Abelian Sheaves

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Introduction

Sheaves on topological spaces were invented by Jean Leray as a tool to deduce global properties from local ones. Then Grothendieck realized that the usual notion of a topological space was not appropriate for algebraic geometry (there being an insufficiency of open subsets), and introduced sites, that is, categories endowed with "Grothendieck topologies" and extended sheaf theory in the framework of sites.

Sheaf theory is an extremely powerful tool and applies to many areas of Mathematics, from Algebraic Geometry to Quantum Field Theory.

The functor $\Gamma(X;\bullet)$, which to a sheaf F on X associates the space $\Gamma(X;F)$ of its global section, is left exact but not right exact in general. The derived functors $H^j(X;F)$ tell us the "obstructions" to pass from local to global. In particular, given a ring $\mathbf{k}$, a topological space X is naturally endowed with the sheaf $\mathbf{k}_X$ of $\mathbf{k}$-valued locally constant functions, and the cohomology of this sheaf is thus a topological invariant of the space.

In these Notes, we shall expose sheaf theory on sites in the framework of derived categories and give some applications. We restrict ourselves to the cases of sites admitting products and fiber products, which makes the theory much easier and very similar to that of sheaves on topological spaces. We also essentially restrict our study to abelian sheaves and we use the results of homological algebra presented in [Sc02]. For further references on homological algebra see [KS06] (and also [GM96], [We94]).

Classical sheaf theory is exposed in particular in [Go58] and [Br67]. For an approach in the language of derived categories, see [Iv87], [GM96], [KS90]. Sheaves on Grothendieck topologies are exposed in [SGA4] and [KS06]. A short presentation in case of the étale topology is given in [Ta94].

Let us briefly describe the contents of these Notes.

Chapters 1 and 2 are devoted to the general theory of sheaves on sites. We first study with some details presheaves on presites with values in an arbitrary category $\mathcal{A}$, then we introduce Grothendieck topologies. Next, we restrict our study to abelian sheaves, that is, to the case where $\mathcal{A} = \mathrm{Mod}(\mathbf{k})$ for a unital commutative ring $\mathbf{k}$. We prove that the category $\mathrm{Mod}(\mathbf{k}_X)$ of abelian sheaves it a Grothendieck category. We define and study the operations of internal hom and tensor product, direct and inverse image, extension and restriction. We also glue sheaves and show how to construct naturally locally constant sheaves.

In **Chapter 3** We study the derived category $\mathrm{D}^+(\mathbf{k}_X)$ of $\mathrm{Mod}(\mathbf{k}_X)$ and the derived operations on sheaves. We describe the Čech complexes associated with a covering and prove the Leray's acyclic theorem. Finally, we make a brief study of ringed sites, that is, sites equipped with a sheaf of rings. We study modules over such sheaves of rings and their natural derived operations.

In **Chapter 4** we study abelian sheaves on topological spaces. We introduce the functors $(\bullet)_Z$ and $\Gamma_Z(\bullet)$ associated to a locally closed subset Z and we study flabby sheaves. Then we study locally constant abelian sheaves. We prove that the cohomology of such sheaves is a homotopy invariant, and using the Čech complex associated to a closed covering, we show how to compute the cohomology of spaces which admit covering by contractible subsets. We apply these techniques to calculate the cohomology of some classical manifolds.

Chapter 5 is devoted to duality on locally compact spaces. We first define the proper direct image functor $f_!$ associated with a morphism $f\colon X \to Y$ of locally compact spaces. (The definition that we propose here, although equivalent, is not the traditional one.) Next, we prove that c-soft sheaves are acyclic for the functor $f_!$ and we study its derived functor $Rf_!$. We prove the two main results of this theory, namely the projection formula and the base change formula. As a byproduct, we get the Künneth formula.

The existence of the right adjoint $f^!$ to $Rf_!$ follows from the Brown representability theorem. We study the properties of this functor and introduce in particular the dualizing complex ω_X that we explicitely calculate when X is a topological manifold. As an application, we expose the De Rham cohomology on real manifolds, the Dolbeault-Grothendieck cohomology on complex manifolds and we construct the Leray-Grothendieck residues morphism.

In these Notes, we use the language of derived categories and follow the notations of [Sc02]. We shall note enter in problems of universes, assuming to be given a universe $\mathcal{U}$ in which we are working, and changing of universe if necessary.

Chapter 1

Presheaves on presites

Presheaves are nothing but contravariant functors, but they play, at least psychologically, a different role than usual functors. In this chapter, we study the natural internal and external operations on presheaves.

In all theses Notes, we denote by $\mathbf{k}$ a commutative unital ring. As far as there is no risk of confusion, we shall write $\otimes$ instead of $\otimes_{\mathbf{k}}$ and Hom instead of $\mathrm{Hom}_{\mathbf{k}}$.

For sheaf theory on sites: see [SGA4] and for an exposition (and a slightly different approach) see [KS06].

1.1 Recollections from category theory

In all these Notes we fix a universe $\mathcal{U}$. A $\mathcal{U}$-set is a set which belongs to $\mathcal{U}$ and a set is $\mathcal{U}$-small if it is isomorphic to a $\mathcal{U}$-set. A category means a $\mathcal{U}$-category, that is, a category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $\mathcal{U}$-small for all $X, Y \in \mathcal{C}$. If $\mathrm{Ob}(\mathcal{C})$ is a $\mathcal{U}$-set, then one says that $\mathcal{C}$ is $\mathcal{U}$-small. By a "big" category, we mean a category in a bigger universe. Note that any category is an $\mathcal{V}$-category for a suitable universe $\mathcal{V}$ and one even can choose $\mathcal{V}$ so that $\mathcal{C}$ is $\mathcal{V}$-small. As far as it has no implication, we shall not always be precise on this matter and the reader may skip the words "small" and "big". The category $\mathbf{Set}$ is the category of $\mathcal{U}$-sets and maps.

Definition 1.1.1. Let $\mathcal{C}$ be a category. One defines the big categories

$$\begin{aligned}\mathcal{C}^{\wedge} &= \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}),\\ \mathcal{C}^{\vee} &= \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}^{\mathrm{op}}) \simeq \mathrm{Fct}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}},\end{aligned}$$

and the functors

$$\begin{aligned}\mathrm{h}_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^{\wedge}, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, X),\\ \mathrm{k}_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^{\vee}, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, \cdot).\end{aligned}$$

Recall that the functors $\mathrm{h}_{\mathcal{C}}$ and $\mathrm{k}_{\mathcal{C}}$ are fully faithful. This is the Yoneda lemma.

Definition 1.1.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, $F\colon \mathcal{C} \to \mathcal{C}'$ a functor and let $Z \in \mathcal{C}'$.

(i) The category $\mathcal{C}_Z$ is defined as follows:

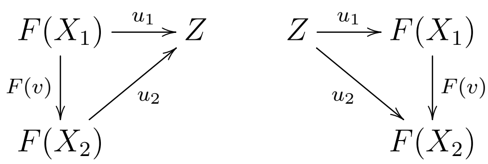
$$\begin{aligned} \mathrm{Ob}(\mathcal{C}_Z) &= \{(X,u); X \in \mathcal{C}, u\colon F(X) \to Y\}, \\ \mathrm{Hom}_{\mathcal{C}_Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_1 = u_2 \circ F(v)\}. \end{aligned}$$

(ii) The category $\mathcal{C}^Z$ is defined as follows:

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^Z) &= \{(X,u); X \in \mathcal{C}, u\colon Z \to F(X))\}, \\ \mathrm{Hom}_{\mathcal{C}^Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_2 = u_1 \circ F(v)\}. \end{aligned}$$

Note that the natural functors $(X,u) \mapsto X$ from $\mathcal{C}_Z$ and $\mathcal{C}^Z$ to $\mathcal{C}$ are faithful.

The morphisms in $\mathcal{C}_Z$ (resp. $\mathcal{C}^Z$) are visualized by the commutative diagram on the left (resp. on the right) below:



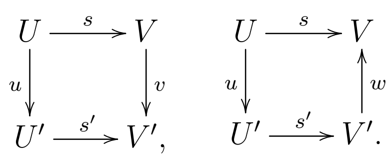
Definition 1.1.3. Let $\mathcal{C}$ be a category. The category $\mathrm{Mor}(\mathcal{C})$ of morphisms in $\mathcal{C}$ is defined as follows.

$$\begin{aligned} &\mathrm{Ob}(\mathrm{Mor}(\mathcal{C})) = \{(U,V,s); U,V \in \mathcal{C}_X, s \in \mathrm{Hom}_{\mathcal{C}}(U,V), \\ &\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((s:U \to V),(s':U' \to V') \\ &\qquad = \{u:U \to U', v:V \to V'; v \circ s = s' \circ u\}. \end{aligned}$$

The category $\mathrm{Mor}_0(\mathcal{C})$ is defined as follows.

$$\begin{aligned} &\mathrm{Ob}(\mathrm{Mor}_0(\mathcal{C})) = \{(U,V,s); U,V \in \mathcal{C}_X, s \in \mathrm{Hom}_{\mathcal{C}}(U,V), \\ &\mathrm{Hom}_{\mathrm{Mor}_0(\mathcal{C})}((s:U \to V),(s':U' \to V') \\ &\qquad = \{u:U \to U', w:V' \to V; s = w \circ s' \circ u\}. \end{aligned}$$

A morphism $(s\colon U \to V) \to (s'\colon U' \to V')$ in $\mathrm{Mor}(\mathcal{C})$ (resp. $\mathrm{Mor}_0(\mathcal{C})$) is visualized by the commutative diagram on the left (resp. on the right) below:



Generators

Recall that a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is conservative if any morphism $f\colon X \to Y$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Definition 1.1.4. Let $\mathcal{C}$ be a category. A family $\{G_i\}_{i\in I}$ of objects of $\mathcal{C}$ is a system of generators if I is a small set and the functor $\prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(G_i, \bullet)\colon \mathcal{C} \to$ **Set** is conservative.

If the family contains a single element, say G, one says that G is a generator. If the category $\mathcal{C}$ admits coproducts and a system of generators as above, then it admits a generator, namely the object $\bigsqcup_{i\in I} G_i$.

Lemma 1.1.5. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories. Then F is conservative if and only if it is faithful.*

The proof is left as an exercise.

Lemma 1.1.6. *Let $\mathcal{A}$ be an abelian category which admits small coproducts and a generator G. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and assume that $\mathrm{Hom}_{\mathcal{A}}(G, X) \to \mathrm{Hom}_{\mathcal{A}}(G, Y)$ is surjective. Then f is an epimorphism.*

The proof is left as an exercise.

Lemma 1.1.7. *Let $\mathcal{A}$ be an abelian category which admits small coproducts and a generator G. Let $X \in \mathcal{A}$. Then there exists a small set I and an epimorphism $G^{\oplus I} \twoheadrightarrow X$.*

Proof. In this proof, we write $\mathrm{Hom}\,(Y, Z)$ instead of $\mathrm{Hom}_{\mathcal{A}}(Y, Z)$.
There is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}\,(G, X), \mathrm{Hom}\,(G, X)) \simeq \mathrm{Hom}\,(G^{\oplus \mathrm{Hom}\,(G,X)}, X).$$

The identity of $\mathrm{Hom}\,(G, X)$ defines the natural morphism $G^{\oplus \mathrm{Hom}\,(G,X)} \to X$ which, to $(g, s) \in G \times \mathrm{Hom}\,(G, X)$, associates $s(g)$. This morphism defines the morphism

$$\mathrm{Hom}\,(G, G^{\oplus \mathrm{Hom}\,(G,X)}) \to \mathrm{Hom}\,(G, X)$$

and this last morphism being obviously surjective, the result follows from Lemma 1.1.6. q.e.d.

Definition 1.1.8. A Grothendieck category is an abelian category which admits small inductive and small projective limits and a generator and such that filtrant small inductive limits are exact.

The Brown representability theorem

In the theorem below, the main result is assertion (b) which is a particular case of the Brown representability theorem for which we refer for example to [KS06, Th 14.3.1]. The other assertions may be easily proved.

Theorem 1.1.9. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two Grothendieck categories and let $\rho\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor. Assume that*

(i) *ρ has finite cohomological dimension,*

(ii) *ρ commutes with small direct sums,*

(iii) *small direct sums of injective objects in $\mathcal{C}$ are acyclic for the functor ρ.*

Then

(a) *the functor $R\rho\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ commutes with small direct sums,*

(b) *the functor $R\rho\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathcal{C}') \to \mathrm{D}(\mathcal{C})$,*

(c) *the functor $\rho^!$ induces a functor $\rho^!\colon \mathrm{D}^+(\mathcal{C}') \to \mathrm{D}^+(\mathcal{C})$.*

(d) *Assume that $\mathcal{C}'$ has finite cohomological dimension. Then the functor $\rho^!$ induces a functor $\rho^!\colon \mathsf{D}^{\mathrm{b}}(\mathcal{C}') \to \mathsf{D}^{\mathrm{b}}(\mathcal{C})$.*

1.2 Presites and presheaves

Presites

Definition 1.2.1. (i) A presite X is a small category $\mathcal{C}_X$.

(ii) Let X and Y be two presites. A morphism of presites $f : X \to Y$ is a functor $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$.

In the sequel, we shall say that a presite X has a property $\mathcal{P}$ if the category $\mathcal{C}_X$ has the property $\mathcal{P}$.

For example, we say that X has a terminal object if so has $\mathcal{C}_X$. In such a case, we denote this object by X.

We denote by X^{op} the presite associated with the category $\mathcal{C}_X^{\mathrm{op}}$.

We denote by $\widehat{X}$ the presite associated with the category $\mathcal{C}_X^\wedge$.

Example 1.2.2. (i) Let X be a topological space and let Op_X denote the family of open subsets of X. This set is ordered, and we keep the same notation for the associated category. Hence:

$$\mathrm{Hom}_{\mathrm{Op}_X}(U,V) = \begin{cases} \{\mathrm{pt}\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that this category admits a terminal object, namely X, and finite products, namely $U \times V = U \cap V$. We shall identify a topological space X to the presite associated with the category Op_X.
(ii) Let $f : X \to Y$ be a continuous map of topological spaces. It defines a morphism of presites by setting

$$f^t(V) := f^{-1}(V) \text{ for } V \in \mathrm{Op}_Y.$$

In particular, for U open in X, there are natural morphisms of presites

$$i_U \colon U \to X, \mathrm{Op}_X \ni V \mapsto (U \cap V) \in \mathrm{Op}_U, \tag{1.1}$$
$$j_U \colon X \to U, \mathrm{Op}_U \ni V \mapsto V \in \mathrm{Op}_X. \tag{1.2}$$

Presheaves

Definition 1.2.3. Let $\mathcal{A}$ be a category.

(i) An $\mathcal{A}$-valued presheaf F on a presite X is a functor $F \colon \mathcal{C}_X^{\mathrm{op}} \to \mathcal{A}$.

(ii) One denotes by $\mathrm{PSh}(X, \mathcal{A})$ the (big) category of presheaves on X with values in $\mathcal{A}$. In other words, $\mathrm{PSh}(X, \mathcal{A}) = \mathrm{Fct}(\mathcal{C}_X^{\mathrm{op}}, \mathcal{A})$.

(iii) One sets $\mathrm{PSh}(X) := \mathrm{PSh}(X, \mathbf{Set})$. In other words, $\mathrm{PSh}(X) = \mathcal{C}_X^{\wedge}$.

(iv) One sets $\mathrm{PSh}(\mathbf{k}_X) := \mathrm{PSh}(X, \mathrm{Mod}(\mathbf{k}))$ and calls an object of $\mathrm{PSh}(\mathbf{k}_X)$ a $\mathbf{k}$-abelian presheaf, or an abelian presheaf, for short.

- A presheaf F on X associates to each object $U \in \mathcal{C}_X$ an object $F(U)$ of $\mathcal{A}$, and to each morphism $u \colon U \to V$, a morphism $\rho_u : F(V) \to F(U)$, such that for $v \colon V \to W$, one has:

$$\rho_{\mathrm{id}_U} = \mathrm{id}_{F(U)}, \qquad \rho_{v \circ u} = \rho_u \circ \rho_v.$$

- The morphism ρ_u is called a restriction morphism. When there is no risk of confusion, we shall not write it.

- A morphism of presheaves $\varphi : F \to G$ is thus the data for any $U \in \mathcal{C}_X$ of a moprhism $\varphi(U) : F(U) \to G(U)$ such that for any morphism $V \to U$, the diagram below commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi(U)} & G(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\varphi(V)} & G(V) \end{array}$$

- The category $\mathrm{PSh}(X, \mathcal{A})$ inherits of most all properties of the category $\mathcal{A}$. For example, if $\mathcal{A}$ admits small inductive (resp. projective) limits then so does $\mathrm{PSh}(X, \mathcal{A})$. If $\mathcal{A}$ is abelian, then $\mathrm{PSh}(X, \mathcal{A})$ is abelian.

- If $\mathcal{A}$ is a subcategory of **Set**, for $U \in \mathcal{C}_X$, an element of $s \in F(U)$ is called a section of F on U.

- In view of the Yoneda lemma, the functor

$$\mathrm{h}_X : \mathcal{C}_X \hookrightarrow \mathrm{PSh}(X), \quad U \mapsto \mathrm{Hom}_{\mathcal{C}_X}(\,\bullet\,, U)$$

  is fully faithful. One shall be aware that, when $\mathcal{C}_X$ admits projective or inductive limits, the functor h_X commutes with projective limits but not with inductive limits in general.

Notation 1.2.4. For $U \in \mathcal{C}_X$, one denotes by $\Gamma(U; \bullet) \colon \mathrm{PSh}(X, \mathcal{A}) \to \mathcal{A}$ the functor $F \mapsto F(U)$.

The functor $\Gamma(U; \bullet)$ commutes to inductive and projective limits. For example, if $\mathcal{A}$ is an abelian category and $\varphi : F \to G$ is a morphism of presheaves, then $(\mathrm{Ker}\,\varphi)(U) \simeq \mathrm{Ker}\,\varphi(U)$ and $(\mathrm{Coker}\,\varphi)(U) \simeq \mathrm{Coker}\,\varphi(U)$, where $\varphi(U) : F(U) \to G(U)$.

Examples 1.2.5. (i) Let $M \in \mathcal{A}$. The correspondence $U \mapsto M$ is a presheaf, called the constant presheaf on X with fiber M.
(ii) Let X denote a topological space and let $\mathcal{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on U. Then $U \mapsto \mathcal{C}^0(U)$ (with the usual restriction morphisms) is a presheaf of $\mathbb{C}$-vector spaces, denoted $\mathcal{C}^0_X$.

