p_D^{2\mathbb{Z}} O_D^*$.

2. $k_D = \mathbb{F}_{81}$, $N_{k_D/k}(z) = z^{1+3+9+27} = z^{40}$ and $x^2 = 1$ for $z \in \mathbb{F}_{81}^*, x \in \mathbb{F}_3^*$. We have $z^{40} = 1$ if and only if z is a square in $\mathbb{F}_{81}^*$.

3. Let $h \in G(F)$ with $d_h = \zeta \in O_D^*$ a root of 1 of order 16. Then $h \in K$ hence h normalizes χ . Let $g \in G(F)$ with $d_g = p_D^2$. For $u \in K$ of image $(z, x, y) \in K/K_+$ the image of $gug^{-1}u^{-1}$ in K/K_+ is $(z^8, 1, 1)$. The order of χ is 8. Hence g normalizes χ . The image of $ghg^{-1}h^{-1}$ in K/K_+ is $(\zeta^8, 1, 1) = (-1, 1, 1)$ which does not belong to the kernel of χ . Hence $\chi(G_\chi, G_\chi) \neq 1$ and the Hecke algebra $\mathcal{H}(G, K, \chi)$ is not commutative (lemma 2.2). □

2.5 G splits over an unramified extension

In this section we prove:

Proposition 2.6. *When G is F -anisotropic modulo the center and splits over an unramified extension of F , then the Hecke algebra $\mathcal{H}(G, K, V)$ is commutative for all (K, V) .*

Proof. □

3 G general

From now on, G is a general reductive connected F -group. We choose a maximal F -split subtorus S of G . The connected center Z_G of G is contained in S . The centralizer $Z := Z_G(S)$ of S in G is a connected reductive F -group which is F -anisotropic modulo the center. Let $N := N_G(S)$ be the normalizer of S in G . Let A be the apartment associated to S in the Bruhat-Tits building of the adjoint group G_{ad} over F . We choose a chamber C of A and a special vertex O of C . Let I be the Iwahori subgroup of $G(F)$ associated to C and K the special parahoric subgroup of $G(F)$ associated to O . Modulo the action of $N(F)$ on A the chamber C is unique. The number of $G(F)$ -conjugacy classes of the special parahoric subgroups of $G(F)$ is finite and can be > 1 . Let Φ be the set of roots of G relative to S , let Δ be the basis of Φ defined by the walls of the chamber C containing O .

3.1 The group Λ

Let $\tilde{Z}_0$ be the unique maximal compact subgroup of $Z(F)$, let Z_0 be the unique parahoric subgroup of $Z(F)$, let Z_{0+} be the pro- p -radical of Z_0 and let

$$\Lambda := Z(F)/Z_0.$$

As in [2], Λ is a commutative finitely generated group with torsion subgroup $\Lambda_{tor} = \tilde{Z}_0/Z_0$ and S_* identifies to a subgroup of finite index of Λ . One denotes by $g \mapsto \lambda_g$ the quotient map $G(F) \rightarrow \Lambda$ and the group law on Λ is denoted additively, hence $\lambda_g + \lambda_{g'} = \lambda_{gg'}$ for $g, g' \in G(F)$.

Let G' be the subgroup of $G(F)$ generated by the parahoric subgroups. For $a \in \Delta$, let U_a be the root subgroup of G (noted $U_{(a)}$ in ([Spr2] 15.4); if $2a$ is a root the $U_{2a} \subset U_a$). The group G' is the normal subgroup of $G(F)$ generated by Z_0 and by $(U_a)_{a \in \Phi}$. We recall Z_G is the connected center of G . The subgroup $G'Z_G(F)$ of $G(F)$ has finite index. Let $Z' := Z(F) \cap G'$. Then

$$\Lambda' := Z'/Z_0$$

is a free commutative finitely generated subgroup of Λ .

Let $\Phi^\vee \subset S_*$ be the set of coroots, let $a^\vee \in \Phi^\vee$ be the coroot of $a \in \Phi$, and let $(\omega_{a^\vee})_{a \in \Delta}$ be the fundamental coweights in $\mathbb{Q}\Phi^\vee = \bigoplus_{a \in \Delta} \mathbb{Q}a^\vee$ defined by $\langle a, \omega_{b^\vee} \rangle = \delta_{a,b}$ for $a, b \in \Delta$. Both $(a^\vee)_{a \in \Delta}$ and $(\omega_{a^\vee})_{a \in \Delta}$ are $\mathbb{Q}$ -basis of $\mathbb{Q}\Phi^\vee$. A root $a \in \Phi$ extends to a linear form $x \mapsto \langle a, x \rangle$ on

$$S_* \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = (S_* \otimes_{\mathbb{Z}} \mathbb{Q})^{\Phi=0} \oplus \mathbb{Q}\Phi^\vee.$$

The kernel of the natural map $\lambda \mapsto \lambda \otimes 1 : \Lambda \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is Λ_{tor} . For $a \in \Phi$ and $\lambda \in \Lambda$, we define $\langle a, \lambda \rangle := \langle a, \lambda \otimes 1 \rangle$. We have the morphism $\omega : \Lambda \rightarrow \mathbb{Q}\Phi^\vee$ defined by

$$\lambda \mapsto \omega(\lambda) := \sum_{a \in \Delta} \langle a, \lambda \rangle \omega_{a^\vee}.$$

We say that $\lambda \in \Lambda$ is dominant when $\langle a, \lambda \rangle \geq 0$ for all $a \in \Delta$, i.e. $\omega(\lambda) \in \sum_{a \in \Delta} \mathbb{Q}_{\geq 0} \omega_{a^\vee}$, and is anti-dominant when $-\lambda$ is dominant. By the Gordan's lemma the submonoid Λ^+ of dominant elements of Λ is finitely generated. The same property is true for the submonoid $\Lambda^- = -\Lambda^+$ of anti-dominant elements. The kernel of ω is

$$\Lambda^{\Phi=0} := \tilde{Z}_0 Z_G(F) / Z_0.$$

We have

$$\Lambda^{\Phi=0} \oplus \Lambda' \subset \Lambda$$

and the index is finite. We have a unique decomposition

$$\Lambda = \Lambda^{\Phi=0} \oplus \Lambda_{der}$$

with $\Lambda' \subset \Lambda_{der}$. The restriction of ω to Λ_{der} is an isomorphism $\Lambda_{der} \simeq \omega(\Lambda)$. We fix a valuation v_F on F with $v_F(F) = \mathbb{Z}$. There exists a positive rational number $e_a \in \mathbb{Q}_{>0}$ such

$$N(F) \cap (U_{-a}(F)(U_a(F) - \{1\})U_{-a}(F))$$

acts on the apartment $A = \mathbb{R}\Phi^\vee$ by reflections at the affine hyperplanes $\{x \in A \mid \langle a, x \rangle \in e_a \mathbb{Z}\}$ ([BTII] 5.2.1). By ([BTI] 6.2.20), we have

$$\omega(\Lambda') = \sum_{a \in \Delta} \mathbb{Z} e_a a^\vee \subset \omega(\Lambda) \subset \sum_{a \in \Delta} \mathbb{Z} e_a \omega_{a^\vee}.$$

Lemma 3.1. $e_a = 1$ when G splits over an unramified extension of F .

Proof. ([L] 4.8). □

Remark 3.2. We have the fundamental weights in $\mathbb{Q}\Phi = \bigoplus_{a \in \Delta} \mathbb{Q}a$ defined by $\langle \omega_a, b^\vee \rangle = \delta_{a,b}$ for $a, b \in \Delta$. We have $a = \sum_{b \in \Delta} \omega_b \langle a, b^\vee \rangle$ because $\sum_{b \in \Delta} \langle \omega_b, c^\vee \rangle \langle a, b^\vee \rangle = \langle a, c^\vee \rangle$ for any $a, c \in \Delta$. Hence

$$\omega(\lambda) = \sum_{b \in \Delta} \langle \omega_b, \lambda \rangle b^\vee.$$

3.2 The Weyl Iwahori group W

The subgroup G' of $G(F)$ generated by the parahoric subgroups has a BN -pair (I, N') , where $N' := N(F) \cap G'$ and I is the Iwahori subgroup of K ([BTII] 5.2.12). The affine Weyl group of the BN -pair is

$$W_{aff} = N'/Z_0 \quad .$$

By the theory of BN -pairs for G' ***, the special parahoric group K has a BN -pair $(I, N(F) \cap K)$ of Weyl group equal to the relative Weyl group W_f of G . We have [Hra]

$$W_f = N(F)/Z(F) \simeq (N(F) \cap K)/Z_0 \simeq N'/Z' \quad , \quad K \cap Z(F) = Z_0 \quad .$$

The finite Weyl group W_f is isomorphic to the finite Coxeter group generated by the set of reflections with respect to the walls of the chamber C containing O ([AB] 10.1.6). The affine Weyl group W_{aff} is a Coxeter group generated by the set S_{aff} of reflections with respect to the walls of the chamber C , and

$$W_{aff} \simeq \Lambda' \rtimes W_f \quad .$$

We have the length ℓ and the Bruhat-Chevalley partial order $\leq_{Br}$ on (W_{aff}, S_{aff}) , defined by $w'_{aff} \leq_{Br} w_{aff}$ if and only if w'_{aff} has a reduced decomposition extracted from a reduced decomposition of w_{aff} . The Weyl Iwahori group

$$W := N(F)/Z_0$$

acts naturally on the apartment $A = \mathbb{R}\Phi^\vee$ and $\Lambda \subset W$ is a normal subgroup of finite index acting on A by translations. The normalizer Ω of the chamber C in W normalizes S_{aff} and normalises I . We have

$$\Lambda^{\Phi=0} = \Lambda \cap \Omega \quad , \quad \Lambda' = \Lambda \cap W_{aff} \quad .$$

The subgroup W_{aff} of W is normal and W is a semidirect product

$$W = W_{aff} \rtimes \Omega \simeq \Lambda \rtimes W_f.$$

The Weyl Iwahori group W is a quasi-Coxeter group (a semidirect product of a normal Coxeter group by an abelian group). We write $w = w_{aff}u$ with $w \in W_{aff}$ and $u \in \Omega$. When $G = G_{sc}$, one has $W = W_{aff}$ ([T] 1.13). The length and the partial Bruhat order of W_{aff} extend trivially to the quasi-Coxeter group W . By definition $\ell(w) = \ell(w_{aff})$, and $w \leq_{Br} w'$ if and only if $u = u'$ and $w_{aff} \leq_{Br} w'_{aff}$.

finite

Lemma 3.3. *Given $w \in W$,*

- a) *the number of $w' \in W$ such that $w' \leq_{Br} w$ is finite,*
- b) *the double class $W_f w W_f$ contains a unique element minimal w_{min} for the order $\leq_{Br}$.*

Proof. a) Obvious.

b) We have $W_f w W_f = W_f w_{aff} W'_f u$ where $W'_f = u W_f u^{-1}$. By the theory of Coxeter groups [?] there exists $w'_{aff} \in W_f w_{aff} W'_f$ such that any $x_{aff} \in W_f w' W'_f$ can be written $x_{aff} = a w'_{aff} b$ with $a \in W_f, b \in W'_f$ and $\ell(x_{aff}) = \ell(a) + \ell(w'_{aff}) + \ell(b)$. Let $w_{min} := w'_{aff} u$. Then any $x \in W_f w W_f$ can be written $x = x_{aff} u = a w_{min} c$ with $a, c \in W_f$ and $\ell(x) = \ell(a) + \ell(w_{min}) + \ell(b)$. This implies that w_{min} is the unique minimal element of $W_f w W_f$ for $\leq_{Br}$. $\square$

By the theory of BN -pairs for G' and for K we have

$$\boxed{a} \quad (3) \quad K = \sqcup_{w \in W_f} I w I \quad , \quad G' = \sqcup_{w \in W_{aff}} I w I \quad .$$

$$\boxed{b} \quad (4) \quad I w I w' \subset \sqcup_{w'' \in W_{aff}, w'' \leq_{Br} w' I w w''} \quad \text{for all} \quad w, w' \in W_{aff} \quad .$$

This remains true for the quasi-Coxeter group $W = W_{aff} \rtimes \Omega$.

Lemma 3.4. *We have*

$$\boxed{\mathbf{e}} \quad (5) \quad G(F) = \sqcup_{w \in W} IwI \quad ,$$

$$\boxed{\mathbf{c}'} \quad (6) \quad IwIw' \subset \sqcup_{w'' \in W, w'' \leq_{Br} w'} Iww'' \quad \text{for all } w, w' \in W \quad .$$

Proof. For $\boxed{\mathbf{e}}$ see ([HRa] Remark 9, p. 193). For $\boxed{\mathbf{c}'}$, we write $w = w_{aff}u, w' = w'_{aff}u'$ as before. By $\boxed{\mathbf{b}}$ $IwIw' = Iw_{aff}uIw'_{aff}u' = Iw_{aff}Iuw'_{aff}u^{-1}uu'$ is contained in

$$\begin{aligned} \sqcup_{w''_{aff} \in W_{aff}, w''_{aff} \leq_{Br} uw'_{aff}u^{-1}} Iw_{aff}w''_{aff}uu' &= \sqcup_{u^{-1}w''_{aff}u \leq_{Br} w'_{aff}} Iw_{aff}uu^{-1}w''_{aff}uu' \\ &= \sqcup_{w''_{aff} \leq_{Br} w'_{aff}} Iw_{aff}uw''_{aff}u' = \sqcup_{w'' \leq_{Br} w'} Iww'' \quad . \end{aligned}$$

□

3.3 Intersection of the Cartan and Iwasawa double classes

1. Cartan decomposition ([HR]): the natural map from $Z(F)$ to $K \backslash G(F) / K$ induces bijections

$$\Lambda^+ \simeq K \backslash G(F) / K \quad , \quad \Lambda^- \simeq K \backslash G(F) / K \quad .$$

2. Iwasawa decomposition ([HR]):

$$G(F) = KB(F) = KZ(F)U(F)$$

where $B = Z.U$ is the minimal parabolic F -subgroup of G associated to Δ .