Proposition 1.2.6. *Let $F, G \in \mathrm{PSh}(X, \mathcal{A})$. There is a natural isomorphism*

$$\lambda \colon \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(F, G) \xrightarrow{\sim} \varprojlim_{(U \to V) \in \mathrm{Mor}_0(\mathcal{C}_X)^{\mathrm{op}}} \mathrm{Hom}_{\,\mathcal{A}}(F(V), G(U)). \tag{1.3}$$

Proof. (i) First, we construct the map λ. Let $\varphi\colon F \to G$ be a morphism in $\mathrm{PSh}(X, \mathcal{A})$ and let $U \to V$ be a morphism in $\mathcal{C}_X$. The morphisms $\varphi(U)\colon F(U) \to G(U)$ and $F(U \to V)\colon F(V) \to F(U)$ define the morphism $\varphi_{U \to V}\colon F(V) \to G(U)$. Moreover a morphism $a\colon (U \to V) \to (U' \to V')$ in $\mathrm{Mor}_0(\mathcal{C}_X)$ defines a morphism

$$\varphi_a\colon \mathrm{Hom}_{\mathcal{A}}(F(V'), G(U')) \to \mathrm{Hom}_{\mathcal{A}}(F(V), G(U))$$

as follows. To $\varphi_{U' \to V'}\colon F(V') \to G(U')$, one associates the composition

$$\varphi_{U \to V}\colon F(V) \to F(V') \xrightarrow{\varphi_{U' \to V'}} G(U') \to G(U).$$

(ii) The map λ is injective. Indeed, $\lambda(\varphi) = \lambda(\psi)$ implies that $\varphi(U) = \psi(U)$ for all $U \in \mathcal{C}_X$.
(iii) The map λ is surjective. Let $\{\varphi(U \to V)\}_{U \to V} \in \varprojlim_{U \to V} \mathrm{Hom}_{\mathcal{A}}(F(V), G(U))$.
To a morphism $s\colon U \to V$ in $\mathcal{C}_X$, one associates the two morphisms in $\mathrm{Mor}_0(\mathcal{C}_X)$:

$$\begin{array}{ccc} U & \xrightarrow{s} & V \\ \downarrow & & \uparrow{\scriptstyle s} \\ U & \longrightarrow & U, \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{s} & V \\ {\scriptstyle s}\downarrow & & \uparrow \\ V & \longrightarrow & V \end{array}$$

In the the diagram below, the two triangles commute. Hence, the square commutes.

(1.4)
$$\begin{array}{ccc} F(V) & \xrightarrow{\varphi(V)} & G(V) \\ \downarrow & \searrow^{\varphi(U \to V)} & \downarrow \\ F(U) & \xrightarrow[\varphi(U)]{} & G(U). \end{array}$$

Therefore, the family $\{\varphi(U \to V)\}_{U \to V}$ defines a morphism of functors $\varphi\colon F \to G$. q.e.d.

1.3 Direct and inverse images

In this section, we shall consider a category $\mathcal{A}$ satisfying

(1.5) $\mathcal{A}$ admits small projective limits and small inductive limits.

Consider a morphism of presites $f : X \to Y$, that is, a functor $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$. We shall use Definition 1.1.2.

Definition 1.3.1. Consider a morphism of presites $f\colon X \to Y$.

(i) Let $F \in \mathrm{PSh}(X,\mathcal{A})$. One defines $f_*F \in \mathrm{PSh}(Y,\mathcal{A})$, the direct image of F by f, by setting for $V \in \mathcal{C}_Y$: $f_*F(V) = F(f^t(V))$.

(ii) Let $G \in \mathrm{PSh}(Y,\mathcal{A})$. One defines $f^\dagger G$ by setting for $U \in \mathcal{C}_X$:

$$f^\dagger G(U) = \varinjlim_{(U \to f^t(V)) \in (\mathcal{C}_Y^U)^{\mathrm{op}}} G(V).$$

(iii) Let $G \in \mathrm{PSh}(Y,\mathcal{A})$. One defines $f^\ddagger G$, by setting for $U \in \mathcal{C}_X$:

$$f^\ddagger G(U) = \varprojlim_{(f^t(V) \to U) \in ((\mathcal{C}_Y)_U)} G(V).$$

Note that $f^\dagger G$ is a well defined presheaf on X. Indeed, consider a morphism $u\colon U \to U'$ in $\mathcal{C}_X$. The morphism $f^\dagger G(U') \to f^\dagger G(U)$ is given by:

$$f^\dagger G(U') = \varinjlim_{(U' \to f^t(V'))} G(V') \to \varinjlim_{(U \to f^t(V))} G(V).$$

There is a similar remark with $f^\ddagger G$.

Theorem 1.3.2. *Let $f\colon X \to Y$ be a morphism of presites.*

(i) *The functor $f^\dagger\colon \mathrm{PSh}(Y,\mathcal{A}) \to \mathrm{PSh}(X,\mathcal{A})$ is left adjoint to the functor $f_*\colon \mathrm{PSh}(X,\mathcal{A}) \to \mathrm{PSh}(Y,\mathcal{A})$. In other words, we have an isomorphism, functorial with respect to $F \in \mathrm{PSh}(X,\mathcal{A})$ and $G \in \mathrm{PSh}(Y,\mathcal{A})$:*

$$\mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(f^\dagger G, F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathcal{A})}(G, f_*F). \tag{1.6}$$

(ii) *The functor $f^\ddagger\colon \mathrm{PSh}(Y,\mathcal{A}) \to \mathrm{PSh}(X,\mathcal{A})$ is right adjoint to the functor $f_*\colon \mathrm{PSh}(X,\mathcal{A}) \to \mathrm{PSh}(Y,\mathcal{A})$. In other words, we have an isomorphism, functorial with respect to $F \in \mathrm{PSh}(X,\mathcal{A})$ and $G \in \mathrm{PSh}(Y,\mathcal{A})$:*

$$\mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(F, f^\ddagger G) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathcal{A})}(f_*F, G). \tag{1.7}$$

Proof. Note that (i) and (ii) are equivalent by reversing the arrows, that is, by considering the morphism of presites $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$. Hence, we shall only prove (i).

(a) First, we construct a map

$$\Phi\colon \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathcal{A})}(G, f_*F) \to \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(f^\dagger G, F).$$

Let $\theta \in \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathcal{A})}(G, f_*F)$ and let $U \in \mathcal{C}_X$. For $V \in \mathcal{C}_Y$ and a morphism $U \to f^t(V)$, the morphism

$$G(V) \xrightarrow{\theta(V)} F(f^t(V)) \to F(U)$$

gives a morphism $\Phi(\theta)(U)\colon \varinjlim_{U\to f^t(V)} G(V) \to F(U)$. The morphism $\Phi(\theta)(U)$ is functorial in U, that is, for any morphism $U' \to U$ in $\mathcal{C}_X$, the diagram below commutes:

$$\begin{array}{ccc} \varinjlim\limits_{U\to f^t(V)} G(V) & \xrightarrow{\Phi(\theta)(U)} & F(U) \\ \downarrow & & \downarrow \\ \varinjlim\limits_{U'\to f^t(V')} G(V') & \xrightarrow{\Phi(\theta)(U')} & F(U'). \end{array}$$

Therefore, the family of morphisms $\{\Phi(\theta)(U)\}_U$ defines the morphism $\Phi(\theta)$.

(b) Next, we construct a map

$$\Psi\colon \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(f^\dagger G, F) \to \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathcal{A})}(G, f_*F).$$

Let $\lambda \in \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathcal{A})}(f^\dagger G, F)$ and let $V \in \mathcal{C}_Y$. The morphism

$$\lambda(f^tV)\colon \varinjlim_{f^tV\to f^tW} G(W) = f^\dagger G(f^tV) \to F(f^tV)$$

together with the morphism $G(V) \to \varinjlim\limits_{f^tV\to f^tW} G(W)$ defines the morphism $\Psi(\lambda)(V)\colon G(V)F(f^tV)$. The morphisms $\Psi(\lambda)(V)$ are functorial in V and define $\Psi(\lambda)$.

(c) The reader will check that Ψ and Φ are inverse one to each other. q.e.d.

Proposition 1.3.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of presites. Let $F \in \mathrm{PSh}(X,\mathcal{A})$ and let $G \in \mathrm{PSh}(Z,\mathcal{A})$. Then*

$$\begin{aligned} (g\circ f)_* &\simeq g_*\circ f_*, \\ (g\circ f)^\dagger &\simeq f^\dagger\circ g^\dagger, \\ (g\circ f)^\ddagger &\simeq f^\ddagger\circ g^\ddagger. \end{aligned}$$

Proof. The first isomorphism is obvious and the others follow by adjunction. q.e.d.

Note that the constructions of the functors $f^\dagger G$ and $f^\ddagger G$ are variant of the so-called Kan extension of functors.

1.4 Restriction and extension of presheaves

Let X be a presite. We shall first make the hypothesis:

(1.8) the presite X admits products of two objects and fiber products.

Notation 1.4.1. For a presite X satisfying (1.8) and $U_1, U_2 \in \mathcal{C}_X$, we shall denote by $U_1 \times_X U_2$ their product in $\mathcal{C}_X$.

Note that a category admits a terminal object and fiber products if and only if it admits finite projective limits. If a category $\mathcal{C}_X$ admits a terminal object X, then $U \times_X V \xrightarrow{\sim} U \times V$ for any $U, V \in \mathcal{C}_X$.

Definition 1.4.2. (i) For $U \in \mathcal{C}_X$, we set $\mathcal{C}_U := (\mathcal{C}_X)_U$ and we still denote by U the presite associated with the category $\mathcal{C}_U$.
(ii) We denote by $\mathrm{j}_U \colon X \to U$ the morphism of presites associated with the functor $\mathrm{j}_U^t \colon \mathcal{C}_U \to \mathcal{C}_X$ which, to $(v \colon V \to U) \in \mathcal{C}_U$, associates $V \in \mathcal{C}_X$.
(iii) We denote by $i_U \colon U \to X$ the morphism of presites associated with the functor $i_U^t \colon \mathcal{C}_X \to \mathcal{C}_U$ which, to $V \in \mathcal{C}_X$, associates $i_U^t(V) = (U \times_X V \to U) \in \mathcal{C}_U$.

Let $F \in \mathrm{PSh}(X, \mathcal{A})$. One sets

$$F|_U = \mathrm{j}_{U*} F$$

and one calls $F|_U$ the restriction of F to U.
More generally, consider a morphism $s \colon V \to U$ in $\mathcal{C}_X$. One denotes by $\mathrm{j}^t_{V \xrightarrow{s} U} \colon \mathcal{C}_V \to \mathcal{C}_U$ the natural functor and by $\mathrm{j}_{V \xrightarrow{s} U} \colon U \to V$ the associated morphism of presites.

Proposition 1.4.3. *Let $U \in \mathcal{C}_X$ and $(V \to U) \in \mathcal{C}_U$. For $F \in \mathrm{PSh}(X, \mathcal{A})$ and $G \in \mathrm{PSh}(U, \mathcal{A})$, we have:*

(i) $\mathrm{j}_{U*} F(V \to U) \simeq F(V)$,

(ii) $\mathrm{j}_U^\dagger\, G(V) \simeq \coprod_{s \in \mathrm{Hom}_{\mathcal{C}_X}(V,U)} G(V \xrightarrow{s} U)$.

(iii) $\mathrm{j}_U^\ddagger\, G(V) \simeq G(U \times_X V \to U)$.

Proof. (i) is obvious.
(ii) By its definition,

$$\begin{aligned} \mathrm{j}_U^\dagger\, G(V) &\simeq \varinjlim_{(V \to \mathrm{j}_U^t(W \to U)) \in ((\mathcal{C}_U)^V)^{\mathrm{op}}} G(W \to U) \\ &\simeq \varinjlim_{V \to W \to U} G(W \to U) \\ &\simeq \varinjlim_{(s \colon V \to U) \in \mathrm{Hom}\,(V,U)} G(V \xrightarrow{s} U). \end{aligned}$$

Here, we use the fact that the category $\mathrm{Hom}_{\mathcal{C}_X}(V, U)$ is cofinal in $((\mathcal{C}_U)^V)^{\mathrm{op}}$. The result follows since the category $\mathrm{Hom}_{\mathcal{C}_X}(V, U)$ is discrete.
(iii) By its definition,

$$\begin{aligned} \mathrm{j}_U^{\ddagger} G(V) &\simeq \varprojlim_{(\mathrm{j}_U^{\ddagger}(W\to U)\to V)\in(\mathcal{C}_U)_V} G(W \to U) \\ &\simeq \varprojlim_{U\leftarrow W\to V} G(W \to U) \\ &\simeq G(U \times_X V \to U). \end{aligned}$$

Here, the last isomorphism follows from the fact that $U \times_X V \to U$ is a terminal object in $(\mathcal{C}_U)_V$. q.e.d.

1.5 The functors hom and tens

Let $s\colon V \to U$ be a morphism and let $F, G \in \mathrm{PSh}(X, \mathcal{A})$. The functor $\mathrm{j}_{V \xrightarrow{s} U *}\colon \mathrm{PSh}(U, \mathcal{A}) \to \mathrm{PSh}(V, \mathcal{A})$ defines the map

$$\mathrm{Hom}_{\,\mathrm{PSh}(U,\mathcal{A})}(F|_U, G|_U) \to \mathrm{Hom}_{\,\mathrm{PSh}(V,\mathcal{A})}(F|_V, G|_V).$$

Definition 1.5.1. Let $F, G \in \mathrm{PSh}(X, \mathcal{A})$. One denotes by $\mathcal{H}om\,(F, G)$ the presheaf of sets on X, $U \mapsto \mathrm{Hom}_{\,\mathrm{PSh}(U,\mathcal{A})}(F|_U, G|_U)$.

By its definition, we have for $U \in \mathcal{C}_X$:

$$\mathrm{Hom}_{\,\mathrm{PSh}(U,\mathcal{A})}(F|_U, G|_U) \simeq \mathcal{H}om\,(F, G)(U). \tag{1.9}$$

From now on and until the end of this section, we assume that $\mathcal{A} = \mathrm{Mod}(\mathbf{k})$. Note that in this case, $\mathcal{H}om\,(F, G)$ belongs to $\mathrm{PSh}(\mathbf{k}_X)$. Then, one calls it the "internal hom" of F and G.

Denote by $\widetilde{\mathbf{k}}_X$ the constant presheaf $U \mapsto \mathbf{k}$. Then

$$\mathcal{H}om\,(\widetilde{\mathbf{k}}_X, F) \simeq F. \tag{1.10}$$

Applying (1.9), we get

$$\mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(\widetilde{\mathbf{k}}_X, F) \simeq F(X). \tag{1.11}$$

Now let $U \in \mathcal{C}_X$. We have the isomorphism

$$\mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(\mathrm{j}_U^{\dagger}\,\mathrm{j}_{U*}\widetilde{\mathbf{k}}_X, F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_U)}(\mathrm{j}_{U*}\widetilde{\mathbf{k}}_X, \mathrm{j}_{U*}F).$$

Since $\widetilde{\mathbf{k}}_U \simeq \mathrm{j}_{U*}\widetilde{\mathbf{k}}_X$ in $\mathrm{PSh}(\mathbf{k}_U)$, we get the isomorphism

$$F(U) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(\mathrm{j}_U^{\dagger}\,\mathrm{j}_{U*}\widetilde{\mathbf{k}}_X, F). \tag{1.12}$$

Theorem 1.5.2. *The category* $\mathrm{PSh}(\mathbf{k}_X)$ *is an abelian Grothendieck category.*

Proof. The fact that this category is abelian, admits small limits and colimits and small colimits are exact is obvious. It remains to show that it admits a small family of generators. By (1.12), we may choose the family $\{\mathrm{j}_U^\dagger\,\mathrm{j}_{U*}\widetilde{\mathbf{k}}_X\}_{U\in\mathcal{C}_X}$. q.e.d.

Definition 1.5.3. Let $F_1, F_2 \in \mathrm{PSh}(\mathbf{k}_X)$. Their tensor product, denoted $F_1 \overset{\mathrm{psh}}{\otimes} F_2$ is the presheaf $U \mapsto F_1(U) \otimes F_2(U)$.

Proposition 1.5.4. *Let* $F_i \in \mathrm{PSh}(k_X)$, $(i = 1, 2, 3)$. *There is a natural isomorphism:*

$$\mathcal{H}om\,(F_1 \overset{\mathrm{psh}}{\otimes} F_2, F_3) \simeq \mathcal{H}om\,(F_1, \mathcal{H}om\,(F_2, F_3)).$$

We skip the proof.

1.6 Presheaves on topological spaces

Definition 1.6.1. Assume X is a topological space and assume that $\mathcal{A}$ admits small inductive limits. Let $x \in X$, and let I_x denote the full subcategory of Op_X consisting of open neighborhoods of x. For a presheaf F on X, one sets:

$$F_x = \varinjlim_{U \in I_x^{\mathrm{op}}} F(U). \tag{1.13}$$

One calls F_x the stalk of F at x.

Proposition 1.6.2. *Assume that* $\mathcal{A}$ *is abelian, admits small inductive limits and that small filtrant inductive limits are exact in* $\mathcal{A}$. *Then the functor* $F \mapsto F_x$ *from* $\mathrm{PSh}(X, \mathcal{A})$ *to* $\mathcal{A}$ *is exact.*

Proof. The functor $F \mapsto F_x$ is the composition

$$\mathrm{PSh}(X, \mathcal{A}) = \mathrm{Fct}(\mathrm{Op}_X^{\mathrm{op}}, \mathcal{A}) \to \mathrm{Fct}(I_x^{\mathrm{op}}, \mathcal{A}) \xrightarrow{\varinjlim} \mathcal{A}.$$

The first functor associates to a presheaf F its restriction to the category I_x^{op}. It is clearly exact. Since $U, V \in I_x$ implies $U \cap V \in I_x$, the category I_x^{op} is filtrant and it follows that the functor $\varinjlim$ is exact. q.e.d.

Assume $\mathcal{A} = \mathbf{Set}$ or $\mathcal{A} = \mathrm{Mod}(\mathbf{k})$. Let $x \in U$ and let $s \in F(U)$. The image $s_x \in F_x$ of s is called the germ of s at x.