$\boxed{\mathbf{e}'}$ **Proposition 3.5.** *1. Let $w \in W$ and $z \in Z(F)$. Then*

$$KwK \cap KzU(F) \neq \emptyset$$

implies $(\lambda_z)_{min} \leq_{Br} w_{min}$.

2. *When $z \in Z(F)$ is antidominant we have:*

$$KzK \cap zU(F) = z(K \cap U(F)) \quad .$$

Proof. 1) We imitate the proof of [HR] *** when F is replaced by the p -adic closure of the maximal unramified extension of F . We have $zu = iw'i'$ for some $w' \in W_f w W_f, i, i' \in I$, using $\boxed{\mathbf{b}}$. Multiplying on the right by $s \in S(F)$ such that $sus^{-1} \in I$ one gets $\lambda_z \lambda_s \in Iw'I\lambda_z I$. Using the lemma $\boxed{\mathbf{b}}$, we deduce $\lambda_z = w'$ for some $w'' \in W$ with $w'' \leq_{Br} w'$. We obtain

$$\boxed{\mathbf{f}} \quad (7) \quad \lambda_z \leq_{Br} w' \quad \text{for some } w' \in W_f w W_f \quad .$$

By the proof of b) in the lemma $\boxed{\mathbf{f}}$, there exists $a, b \in W_f$ such that $w' = aw_{min}b$ with $\ell(w') = \ell(a) + \ell(w_{min}) + \ell(b)$. We deduce from $\boxed{\mathbf{g}}$ that $\lambda_z = a'w''b'$ for some $a', b' \in W_f$ and $w'' \in W$ with $a' \leq_{Br} a, w'' \leq_{Br} w_{min}, b' \leq_{Br} b$. We obtain

$$\boxed{\mathbf{f}} \quad (8) \quad w'' \leq_{Br} w'_{min} \quad \text{for some } w'' \in W_f \lambda_z W_f \quad .$$

This implies $w_{\lambda_z, min} \leq_{Br} w_{min}$.

2) The condition that z is antidominant is necessary to have $Kz \cap zU(F) \subset z(K \cap U(F))$. Let $\tilde{K}$ be the special maximal subgroup containing K . We have by ([BTII] 5.2.4)

$$\tilde{K} \cap U(F) = K \cap U(F) \quad .$$

We have by ([BTII] 4.4.4 (ii))

$$\tilde{K}z\tilde{K} \cap zU(F) \subset z\tilde{K} \cap zU(F) = z(\tilde{K} \cap U(F)) \quad .$$

Hence $KzK \cap zU(F) \subset z(K \cap U(F))$. The opposite containment is obvious. □

Let $\lambda \in \Lambda$. We write $K\lambda K := KzK$ when $\lambda = \lambda_z$ with $z \in Z(F)$. Let X_λ be the set of $\lambda' \in \Lambda$ such that

$$K\lambda K \cap K\lambda'U(F) \neq \emptyset .$$

inclusion

Lemma 3.6. *The set X_λ is finite and the map $\lambda \mapsto \omega(\lambda)$ is injective on X_λ .*

Proof. The finiteness results from $\lambda'_{min} \leq_{Br} \lambda_{min}$ and the lemma ^{finite} 3.3. If the map is not injective, there exists $\mu_0 \in \Lambda^{\Phi=0}$ non zero such that $\lambda'_{min} \leq_{Br} \lambda_{min}$ and $(\mu_0 + \lambda')_{min} \leq_{Br} \lambda_{min}$. By definition of $\leq_{Br}$ for W , this is impossible because $(\mu_0 + \lambda')_{min} = \mu_0 \lambda'_{min}$. $\square$

Using the finiteness of X_λ and the fact that Λ^- (or Λ^+) is a system of representatives of the orbits of W_f in Λ , one can choose a total ordering $\leq$ on Λ^- (or Λ^+) such that for all $\lambda, \lambda' \in \Lambda^-$ (or Λ^+) ([HR])

$$K\lambda K \cap K\lambda'U(F) \neq \emptyset \text{ implies } \lambda' \leq \lambda .$$

Remark 3.7. Hitzelberger ^{Hit} [?] proved that $z, z' \in Z'$ satisfy $KzK \cap Kz'U(F) \neq \emptyset$ if and only if the point $z'.O$ of the apartment A belongs to the convex envelope of the conjugates of $z.O$ by W_f .

When G is unramified and K hyperspecial, Rapoport ^{Ra} [Ra] proved that for $\lambda, \lambda' \in \Lambda^+$ the relation $K\lambda K \cap K\lambda'U(F) \neq \emptyset$ is equivalent to $\lambda - \lambda' \in \sum_{a \in \Delta} \mathbb{N} a^\vee$.

Stembridge ^{Stem} [?] proved that for $\lambda, \lambda' \in \Lambda^- \cap \Lambda^+$ the relation $\lambda' \leq_{Br} \lambda$ is equivalent to $\lambda - \lambda' \in \sum_{a \in \Delta} \mathbb{N} e_a a^\vee$.

3.4 Parabolic subgroups

3.4

To any point $z \in Z(F)$, we have the parabolic subgroup P_z of G generated by Z and the root subgroups U_a ([BTII] 5.1) for all roots $a \in \Phi$ with $\langle a, z \rangle \geq 0$. The unipotent radical N_z is generated by the U_a for all $a \in \Phi$ such that $\langle a, z \rangle > 0$ and the Levi component M_z is generated by Z and the U_a for all $a \in \Phi$ such that $\langle a, z \rangle = 0$. The parabolic subgroups $P_{z^{-1}}$ and P_z are opposite of intersection M_z . The parabolic subgroup P_z contains B if and only if z is dominant, $P_z = B$ if and only if $\langle z, a \rangle > 0$ for all $a \in \Delta$, and $P_z = G$ if and only if $\langle z, a \rangle = 0$ for all $a \in \Delta$, i.e. $z \in \tilde{Z}_0 Z_G(F)$. Let $\Phi_{red} \subset \Phi$ be the subset of non divisible roots.

The group K is generated by $Z_0 = K \cap Z(F)$ and $U_{a,0} := K \cap U_a(F)$ for all $a \in \Phi_{red}$ and the pro- p -radical ^{SS} K_+ of K is generated by $Z_{0+} = K_+ \cap Z(F)$ and $U_{a,0+} := K_+ \cap U_a(F)$ for all $a \in \Phi_{red}$ [SS] ***. The maximal compact subgroup $\tilde{K}$ contenant K is equal to $\tilde{K} = \tilde{Z}_0 K$. Let

$$P_{z,0} := K \cap P_z \quad , \quad M_{z,0} := K \cap M_z \quad , \quad N_{z,0} := K \cap N_z \quad , \quad U_0^\pm := K \cap U^\pm(F) \quad .$$

$$P_{z,0+} := K_+ \cap P_z \quad , \quad M_{z,0+} := K_+ \cap M_z \quad , \quad N_{z,0+} := K_+ \cap N_z \quad , \quad U_{0+}^\pm := K_+ \cap U^\pm(F) \quad .$$

pab

Lemma 3.8. $M_{z,0} = z^{-1}Kz \cap M_z(F)$ is the special parahoric subgroup of $M_z(F)$ associated to O and $M_{z,0+} = z^{-1}K_+z \cap M_z(F)$ is the pro- p -radical of $M_{z,0}$.

Proof. The group $K \cap M_z(F)_1$ is the special parahoric subgroup of $M_z(F)$ associated to O [HRa], and the maximal compact subgroup of $M_z(F)$ fixing O is

$$\tilde{Z}_0(K \cap M_z(F)_1) = M_z(F) \cap \tilde{Z}_0 K = \tilde{Z}_0 M_{z,0} .$$

Using that $K \cap \tilde{Z}_0 = Z_0$ we deduce

$$K \cap M_z(F)_1 = M_{z,0} .$$

This can also be deduced from the analogous result over the completion L of the maximal unramified extension of F ([HR] 4.1.1). The group $M_{z,0} = K \cap M_z(F)_1$ is generated by Z_0 and $U_{a,0} := K \cap U_a(F)$ for all $a \in \Phi_{red}$ such that $\langle a, z \rangle = 0$. When $\langle a, z \rangle = 0$ we have $zU_{a,0}z^{-1} = U_{a,0}$. The same arguments imply the result for K_+ instead of K . $\square$

conj

Lemma 3.9. $M_{z,0}, M_{z,0+}, Z_0, Z_{0+}$ and $U_{a,0}, U_{a,0+}$ for each root $a \in \Phi_{red}$ such that $\langle a, z \rangle = 0$, are stable by conjugation by z .

Proof. Obvious for $Z_0, Z_{0+}, U_{a,0}, U_{a,0+}$ and using the lemma 3.8 for $M_{z,0}, M_{z,0+}$. $\square$

3.5 Reduction

By ([T] 3.5), there exists a reductive connected k -group $\overline{G}$ such that

$$\overline{G}(k) = K/K_+ .$$

Let

$$\text{red} : K \rightarrow \overline{G}(k)$$

be the quotient map, called the reduction map. The group $\overline{S}$ is a maximal k -split subtorus of $\overline{G}$ of centralizer $\overline{Z}$. The root system $\Phi(\overline{G}, \overline{S})$ is contained in Φ and contains the subset Φ_{nm} of roots $a \in \Phi$ such that $2a \notin \Phi$; when G is F -quasisplit, it is equal to Φ_{nm} hence is reduced ([L] 6.9, 10.36), [BTII] 4.6.12, 5.1.31, [T] 1.9, 3.5.1). The root subgroup $\overline{U}_a(k)$ of $\overline{G}(k)$ associated to $a \in \Phi(\overline{G}, \overline{S})_{red}$ is equal to ([SS] ***)

$$\overline{U}_a(k) = \text{red}(U_{a',0}) \text{ for } a' \in \Phi_{red} \cap \mathbb{Q}_{>0} a .$$

For $z \in Z(F)$, the group $\text{red}(K \cap z^{-1}Kz)$ is the image in $\overline{G}(k)$ of the parahoric subgroup fixing a facet of the star of A containing O . There exists a parabolic k -subgroup $\overline{P}_z$ of $\overline{G}$ such that ([BTII] 5.1.32)

$$\overline{P}_z(k) = \text{red}(K \cap z^{-1}Kz) .$$

The unipotent radical $\overline{N}_z$ of $\overline{P}_z$ is generated by the $\overline{U}_a$ for $a \in \Phi(\overline{G}, \overline{S})_{red}$ with $\langle a, z \rangle > 0$. For $a \in \Phi_{red}$ with $\langle a, z \rangle < 0$ the group $zU_{a,0+}z^{-1}$ contains $U_{a,0}$, and when $\langle a, z \rangle > 0$ the group $zU_{a,0}z^{-1}$ is contained in $U_{a,0+}$. We deduce that

$$\text{red}(P_{z,0}) = \text{red}(M_{z,0}).\text{red}(N_{z,0}) = \text{red}(K \cap z^{-1}Kz) .$$

We deduce

$$\text{red}(P_{z,0}) = \overline{P}_z(k) , \quad \text{red}(M_{z,0}) = \overline{M}_z(k) , \quad \text{red}(N_{z,0}) = \overline{N}_z(k) .$$

The k -parabolic subgroups $\overline{P}_{z^{-1}}$ and $\overline{P}_z$ of $\overline{G}$ are opposite of intersection $\overline{M}_z$. We denote by $[g] \in \overline{M}_z(k)$ the component of $g \in \overline{P}_z(k)$ or $g \in \overline{P}_{z^{-1}}(k)$, and by $\sigma(z)$ the automorphism of $\overline{M}_z(k)$ induced by the conjugation by z (lemma 3.9). We consider the morphism

$$\begin{aligned} \psi : K \cap z^{-1}Kz &\rightarrow \overline{G}(k) \times \overline{G}(k) \\ \psi(g) &= (\text{red}(g), \text{red}(zgz^{-1})) . \end{aligned}$$

fa

Lemma 3.10. *The image of the morphism ψ is*

$$\{ (g_+, g_-) \in \overline{P}_z(k) \times \overline{P}_{z^{-1}}(k) \text{ with } [g_-] = \sigma(z)[g_+] \} .$$

Proof. ([H] 3.8) Clearly

$$\psi(K \cap z^{-1}Kz \cap M(F)) = \{ (g_+, \sigma_z(g_+))_{g_+ \in \overline{M}_z(k)} \} .$$

For any $a \in \Phi_{red}$ such that $\langle a, z \rangle > 0$, the groups $U_{a,0}$ and $z^{-1}U_{-a,0}z$ are contained in $K \cap z^{-1}Kz$. We deduce that the image of ψ contains $\overline{N}_z(k) \times \overline{N}_{z^{-1}}(k)$. Hence the set given in the lemma is contained in the image of ψ . There is nothing else because

$$\text{red}(K \cap z^{-1}Kz) = \text{red}(K \cap P_z(F)) \quad , \quad \text{red}(K \cap zKz^{-1}) = \text{red}(K \cap P_{z^{-1}}(F)) \quad .$$

□

4 Irreducible representations of the finite group $G(k)$

In this section, k is a finite field of characteristic p , G is a connected reductive k -group. Let $B = T.U$ be a minimal parabolic k -subgroup of G of unipotent radical U and Levi component T (a maximal k -torus), let S be the k -split maximal subtorus of T with normalizer $N := N_G(S)$ in G . Let $W := N(k)/T(k)$ be the relative Weyl group with maximal element w_0 , let $\Phi = \Phi(G, S)$ be the group of roots of G relative to S , let Δ be the subset of simple roots defined by B and let $s_a \in W$ be the reflection associated to $a \in \Delta$. Let $B^- = w_0 B w_0^{-1}$ the opposite parabolic subgroup of B such that $B \cap B^- = T$. Let k' be a field of characteristic p containing a root of 1 of order the exponent of $T(k)$.