Since I_x^{op} is filtrant, a germ $s_x \in F_x$ is represented by a section $s \in F(U)$ for some open neighborhood U of x, and for $s \in F(U), t \in F(V)$, $s_x = t_x$ means that there exists an open neighborhood W of x with $W \subset U \cap V$ such that $\rho_{WU}(s) = \rho_{WV}(t)$.

Exercises to Chapter 1

Exercise 1.1. Let X be a presite, let $U \in \mathcal{C}_X$ and let F, G, $H \in \mathrm{PSh}(X)$. Prove the isomorphisms

$$\begin{aligned}
\mathcal{H}om\,(F,G)(U) &\simeq \mathrm{Hom}\,_{\mathrm{PSh}(X)}(F \times U, G),\\
\mathcal{H}om\,(F \times H, G) &\simeq \mathcal{H}om\,(F, \mathcal{H}om\,(H,G)),\\
\mathrm{Hom}\,_{\mathrm{PSh}(X)}(F \times H, G) &\simeq \mathrm{Hom}\,_{\mathrm{PSh}(X)}(F, \mathcal{H}om\,(H,G)).
\end{aligned}$$

Exercise 1.2. Let X be a presite. Prove that a morphism $u\colon A \to B$ in $\mathrm{PSh}(X)$ is a monomorphism (resp. an epimorphism) if and only if the morphism $u(U)\colon A(U) \to B(U)$ in **Set** is a monomorphism (resp. an epimorphism) for any $U \in \mathcal{C}_X$.

Exercise 1.3. Let X be a presite.
Consider morphisms $u\colon A \to C$ and $v\colon B \to C$ in $\mathrm{PSh}(X)$. Prove that $(A \times_C B)(U) \simeq A(U) \times_{C(U)} B(U)$ for any $U \in \mathcal{C}_X$.

Exercise 1.4. Assume X is a topological space and let $U \in \mathrm{Op}_X$. Prove that the composition of morphisms of presites $U \xrightarrow{i_U} X \xrightarrow{j_U} U$ is isomorphic to the identity functor of the presite U. Show that this result is no more true in general.

Exercise 1.5. Let $\alpha\colon \mathcal{J} \to \mathcal{I}$ be a functor of small categories and let $\mathcal{A}$ be a category which admits small inductive limits. Define the functor $\alpha_*\colon \mathrm{Fct}(\mathcal{I},\mathcal{A}) \to \mathrm{Fct}(\mathcal{J},\mathcal{A})$ by setting $\alpha_*(F) = F \circ \alpha$, $F \in \mathrm{Fct}(\mathcal{I},\mathcal{A})$.
(i) Prove that α_* admits a left adjoint.
(ii) Let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor. We assume that $\mathcal{C}$ is small and $\mathcal{A}$ admits small inductive limits. Prove that there exists a unique (up to isomorphism) functor $\widehat{F}\colon \mathcal{C}^\wedge \to \mathcal{A}$ which extends F and which commutes with small inductive limits in $\mathcal{C}^\wedge$.

Chapter 2

Sheaves on sites

A site X is a small category $\mathcal{C}_X$ endowed with a Grothendieck topology. The objects of the category play the role of the open subsets of a topological space and one axiomatizes the notion of a covering. The theory is much easier when assuming, as we do here, that the category $\mathcal{C}_X$ admits finite products and fiber products. We study abelian sheaves on such sites, constructing the sheaf associated with a presheaf and the usual internal and external operations on sheaves. We also have a glance to locally constant sheaves. Finally, we glue sheaves, that is, given a covering of X and sheaves defined on the open sets of this coverings satisfying a natural cocycle condition, we prove the existence and unicity of a sheaf on X locally isomorphic to these locally defined sheaves.
Some references: [SGA4, Ta94, KS06].

2.1 Grothendieck topologies

We shall axiomatize the classical notion of a covering in a topological space.

Let X be a presite. All along theses Notes, we assume

(2.1) the presite X admits products of two objects and fiber products

Recall Notation 1.4.1.

In the sequel, we shall often write $\mathcal{S} \subset \mathcal{C}_U$ instead of $\mathcal{S} \subset \mathrm{Ob}(\mathcal{C}_U)$. We shall also often write $V \in \mathcal{C}_U$ instead of $(V \to U) \in \mathcal{C}_U$. For $\mathcal{S} \subset \mathcal{C}_U$ and $V \in \mathcal{C}_U$, we set

$$V \times_U \mathcal{S} := \{V \times_U W; W \in \mathcal{S}\},$$

a subset of $\mathcal{C}_V$.

For $\mathcal{S}_1 \subset \mathcal{C}_U$ and $\mathcal{S}_2 \subset \mathcal{C}_U$, we set

$$\mathcal{S}_1 \times_U \mathcal{S}_2 := \{V_1 \times_U V_2; V_1 \in \mathcal{S}_1, V_2 \in \mathcal{S}_2\},$$

a subset of $\mathcal{C}_U$.

For a morphism of presites $f\colon X \to Y$, $V \in \mathcal{C}_Y$ and $\mathcal{S} \subset \mathcal{C}_V$, we set

$$f^t(\mathcal{S}) := \{f^t(W); W \in \mathcal{S}\},$$

a subset of $\mathcal{C}_{f^t(V)}$.

Definition 2.1.1. Let $U \in \mathcal{C}_X$. Consider two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathrm{Ob}(\mathcal{C}_U)$. One says that $\mathcal{S}_1$ is a refinement of $\mathcal{S}_2$ if for any $U_1 \in \mathcal{S}_1$ there exists $U_2 \in \mathcal{S}_2$ and a morphism $U_1 \to U_2$ in $\mathcal{C}_U$. In such a case, we write $\mathcal{S}_1 \preceq \mathcal{S}_2$.

Remark 2.1.2. Instead of considering a subset $\mathcal{S}$ of $\mathrm{Ob}(\mathcal{C}_U)$, one may also consider a family $\mathcal{U} = \{U_i\}_{i\in I}$ of objects of $\mathcal{C}_U$ indexed by a set I. To such a family one may associate $\mathcal{S} = \mathrm{Im}(\mathcal{U}) \subset \mathrm{Ob}(\mathcal{C}_U)$. Then for $\mathcal{U}_1 = \{U_i\}_{i\in I}$ and $\mathcal{U}_2 = \{V_j\}_{j\in J}$, we say that $\mathcal{U}_1$ is a refinement of $\mathcal{U}_2$ and write $\mathcal{U}_1 \preceq \mathcal{U}_2$ if for any $i \in I$ there exists $j \in J$ and a morphism $U_i \to V_j$ in $\mathcal{C}_U$. This is equivalent to saying that $\mathrm{Im}\,\mathcal{U}_1 \preceq \mathrm{Im}\,\mathcal{U}_2$.

Of course, if the map $I \to \mathrm{Ob}(\mathcal{C}_U)$, $i \mapsto U_i$ is injective, it is equivalent to work with $\mathcal{U} = \{U_i\}_{i\in I}$ or with $\mathcal{S} = \mathrm{Im}(\mathcal{U})$.

Definition 2.1.3. Let X be a presite satisfying hypothesis (1.8). A Grothendieck topology (or simply "a topology") on X is the data for each $U \in \mathcal{C}_X$ of a family $\mathrm{Cov}(U)$ of subsets of $\mathrm{Ob}(\mathcal{C}_U)$ satisfying the axioms COV1–COV4 below.

COV1 $\{U\}$ belongs to Cov_U.

COV2 If $\mathcal{S}_1 \in \mathrm{Cov}_U$ is a refinement of $\mathcal{S}_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, then $\mathcal{S}_2 \in \mathrm{Cov}_U$.

COV3 If $\mathcal{S}$ belongs to Cov_U, then $\mathcal{S} \times_U V$ belongs to $\mathrm{Cov}(V)$ for any $(V \to U) \in \mathcal{C}_U$induced

COV4 If $\mathcal{S}_1$ belongs to Cov_U, $\mathcal{S}_2 \subset \mathcal{C}_U$, and $\mathcal{S}_2 \times_U V$ belongs to $\mathrm{Cov}(V)$ for any $V \in \mathcal{S}_1$, then $\mathcal{S}_2$ belongs to Cov_U.

An element of $\mathrm{Cov}(U)$ is called a covering of U.

Intuitively, COV3 means that a covering of an open set U induces a covering on any open subset $V \subset U$, and COV4 means that if a family of open subsets of U induces a covering on each subset of a covering of U, then this family is a covering of U.

Since the category $\mathcal{C}_X$ does not necessarily admit a terminal object, the following definition is useful.

Definition 2.1.4. Let X be a presite endowed with a Grothendieck topology. A covering of X is a subset $\mathcal{S}$ of $\mathrm{Ob}(\mathcal{C}_X)$ such that $\mathcal{S} \times_X U$ belongs to $\mathrm{Cov}(U)$ for any $U \in \mathcal{C}_X$.

Definition 2.1.5. (i) A site X is a presite X satisfying hypothesis (1.8) and endowed with a Grothendieck topology.

(ii) A morphism of sites $f\colon X \to Y$ is a morphism of presites such that

(a) $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$ commutes with products and fiber products,

(b) for any $V \in \mathcal{C}_Y$ and $\mathcal{S} \in \mathrm{Cov}(V)$, $f^t(\mathcal{S}) \in \mathrm{Cov}(f^t(V))$.

Examples 2.1.6. (i) The classical notion of a covering on a topological space X is as follows. A family $\mathcal{S} \subset \mathrm{Op}_U$ is a covering if $\bigcup_{V\in\mathcal{S}} V = U$. Axioms COV1–COV4 are clearly satisfied, and we still denote by X the site so obtained. If $f\colon X \to Y$ is a continuous map of topological spaces, it defines a morphism of sites.
(ii) Let X be a presite. The initial topology on X is defined as follows. Any subset of $\mathrm{Ob}(\mathcal{C}_U)$ is a covering. We shall denote by X_{ini} this site. Note that if X is a site, the morphism of presites $\mathrm{id}_X\colon X \to X$ induces a morphism of sites $X_{\mathrm{ini}} \to X$.
(iii) Let X be a presite. The final topology on X is defined as follows. A family $\mathcal{S} \subset \mathrm{Ob}(\mathcal{C}_U)$ is a covering of U if and only if $\{U\} \in \mathcal{S}$. Note that if X is a site, the morphism of presites $\mathrm{id}_X\colon X \to X$ induces a morphism of sites $X \to X_{\mathrm{fin}}$.
(iv) We shall denote by $\{\mathrm{pt}\}$ the set with one element and we denote this element by pt. We endow $\{\mathrm{pt}\}$ with the discrete topology. Hence, the category $\mathcal{C}_{\{\mathrm{pt}\}}$ associated with the presite $\{\mathrm{pt}\}$ has two objects, $\emptyset$ and pt and $\{\mathrm{pt}\}$ is a site. The Grothendieck topology so defined is the final topology. If X is a toplogical space, we shall usually denote by $a_X\colon X \to \{\mathrm{pt}\}$ the unique continuous map from X to $\{\mathrm{pt}\}$.
(v) Let Pt be the category with one object (let us say c) and one morphism, id_c. Then the initial and final topology on Pt differs. The empty covering is a covering of c for the initial topology, not for the final one. In the sequel, we endow Pt with the final topology. If X is a site with a terminal object X, there is a natural morphism of sites $X \to \mathrm{Pt}$, which associates the object $X \in \mathcal{C}_X$ to $c \in \mathrm{Pt}$.
(vi) Let X be a topological space. Let us endow Op_X with the following Grothendieck topology: $\mathcal{S} \subset \mathrm{Op}_U$ is a covering of U if there exists a finite subset $\mathcal{S}' \subset \mathcal{S}$ such that $\bigcup_{V\in\mathcal{S}'} V = U$. Axioms COV1–COV4 are clearly satisfied. We denote by X_{finite} the site so obtained.

(vii) Let X be a real analytic manifold. The subanalytic site X_{sa} is defined in [?] as follows: the objects of $\mathcal{C}_{X_{\mathrm{sa}}}$ are the relatively compact subanalytic open subsets of X and the topology is that of X_{finite}, that is, a covering of $U \in \mathcal{C}_{X_{\mathrm{sa}}}$ is a covering of U in X_{finite}.
(viii) Let X be a topological space endowed with an equivalence relation $\sim$. Let $\mathcal{C}_X$ be the category of saturated open subsets (U is saturated if $x \in U$ and $x \sim y$ implies $y \in U$). We endow $\mathcal{C}_X$ with the induced topology, that is, the coverings of $U \in \mathcal{C}_X$ are the saturated coverings of U in X.
(ix) Let $\mathcal{V}$ be a universe with $\mathcal{V} \in \mathcal{U}$. Denote by $C^\infty_{\mathcal{V}}$ be the small $\mathcal{U}$-category whose objects are the real manifolds of class C^∞ belonging to $\mathcal{V}$ and morphisms are morphisms of such manifolds. Let $X \in C^\infty_{\mathcal{V}}$ and define the category $\mathcal{C}_X$ as follows. An object of $\mathcal{C}_X$ is an étale morphism $f\colon Y \to X$ in $C^\infty_{\mathcal{V}}$. (Recall that a morphism $f\colon Y \to X$ is étale if f is open and, locally on Y, f is an isomorphism onto its image.) A morphism $u\colon (Y_1 \xrightarrow{f_1} X) \to (Y_2 \xrightarrow{f_2} X)$ is a morphism $g\colon Y_1 \to Y_2$ such that $f_2 \circ g = f_1$. Necessarily, g is étale. Let us denote by X_{et} the presite so defined. We endowed it with the following topology: a family of morphism $\{U_i \xrightarrow{f_i} U\}_i$ is a covering of $U \in \mathcal{C}_X$ if U is the union of the $f_i(U_i)$'s.

Let X be a site and let $U \in \mathcal{C}_X$. To U is associated the category $\mathcal{C}_U$. Denoting again by U the presite associated with $\mathcal{C}_U$, the presite U satisfies (1.8).

The functor $\mathrm{j}^t_U\colon \mathcal{C}_U \to \mathcal{C}_X$ given by

$$\mathrm{j}^t_U(V \to U) = V$$

defines a morphism of presites:

$$\mathrm{j}_U\colon X \to U. \tag{2.2}$$

The functor $i^t_U\colon \mathcal{C}_X \to \mathcal{C}_U$ given by

$$i^t_U(V) = U \times_X V \to U$$

defines a morphism of presites

$$i_U\colon U \to X. \tag{2.3}$$

Definition 2.1.7. The induced topology by X on the presite U is defined as follows. Let $(V \to U) \in \mathcal{C}_U$. A subset $\mathcal{S} \subset \mathcal{C}_V$ is a covering of $(V \to U)$ if $\mathrm{j}^t_U(\mathcal{S})$ is a covering of V in X.

Clearly this family satisfies the axioms COV1–COV4, and thus defines a topology on the presite U.

Lemma 2.1.8. *The morphisms of presites* (2.2) *and* (2.3) *are morphisms of sites.*

The obvious verifications are left to the reader.

Example 2.1.9. Let X be a topological space, U an open subset. Note that both Op_X and Op_U admit finite projective limits, but in general j_U does not commute with such limits since it does not send the terminal object U of Op_U to the terminal object X of Op_X.

2.2 Sheaves

Let $\mathcal{A}$ be a category satisfying

(2.4) $$\mathcal{A} \text{ admits small projective limits.}$$

Let $\mathcal{S} \subset \mathcal{C}_U$ and let $F \in \mathrm{PSh}(X, \mathcal{A})$. One defines $F(\mathcal{S})$ by the exact sequence (i.e., $F(\mathcal{S})$ is the kernel of the double arrow):

(2.5) $$F(\mathcal{S}) \to \prod_{V \in \mathcal{S}} F(V) \rightrightarrows \prod_{V', V'' \in \mathcal{S}} F(V' \times_U V'').$$

Here the two arrows are associated with $\prod_{V \in \mathcal{S}} F(V) \to F(V') \to F(V' \times_X V'')$ and $\prod_{V \in \mathcal{S}} F(V) \to F(V'') \to F(V' \times_X V'')$.

Assume that $\mathcal{S}$ is stable by product, that is, if $V \to U$ and $W \to U$ belong to $\mathcal{S}$ then $V \times_U W \to U$ belongs to $\mathcal{S}$. In this case, looking at $\mathcal{S}$ as a full subcategory of $\mathcal{C}_U$, we have:

(2.6) $$F(\mathcal{S}) \simeq \varprojlim_{(V \to U) \in \mathcal{S}} F(V).$$

Note that, if $\mathcal{A} = \mathbf{Set}$, a section $s \in F(\mathcal{S})$ is the data of a family of sections $\{s_V \in F(V)\}_{V \in \mathcal{S}}$ such that for any $V', V'' \in \mathcal{S}$,

$$s_{V'}|_{V' \times_X V''} = s_{V''}|_{V' \times_X V''}.$$

For a presheaf F, there is a natural map

(2.7) $$F(U) \to F(\mathcal{S}).$$

Definition 2.2.1. (i) One says that a presheaf F is separated if for any $U \in \mathcal{C}_X$ and any covering $\mathcal{S}$ of U, the natural morphism $F(U) \to F(\mathcal{S})$ is a monomorphism.

(ii) One says that a presheaf F is a sheaf if for any $U \in \mathcal{C}_X$ and any covering $\mathcal{S}$ of U, the natural map $F(U) \to F(\mathcal{S})$ is an isomorphism.

(iii) One denotes by $\mathrm{Sh}(X, \mathcal{A})$ the full subcategory of $\mathrm{PSh}(X, \mathcal{A})$ whose objects are sheaves and by $\iota_X : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{PSh}(X, \mathcal{A})$ the forgetful functor. If there is no risk of confusion, we write ι instead of ι_X, or even, we do not write ι.

(iv) One sets $\mathrm{Sh}(X) = \mathrm{Sh}(X, \mathbf{Set})$ and $\mathrm{Mod}(\mathbf{k}_X) = \mathrm{Sh}(X, \mathrm{Mod}(\mathbf{k}))$. One calls an object of $\mathrm{Mod}(\mathbf{k}_X)$ a $\mathbf{k}$-abelian sheaf, or an abelian sheaf, for short.

Assume that $\mathcal{A}$ is either the category $\mathbf{Set}$ or the category $\mathrm{Mod}(\mathbf{k})$. Let F be a presheaf on X and consider the two conditions below.

S1 For any $U \in \mathcal{C}_X$, any covering $\mathcal{S}$ of U, any $s, t \in F(U)$ satisfying $s|_V = t|_V$ for all $V \in \mathcal{S}$, one has $s = t$.