4.1 BN -pair

For the definition and the properties of a Tits system, or equivalently a BN -pair, see ([Bor] IV, §2). For the definition of the root system of a Coxeter system, see ([CE] 2.1). For the definition of the root groups associated to a Tits system with a finite Weyl group, see ([CE] 2.3). For the definition of a strongly split BN -pair of characteristic p , see ([CE] 2.20, exercise 5 of the chapter 2).

Lemma 4.1. *The system $(G(k), B(k), N(k), (s_a)_{a \in \Delta})$ forms a strongly split BN -pair of characteristic p .*

Proof. The system $(G(k), B(k), N(k), (s_a)_{a \in \Delta})$ is a Tits system ([Bor] 21.15).

The root system of the Coxeter system $(W, (s_a)_{a \in \Delta})$ i.e. the image of Δ by W , is equal to the subset Φ_{red} of reduced roots in Φ ([Bor] proposition 15).

The root subgroups $(U_a(k))_{a \in \Phi_{red}}$ of the BN -pair coincide with the usual root groups ([Spr2] 15.4). They satisfy the the commutator relations ([L] Proposition 5.2). □

4.2 Characters of the Hecke ring $\mathcal{H}(G(k), U(k))$

The Bruhat decomposition

$$G(k) = \cup_{n \in N(k)} U(k)nU(k)$$

shows that the Hecke ring $\mathcal{H}(G(k), U(k))$ is a free $\mathbb{Z}$ -module of basis the characteristic functions T_n of $U(k)nU(k)$ for all $n \in N(k)$. The Hecke ring is commutative if and only if W is commutative. However the simple modules of the k' -algebra $k' \otimes_{\mathbb{Z}} \mathcal{H}(G(k), U(k))$ are all one dimensional. They are given by characters $\mathcal{H}(G(k), U(k)) \rightarrow k'$ which are parametrized as follows ([CE] Th. 6.10).

For any simple root $a \in \Delta$ let $T_a := T \cap \langle U_a, U_{-a} \rangle$. For any character $\chi \in \text{Hom}(T(k), k'^*)$ one associates a subset Δ_χ of Δ defined by

$$\Delta_\chi := \{a \in \Delta \mid \chi(T_a(k)) = 1\} .$$

The pairs (χ, I) for $\chi \in \text{Hom}(T(k), k'^*)$ and $I \subset \Delta_\chi$ parametrize the characters $\mathcal{H}(G(k), U(k)) \rightarrow k'$. The character $\psi_{\chi, I}$ parametrized by (χ, I) is defined by

$$\psi_{\chi, I}(T_n) = (-1)^r \chi(t) \text{ when } a_1 \dots a_r \in I \quad \text{otherwise} \quad \psi_{\chi, I}(T_n) = 0 \quad ,$$

where $n = n_{a_1} \dots n_{a_r} t$ with $t \in T(k)$ and $w = s_{a_1} \dots s_{a_r}$ is a reduced decomposition of the image w of n in W and $s_a \in W$ is the image of $n_a \in N(k)$ for $a \in \Delta$. The character $\psi_{\chi, I}$ does not depend on the choices of n_a for $a \in \Delta$. The classification does not depend on the choice of k' .

4.3 Irreducible representations of $G(k)$

We introduce more notation. For any subset $J \subset \Delta$ set $\mathbb{N}J := \bigoplus_{a \in J} \mathbb{N}a$. We denote by $P_J = M_J N_J$ the standard parabolic k -subgroup of G with unipotent radical N_J generated by the root subgroups $(U_a)_{a \in \Phi \cap (\mathbb{N}(\Delta \setminus J))}$, and Levi subgroup M_J generated by T and the root subgroups $(U_a)_{a \in \Phi \cap \mathbb{N}J}$. The relative Weyl group W_J of M_J is generated by the reflections s_a for all $a \in J$. The group $B_J := M_J \cap B$ is a minimal parabolic subgroup $B_J = TU_J$ of M_J and U_J is the subgroup of U generated by the roots subgroups $(U_a)_{a \in \Phi \cap \mathbb{N}J}$. The opposite parabolic subgroup P_J^- of P_J is $P_J^- = M_J N_J^-$. When $J = \Delta$ we have $P_\Delta = M_\Delta = G, U_\Delta = U, N_\Delta = \{1\}$; when $J = \emptyset$ we have $P_\emptyset = B, M_\emptyset = T, U_\emptyset = \{1\}, N_\emptyset = U$.

1H

1. The functor $V \mapsto V^{U(k)}$ induces a bijection from the isomorphism classes of the irreducible k' -representations of $G(k)$ to the characters $\mathcal{H}(G(k), U(k)) \rightarrow k'$ ([CE] 6.12).

This implies that the irreducible representations of $G(k)$ in an algebraically closed field E containing k are defined over the subfield $k(\mu)$ where $\mu \in E$ is root of 1 of order the exponent of $T(k)$.

We denote by V any irreducible k' -representation of $G(k)$, and by $V(\chi, I) = V_G(\chi, I)$ an irreducible k' -representation of $G(k)$ such that $\mathcal{H}(G(k), U(k))$ acts on the one dimensional k' -vector space $V(\chi, I)^{U(k)}$ by the character $\psi_{\chi, I}$.

2. Let $I \subset \Delta$. The representation of $M_J(k)$ on $V(\chi, I)^{N_J(k)}$ is isomorphic to $V_{M_J}(\chi, I \cap J)$ ([C] 3.6).
3. $V = V^{N_J(k)} \oplus (1 - N_{-J}(k)).V^{N_J(k)}$ ([C] 3.2).
4. $V = U^-(k).V^{U(k)}$ ([C] 1.2).
5. The dimension of $V(\chi, I)$ is $\leq |U(k)|$ with equality if and only if $I = \Delta$ ([CE] 6.12).

car1

Proposition 4.2. *The parameter (χ, I) of V can be found as follows: χ is the character giving the action of $T(k)$ on the line $V^{U(k)}$ and $P_{\Delta \setminus I}(k)$ is the stabilizer of $V^{U(k)}$ in $G(k)$.*

Proof. This is clear for χ . For I see ([Cur] 6.15); the stabilizer of $V^{U(k)}$ in $G(k)$ is a standard parabolic subgroup and a non zero element of $V(\chi, I)^{U(k)}$ is a weight element of weight $(\chi, \Delta \setminus I)$ in the sense of Curtis. $\square$

Corollary 4.3. *The dimension of $V(\chi, I)$ is 1 if and only if $I = \emptyset$ and $\Delta_\chi = \Delta$.*

Proof. Obvious. $\square$

4.4 Action of an automorphism

Let σ be an automorphism of $G(k)$ which stabilizes $T(k)$ and $U_a(k)$ for any root $a \in \Phi$. The automorphism σ induces an automorphism of the Hecke ring $\mathcal{H}(G(k), U(k))$, still denoted by σ , such that $\sigma(T_n) = T_{\sigma(n)}$ for all $n \in N(k)$. The automorphism σ permutes the k' -irreducible representations of $G(k)$ and of $\mathcal{H}(G(k), U(k))$.

0.9

Lemma 4.4. $\psi_{\chi, I} \circ \sigma = \psi_{\chi \circ \sigma, I}$.

Proof. $(\sigma(n_a))_{a \in \Phi}$ is another possible choice of $(n_a)_{a \in \Phi}$ and we have $\sigma(n) = \sigma(n_{a_1}) \dots \sigma(n_{a_r}) \sigma(t)$. $\square$

inv

Lemma 4.5. $V(\chi, I) \circ \sigma \simeq V(\chi \circ \sigma, I)$.

Proof. a) Computing the action of the Hecke algebra $\mathcal{H}(G(k), U(k))$ on $V(\chi, I) \circ \sigma$, one sees that T_t acts by the character $\chi \circ \sigma$ and that the set I does not change. One gets

$$V(\chi, I) \circ \sigma \simeq V(\chi \circ \sigma, I) \quad .$$

A variant of the proof uses the proposition ^{|car1}4.2. It is clear that $T(k)$ acts on $(V(\chi, I) \circ \sigma)^{U(k)}$ by the $\chi \circ \sigma$. The representations $V(\chi, I) \circ \sigma$ and $V(\chi, I)$ have the same stabiliser in $G(k)$ because the stabilizers are standard parabolic subgroups generated by $B(k)$ and $U_a(k)$ for some negative roots $a \in \Phi$, and $U_a(k)$ stabilizes $V(\chi, I)^{U(k)}$ if and only if it stabilizes $(V(\chi, I) \circ \sigma)^{U(k)}$, because σ restricts to an automorphism of $U_a(k)$. $\square$

5 The Satake isomorphism

Let K be the special parahoric subgroup of $G(F)$ fixing the special point 0 of the apartment A , let k' be a field of characteristic p containing a root of 1 of order $\overline{G}(k)$, and let V be an irreducible k' -representation of K , identified to a representation of $\overline{G}(k)$. The representations of Z_0 on V^{U_0} and $V_{U_0^-}$ identify with the representations of $\overline{Z}(k)$ on $V^{\overline{U}(k)}$ and $V_{\overline{U}^-(k)}$.

5.1 Basis of $\mathcal{H}(G, K, V)$

The group Z_0 acts on the line $V^{U_0} \simeq V_{U_0^-}$ by a character χ_V . When $z \in Z(F)$ is antidominant, $N_{z^{-1}, 0}$ contains U_0 and $M_{z, 0} = M_{z^{-1}, 0}$.

Z=

Lemma 5.1. *Let $z \in Z(F)$ antidominant. The irreducible representations $V^{N_{z^{-1}, 0}}$ and $V^{N_{z^{-1}, 0}} \circ \sigma(z)$ of $M_{z, 0}$ are isomorphic if and only if $\chi_V = \chi_V \circ \sigma(z)$.*

Proof. Lemma ^{inv}4.5 and proposition ^{|car1}4.2. $\square$

We identify χ_V with a character of Z_0 . Let

$$Z_V := \{ z \in Z(F) \mid \chi_V(zxz^{-1}x^{-1}) = 1 \text{ for all } x \in Z_0 \} \quad .$$

and Z_V^- the antidominant elements of Z_V . For $z \in Z_V^-$, there exists a unique $M_{z, 0}$ -equivariant morphism

$$a_z : V^{N_{z^{-1}, 0}} \rightarrow V^{N_{z^{-1}, 0}} \circ \sigma(z)$$

fixing pointwise V^{U_0} , by the lemma ^{Z=}5.1. We have a natural injection

$$i_z : V^{N_{z^{-1}, 0}} \circ \sigma(z) \rightarrow V$$

and a canonical $\overline{M}_z(k)$ -equivariant morphism

$$b_z : V \rightarrow V_{N_{z,0}} \rightarrow V^{N_{z^{-1},0}}$$

fixing pointwise $V^{\overline{N}_{z^{-1},0}}$. The morphism $\phi_z := i_z \circ a_z \circ b_z : V \rightarrow V$ fixes pointwise the line $V^{\overline{U}}$. These morphisms depend only on $\lambda_z \in \Lambda$. Let Λ_V be the image of Z_V in Λ and Λ_V^- the antidominant elements of Λ_V .

Proposition 5.2. *For $z \in Z_V^-$, there exists a function $T_z \in \mathcal{H}(G, K, V)$ of support KzK with*

$$T_z(z) = \phi_z.$$

A k' -basis of $\mathcal{H}(G, K, V)$ is $(T_{z_\lambda})_{\lambda \in \Lambda_V^-}$.

Proof. Suppose that $f \in \mathcal{H}(G, K, V)$ is supported on KzK for some $z \in Z(F)$. By the Cartan decomposition we can suppose z antidominant. It follows immediately from the definitions that the possible values of $f(z)$ are all $\phi \in \text{End}_{k'}(V)$ such that

$$k_1 \phi = \phi k_2 \quad \text{whenever } k_1, k_2 \in K \text{ and } k_1 z = z k_2.$$

Note that $k_2 \in K \cap z^{-1}Kz$ and $k_1 = zk_2z^{-1}$. The proposition 5.10 implies that equivalently ϕ factors

$$V \rightarrow V_{N_{z,0}} \rightarrow V^{N_{z^{-1},0}} \rightarrow V^{N_{z^{-1},0}} \circ \sigma(z) \rightarrow V.$$

If $z \in Z_V^-$, then ϕ is a multiple of ϕ_z ; otherwise there are no such ϕ (lemma 5.1). $\square$

5.2 The Satake morphism

The Satake morphism is a consequence of the Iwasawa decomposition $G(F) = KZ(F)U(F)$ for the special parahoric subgroup K and of the structure of the induced representation $\text{Ind}_K^{G(F)}V$ as a $G(F) \times \mathcal{H}(G, K, V)$ -module (lemma 4.1).

The subspace $(\text{Ind}_K^{G(F)}V)_c^{U(F)}$ of $f \in (\text{Ind}_K^{G(F)}V)^{U(F)}$ such that $f|_{Z(F)}$ is compactly supported is a $Z(F) \times \mathcal{H}(G, K, V)$ -submodule of $\text{Ind}_K^{G(F)}V$. The restriction to $Z(F)$ is a $Z(F)$ -equivariant isomorphism

$$\text{res}_{Z(F)} : (\text{Ind}_K^{G(F)}V)_c^{U(F)} \rightarrow \text{c-ind}_{Z_0}^{Z(F)}V^{U_0}$$

by the Iwasawa decomposition. The action of $\mathcal{H}(G, K, V)$ on $(\text{Ind}_K^{G(F)}V)_c^{U(F)}$ induces a $Z(F)$ -equivariant action of $\mathcal{H}(G, K, V)$ on $\text{c-ind}_{Z_0}^{Z(F)}V^{U_0}$.