S2 For any $U \in \mathcal{C}_X$, any covering $\mathcal{S}$ of U, any family $\{s_V \in F(V)\}_{V \in \mathcal{S}}$ satisfying $s_V|_{V \times_U W} = s_W|_{V \times_U W}$ for all $U, V \in \mathcal{S}$, there exists $s \in F(U)$ with $s|_V = s_V$ for all $V \in \mathcal{S}$.

The next results are obvious.

Proposition 2.2.2. *Assume that $\mathcal{A}$ is either the category* $\mathbf{Set}$*, or the category* $\mathrm{Mod}(\mathbf{k})$ *for a ring* $\mathbf{k}$*. A presheaf F is separated (resp. is a sheaf) if and only if it satisfies* S1 *(resp.* S1 *and* S2*).*

Proposition 2.2.3. *Let F be a sheaf on X. Then for $U \in \mathcal{C}_X$, $F|_U$ is a sheaf on U.*

Example 2.2.4. If X is a topological space, F is a abelian sheaf on X and $\{U_i\}_{i \in I}$ is a family of disjoint open subsets, then $F(\bigsqcup_i U_i) = \prod_i F(U_i)$. In particular, $F(\emptyset) = 0$.

Remark 2.2.5. Let F be an abelian sheaf on a topological space X.
(i) One defines its support, denoted by $\mathrm{supp}\, F$, as the complementary of the union of all open subsets U of X such that $F|_U = 0$. Note that $F|_{X \setminus \mathrm{supp}\, F} = 0$.
(ii) Let $s \in F(U)$. One defines its support, denoted by $\mathrm{supp}\, s$, as the complementary of the union of all open subsets U of X such that $s|_U = 0$.
Of course, the notion of support has no meaning on a site in general.

Theorem 2.2.6. *Let $F \in \mathrm{PSh}(X, \mathcal{A})$ and $G \in \mathrm{Sh}(X, \mathcal{A})$. The presheaf $\mathcal{H}om\,(F, \iota_X G)$ is a sheaf of sets on X. (A sheaf of* $\mathbf{k}$*-modules in case $\mathcal{A} = \mathrm{Mod}(\mathbf{k})$.)*

In the sequel, we shall not write ι_X.

Proof. Let $U \in \mathcal{C}_X$ and let $\mathcal{S}$ be a covering of U. We shall check conditions S1 and S2 as in Proposition 2.2.2. Consider the diagram

$$\begin{array}{ccccccc}
F(U) & \longrightarrow & F(\mathcal{S}) & \longrightarrow & \prod_{V\in\mathcal{S}} F(V) & \rightrightarrows & \prod_{V',V''\in\mathcal{S}} F(V'\times_U V'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G(U) & \xrightarrow{\sim} & G(\mathcal{S}) & \longrightarrow & \prod_{V\in\mathcal{S}} G(V) & \rightrightarrows & \prod_{V',V''\in\mathcal{S}} G(V'\times_U V'')
\end{array}$$

(S1) Let $\varphi, \psi \colon F|_U \to G|_U$ be two morphisms defined on U. Denote by φ_V, ψ_V their restriction to $V \in \mathcal{S}$. These families of morphisms define the morphisms $\varphi_{\mathcal{S}}, \psi_{\mathcal{S}} \colon F(\mathcal{S}) \to G(\mathcal{S})$. Assuming that $\varphi_V = \psi_V$ for all V, we get $\varphi_{\mathcal{S}} = \psi_{\mathcal{S}}$ hence $\varphi(U) = \psi(U)$ and by the same argument, $\varphi(V) = \psi(V)$ for any $V \to U$.

(S2) Let $\{\varphi_V\}_V$ be a family of morphisms $\varphi_V \colon F|_V \to G|_V$ and assume that $\varphi_V = \varphi_W$ on $V \times_U W$. Then this family of morphisms defines a morphism $\varphi_{\mathcal{S}} \colon F(\mathcal{S}) \to G(\mathcal{S})$. One constructs $\varphi(U)$ as the composition $F(U) \to F(\mathcal{S}) \xrightarrow{\varphi_{\mathcal{S}}} G(\mathcal{S}) \xleftarrow{\sim} G(U)$. Replacing U with $V \to U$, one checks easily that the family of morphisms $\{\varphi(V)\}_{V\to U}$ so constructed defines a morphism of presheaves $F|_U \to G|_U$. q.e.d.

We shall still denote by $\mathcal{H}om\,(F, G)$ the sheaf given by Theorem 2.2.6.

Corollary 2.2.7. *Let $\varphi \colon F \to G$ be a morphism in* $\mathrm{Sh}(X, \mathcal{A})$. *Assume that there is a covering $\mathcal{S}$ of X such that $\varphi_V \colon F|_V \to G|_V$ is an isomorphism for any $V \in \mathcal{S}$. Then φ is an isomorphism.*

Proof. For $V \in \mathcal{S}$, denote by ψ_V the inverse of φ_V. Then for any $V, W \in \mathcal{S}$, $\psi_V|_{V\times_X W} = \psi_W|_{V\times_X W}$. By Theorem 2.2.6, there exists $\psi \colon G \to F$ such that $\psi|_V = \psi_V$ for all $V \in \mathcal{S}$. Clearly $\psi \circ \varphi = \mathrm{id}_F$ and $\varphi \circ \psi = \mathrm{id}_G$. q.e.d.

In § 2.8 we shall construct sheaves which are locally isomorphic without being isomorphic.

Examples 2.2.8. (i) Let X be a topological space.

(a) The presheaf $\mathcal{C}^0_X$ of complex valued continuous functions is a sheaf.

(b) Let $M \in \mathrm{Mod}(\mathbf{k})$. The presheaf M_X of locally constant functions on X with values in M is a sheaf. Note that the constant presheaf with stalk M is not a sheaf except if $M = 0$.

(ii) Let X be a real analytic manifold.

(a) The presheaf $\mathcal{C}_X^\omega$ of complex valued real analytic functions is a sheaf,

(b) the presheaf $\mathcal{C}_X^\infty$ of complex valued functions of class $\mathcal{C}^\infty$ is a sheaf as well as $\mathcal{C}_X^{\infty,(p)}$, the presheaf of p-forms of class $\mathcal{C}^\infty$,

(c) the presheaf $\mathcal{D}b_X$ of complex valued distributions is a sheaf, as well as the presheaf $\mathcal{B}_X$ of complex valued hyperfunctions.

(iii) Let X be a complex manifold.

(a) The presheaf $\mathcal{O}_X$ of holomorphic functions is a sheaf as well as the presheaf Ω_X^p of holomorphic p-forms (hence, $\Omega_X^0 = \mathcal{O}_X$),

(b) the presheaf $\mathcal{D}_X$ of (finite order) holomorphic differential operators is a sheaf.

(iv) On a topological space X, the presheaf $U \mapsto \mathcal{C}_X^{0,b}(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property and axiom (S2) is not satisfied. However, this presheaf is a sheaf on the site X_{finite} defined in Examples 2.1.6.

(v) Let $X = \mathbb{C}$, and denote by z the holomorphic coordinate. The holomorphic derivation $\frac{\partial}{\partial z}$ is a morphism from $\mathcal{O}_X$ to $\mathcal{O}_X$. Consider the presheaf:

$$F \colon U \mapsto \mathcal{O}(U)/\frac{\partial}{\partial z}\mathcal{O}(U),$$

that is, the presheaf $\mathrm{Coker}(\frac{\partial}{\partial z} : \mathcal{O}_X \to \mathcal{O}_X)$. For U an open disc, $F(U) = 0$ since the equation $\frac{\partial}{\partial z} f = g$ is always solvable. However, if $U = \mathbb{C} \setminus \{0\}$, $F(U) \neq 0$. Hence the presheaf F does not satisfy axiom (S1).

2.3 Sheaf associated with a presheaf

From now on, and until the end of these Notes, with the exception of § 3.3, we restrict ourselves to the case where $\mathcal{A} = \mathrm{Mod}(\mathbf{k})$. However, many constructions and results still hold in other situations, in particular when chooseing $\mathcal{A} = \mathbf{Set}$. References are made to [KS06].

Recall the notations $\mathrm{Sh}(X, \mathrm{Mod}(\mathbf{k})) = \mathrm{Mod}(\mathbf{k}_X)$ and $\mathrm{PSh}(X, \mathrm{Mod}(\mathbf{k})) = \mathrm{PSh}(\mathbf{k}_X)$.

In this section, we shall explain how to construct the "sheaf associated with a presheaf". More precisely, we shall show that the natural forgetful functor $\iota_X : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ which, to a sheaf F, associates the underlying presheaf, admits a left adjoint. Let $U \in \mathcal{C}_X$ and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subsets of $\mathcal{C}_U$. Notice first that the relation $\mathcal{S}_1 \preceq \mathcal{S}_2$ is a pre-order on

$\mathrm{Cov}(U)$. Hence, $\mathrm{Cov}(U)$ inherits a structure of a category:

$$\mathrm{Hom}_{\,\mathrm{Cov}(U)}(\mathcal{S}_1, \mathcal{S}_2) = \begin{cases} \{\mathrm{pt}\} & \text{if } \mathcal{S}_1 \text{ is a refinement of } \mathcal{S}_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

For $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Cov}(U)$, $\mathcal{S}_1 \times_U \mathcal{S}_2$ again belongs to $\mathrm{Cov}(U)$. Therefore:

Lemma 2.3.1. *The category* $\mathrm{Cov}(U)$ *is cofiltrant* (*i.e., the opposite category is filtrant*)*.*

Lemma 2.3.2. *Let* $F \in \mathrm{PSh}(\mathbf{k}_X)$ *and let* $U \in \mathcal{C}_X$*. Then* F *naturally defines a functor* $\mathrm{Cov}(U)^{\mathrm{op}} \to \mathcal{A}$*.*

Proof. Let $\mathcal{S}_1 \preceq \mathcal{S}_2$. We shall construct a natural morphism $F(\mathcal{S}_2) \to F(\mathcal{S}_1)$. For $V_1 \in \mathcal{S}_1$ we construct $F(\mathcal{S}_2) \to F(V_1)$ by choosing $V_2 \in \mathcal{S}_2$ and a morphism $V_1 \to V_2$. The composition $F(\mathcal{S}_2) \to \prod_{V \in \mathcal{S}_2} F(V) \to F(V_2) \to F(V_1)$ does not depend on the choice of $V_1 \to V_2$. Indeed, if we have two morphisms $V_1 \to V_2'$ and $V_1 \to V_2''$, these morphisms factorize through $V_1 \to V_2' \times_{V_1} V_2''$ and the composition $F(\mathcal{S}_2) \to F(V_2') \to F(V_2' \times_{V_1} V_2'') \to F(V_1)$ is the same as the composition $F(\mathcal{S}_2) \to F(V_2'') \to F(V_2' \times_{V_1} V_2'') \to F(V_1)$.

The family of morphisms $F(\mathcal{S}_2) \to F(V_1)$, $V_1 \in \mathcal{S}_1$, defines $F(\mathcal{S}_2) \to F(\mathcal{S}_1)$ and one checks easily the functoriality of this construction. q.e.d.

One defines the presheaf F^+ by setting for all $U \in \mathcal{C}_X$:

$$F^+(U) = \varinjlim_{\mathcal{S} \in \mathrm{Cov}(U)} F(\mathcal{S}). \tag{2.8}$$

For any $V \to U$, the morphism $F^+(U) \to F^+(V)$ is defined by the sequence of morphisms

$$F^+(U) = \varinjlim_{\mathcal{S} \in \mathrm{Cov}(U)} F(\mathcal{S}) \to \varinjlim_{\mathcal{S} \in \mathrm{Cov}(U)} F(V \times_U \mathcal{S}) \to \varinjlim_{\mathcal{T} \in \mathrm{Cov}(V)} F(\mathcal{T}) = F^+(V).$$

The second arrow is well-defined since $V \times_U \mathcal{S} \in \mathrm{Cov}(V)$.

Clearly the correspondence $F \mapsto F^+$ defines a functor ${}^+ : \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$. Moreover for each $U \in \mathcal{C}_X$, the maps $F(U) \to F(\mathcal{S}), \mathcal{S} \in \mathrm{Cov}(U)$ define $F(U) \to \varinjlim_{\mathcal{S} \in \mathrm{Cov}(U)} F(\mathcal{S}) = F^+(U)$. Hence, there is a morphism of functors $\alpha : \mathrm{id} \to {}^+$.

Theorem 2.3.3. (i) *If* F *is a separated presheaf, then* $F \to F^+$ *is a monomorphism.*

(ii) *If* F *is a sheaf, then* $F \to F^+$ *is an isomorphism.*

(iii) *For any presheaf F, F^+ is a separated presheaf.*

(iv) *For any separated presheaf F, F^+ is a sheaf.*

(v) *The functor* ${}^a := {}^{++}\colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$ *is a left adjoint to the embedding functor* $\iota_X\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$.

(vi) *The functor* ${}^+\colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ *is left exact.*

Proof. (i) By the hypothesis, for any open set U and any covering $\mathcal{S}$ of U, the morphism $F(U) \to F(\mathcal{S})$ is a monomorphism. Since $\mathrm{Cov}(U)$ is cofiltrant, $F(U) \to F^+(U)$ is a monomorphism.
(ii) By the hypothesis, for any open set U and any covering $\mathcal{S}$ of U, the morphism $F(U) \to F(\mathcal{S})$ is an isomorphism. The result follows.
(iii)–(iv) We shall not give the proof here.
(v) Let $G \in \mathrm{Mod}(\mathbf{k}_X)$. The morphism $F \to F^+$ defines the morphism

$$\lambda\colon \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F^+, G) \to \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F, G)$$

and the functor ${}^+$ defines the morphism

$$\begin{aligned} \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F, G) &\to \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F^+, G^+) \\ &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F^+, G). \end{aligned}$$

One checks that these two morphisms are inverse one to each other. Therefore λ is an isomorphism. Replacing F with F^+, the result follows.
(vi) It is enough to prove that for each $U \in \mathcal{C}_X$, the functor $F \to F^+(U)$ is left exact. Since $F^+(U) = \varinjlim F(\mathcal{S})$, where $\mathcal{S}$ ranges over the cofiltrant category $\mathrm{Cov}(U)$, it is enough to check that the functor $F \to F(\mathcal{S})$ is left exact. This follows from (2.5) and the fact that $F \mapsto F(V)$ is exact. q.e.d.

In the sequel, we shall often omit to write the symbol ι_X. Hence, (v) may be written as follows with $F \in \mathrm{PSh}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_X)$

$$\mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F, G) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}(F^a, G). \tag{2.9}$$

Definition 2.3.4. (i) If F is a presheaf on X, the sheaf F^a is called the sheaf associated with F.

(ii) We denote by $\theta\colon \mathrm{id} \to \iota_X \circ {}^a$ the natural morphism of functor associated with the pair of adjoint functor $({}^a, \iota_X)$.

Hence, Theorem 2.3.3 (vi) may be formulated as follows: any morphism of presheaves $\varphi\colon F \to G$ factorizes uniquely as

(2.10)
$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G. \\ {\scriptstyle\theta}\downarrow & \nearrow & \\ F^a & & \end{array}$$

Remark 2.3.5. When X is a topological space the construction of the sheaf F^a is much easier. Define:

$$\begin{aligned} F^a(U) \;=\; & \{s : U \to \bigsqcup_{x\in U} F_x \;;\; s(x) \in F_x, \\ & \text{for all } x \in U,\ \text{there exists } V \text{ open in } U, \text{ with } x \in V, \\ & \text{there exists } t \in F(V) \text{ with } t_y = s(y),\ \text{for all } y \in V\}. \end{aligned}$$

Define $\theta : F \to F^a$ as follows. To $s \in F(U)$, one associates the section of F^a:

$$(x \mapsto s_x) \in F^a(U).$$

One checks that (F^a, θ) has the required properties, that is, any morphism of presheaves $\varphi\colon F \to G$ factorizes uniquely as in (2.10). Details are left to the reader.

Example 2.3.6. One denotes by $\mathbf{k}_X$ the sheaf associated with the constant presheaf $U \mapsto \mathbf{k}$. Mor generally, one defines similarly the constant sheaf M_X for $M \in \mathrm{Mod}(\mathbf{k})$. It follows from (1.10) and (1.11) that

(2.11) $$\mathcal{H}om\,(\mathbf{k}_X, F) \simeq F, \quad \mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_X, F) \simeq F(X).$$

2.4 The category of abelian sheaves

Notation 2.4.1. In the sequel, as far as there is no risk of confusion, we shall write $\mathrm{Hom}_{\mathbf{k}_X}$, or sometimes simply Hom, instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}$.

Theorem 2.4.2. (i) *The category* $\mathrm{Mod}(\mathbf{k}_X)$ *admits small projective limits and such limits commute with the functor* ι_X.

(ii) *The category* $\mathrm{Mod}(\mathbf{k}_X)$ *admits small inductive limits. More precisely, if* $\{F_i\}_{i\in I}$ *is an inductive system of sheaves, its inductive limit is the sheaf associated with its inductive limit in* $\mathrm{PSh}(\mathbf{k}_X)$.

(iii) *The functor* $\iota_X : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ *is fully faithful and commutes with small projective limits (in particular, it is left exact). The functor* ${}^a : \mathrm{PSh}(\mathbf{k}_X)) \to \mathrm{Mod}(\mathbf{k}_X)$ *commutes with small inductive limits and is exact.*

(iv) *Small filtrant inductive limits are exact in* $\mathrm{Mod}(\mathbf{k}_X)$.

(v) *Let* $\varphi \colon F \to G$ *be a morphism in* $\mathrm{Mod}(\mathbf{k}_X)$. *Denote by* "Im"$\varphi$ *and* "Coim"φ *the image and coimage of this morphism in the category* $\mathrm{PSh}(\mathbf{k}_X)$ (i.e., *the image and coimage of* $\iota_X(\varphi)$). *Then* $\mathrm{Im}\,\varphi \simeq (\text{“Im”}\varphi)^a$ *and* $\mathrm{Coim}\,\varphi \simeq (\text{“Coim”}\varphi)^a$.