Definition 5.3. *The Satake morphism $\mathcal{S} : \mathcal{H}(G, K, V) \rightarrow \mathcal{H}(Z, Z_0, V^{U_0})$ is the algebra morphism satisfying*

$$\text{res}_{Z(F)} \circ f = \mathcal{S}(f) \circ \text{res}_{Z(F)} \quad \text{for all } f \in \mathcal{H}(G, K, V).$$

Proposition 5.4. *The Satake morphism $\mathcal{S}$ is given by*

$$\mathcal{S}(f)(z) = \sum_{u \in U(F)/U_0} f(zu)|_{V^{U_0}}.$$

Proof. By linearity it is enough to prove the formula for an arbitrary element $T \in \mathcal{H}(G, K, V)$ of support KzK for an arbitrary $z \in Z(F)$. We denote by i_B the inverse of $\text{res}_{Z(F)}$ and by $[1, m]_{Z_0}$ the function in $\text{c-ind}_{Z_0}^{Z(F)}V^{U_0}$ with support Z_0 and value $m \in V^{U_0}$ at 1. We have $T \circ i_B = i_B \circ \mathcal{S}(T)$ and

$$i_B([1, m]_{Z_0}) = \sum_{u \in U(K)/U_0} u.[1, m]_K.$$

We compute

$$(T \circ i_B)([1, m]_{Z_0})(z')$$

for any $z' \in Z(F)$. We write KzK as a disjoint union $\cup_x Kzx$. For any $g \in G(F), v \in V$, we have (for $\frac{1}{1}$):

$$T(g.[1, v]_K) = g. \sum_x (zx)^{-1}.[1, (T(z) \circ x)v]_K .$$

Hence

$$(T \circ i_B)([1, m]_{Z_0}) = \sum_{u \in U(K)/U_0} u. \sum_x (zx)^{-1}.[1, T(z)(x.m)]_K .$$

The support of $g[1, v]_K$ is Kg^{-1} ; its value at kg^{-1} is kv . The value at z' of $u(zx)^{-1}.[1, T(z)(x.m)]_K$ is 0 if z' does not belong to $Kzxu^{-1}$, and is $kT(z)(x.m)$ if $k = z'u(zx)^{-1} \in K$. We have

$$(T_z \circ i_B)([1, m]_{Z_0})(z') = \sum_{u \in U(F)/U_0, z'u(zx)^{-1} \in K} (z'u(zx)^{-1} \circ T(z) \circ x) m .$$

We have $T(z'u) = 0$ if $z'u$ does not belong to KzK and $T(z'u) = k \circ T(z) \circ x$ if $z'u = kzx$ with $k, x \in K$. We deduce that

$$(T \circ i_B)([1, m]_{Z_0})(z') = \sum_{u \in U(F)/U_0} T(z'u) m .$$

Equivalently $\mathcal{S}(T)(z') = (i_B \circ \mathcal{S}(T))(z') = (T \circ i_B)(z') \in \text{End}_k V^{U_0}$ is the morphism

$$\sum_{u \in U(F)/U_0} T(z'u) .$$

□

support

Proposition 5.5. *The image of the Satake morphism vanishes outside the antidominant elements of $Z(F)$.*

Proof. Let $z \in Z(F)$ not antidominant, let $a \in \Delta$ be a simple root such that $\langle a, z \rangle > 0$, let $f \in \mathcal{H}(G, K, V)$, and let $x \in V^{U_0}$. Using the decomposition of U as a semidirect product

$$U = U^{(a)} \rtimes U_a$$

where the normal subgroup $U^{(a)}$ is generated by the root groups U_b with b positive and not multiple of a , and the fact that $zU_{a,0}z^{-1} \subset U_{a,0+}$ (proof of prop. 5.10), we obtain:

$$\begin{aligned} \mathcal{S}f(z) x &= \sum_{u_a \in U_a(F)/U_{a,0}} \sum_{u^{(a)} \in U^{(a)}(F)/U_0^{(a)}} f(zu_a u^{(a)}) x \\ &= \sum_{u_a \in U_a(F)/U_{a,0}} \sum_{u^{(a)} \in U^{(a)}(F)/U_0^{(a)}} f(zu_a z^{-1} z u^{(a)}) x \\ &= \sum_{u_a \in U_a(F)/zU_{a,0}z^{-1}} \sum_{u^{(a)} \in U^{(a)}(F)/U_0^{(a)}} f(u_a z u^{(a)}) x . \end{aligned}$$

When $U_{a,0}$ acts trivially on V , the function $f_a : u_a \mapsto f(u_a z u^{(a)})(v)$ is invariant by left translation by $U_{a,0}$, the sum $\mathcal{S}f(z) x$ is divisible by $[U_{a,0} : U_{a,0+}]$ which is a non zero power of p . For a general V only $U_{a,0+}$ acts trivially on V .

a) If $zU_{a,0}z^{-1} \neq U_{a,0+}$ then the sum $\mathcal{S}f(z) x$ is divisible by $[U_{a,0+} : zU_{a,0}z^{-1}]$ which is a non zero power of p .

b) Otherwise, we use Herzig's trick ([H] Step 5 in the proof of theorem 1.2). Let $\lambda \in \Lambda^-$. Let Y_λ be the image in Λ of the set of $z \in Z(F)$ such that $\mathcal{S}T_{z_\lambda}(z) \neq 0$, is contained in Λ^- . We want to prove $Y_\lambda \subset \Lambda^-$. Otherwise there exists $a_1 \in \Delta$ and $\lambda_z \in Y_\lambda$ such that $\langle a_1, z \rangle > 0$. We choose a total ordering on Δ starting with a_1 . We choose $z' \in Z(F)$ such that $\lambda_{z'} \in Y_\lambda$ and $\langle a, \lambda_{z'} \rangle_{a \in \Delta}$ is the greatest possible for the lexicographic order on $\mathbb{Z}^\Delta$. Then $\langle a_1, z' \rangle > 0$. This implies $z'^2 U_{a,0} z'^{-2} \subset z' U_{a,0} z'^{-1} \subset U_{a,0+}$ and $z'^2 U_{a,0} z'^{-2} \neq U_{a,0+}$. We will show that $\mathcal{S}(T_z^2)(z'^2) \neq 0$. This contradicts a) hence $Y_\lambda \subset \Lambda^-$.

We have $(\mathcal{S}T_z)^2 = \mathcal{S}(T_z^2)$. We show that

$$(\mathcal{S}T_z)^2(z'^2) = \sum_{\lambda \in \Lambda} \mathcal{S}T_z(z_\lambda) \circ \mathcal{S}T_z(z_\lambda^{-1} z'^2)$$

is not 0. If the term indexed by λ is not 0 then $\langle a, \lambda \rangle_{a \in \Delta}$ and $\langle a, 2\lambda_{z'} - \lambda \rangle_{a \in \Delta}$ are smaller or equal to $\langle a, \lambda_{z'} \rangle_{a \in \Delta}$. This yields the equality $\omega(\lambda) \stackrel{\text{inclusion}}{=} \omega(\lambda_{z'})$. The set Y_λ is contained in the finite set X_λ considered in the lemma 3.6 on which the map ω is injective. Hence $\lambda_{z'} = \lambda$. So $(\mathcal{S}T_z)^2(z'^2) \neq 0$. $\square$

Sat1

Theorem 5.6. *The Satake morphism induces an isomorphism*

$$\mathcal{H}(G, K, V) \simeq \mathcal{H}(Z_V^-, Z_0, V^{U_0})$$

where $\mathcal{H}(Z_V^-, V_0, V^{U_0})$ is the subalgebra of $\mathcal{H}(Z, Z_0, V^{U_0}) = \oplus_{\lambda \in \Lambda_V^-} k' \tau_{z_\lambda}$ of functions with support contained in Z_V^- .

Proof. We know that $(T_{z_\lambda})_{\lambda \in \Lambda_V^-}$ is a k' -basis of $\mathcal{H}(G, V)$ and that $(\tau_{z_\lambda})_{z \in \Lambda_V}$ is a basis of $\mathcal{H}(Z, Z_0, V^{\overline{U}(k)}) = \mathcal{H}(Z, \chi_V)$. We know by the proposition 5.5 and the lemma 5.6 that

$$\mathcal{S}T_{z_\lambda} = \tau_{z_\lambda} + \sum_{\lambda' \in \Lambda_V^-, \lambda' \leq \lambda} a_{\lambda'} \tau_{z_{\lambda'}}$$

and that the set of $\lambda' \in \Lambda^-$ with $\lambda' \leq \lambda$ is finite. We deduce the theorem by the usual triangular argument. $\square$

Remark 5.7. The same arguments apply to the ring

$$\mathcal{H}(G, K) := \text{End}_{\mathbb{Z}G} \mathbb{Z}[K \backslash G] \quad .$$

The ring morphism $\mathcal{S} : \mathcal{H}(G, K) \rightarrow \mathcal{H}(Z, Z_0) = \mathbb{Z}[\Lambda]$ defined using $C_c(K \backslash G(F)/U(F), \mathbb{Z})$ satisfies

$$\mathcal{S}(f)(z) = \sum_{u \in U(F)/U_0} f(zu) \quad ,$$

and is injective of image $\oplus_{\lambda \in \Lambda^-} \mathbb{Z} \tau_\lambda$ modulo $p \mathcal{H}(Z, Z_0)$.

5.3 Finiteness

One identifies χ_V to a k' -character of Z_0 . Let

$$Z'_V := \{x \in Z(F) \mid \chi_V(xyx^{-1}y^{-1}) = 1 \text{ for all } y \in Z_V\} \quad .$$

By the lemma 2.1, Z'_V is a normal subgroup of Z_V of finite index containing Z_0 . Let $\Lambda'_V := Z'_V/Z_0$ and let $Z_V'^-, \Lambda_V'^-$ be the antidominant elements of Z'_V, Λ'_V .

fg

Lemma 5.8. *The antidominant monoid Λ_V^- is finitely generated, generates the group Λ_V' and Λ_V^-/Λ_V' is finite.*

Proof. We choose $s \in S(F)$ satisfying $\langle a, s \rangle < 0$ for each $a \in \Delta$. For any element $z \in Z(F)$ there exists an integer $n \in \mathbb{N}$ such that $s^n z$ is antidominant. We have $s^n z \in Z_V'$ if $z \in Z_V'$ because $S(F) \subset Z_V'$. We deduce that the group Z_V' is generated by its antidominant elements and that the same is true for the group Λ_V' .

The group $\Lambda_V' \subset \Lambda$ is commutative and finitely generated. The intersection of $\Lambda_V' \otimes 1$ with the convex polyhedral cone of

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = S_* \otimes_{\mathbb{Z}} \mathbb{R}$$

defined by the equations $\langle x, a \rangle \leq 0$ for all $a \in \Delta$ is a finitely generated monoid by Gordan's lemma [Fu]. The kernel of $\lambda \mapsto \lambda \otimes 1$ is the torsion subgroup which is finite. Hence the monoid Λ_V^- is finitely generated.

Let d be the smallest positive integer such that $\chi_V^d = 1$. Then $z^d \in Z_V'$ for any $z \in Z_V$ by the proof of the lemma 2.1, and z is antidominant if and only if z^d is antidominant. Hence $d\Lambda_V^- \subset \Lambda_V'$. As the set $\Lambda_V^-/d\Lambda_V^-$ is finite, the set Λ_V^-/Λ_V' is finite. $\square$

center

Lemma 5.9. *The center of the algebra $\mathcal{H}(Z_V^-, Z_0, V^{U_0})$ is $\mathcal{H}(Z_V'^-, Z_0, V^{U_0})$ and is isomorphic to $k'[\Lambda_V'^-]$.*

Proof. By the product formula (proof of the lemma 2.2), the center of the algebra $\mathcal{H}(Z_V^-, Z_0, V^{U_0})$ is formed by the functions with support in the set $Z_V''^-$ of elements $x \in Z_V^-$ such that $\chi_V(xy x^{-1} y^{-1}) = 1$ for all $y \in Z_V^-$, and is isomorphic to $k'[Z_V''^-/Z_0]$. Let $s \in S(F)$ as in the proof of the lemma 5.8 and let z be an arbitrary element of Z_V . There exists an integer $n \in \mathbb{N}$ such that $s^n z \in Z_V^-$. We have $\chi_V(xy x^{-1} y^{-1}) = 1$ for $y = s^n z$. As $S(F)$ is central in $Z(F)$ we deduce $\chi_V(x z x^{-1} z^{-1}) = 1$. Hence $Z_V''^- = Z_V'^-$. $\square$

We deduce from the theorem 5.6 and the lemma 5.8 and 5.9:

Corollary 5.10. *The center of the Hecke algebra $\mathcal{H}(G, K, V)$ is a finitely generated algebra isomorphic to the group algebra $k'[\Lambda_V'^-]$, and $\mathcal{H}(G, K, V)$ is a finitely generated module over its center.*

We denote by $\mathcal{Z}(G, K, V)$ the center of $\mathcal{H}(G, K, V)$. By a general result [Vig] 5.3, this implies :

Corollary 5.11. *When k' is an algebraically closed field, any simple module of $\mathcal{H}(G, K, V)$ is finite dimensional, has a central character, and is a quotient of*

$$\mathcal{H}(G, K, V) \otimes_{\mathcal{Z}(G, K, V), \chi} k'$$

for the central character $\chi : \mathcal{Z}(G, K, V) \rightarrow k'$.

A character of $\mathcal{Z}(G, K, V) \simeq k'[\Lambda_V'^-] \rightarrow k'$ identifies with a character $\Lambda_V'^- \rightarrow k'$. The subgroup of the invertible elements of $\Lambda_V'^-$ is $\Lambda^{\Phi=0} \cap \Lambda_V'^-$.

Definition 5.12. *A character $\mathcal{Z}(G, K, V) \rightarrow k'$ corresponding to a character of $\Lambda_V'^-$ vanishing outside $\Lambda^{\Phi=0} \cap \Lambda_V'^-$ is called supersingular. A simple module of $\mathcal{H}(G, K, V)$ with a supersingular central character is called supersingular.*

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