(vi) *The category* $\mathrm{Mod}(\mathbf{k}_X)$ *is an abelian Grothendieck category.*

Proof. (i) Let $\{F_i\}_{i\in I}$ be a small projective system of sheaves, let $U \in \mathcal{C}_X$ and let $\mathcal{S} \in \mathrm{Cov}(U)$. By the definition of $F(\mathcal{S})$, one sees that the morphism $F(U) \to F(\mathcal{S})$ commutes with projective limits, that is, $(\varprojlim_i F_i)(U) \xrightarrow{\sim} (\varprojlim_i F_i)(\mathcal{S})$. Hence a projective limit of sheaves in the category $\mathrm{PSh}(\mathbf{k}_X)$ is a sheaf. The fact that this sheaf is a projective limit in $\mathrm{Mod}(\mathbf{k}_X)$ of the projective system $\{F_i\}_{i\in I}$ follows from the fact that the forgetful functor $\mathrm{PSh}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$ is fully faithful:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{k}_X}(G, \varprojlim_i F_i) &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathbf{k}_X)}(G, \varprojlim_i F_i) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(\mathbf{k}_X)}(G, F_i) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathbf{k}_X}(G, F_i).
\end{aligned}$$

(ii) Let $\{F_i\}_{i\in I}$ is a small inductive system of sheaves. Let us denote by $\text{“}\varinjlim_i\text{”}\, F_i$ its inductive limit in the category $\mathrm{PSh}(\mathbf{k}_X)$ and let $G \in \mathrm{Mod}(\mathbf{k}_X)$. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{k}_X}((\text{“}\varinjlim_i\text{”}\, F_i)^a, G) &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathbf{k}_X)}(\text{“}\varinjlim_i\text{”}\, F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(\mathbf{k}_X)}(F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathbf{k}_X}(F_i, G).
\end{aligned}$$

(iii) The functor ι_X is fully faithful by definition. By adjunction, ι_X commutes with small projective limits and ^{a} commutes with small inductive limits. It remains to prove that ^{a} is left exact.

By Theorem 2.3.3 (vi), the functor $\iota_X \circ {}^a \colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ is left exact. Since this functor takes its values in $\mathrm{Mod}(\mathbf{k}_X)$, and $\iota_X \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ is conservative and left exact, the functor ${}^a \colon \mathrm{PSh}(\mathbf{k}_X)) \to \mathrm{Mod}(\mathbf{k}_X)$ is left exact.
(iv) Small filtrant inductive limits are exact in the category $\mathrm{Mod}(\mathbf{k})$, whence in the category $\mathrm{PSh}(\mathbf{k}_X)$. Then the result follows since ^{a} is exact.
(v) By (i) and (ii), the category $\mathrm{Mod}(\mathbf{k}_X)$ admits small projective and inductive limits. Denote by "$\oplus$" and "Coker" the coproduct and the cokernel in the category $\mathrm{PSh}(\mathbf{k}_X)$. Then

$$\begin{aligned}
\mathrm{Coim}\,\varphi &= \mathrm{Coker}\big(F \times_G F \rightrightarrows F\big) \\
&\simeq \big(\text{“Coker”}\big(F \times_G F \rightrightarrows F\big)\big)^a \simeq \big(\text{“Coim”}(\varphi)\big)^a, \\
\mathrm{Im}\,\varphi &= \mathrm{Ker}\big(G \rightrightarrows G \oplus_F G\big)\big) \simeq \mathrm{Ker}\big(G \rightrightarrows (G \text{ “}\underset{F}{\oplus}\text{” } G)^a)\big) \\
&\simeq \big(\mathrm{Ker}(G \rightrightarrows G \text{ “}\oplus\text{” } G)\big)^a \simeq \big(\text{“Im”}(\varphi)\big)^a.
\end{aligned}$$

Here, the fourth isomorphism follows from the fact that the functor ^{a} being exact, it commutes with kernels. It follows that for a morphism $\varphi \colon F \to G$ in $\mathrm{Mod}(\mathbf{k}_X)$, the natural morphism $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. Therefore $\mathrm{Mod}(\mathbf{k}_X)$ is abelian.
(vi) It remains to prove that this category admits a system of generators. Set $\mathbf{k}_{XU} := (\mathrm{j}_U^\dagger \, \mathrm{j}_{U*} \widetilde{\mathbf{k}}_X)^a$. Then

$$\mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{XU}, F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(\mathrm{j}_U^\dagger \, \mathrm{j}_{U*} \widetilde{\mathbf{k}}_X, F)$$

and it follows from Theorem 1.5.2 that the family $\{\mathbf{k}_{XU}\}_{U \in \mathcal{C}_X}$ is a system of generators. (We shall recover the sheaves $\mathbf{k}_{XU}$ with Notation 2.7.4 and (2.22).) q.e.d.

Remark 2.4.3. The object $\mathcal{G} := \bigoplus_{U \in \mathcal{C}_X} \mathbf{k}_{XU}$ is a generator of the abelian category $\mathrm{Mod}(\mathbf{k}_X)$.

Corollary 2.4.4. *Let $\varphi \colon F \to G$ be a morphism in $\mathrm{Mod}(\mathbf{k}_X)$. Then φ is an epimorphism if and only if, for any $U \in \mathcal{C}_X$ and any $t \in G(U)$, there exists $\mathcal{S} \in \mathrm{Cov}(U)$ and for any $V \in \mathcal{S}$ there exists $s \in F(V)$ with $\varphi(s) = t$ in $G(V)$.*

Proof. As above, denote by "Im"φ the image of φ in the category $\mathrm{PSh}(X)$. Since this presheaf is a subpresheaf of the sheaf G, it is separated. Hence $\mathrm{Im}(\varphi) \simeq (\text{“Im”}(\varphi))^+$ and

$$\mathrm{Im}(\varphi)(U) \simeq \varinjlim_{\mathcal{S} \in \mathrm{Cov}(U)} \text{“Im”}(\varphi)(\mathcal{S}).$$

Now φ is an epimorphism if and only if $\operatorname{Im}\varphi \xrightarrow{\sim} G$. Let $t \in G(U)$. Since $\operatorname{Cov}(U)^{\mathrm{op}}$ is filtrant, $t \in \operatorname{Im}(\varphi)(U)$ if and only if there exists $\mathcal{S} \in \operatorname{Cov}(U)$ with $t \in$ "Im"$(\varphi)(\mathcal{S})$. The result follows. q.e.d.

Corollary 2.4.5. *Let* $F' \xrightarrow{\varphi} F \xrightarrow{\psi} F''$ *be a complex in* $\operatorname{Mod}(\mathbf{k}_X)$. *Then the conditions below are equivalent:*

(a) *this complex is exact,*

(b) *for any* $U \in \mathcal{C}_X$ *and any* $s \in F(U)$ *such that* $\psi(s) = 0$, *there exist a covering* $\mathcal{S} \in \operatorname{Cov}(U)$ *and for each* $V \in \mathcal{S}$ *there exists* $t \in F'(V)$ *such that* $\varphi(t) = s|_V$,

(c) *there exists a covering* $\mathcal{S}$ *of* X *such that the sequence* $F'|_U \xrightarrow{\varphi} F|_U \xrightarrow{\psi} F''|_U$ *is exact for any* $U \in \mathcal{S}$.

Proof. (a) is equivalent to saying that the natural morphism $\operatorname{Im}\varphi \to \operatorname{Ker}\psi$ is an epimorphism, and this last condition is equivalent to (b) by Corollary 2.4.4.
(a) $\Rightarrow$ (c) follows from $(\operatorname{Im}\varphi)|_U \simeq \operatorname{Im}(\varphi|_U)$ and $(\operatorname{Ker}\psi)|_U \simeq \operatorname{Ker}(\psi|_U)$.
(c) $\Rightarrow$ (b) is obvious. q.e.d.

Example 2.4.6. Assume that X is a topological space. Then a complex $F' \to F \to F''$ in $\operatorname{Mod}(\mathbf{k}_X)$ is exact if and only if the sequence $F'_x \to F_x \to F''_x$ is exact in $\operatorname{Mod}(\mathbf{k})$ for any $x \in X$.

Examples 2.4.7. Let X be a real analytic manifold of dimension n. The (augmented) de Rham complex is

$$0 \to \mathbb{C}_X \to \mathcal{C}_X^{\infty,(0)} \xrightarrow{d} \cdots \to \mathcal{C}_X^{\infty,(n)} \to 0 \tag{2.12}$$

where d is the differential. This complex of sheaves is exact. The same result holds with the sheaf $\mathcal{C}_X^{\infty}$ replaced with the sheaf $\mathcal{C}_X^{\omega}$ or the sheaf $\mathcal{D}b_X$.
(ii) Let X be a complex manifold of dimension n. The (augmented) holomorphic de Rham complex is

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0 \tag{2.13}$$

where d is the holomorphic differential. This complex of sheaves is exact.

The functor $\Gamma(U; \bullet)$

We have introduce the functor $\Gamma(U; \bullet)$ on presheaves in Notation 1.2.4. We keep the same notation for the restriction of this functor to the category $\mathrm{Mod}(\mathbf{k}_X)$.

Definition 2.4.8. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$.

(i) For $U \in \mathcal{C}_X$, one sets $\Gamma(U; F) = F(U)$.

(ii) One sets $\Gamma(X; F) = \varprojlim_{U \in \mathcal{C}_X} F(U)$.

Of course, if $\mathcal{C}_X$ admits a terminal object X, (i) and (ii) in Definition 2.4.8 are compatible. Moreover, one has for $U \in \mathcal{C}_X$

$$\Gamma(U; F) \simeq \Gamma(U; F|_U).$$

Since ι_X is left exact and the functor $F \mapsto F(U)$ is exact on $\mathrm{PSh}(\mathbf{k}_X)$, the functor $\Gamma(U; \bullet)\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathcal{A}$ is left exact.

Remark 2.4.9. As usual, one endows the set pt is with its natural topology. Then the functor

$$\Gamma(\mathrm{pt}; \bullet)\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k})$$

is an equivalence of categories. In the sequel, we shall identify these two categories.

The functor $\Gamma(X; \bullet)$ is not exact in general, as shown by the example below, a variant of Example 2.2.8 (iv).

Example 2.4.10. Let X be a complex curve. The holomorphic De Rham complex reads as $0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega_X \to 0$. Applying the functor $\Gamma(U; \bullet)$ for an open subset U of X, we find the complex $0 \to \mathbb{C}_X(U) \to \mathcal{O}_X(U) \xrightarrow{d} \Omega_X(U) \to 0$. Choosing for example $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$, this complex is no more exact.

Injective sheaves

Of course, an abelian sheaf is called injective (resp. projective) if it is so in the category $\mathrm{Mod}(\mathbf{k}_X)$.

Examples 2.4.11. (i) Let X denote a real analytic manifold. The sheaf $\mathcal{B}_X$ of Sato's hyperfunctions is injective, contrarily to the sheaf $\mathcal{D}b_X$ of Schwartz's distributions.
(ii) When X is endowed with the subanalytic topology X_{an}, the sheaf $\mathcal{D}b^t_{X_{\mathrm{an}}}$ of Schwartz's tempered distributions is injective. (See [**?**].)

2.5 Internal hom and tens

It follows from Theorem 2.2.6 that for $F \in \mathrm{PSh}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_X)$, the presheaf $\mathcal{H}om\,(F, \iota_X G)$ belongs to $\mathrm{Mod}(\mathbf{k}_X)$. Clearly, the bifunctor

$$\mathcal{H}om : (\mathrm{Mod}(\mathbf{k}_X))^{\mathrm{op}} \times \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$$

is left exact.

Definition 2.5.1. Let $F_1, F_2 \in \mathrm{Mod}(\mathbf{k}_X)$. Their tensor product, denoted $F_1 \otimes F_2$ is the sheaf associated with the presheaf $F_1 \overset{\mathrm{psh}}{\otimes} F_2$.

Clearly, the bifunctor

$$\otimes\colon (\mathrm{Mod}(\mathbf{k}_X))^{\mathrm{op}} \times \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$$

is right exact. If $\mathbf{k}$ is a field, this functor is exact.

Proposition 2.5.2. *Let $F_i \in \mathrm{PSh}(k_X)$, $(i = 1, 2, 3)$. There is a natural isomorphism:*

$$\mathcal{H}om\,(F_1 \otimes F_2, F_3) \simeq \mathcal{H}om\,(F_1, \mathcal{H}om\,(F_2, F_3)).$$

Proof. This follows immediately from Proposition 1.5.4. q.e.d.

Definition 2.5.3. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. One says that F is flat if the functor $F \otimes \bullet$ is exact.

Remark that, if X is a topological space and $x \in X$, $(F_1 \otimes F_2)_x \simeq (F_1)_x \otimes (F_2)_x$ and it follows that F is flat if and only if F_x is a $\mathbf{k}$-flat module for any $x \in X$.

Although the category $\mathrm{Mod}(\mathbf{k}_X)$ does not have enough projectives, Proposition 2.5.4 is sufficient to derive the tensor product.

Proposition 2.5.4. *Let $G \in \mathrm{Mod}(\mathbf{k}_X)$. Then the category of flat sheaves is projective with respect to the functor $G \otimes \bullet$.*

Proof. (i) We have already seen in Remark 2.4.3 that $\mathcal{G} = \bigoplus_{U \in \mathcal{C}_X} \mathbf{k}_{XU}$ is a generator of $\mathrm{Mod}(\mathbf{k}_X)$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. By Lemma 1.1.7, there exists a small set I and an epimorphism $\mathcal{G}^{\oplus I} \twoheadrightarrow F$. Since the sheaves $\mathbf{k}_{XU}$ are flat and small direct sums of flat sheaves are flat, the sheaf $\mathcal{G}^{\oplus I}$ is flat. Hence, there are enough flat objects.
(ii) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. If F'' and F are flat, then F' is flat. This is checked as in the case of usual $\mathbf{k}$-modules and left as an exercise.
(iii) Consider the exact sequence in (ii). If F'' is flat, then the sequence obtained by applying the functor $G \otimes \bullet$ will remain exact by the definition of flatness. q.e.d.

2.6 Direct and inverse images

Let $f\colon X \to Y$ be a morphism of sites. We have already defined the direct and inverse images of presheaves.

Proposition 2.6.1. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then the presheaf f_*F is a sheaf on Y.*

Proof. Let $V \in \mathcal{C}_Y$ and let $\mathcal{S}$ be a covering of V. Since $f^t\mathcal{S}$ is a covering of f^tV, we get the chain of isomorphisms

$$f_*F(V) = F(f^t(V)) \simeq F(f^t(\mathcal{S})) = f_*F(\mathcal{S}).$$

q.e.d.

Hence, the functor $f_*\colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{PSh}(Y, \mathcal{A})$ induces a functor (we keep the same notation)

$$f_*\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y).$$

We shall see in §2.7 that the functor j_{U*} is exact.

Definition 2.6.2. Let $G \in \mathrm{Mod}(\mathbf{k}_Y)$. One denotes by $f^{-1}G$ the sheaf on X associated with the presheaf $f^\dagger G$ and calls it the inverse image of G. In other words, $f^{-1}G = (f^\dagger G)^a$.

Theorem 2.6.3. *Let $f\colon X \to Y$ be a morphism of sites.*

(i) *The functor $f^{-1}\colon \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{Mod}(\mathbf{k}_X)$ is left adjoint to f_*. In other words, there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{k}_X}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathbf{k}_Y}(G, f_*F)$$

functorial with respect to $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$.

(ii) *The functor f_* is left exact and commutes with small projective limits.*

(iii) *The functor f^{-1} is right exact and commutes with small inductive limits.*

(iv) *There are natural morphisms of functors $\mathrm{id} \to f_*f^{-1}$ and $f^{-1}f_* \to \mathrm{id}$.*

(v) *Assume that for any $U \in \mathcal{C}_X$, the category $(\mathcal{C}_Y^U)^{\mathrm{op}}$ is either filtrant or empty. Then the functor f^{-1} is exact.*

Proof. (i) Denote for a while by "f_*" the direct image in the categories of presheaves. Since $f^\dagger$ is left adjoint to "f_*" and ^{a} is left adjoint to ι_X, $f^{-1} = {}^a \circ f^\dagger$ is left adjoint to $f_* =$ "f_*" $\circ\, \iota_X$.
(ii)–(iv) follow from the adjunction property.
(v) Since the functor ^{a} is exact, it is enough to prove that the functor $f^\dagger \colon \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{PSh}(\mathbf{k}_X)$ is left exact. Let

$$0 \to G' \to G \to G'' \tag{2.14}$$

be an exact sequence of sheaves on Y. For each $V \in \mathrm{Mod}(\mathbf{k}_Y)$, the sequence

$$0 \to G'(V) \to G(V) \to G''(V) \tag{2.15}$$

is exact. By Definition 1.3.1, the sequence

$$0 \to (f^\dagger G')(U) \to (f^\dagger G)(U) \to (f^\dagger G'')(U) \tag{2.16}$$

is obtained by applying the functor $\varinjlim\limits_{(U\to f^t(V))\in\mathcal{C}_Y^U}$ to the sequence (2.15). This functor is exact if the category $(\mathcal{C}_Y^U)^{\mathrm{op}}$ is either filtrant or empty. q.e.d.

Corollary 2.6.4. *Let $G \in \mathrm{Mod}(\mathbf{k}_Y)$. Then $(f^\dagger G)^a \xrightarrow{\sim} f^{-1}(G^a)$.*

Proof. One has the chain of isomorphisms, functorial with respect to $F \in \mathrm{Mod}(\mathbf{k}_X)$:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{k}_X}((f^\dagger G)^a, F) &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathbf{k}_X)}(f^\dagger G, F) \simeq \mathrm{Hom}_{\mathbf{k}_Y}(G, f_*F) \\ &\simeq \mathrm{Hom}_{\mathbf{k}_Y}(G^a, f_*F) \simeq \mathrm{Hom}_{\mathbf{k}_X}(f^{-1}(G^a), F).\end{aligned}$$

Hence, the result follows from the Yoneda lemma. q.e.d.

Consider two morphisms of sites $f \colon X \to Y$ and $g \colon Y \to Z$.

Proposition 2.6.5. (i) *$g \circ f \colon X \to Z$ is a morphism of sites.*

(ii) *One has natural isomorphisms of functors*

$$g_* \circ f_* \simeq (g \circ f)_*, \qquad f^{-1} \circ g^{-1} \simeq (g \circ f)^{-1}.$$

Proof. (i) is obvious.
(ii) The functoriality of direct images for presheaves is clear (see Proposition 1.3.3). It then follows for sheaves from Proposition 2.6.1. The functoriality of inverse images follows by adjunction. q.e.d.

Examples 2.6.6. Assume that $f\colon X\to Y$ is a morphism of topological spaces. Then for $x\in X$,

$$(f^{-1}G)_x \simeq (f^{\dagger}G)_x \simeq G_{f(x)} \tag{2.17}$$

and f^{-1} is exact. In particular, denote by $i_x\colon\{x\}\hookrightarrow X$ the embedding of $x\in X$ into X. Then, for $F\in \mathrm{Mod}(\mathbf{k}_X)$

$$F_x \simeq i_x^{-1}F.$$

(ii) Let $M\in\mathrm{Mod}(\mathbf{k})$. Recall that M_X denotes the sheaf associated with the constant presheaf $U\mapsto M$. Hence, if X has a terminal object and $a_X\colon X\to \mathrm{Pt}$ is the canonical map:

$$M_X\simeq a_X^{-1}M_{\mathrm{Pt}}.$$

(iv) Let $Z=\{a,b\}$ be a set with two elements, let Y be a topological space and let $X=Z\times Y\simeq Y\bigsqcup Y$, the disjoint union of two copies of Y. Let $f: X\to Y$ be the projection. Then $f_*f^{-1}G\simeq F\oplus F$. In fact, if V is open in Y, then $\Gamma(V;f_*f^{-1}G)\simeq\Gamma(V\bigsqcup V;f^{-1}G)\simeq\Gamma(V;G)\oplus\Gamma(V;G)$.
(v) Let $X=Y=\mathbb{C}\setminus\{0\}$, and let $f: X\to Y$ be the map $z\mapsto z^2$, where z denotes a holomorphic coordinate on $\mathbb{C}$. If D is an open disk in Y, $f^{-1}D$ is isomorphic to the disjoint union of two copies of D. Hence, the sheaf $f_*\mathbf{k}_X|_D$ is isomorphic to $\mathbf{k}_D^2$, the constant sheaf of rank two on D. However, $\Gamma(Y;f_*\mathbf{k}_X)=\Gamma(X;\mathbf{k}_X)=\mathbf{k}$, which shows that the sheaf $f_*\mathbf{k}_X$ is not isomorphic to $\mathbf{k}_X^2$.
(vi) Let $f: X\to Y$ be a morphism of topological spaces. To each open subset $V\subset Y$ is associated a natural "pull-back" map: $\Gamma(V;\mathcal{C}_Y^0)\to\Gamma(V;f_*\mathcal{C}_X^0)$ defined by $\varphi\mapsto\varphi\circ f$. We obtain a morphism $\mathcal{C}_Y^0\to f_*\mathcal{C}_X^0$, hence a morphism:

$$f^{-1}\mathcal{C}_Y^0\to\mathcal{C}_X^0.$$

For example, if X is closed in Y and f is the injection, $f^{-1}\mathcal{C}_Y^0$ will be the sheaf on X of continuous functions on Y defined in a neighborhood of X. If f is smooth (locally on X, f is isomorphic to a projection $Y\times Z\to Y$), then $f^{-1}\mathcal{C}_Y^0$ will be the subsheaf of $\mathcal{C}_X^0$ consisting of functions locally constant on the fibers of f.
(vii) Let $i_S: S\hookrightarrow X$ be the embedding of a closed subset S of a topological space X. Then the functor i_{S*} is exact.

2.7 Restriction and extension of sheaves

Let X be a site and let $U\in\mathcal{C}_X$. We have already defined the morphisms of sites $\mathrm{j}_U\colon X\to U$ and $i_U\colon U\to X$.

Proposition 2.7.1. *Let $U \in \mathcal{C}_X$.*

(i) *One has the isomorphisms of functors of presheaves*

$$\mathrm{j}_{U*} \simeq i_U^\dagger, \qquad \mathrm{j}_U^\ddagger \simeq i_{U*}.$$

In particular, the functor $i_U^\dagger$ sends $\mathrm{Mod}(\mathbf{k}_X)$ to $\mathrm{Mod}(\mathbf{k}_U)$ and the functor $\mathrm{j}_U^\ddagger$ sends $\mathrm{Mod}(\mathbf{k}_U)$ to $\mathrm{Mod}(\mathbf{k}_X)$.

(ii) *The functor $\mathrm{j}_{U*}\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_U)$ commutes with small inductive and projective limits and in particular is exact. Moreover, $j_{U*} \simeq i_U^{-1}$.*

(iii) *The functor $\mathrm{j}_U^{-1}\colon \mathrm{Mod}(\mathbf{k}_U) \to \mathrm{Mod}(\mathbf{k}_X)$ is exact.*

Proof. (i) Let $F \in \mathrm{PSh}(\mathbf{k}_X)$ and let $(V \to U) \in \mathcal{C}_U$. One has

$$\begin{aligned} i_U^\dagger F(V \to U) &\simeq \varinjlim_{W \to V \to U} F(W) \\ &\simeq F(V) \simeq \mathrm{j}_{U*}F(V \to U). \end{aligned}$$

The isomorphism $\mathrm{j}_U^\ddagger \simeq i_{U*}$ follows by adjunction.
(ii) The functor j_{U*} admits both a right and a left adjoint. The isomorphism $j_{U*} \simeq i_U^{-1}$ follows from (i).
(iii) Since direct sums are exact in $\mathrm{Mod}(\mathbf{k})$, the functor $\mathrm{j}_U^\dagger$ is exact by Proposition 1.4.3. Then the result follows since ^{a} is exact. q.e.d.

Notation 2.7.2. One sets for $F \in \mathrm{Mod}(\mathbf{k}_X)$:

$$F|_U = \mathrm{j}_{U*}F \simeq i_U^{-1}F.$$

One usually sets

$$i_{U!} := \mathrm{j}_U^{-1}. \tag{2.18}$$

Hence, $i_{U!}$ is exact.

Hence, we have two pairs of adjoint functors $(\mathrm{j}_U^{-1}, \mathrm{j}_{U*})$, $(\mathrm{j}_{U*}, \mathrm{j}_U^\ddagger)$:

$$\mathrm{Mod}(\mathbf{k}_U) \underset{\mathrm{j}_U^\ddagger}{\overset{\mathrm{j}_U^{-1}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_X), \quad (\leftarrow \mathrm{j}_{U*}) \tag{2.19}$$

which are also written as two pairs of adjoint functors (i_U^{-1}, i_{U*}), $(i_{U!}, i_U^{-1})$:

$$\mathrm{Mod}(\mathbf{k}_U) \underset{i_{U*}}{\overset{i_{U!}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_X). \quad (\leftarrow i_U^{-1}) \tag{2.20}$$

Let $f\colon X \to Y$ be a morphism of sites. Let $V \in \mathcal{C}_Y$, set $U = f^t(V)$ and denote by $f_V\colon U \to V$ the morphism of sites associated with the functor $f^t_V\colon \mathcal{C}_V \to \mathcal{C}_U$ deduced from f^t. We get the commutative diagram of sites

$$\begin{array}{ccccc} U & \xrightarrow{i_U} & X & \xrightarrow{j_U} & U \\ \downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_V} \\ V & \xrightarrow[i_V]{} & Y & \xrightarrow[j_V]{} & V. \end{array} \tag{2.21}$$

Proposition 2.7.3. *There are natural isomorphisms of functors*

$$\begin{aligned} i_V^{-1} f_* \simeq f_{V*} i_U^{-1}, &\qquad f^{-1} i_{V!} \simeq i_{U!} f_V^{-1}, \\ f_* \, \mathrm{j}_U^\ddagger \simeq \mathrm{j}_V^\ddagger \, f_{V*}, &\qquad \mathrm{j}_{U*} f^{-1} \simeq f_V^{-1} \mathrm{j}_{V*}. \end{aligned}$$

Proof. This follows from the isomorphisms $\mathrm{j}_{V*} f_* \simeq f_{V*} \mathrm{j}_{U*}$, $f^{-1}\,\mathrm{j}_V^{-1} \simeq \mathrm{j}_U^{-1}\, f_V^{-1}$, $f_* i_{U*} \simeq i_{V*} f_{V*}$ and $i_U^{-1} f^{-1} \simeq i_V^{-1} f_V^{-1}$. q.e.d.

Notation 2.7.4. For $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $U \in \mathcal{C}_X$, one sets

$$\begin{aligned} F_U &= \mathrm{j}_U^{-1}\, \mathrm{j}_{U*} F \simeq i_{U!} i_U^{-1} F, \\ \Gamma_U F &= \mathrm{j}_U^\ddagger\, \mathrm{j}_{U*} F \simeq i_{U*} i_U^{-1} F. \end{aligned}$$

In case $F = \mathbf{k}_X$, one writes for short $\mathbf{k}_{XU}$ instead of $(\mathbf{k}_X)_U$.

Recall (see § 1.5) that we have denoted by $\widetilde{\mathbf{k}}_X$ the constant presheaf $U \mapsto \mathbf{k}$. Then (see Exercises 2.6 and 2.5):

$$\mathbf{k}_{XU} \simeq \mathrm{j}_U^{-1}\, \mathrm{j}_{U*} \widetilde{\mathbf{k}}_X. \tag{2.22}$$

For $V \to U$ a morphism in $\mathcal{C}_X$, there are natural morphisms :

$$F_V \to F_U \to F \to \Gamma_U F \to \Gamma_V F.$$

Also note that $((\,\bullet\,)_U, \Gamma_U(\,\bullet\,))$ is a pair of adjoint functors.

Proposition 2.7.5. *For $U, V \in \mathcal{C}_X$ there are natural isomorphisms*

$$(F_V)_U \simeq F_{U \times_X V}, \qquad \Gamma_U(\Gamma_V F) \simeq \Gamma_{U \times_X V}(F).$$

Proof. By adjunction, it is enough to prove the second isomorphism. One has for $W \in \mathcal{C}_X$:

$$\begin{aligned} \Gamma_U(\Gamma_V(F))(W) &\simeq \Gamma_V(F)(U \times_X W) \\ &\simeq F(V \times_X U \times_X W) \simeq \Gamma_{U \times_X V}(F)(W). \end{aligned}$$

q.e.d.

Proposition 2.7.6. *Let $f\colon X \to Y$ be a morphism of sites. For $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$, there is a natural isomorphism in $\mathrm{Mod}(\mathbf{k}_Y)$*

$$\mathcal{H}om\,(G, f_*F) \xrightarrow{\sim} f_*\mathcal{H}om\,(f^{-1}G, F). \tag{2.23}$$

Proof. Let $V \in \mathcal{C}_Y$ and set $U = f^t(V)$. Denote by $f_U\colon U \to V$ the morphism of sites associated with f. Using Proposition 2.7.3, we get the chain of isomorphisms

$$\begin{aligned}
\Gamma(V; f_*\mathcal{H}om\,(f^{-1}G, F)) &\simeq \Gamma(U; \mathcal{H}om\,(f^{-1}G, F)) \\
&\simeq \mathrm{Hom}\,(f^{-1}G|_U, F|_U) \\
&\simeq \mathrm{Hom}\,((f_U)^{-1}(G|_V), F|_U) \\
&\simeq \mathrm{Hom}\,(G|_V, (f_U)_*F|_U) \\
&\simeq \mathrm{Hom}\,(G|_V, (f_*F)|_V) \\
&\simeq \Gamma(V; \mathcal{H}om\,(G, f_*F)).
\end{aligned}$$

These isomorphisms being functorial with respect to V, the isomorphism (2.23) follows. q.e.d.

Proposition 2.7.7. *Let $U \in \mathcal{C}_X$, let $G \in \mathrm{Mod}(\mathbf{k}_U)$ and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. There is a natural isomorphism*

$$j_U^{-1}(G \otimes j_{U*}F) \simeq j_U^{-1}G \otimes F. \tag{2.24}$$

Note that isomorphism (2.24) may also be written as

$$i_{U!}(G \otimes i_U^{-1}F) \simeq i_{U!}G \otimes F. \tag{2.25}$$

Proof. The right hand side of (2.24) is the sheaf associated with the presheaf

$$V \mapsto (\bigoplus_{s \in \mathrm{Hom}\,(V,U)} G(V \xrightarrow{s} U)) \otimes F(V),$$

and the left hand side is the sheaf associated with the presheaf

$$V \mapsto (\bigoplus_{s \in \mathrm{Hom}\,(V,U)} G(V \xrightarrow{s} U) \otimes F(V)).$$

q.e.d.

Proposition 2.7.8. *Let $U \in \mathcal{C}_X$. There are natural isomorphisms, functorial in $F \in \mathrm{Mod}(\mathbf{k}_X)$:*

$$\begin{aligned}
&\Gamma(U; F) \simeq \mathrm{Hom}\,(\mathbf{k}_{XU}, F), \\
&\Gamma_U(F) \simeq \mathcal{H}om\,(\mathbf{k}_{XU}, F), \\
&F_U \simeq \mathbf{k}_{XU} \otimes F.
\end{aligned}$$

Proof. The first isomorphism follows from (1.12) thanks to (2.22) and the second isomorphism follows. Let us prove the third one. We have by Proposition 2.7.7

$$\begin{aligned} F_U &\simeq j_U^{-1} i_U^{-1} F \simeq j_U^{-1}(j_{U*}F \otimes j_{U*}\mathbf{k}_X) \\ &\simeq F \otimes j_U^{-1} j_{U*}\mathbf{k}_X. \end{aligned}$$

q.e.d.

The case of a topological space

When X is a topological space, one better uses the functors i_U^{-1} and $i_{U!}$ rather that j_{U*} and j_U^{-1}, respectively.

If U, V are open subsets, then $U \times_X V = U \cap V$. It follows that the morphism of sites i_U corresponds to the continuous embedding $U \hookrightarrow X$. Since the composition of morphisms of sites

$$U \xrightarrow{i_U} X \xrightarrow{\mathrm{j}_U} U \tag{2.26}$$

is the identity, we obtain:

$$i_U^{-1} \circ i_{U*} \simeq \mathrm{id}, \qquad i_U^{-1} \circ i_{U!} \simeq \mathrm{id}\,. \tag{2.27}$$

Hence, i_{U*} and $i_{U!}$ are fully faithful in this case.

2.8 Locally constant sheaves

Definition 2.8.1. (i) Let X be a site and let $M \in \mathcal{A}$. The constant sheaf M_X with stalk M is the sheaf associated with the constant presheaf with values M.

(ii) A constant sheaf is a sheaf isomorphic to a sheaf M_X for some $M \in \mathcal{A}$.

(iii) A sheaf F on X is locally constant if there exists a covering $\mathcal{S} \in \mathrm{Cov}(X)$ such that $F|_U$ is a constant sheaf on U for each $U \in \mathcal{S}$.

(iv) If $\mathbf{k}$ is a field, a local system over $\mathbf{k}$ is a locally constant sheaf of finite rank (*i.e.,* locally isomorphic to $\mathbf{k}_X^m$ for some integer m).

If X is a topological space and $M \in \mathbf{Set}$, the constant sheaf M_X is the sheaf of locally constant M-valued functions on X.

Locally constant sheaves, and their generalization, constructible sheaves, play an important role in various fields of mathematics.

Examples 2.8.2. (i) Let $X = \mathbb{R}$, the real line with coordinate t. The sheaf $\mathbb{C}_X \cdot \exp(t)$ of functions which are locally a constant multiple of the function $t \mapsto \exp(t)$ is isomorphic to the sheaf $\mathbb{C}_X$, hence is a constant sheaf.
(ii) Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z. Consider the differential operator $P = z\frac{\partial}{\partial z} - \alpha$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let us denote by K_α the kernel of P acting on $\mathcal{O}_X$.

Let U be an open disk in X centered at z_0, and let $A(z)$ denote a primitive of α/z in U. We have a commutative diagram of sheaves on U:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{z\frac{\partial}{\partial z} - \alpha} & \mathcal{O}_X \\ {\scriptstyle \exp(-A(z))}\downarrow & & \downarrow{\scriptstyle \frac{1}{z}\exp(-A(z))} \\ \mathcal{O}_X & \xrightarrow{\frac{\partial}{\partial z}} & \mathcal{O}_X \end{array}$$

Therefore, one gets an isomorphism of sheaves $K_\alpha|_U \xrightarrow{\sim} \mathbb{C}_X|_U$, which shows that K_α is locally constant, of rank one.

On the other hand, $f \in \mathcal{O}(X)$ and $Pf = 0$ implies $f = 0$. Hence $\Gamma(X; K_\alpha) = 0$, and K_α is a locally constant sheaf of rank one on $\mathbb{C} \setminus \{0\}$ which is not constant.
(iii) With the notations of Example 2.6.6 (v), the sheaf $f_*\mathbf{k}_X$ is locally constant of rank 2.

We shall construct locally constant sheaves in Section 2.9.

2.9 Glueing sheaves

One often encounters sheaves which are only defined locally, and it is natural to try to glue them.

For notational convenience, we shall often denote in the sequel by $\mathcal{S} = \{U_i\}_{i \in I}$ a covering of $U \in \mathcal{C}_X$ indexed by a set I (see Remark 2.1.2). In this case, we set

$$U_{ij} = U_i \times_U U_j, \quad U_{ijk} = U_i \times_U U_j \times_U U_k, \text{ etc.} \tag{2.28}$$

Theorem 2.9.1. *Let $\mathcal{S} = \{U_i\}_i$ be a covering of X. Assume to be given, for each $U_i \in \mathcal{S}$, an object $F_i \in \mathrm{Sh}(U_i, \mathcal{A})$ and, for each pair $U_i, U_j \in \mathcal{S}$, an isomorphism $\theta_{ji} \colon F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$ in $\mathrm{Sh}(U_{ij}, \mathcal{A})$, these isomorphisms satisfying the condition that for all $U_i, U_j, U_k \in \mathcal{S}$:*

$$\theta_{ij} \circ \theta_{jk} = \theta_{ik} \text{ on } U_{ijk}. \tag{2.29}$$

Then there exists a sheaf F on X and for each $U_i \in \mathcal{S}$ an isomorphism $\theta_i \colon F|_{U_i} \xrightarrow{\sim} F_i$ such that $\theta_j = \theta_{ji} \circ \theta_i$ for $U_i, U_j \in \mathcal{S}$. Moreover, $(F, \{\theta_i\}_i$ is unique up to unique isomorphism.

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (2.29) is called a 1-cocycle.

Proof. (i) Unicity. Let $\theta_i\colon F|_{U_i} \xrightarrow{\sim} F_i$ and $\lambda_i\colon G|_{U_i} \xrightarrow{\sim} F_i$. Hence, $\theta_j = \theta_{ji} \circ \theta_i$ and $\lambda_j = \theta_{ji} \circ \lambda_i$ on U_j. Consider the isomorphisms

$$\rho_i := \lambda_i^{-1} \circ \theta_i\colon F|_{U_i} \to G|_{U_i}.$$

On U_{ij} we have:

$$\begin{aligned} \rho_j &= \lambda_j^{-1} \circ \theta_j = \lambda_j^{-1} \circ \theta_{ji} \circ \theta_i \\ &= \lambda_i^{-1} \circ \theta_i = \rho_i. \end{aligned}$$

Therefore, the isomorphisms ρ_i's will glue as a unique isomorphism $\rho\colon G \xrightarrow{\sim} F$ on X, by Theorem 2.2.6.

(ii) Existence of a presheaf F. For each open subset V of X, define $F(V)$ by the exact sequence

$$F(V) \longrightarrow \prod_{i\in I} F_i(U_i \times_X V) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{j,k\in I} F_j(U_{jk} \times_X V).$$

Here, the two arrows a, b are defined as follows. Let $U_j, U_k \in \mathcal{S}$. Then a is associated with the composition

$$\prod_{i\in I} F_i(U_i \times_X V) \to F_j(U_j \times_X V) \to F_j(U_{jk} \times_X V)$$

and b is associated with the composition

$$\prod_{i\in I} F_i(U_i \times_X V) \to F_k(U_k \times_X V) \to F_k(U_{jk} \times_X V) \xrightarrow{\theta_{jk}} F_j(U_{jk} \times_X V).$$

(iii) F is a sheaf. Indeed, let $V \in \mathcal{C}_X$ and let $\mathcal{V} \in \mathrm{Cov}(V)$. We may assume that $\mathcal{V}$ is stable by fiber products. Then

$$F_i(U_i \times_X V) \xrightarrow{\sim} \varprojlim_{W\in\mathcal{T}} F_i(U_i \times_X W),$$

and similarly with $F_j(U_{jk} \times_X V)$. Since products commute with projective limits, we get the isomorphism $F(V) \xrightarrow{\sim} \varprojlim_{W\in\mathcal{V}} F(W) = F(\mathcal{V})$.

(iv) The morphisms θ_i's are induced by the projections $F(V) \to \prod_{j\in I} F_j(V \times_X U_j) \to F_i(V \times_X U_i)$. Let us prove they are isomorphisms. Let $l \in I$. We can construct a commutative diagram

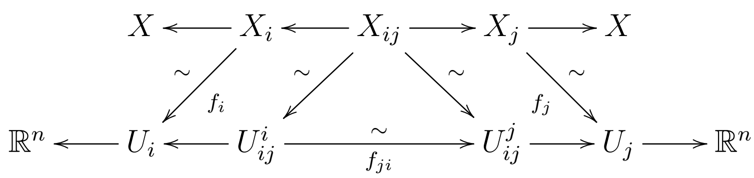
$$\begin{array}{ccccc}
F_l(U_l) & \xrightarrow{\alpha} & \prod_{i\in I} F_i(U_i \times_X U_l) & \overset{a}{\underset{b}{\rightrightarrows}} & \prod_{j,k\in I} F_j(U_{jk} \times_X U_l) \\
\uparrow{\scriptstyle \theta_l(U_i)} & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\
F(U_l) & \longrightarrow & \prod_{i\in I} F_i(U_i \times_X U_l) & \overset{a}{\underset{b}{\rightrightarrows}} & \prod_{j,k\in I} F_j(U_{jk} \times_X U_l)
\end{array}$$

where $\alpha = \{\theta_{il}\}_i$. One checks easily that the sequence on the top is exact, and it follows that $\theta_l(U_l)\colon F(U_l) \to F_l(U_l)$ is an isomorphism. Replacing U_l with any $V \to U_l$, we get the result. q.e.d.

Example 2.9.2. Consider an n-dimensional real manifold X of class $\mathcal{C}^\infty$, and let $\{X_i, f_i\}_{i\in I}$ be an atlas. Recall what it means. The family $\{X_i\}_{i\in I}$ is an open covering of X and $f_i : X_i \xrightarrow{\sim} U_i$ is a topological isomorphism with an open subset U_i of $\mathbb{R}^n$ such that, setting $U^i_{ij} = f_i(X_{ij}) \subset \mathbb{R}^n$, the maps

$$f_{ji} := f_j|_{X_{ij}} \circ f_i^{-1}|_{U^i_{ij}} : U^i_{ij} \to U^j_{ij}, \tag{2.30}$$

are isomorphisms of class $\mathcal{C}^\infty$.



The maps f_{ji} are called the transition functions. The locally constant function on X_{ij} defined as the sign of the Jacobian determinant of the f_{ji}'s is a 1-cocycle. It defines a sheaf locally isomorphic to $\mathbb{Z}_X$ called the orientation sheaf on X and denoted by or_X.

Example 2.9.3. Let $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Consider the covering of X by the two open sets $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$. One can glue $\mathcal{O}_X|_{U_1}$ and $\mathcal{O}_X|_{U_2}$ on $U_1 \cap U_2$ by using the isomorphism $f \mapsto z^m f$ $(k \in \mathbb{Z})$. One gets a locally free sheaf of rank one denoted by $\mathcal{O}_{\mathbb{P}^1}(m)$. For $m \neq 0$, this sheaf is not free.

Exercises to Chapter 2

Exercise 2.1. Let X be a topological space. Prove that the natural morphism $(\mathcal{H}om\,(F,G))_x \to \mathrm{Hom}\,(F_x, G_x)$ is not an isomorphism in general. (Hint: choose $F = \mathbf{k}_{XU}$ with U open.)

Exercise 2.2. Let X be a site satisfying (1.8) and let $u\colon A \to B$ and $v\colon C \to B$ be morphisms in $\mathrm{Sh}(X)$. Assume that u is an epimorphism. Prove that $w\colon A \times_B C \to C$ is an epimorphism. In other words, epimorphisms are stable by base change in $\mathrm{Sh}(X)$.

Exercise 2.3. Let X be a site satisfying (1.8) and let $\mathbf{k}$ be a field. For $F, G \in \mathrm{Mod}(\mathbf{k}_X)$ one defines their tensor product $F \otimes G$ as the sheaf associated with the presheaf $U \mapsto F(U) \otimes G(U)$. For $F, G, H \in \mathrm{Mod}(\mathbf{k}_X)$, prove the isomorphism

$$\mathcal{H}om\,(F, \mathcal{H}om\,(G, H)) \simeq \mathcal{H}om\,(F \otimes G, H).$$

Exercise 2.4. Let $\mathbf{k}$ be a field and X a connected topological space. Let L be a locally free sheaf of rank one on X and set $L^{\otimes -1} := \mathcal{H}om\,(L, \mathbf{k}_X)$.
(i) Prove the isomorphism $L \otimes L^{\otimes -1} \simeq \mathbf{k}_X$.
(ii) Assume that there exists $s \in \Gamma(X; L)$ with $s \neq 0$. Prove that s defines an isomorphism $\mathbf{k}_X \xrightarrow{\sim} L$.

Exercise 2.5. Let X be a site, let $U \in \mathcal{C}_X$ and let $F \in \mathrm{PSh}(\mathbf{k}_X)$. Prove that $(\mathrm{j}_{U*}F)^a \to \mathrm{j}_{U*}(F^a)$ is an isomorphism.

Exercise 2.6. Let $f\colon X \to Y$ be a morphism of sites. Let $G \in \mathrm{PSh}(\mathbf{k}_Y)$. Prove that $(f^\dagger G)^a \to f^{-1}(G^a)$ is an isomorphism.

Exercise 2.7. Let X be a presite satisfying (1.8) and let X_{fin} be the presite X endowed with the final topology. Prove the equivalence of categories $\mathrm{PSh}(X) \simeq \mathrm{Sh}(X_{\mathrm{fin}})$.

Exercise 2.8. Let $f\colon X \to Y$ be a morphism of sites and assume that the functor $f^{-1}\colon \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{Mod}(\mathbf{k}_X)$ is exact. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ be injective. Prove that f_*F is injective. Deduce that for $U \in \mathcal{C}_X$ and F injective in $\mathrm{Mod}(\mathbf{k}_X)$, $F|_U$ is injective in $\mathrm{Mod}(\mathbf{k}_U)$.

Exercise 2.9. Let X be a topological space and let $\mathcal{U}$ be an open covering stable by finite intersections and which is a basis for the topolopy of X (that is, for any $x \in V \in \mathrm{Op}_X$ there exists $x \in U \subset V$ with $U \in \mathcal{U}$). Denote by Y the site such that $\mathcal{C}_Y = \mathcal{U}$, the coverings in Y being the coverings in X. Denote by $f\colon X \to Y$ the natural morphism of sites. Prove that $f_*\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y)$ is an equivalence of categories.

Exercise 2.10. Let $\mathbb{P}^n(\mathbb{C})$ be the complex projective space of dimension n.
(i) Construct an isomorphism of line bundles $\Omega^{(n)}_{\mathbb{P}^n(\mathbb{C})} \simeq \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-n-1)$.
(ii) Prove that the line bundles $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(l)$ and $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(m)$ are not isomorphic for $l \neq m$. (Hint: reduce to the case where $l = 0$.)

Exercise 2.11. Consider a 1-cocycle $(\mathbf{c}, \mathcal{S})$ on a ringed site $(X, \mathcal{O}_X)$. Assume that for each $U_i \in \mathcal{S}$, there exists $c_i \in \Gamma(U_i; \mathcal{O}_X^\times)$ satisfying:

$$c_{ij} = c_i \circ c_j^{-1}$$

on each U_{ij}. Prove that the locally free sheaf $\mathcal{L}_{\mathbf{c}}$ of Corollary 3.3.9 is globally free.

Chapter 3

Derived category of abelian sheaves

From now on and until the end of these Notes, we shall concentrate on abelian sheaves on sites satisfying Hypothesis 2.1. We shall assume moreover that $\mathbf{k}$ has finite global dimension (that is, finite injective, or equivalently projective, dimension).

Notation. As already mentioned, we shall write $\otimes$ instead of $\otimes_{\mathbf{k}}$, Hom instead of $\mathrm{Hom}_{\mathbf{k}}$. We proceed similarly with $\mathbf{k}$ replaced with $\mathbf{k}_X$, $\mathbf{k}_Y$ etc. and $\otimes$ replaced with $\overset{\mathrm{L}}{\otimes}$, Hom with RHom and $\mathcal{H}om$ with $R\mathcal{H}om$.

3.1 The derived category of sheaves

Recall that $\mathrm{Mod}(\mathbf{k}_X)$ is an abelian Grothendieck category and in particular, admits enough injectives. Hence we may derive all left exact functors defined on $\mathrm{Mod}(\mathbf{k}_X)$. Using Proposition 2.5.4 we get that $\mathrm{Mod}(\mathbf{k}_X)$ has enough flat objects and this allows us to derive the tensor product functor. We denote by $\mathrm{D}^*(\mathbf{k}_X)$ the derived category $\mathrm{D}^*(\mathrm{Mod}(\mathbf{k}_X))$, with $^* = +, -, \mathrm{b}, \mathrm{ub}$.

Let $f\colon X \to Y$ be a morphism of sites and let $U \in \mathcal{C}_X$.

Theorem 3.1.1. *The functors below are well defined:*

$$
\begin{aligned}
&\mathrm{RHom}\,(\,\bullet\,,\,\bullet\,) : \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}),\\
&R\mathcal{H}om\,(\,\bullet\,,\,\bullet\,) : \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_X),\\
&f^{-1} : \mathrm{D}^*(\mathbf{k}_Y) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&Rf_* : \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y),\\
&\bullet \overset{\mathrm{L}}{\otimes} \bullet\ : \mathrm{D}^*(\mathbf{k}_X) \times \mathrm{D}^*(\mathbf{k}_X) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{j}_{U*} : \mathrm{D}^*(\mathbf{k}_X) \to \mathrm{D}^*(\mathbf{k}_U) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{j}_U^{-1} : \mathrm{D}^*(\mathbf{k}_U) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&R\,\mathrm{j}_U^! : \mathrm{D}^+(\mathbf{k}_U) \to \mathrm{D}^+(\mathbf{k}_X).
\end{aligned}
$$

Of course, there are other functors which are combinations or particular cases of the preceding ones, such as the derived functor of the exact functor $F \mapsto F_U$ defined on $\mathrm{D}^*(\mathbf{k}_X)$ $(* = \mathrm{b}, +, -)$ or the right derived functor $\mathrm{R}\Gamma_U(\,\bullet\,)\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_X)$.

Then one can extend much of the preceding formulas to the derived functors and obtain the next important formulas that we state without proofs.

$$
\begin{aligned}
&\mathrm{RHom}\,(F_1, F_2) \simeq \mathrm{R}\Gamma(X; R\mathcal{H}om\,(F_1, F_2)),\\
&R\mathcal{H}om\,(F_1 \overset{\mathrm{L}}{\otimes} F_2, F_3) \simeq R\mathcal{H}om\,(F_1, R\mathcal{H}om\,(F_2, F_3)),\\
&R\mathcal{H}om\,(F, Rf_* G) \simeq Rf_* R\mathcal{H}om\,(f^{-1}F, G),\\
&f^{-1}F_1 \overset{\mathrm{L}}{\otimes} f^{-1}F_2 \simeq f^{-1}(F_1 \otimes^L F_2).
\end{aligned}
$$

Here, F, F_1, F_2, F_3 belong to $\mathrm{D}^*(\mathbf{k}_X)$ and G belong sto $\mathrm{D}^*(\mathbf{k}_Y)$.

Of course, there are many other important formulas, such as:

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^*(\mathbf{k}_X)}(\,\bullet\,,\,\bullet\,) &\simeq H^0\mathrm{RHom}\,(\,\bullet\,,\,\bullet\,)\\
R(f \circ g)_* &\simeq Rf_* \circ Rg_*,\\
(f \circ g)^{-1} &\simeq g^{-1} \circ f^{-1},\\
\mathrm{R}\Gamma(U; F) &\simeq \mathrm{RHom}\,(\mathbf{k}_{XU}, F).
\end{aligned}
$$

Notation 3.1.2. In the literature, one often encounters the following notations:

$$
\begin{aligned}
&\mathrm{Ext}^j(F, G) = H^j\mathrm{RHom}\,(F, G),\\
&\mathcal{E}xt^j(F, G) = H^j R\mathcal{H}om\,(F, G),\\
&H^j(X; F) = H^j(\mathrm{R}\Gamma(X; F)),\\
&\mathrm{Tor}_j(F, G) = H^{-j}(F \overset{\mathrm{L}}{\otimes} G).
\end{aligned}
$$

3.2 Čech complexes

Let I_{ord} be a total ordered set and denote by I the underlying set. For $J \subset I$, J finite, we denote by $|J|$ its cardinal. If J is endowed with the order induced by I, we write $J \subset I_{\mathrm{ord}}$.

Denote by $\{e_i\}_{i\in I}$ the canonical basis of $\mathbb{Z}^{\oplus I}$. For $J \subset I_{\mathrm{ord}}$, $J = \{i_0 < i_1 < \cdots < i_p\} \subset I_{\mathrm{ord}}$, we denote by e_J the element $e_{i_0} \wedge \cdots \wedge e_{i_p}$ of $\bigwedge^{p+1} \mathbb{Z}^{\oplus I}$. For σ a permutation of the set J with signature ε_σ, we have in $\bigwedge^{p+1} \mathbb{Z}^{\oplus I}$:

$$e_{\sigma(J)} = \varepsilon_\sigma e_J.$$

Consider a family $\mathcal{U} := \{U_i\}_{i\in I}$ of objects of $\mathcal{C}_X$. For $\emptyset \neq J \subset I$, J finite, set

$$U_J = \prod_{i\in J} U_i.$$

Now let $F \in \mathrm{Mod}(\mathbf{k}_X)$. For J as above and $i \in J$, we denote by $\beta_{(i,J)}\colon F_{U_J} \to F_{U_{J\setminus\{i\}}}$ the natural morphism. We set for $p \geq 0$

$$F_p^{\mathcal{U}} := \bigoplus_{J\subset I_{\mathrm{ord}}, |J|=p+1} F_{U_J} \otimes e_J$$

and we define the differential

$$d\colon F_p^{\mathcal{U}} \to F_{p-1}^{\mathcal{U}} \tag{3.1}$$

by setting for $s_J \in F_{U_J}$:

$$d(s_J \otimes e_J) = \sum_{i\in J} \beta_{(i,J)}(s_J) \otimes e_i \lfloor e_J.$$

One easily checks that

$$d \circ d = 0.$$

Then we have the Čech complex in $\mathrm{Mod}(\mathbf{k}_X)$ in which the term $F_p^{\mathcal{U}}$ is in degree $-p$.

$$F_\bullet^{\mathcal{U}} := \cdots \to F_p^{\mathcal{U}} \xrightarrow{d} \cdots \xrightarrow{d} F_1^{\mathcal{U}} \xrightarrow{d} F_0^{\mathcal{U}} \to 0. \tag{3.2}$$

We also consider the augmented complex

$$F_{\bullet,+}^{\mathcal{U}} := \quad \cdots \to F_p^{\mathcal{U}} \xrightarrow{d} \cdots \xrightarrow{d} F_1^{\mathcal{U}} \xrightarrow{d} F_{\mathcal{U}}^0 \to F \to 0. \tag{3.3}$$

Proposition 3.2.1. *Assume that $\mathcal{U}$ is a covering of X. Then for each $i_0 \in I$, the restriction to U_{i_0} of the complex* (3.3) *is homotopic to zero. Equivalently, $F_\bullet^{\mathcal{U}} \to F$ is a qis.*

Proof. Replacing all U_i by $U_i \times_X U_{i_0}$, we may assume from the beginning that $X = U_{i_0}$. For each finite set $J \subset I$, the two morphisms $U_J \to U_J$ and $U_J \to U_{i_0}$ define the "diagonal" morphism $U_J \to U_J \times U_{i_0}$. Let $\gamma_J \colon F_{U_J} \to F_{U_{\{i_0\} \cup J}}$ denote the morphism associated with this diagonal morphism. Define the homotopy $\lambda : F_p^{\mathcal{U}} \to F_{p+1}^{\mathcal{U}}$ by setting for $s_J \in F_{U_J}$,

$$\lambda(s_J \otimes e_J) = \gamma_J(s_J) \otimes e_{i_0} \wedge e_J.$$

Let us check the relation $d \circ \lambda + \lambda \circ d = \mathrm{id}$. Let $s_J \otimes e_J$ be a section of F_{U_J}. Then

$$\begin{aligned} d \circ \lambda(s_J \otimes e_J) &= \sum_{i \in i_0 \wedge J} \beta_{(i, \{i_0\} \cup J)} \gamma_J(s_J) \otimes e_i \lfloor e_{i_0} \wedge e_J, \\ \lambda \circ d(s_J \otimes e_J) &= \sum_{i \in J} \gamma_{i,J} \beta_{(i,J)}(s_J) \otimes e_{i_0} \wedge e_i \lfloor e_J. \end{aligned}$$

By summing these two relations, we find on the right hand side $\beta_{(i_0,J)} \gamma_J(s_J) \otimes e_{i_0} \lfloor e_{i_0} \wedge e_J = s_J \cdot e_J$. q.e.d.

Example 3.2.2. Let $\{U_j\}_{j=0,1,2}$ be a covering of X. We get the exact complex:

$$0 \to F_{U_{012}} \xrightarrow{d_1} F_{U_{12}} \oplus F_{U_{02}} \oplus F_{U_{01}} \xrightarrow{d_0} F_{U_0} \oplus F_{U_1} \oplus F_{U_2} \xrightarrow{d_{-1}} F \to 0$$

with for example, $d_1(s_{012}) = \beta_{0,12}(s_{012}) - \beta_{1,02}(s_{012}) + \beta_{2,01}(s_{012})$.

Applying Proposition 3.2.1 with $F = \mathbf{k}_X$, we find the complex

$$\mathbf{k}_{X\bullet}^{\mathcal{U}} := \cdots \to (\mathbf{k}_X^{\mathcal{U}})_1 \xrightarrow{d} (\mathbf{k}_X^{\mathcal{U}})_0 \to 0 \tag{3.4}$$

and we have the isomorphism

$$F_\bullet^{\mathcal{U}} \simeq \mathbf{k}_{X\bullet}^{\mathcal{U}} \otimes F.$$

It is then natural to consider the complex of sheaves

$$\mathcal{C}^\bullet(\mathcal{U}, F) := \mathcal{H}om\,(\mathbf{k}_{X\bullet}^{\mathcal{U}}, F) \tag{3.5}$$

and the morphism

$$F \to \mathcal{C}^\bullet(\mathcal{U}, F), \tag{3.6}$$

that is, the complex

$$0 \to F \to \mathcal{C}^0(\mathcal{U}, F) \xrightarrow{d} \mathcal{C}^1(\mathcal{U}, F) \to \cdots ,$$

where

$$\mathcal{C}^p(\mathcal{U}, F) = \prod_{|J|=p+1} \Gamma_{U_J}(F) \otimes e_J \text{ with } \mathcal{C}^{-1}(\mathcal{U}, F) = F,$$

and the morphisms: $d^p \colon \mathcal{C}^p(\mathcal{U}, F) \to \mathcal{C}^{p+1}(\mathcal{U}, F)$ $(p \geq -1)$ are defined by

$$d^p(s_J \otimes e_J) = \sum_{i \in I} \alpha_{(i,J)}(s_J) \otimes e_i \wedge e_J,$$

where $\alpha_{(i,J)}$ is the natural morphism $\Gamma_{U_J}(F) \to \Gamma_{U_{J\cup\{i\}}}(F)$. Moreover, $d^{-1} \colon F \to \prod_i \Gamma_{U_i}(F)$ is the natural morphism.

Proposition 3.2.3. *Assume that $\mathcal{U}$ is a covering of X. The sequence of sheaves* (3.6) *is exact.*

Proof. It is enough to check that this sequence is exact on each $U \in \mathcal{U}$. The additive functor $\mathcal{H}om\,(\cdot, F)$ sends a complex homotopic to zero to a complex homotopic to zero. Applying this functor to the complex (3.3) in which one chooses $F = \mathbf{k}_X$, the result follows from Proposition 3.2.1. q.e.d.

Theorem 3.2.4. (The Leray's acyclic covering theorem.) *Assume that $\mathcal{U}$ is a covering of X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X and assume that for any finite subset J of I, and any $p > 0$, one has $H^p(U_J; F) = 0$. Then* $\mathrm{R}\Gamma(X; \mathcal{C}^\bullet(\mathcal{U}, F)) \simeq \mathrm{R}\Gamma(X; F)$.

Proof. Let $F^\bullet$ be an injective resolution of F and consider the double complex:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X; F) & \longrightarrow & \Gamma(X; \mathcal{C}^\bullet(\mathcal{U}, F)) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X; F^\bullet) & \longrightarrow & \Gamma(X; \mathcal{C}^\bullet(\mathcal{U}, F^\bullet))
\end{array}$$

(i) All rows, except the first one, are exact. In fact, they are obtained by applying the functor $\Gamma(X; \cdot)$ to the exact complex of injective sheaves $0 \to F^j \to \mathcal{C}^\bullet(\mathcal{U}, F^j)$.
(ii) All columns, except the first one, are exact. In fact, it is enough to prove that for $J \subset I$, J finite, the complex $0 \to \Gamma(X; \Gamma_{U_J}(F)) \to \Gamma(X; \Gamma_{U_J}(F^\bullet))$ is

exact. This complex is isomorphic to the complex $0 \to \Gamma(U_J; F) \to \Gamma(U_J; F^\bullet)$ which is exact by the hypothesis.
(iii) From (i) and (ii), one deduces that the cohomology of the first row is isomorphic to that of the first column. q.e.d.

Example 3.2.5. Let $\{U_j\}_{j=0,1,2}$ be an open covering of X, and assume that $H^p(U_J; F) = 0$ for all $p > 0$ and all $J \subset \{0, 1, 2\}$. Then $H^j(X; F)$ is isomorphic to the j-th cohomology group of the complex

$$0 \to \bigoplus_{j=0,1,2} \Gamma(U_j; F) \xrightarrow{d^0} \bigoplus_{J=01,12,02} \Gamma(U_J; F) \xrightarrow{d^1} \Gamma(U_{012}; F) \to 0$$

where the d^j's are linear combinations of the restriction morphisms affected with the sign $\pm$. For example, $d^1|_{\Gamma(U_{02};F)}$ is affected with the sign $-$.

3.3 Ringed sites

It is possible to generalise the preceding constructions by replacing the constant sheaf $\mathbf{k}_X$ with a sheaf of rings $\mathcal{R}$. We shall only present here the main ideas of this theory, skipping the details.

Sheaves of rings and modules

A sheaf of $\mathbf{k}$-algebras (or, equivalently, a $\mathbf{k}_X$-algebra) $\mathcal{R}$ on a site X is a sheaf of $\mathbf{k}$-modules such that for each $U \in \mathcal{C}_X$, $\mathcal{R}(U)$ is endowed with a structure of a $\mathbf{k}$-algebra and the operations (addition, multiplication) commute to the restriction morphisms. A sheaf of $\mathbb{Z}$-algebras is simply called a sheaf of rings. If $\mathcal{R}$ is a sheaf of rings, one defines in an obvious way the notion of a sheaf F of (left) $\mathcal{R}$-modules (or simply, an $\mathcal{R}$-module) as follows: for each $U \in \mathcal{C}_X$, $F(U)$ is an $\mathcal{R}(U)$-module and the action of $\mathcal{R}(U)$ on $F(U)$ commutes to the restriction morphisms. One also naturally defines the notion of an $\mathcal{R}$-linear morphism of $\mathcal{R}$-modules. Hence we have defined the category $\mathrm{Mod}(\mathcal{R})$ of $\mathcal{R}$-modules. If $\mathcal{R}$ is a sheaf of rings, $\mathcal{R}^{\mathrm{op}}$ is the sheaf of rings $U \mapsto \mathcal{R}(U)^{\mathrm{op}}$.

Examples 3.3.1. (i) Let R be a $\mathbf{k}$-algebra. The constant sheaf R_X is a sheaf of $\mathbf{k}$-algebras. In particular, $\mathbf{k}_X$ is a sheaf of $\mathbf{k}$-algebras.
(ii) On a topological space, the sheaf $\mathcal{C}^0_X$ is a $\mathbb{C}_X$-algebra. If X is a real differentiable manifold, the sheaf $\mathcal{C}^\infty_X$ is a $\mathbb{C}_X$-algebra. The sheaf $\mathcal{D}b_X$ is a $\mathcal{C}^\infty_X$-module.
(iii) If X is complex manifold, the sheaves $\mathcal{O}_X$ and $\mathcal{D}_X$ are $\mathbb{C}_X$-algebras and $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

One proves easily the analogue of Theorem 2.4.2 for the category $\mathrm{Mod}(\mathcal{R})$.

Definition 3.3.2. Let $\mathcal{R}$ be a sheaf of $\mathbf{k}$-algebras on the site X and let F be a sheaf of $\mathcal{R}$-modules on X.

(i) F is injective (resp. projective) if F is an injective (resp. a projective) object in the category $\mathrm{Mod}(\mathcal{R})$.

(ii) F is flat if the functor $\bullet \otimes_{\mathcal{R}} F$ is exact,

Remark that, if X is a topological space, F is flat if and only if F_x is a $\mathcal{R}_x$-flat module for any $x \in X$.

Proposition 3.3.3. *Let $\mathcal{R}_Y$ be a sheaf of $\mathbf{k}$-algebras on Y. Let $f\colon X \to Y$ be a morphism of sites and assume that the functor $f^{-1}\colon \mathrm{Mod}(\mathcal{R}_Y) \to \mathrm{Mod}(f^{-1}\mathcal{R}_Y)$ is exact. Let $F \in \mathrm{Mod}(f^{-1}\mathcal{R}_Y)$. Then f_*F is injective in $\mathrm{Mod}(\mathcal{R}_Y)$.*

Proof. This follows immediately from the adjunction formula in Theorem 2.6.3 and the hypothesis that the functor f^{-1} is exact. q.e.d.

Theorem 3.3.4. *Let $\mathcal{R}$ be a sheaf of $\mathbf{k}$-algebras on the site X. The category $\mathrm{Mod}(\mathcal{R})$ is a Grothendieck category. In particular, it is abelian, it admits small projective limits and small inductive limits, small filtrant inductive limits are exact and $\mathrm{Mod}(\mathcal{R})$ has enough injective objects.*

The proof goes as for Theorem **??**

Proposition 3.3.5. *Let $\mathcal{R}$ be a sheaf of $\mathbf{k}$-algebras on the site X and let $G \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$. Then the category of flat $\mathcal{R}$-modules is projective with respect to the functor $G \otimes \bullet$.*

The proof goes as for Proposition 2.5.4.

Ringed sites

Definition 3.3.6. (i) A $\mathbf{k}$-ringed site $(X, \mathcal{O}_X)$ is a site X endowed with a sheaf of commutative $\mathbf{k}$-algebras $\mathcal{O}_X$ on X. (If there is no risk of confusion, we shall omit to mention $\mathbf{k}$.)

(ii) Let $(X, \mathcal{O}_X)$ be a ringed site. A locally free $\mathcal{O}_X$-module $\mathcal{M}$ of rank m is an $\mathcal{O}_X$-module such that there is a covering $\mathcal{S}$ of X and for each $V \in \mathcal{S}$, $\mathcal{O}_X$-linear isomorphisms $\mathcal{M}|_V \xrightarrow{\sim} (\mathcal{O}_X|_V)^m$.

(iii) If $m = 1$, one says that $\mathcal{M}$ is a line bundle.

(iv) If there exists a globally defined isomorphism $\mathcal{M} \xrightarrow{\sim} (\mathcal{O}_X)^m$ on X, one says that $\mathcal{M}$ is globally free.

The next result is obvious.

Proposition 3.3.7. *Let $\mathcal{R}$ be a sheaf of* **k***-algebras on the site X. The forgetful functor* for: $\mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(\mathbf{k}_X)$ *is faithful and exact.*

One shall be aware that this functor is not fully faithful in general

Glueing sheaves on ringed sites

Let $\mathcal{R}$ be a sheaf of **k**-algebras on X and consider the situation of Theorem 2.9.1. If all F_i's are sheaves of $\mathcal{R}|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathcal{R}|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 2.9.1 will be naturally endowed with a structure of a sheaf of $\mathcal{R}$-modules.

Definition 3.3.8. Let $(X, \mathcal{O}_X)$ be a ringed site. Denote by $\mathcal{O}_X^\times$ the abelian sheaf of invertible sections of $\mathcal{O}_X$. A 1-cocycle $(\mathbf{c}, \mathcal{S})$ on X with values in $\mathcal{O}_X^\times$ is the data of a covering $\mathcal{S} = \{U_i\}_i$ of X and for each pair $U_i, U_j \in \mathcal{S}$ a section $c_{ij} \in \Gamma(U_{ij}; \mathcal{O}_X^\times)$, these data satisfying:

$$c_{ij} \cdot c_{jk} = c_{ik} \text{ on } U_{ijk}. \tag{3.7}$$

Applying Theorem 2.9.1, we get:

Corollary 3.3.9. *Consider a* 1*-cocycle* $(\mathbf{c}, \mathcal{S})$ *on X with values in $\mathcal{O}_X^\times$. There exists a unique locally free sheaf $\mathcal{L}_\mathbf{c}$ of rank one with the following property: for each $U_i \in \mathcal{S}$ there exists an isomorphism $\theta_i \colon \mathcal{O}_X|_{U_i} \xrightarrow{\sim} \mathcal{L}_\mathbf{c}|_{U_i}$ and $(\theta_j)^{-1} \circ \theta_i = c_{ij}$ on U_{ij} for any $U_i, U_j \in \mathcal{S}$.*

Example 3.3.10. Recall that $\mathbf{k}^\times$ denote the multiplicative group of invertible elements of **k**. Let $X = \mathbb{S}^1$ be the 1-sphere, and consider a covering of X by two open connected intervals U_1 and U_2. Let $U_{12}^\pm$ denote the two connected components of $U_1 \cap U_2$. Let $\alpha \in \mathbf{k}^\times$. One defines a locally constant sheaf L_α on X of rank one over **k** by glueing $\mathbf{k}_{U_1}$ and $\mathbf{k}_{U_2}$ as follows. Let $\theta_\varepsilon : \mathbf{k}_{U_1}|_{U_{12}^\varepsilon} \to \mathbf{k}_{U_2}|_{U_{12}^\varepsilon}$ $(\varepsilon = \pm)$ be defined by $\theta_+ = 1$, $\theta_- = \alpha$.

If $\mathbf{k} = \mathbb{C}$ there is a more intuitive description of the sheaf L_α. Let us identify $\mathbb{S}^1$ with $[0, 2\pi]/\sim$, where $\sim$ is the relation which identifies 0 and 2π and let t denotes the coordinate. Choose $\beta \in \mathbb{C}$ such that $\exp(i\beta) = \alpha$. Then $L_\alpha \simeq \mathbb{C}_X \cdot exp(i\beta t)$.

Exercises to Chapter 3

Exercise 3.1. Let $Z = \{(x, y) \in \mathbb{R}^2; xy > 1, x > 0\}$ and let $f : Z \to \mathbb{R}$ be the map $(x, y) \mapsto xy$. Calculate $Rf_*\mathbf{k}_Z$.

Exercise 3.2. Let $Z = \{(x, y) \in \mathbb{R}^2; xy \geq 1\}$, and let $f : X \to \mathbb{R}$ be the map $(x, y) \mapsto y$. calculate $Rf_*\mathbf{k}_Z$.

Exercise 3.3. Let $X = \mathbb{R}^4$, $S = \{(x, y, z, t) \in \mathbb{R}^4; t^4 = x^2 + y^2 + z^2; t > 0\}$ and let $f \colon S \hookrightarrow X$ be the natural injection. Calculate $(Rf_*\mathbf{k}_S)_0$.

Exercise 3.4. In this exercise, we shall admit the following theorem: for any open subset U of the complex line $\mathbb{C}$, one has $H^j(U; \mathcal{O}_\mathbb{C}) \simeq 0$ for $j > 0$.

Let ω be an open subset of $\mathbb{R}$, and let $U_1 \subset U_2$ be two open subsets of $\mathbb{C}$ containing ω as a closed subset.
(i) Prove that the natural map $\mathcal{O}(U_2 \setminus \omega)/\mathcal{O}(U_2) \to \mathcal{O}(U_1 \setminus \omega)/\mathcal{O}(U_1)$ is an isomorphism. One denote by $\mathcal{B}(\omega)$ this quotient.
(ii) Construct the restriction morphism to get the presheaf $\omega \to \mathcal{B}(\omega)$, and prove that this presheaf is a sheaf (the sheaf $\mathcal{B}_\mathbb{R}$ of Sato's hyperfunctions on $\mathbb{R}$).
(iii) Prove that the restriction morphisms $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\omega)$ are surjective (i.e. the sheaf $\mathcal{B}_\mathbb{R}$ is flabby).
(iv) Let K be a compact subset of $\mathbb{R}$ and let U be an open subset of $\mathbb{C}$ containing K. Prove the isomorphism $\Gamma_K(\mathbb{R}; \mathcal{B}_\mathbb{R}) \simeq \mathcal{O}(U \setminus K)/\mathcal{O}(U)$.
(v) Let Ω an open subset of $\mathbb{C}$ and let $P = \sum_{j=1}^{m} a_j(z) \frac{\partial}{\partial z}^j$ be a holomorphic differential operator (the coefficients are holomorphic in Ω). Recall the Cauchy theorem which asserts that if Ω is simply connected and if $a_m(z)$ does not vanish on Ω, then P acting on $\mathcal{O}(\Omega)$ is surjective. Prove that if ω is an open subset of $\mathbb{R}$ and if P is a holomorphic differential operator defined in a open neighborhood of ω, then P acting on $\mathcal{B}(\omega)$ is surjective

Exercise 3.5. Let $\mathcal{R}$ be a sheaf of commutative rings on a topological space X. Prove that a sheaf F of $\mathcal{R}$-modules is injective in the category $\mathrm{Mod}(\mathcal{R})$ if and only if, for any sheaf of ideals $\mathcal{I}$ of $\mathcal{R}$, the natural morphism $\Gamma(X; F) \to \mathrm{Hom}_\mathcal{R}(\mathcal{I}, F)$ is surjective.

Chapter 4

Sheaves on topological spaces

In this chapter we restrict our study of sheaves of $\mathbf{k}$-modules to the case of topological spaces. If A is a subset of a topological space X, we denote by $\overline{A}$ its closure, by $\mathrm{Int}A$ its interior and we set $\partial A = \overline{A} \setminus \mathrm{Int}A$.

4.1 Restriction of sheaves

Let Z be a subset of X, $i_Z\colon Z \hookrightarrow X$ the inclusion. One endows Z with the induced topology and for $F \in \mathrm{Mod}(\mathbf{k}_X)$, one sets:

$$\begin{aligned} F|_Z &= i_Z^{-1}F, \\ \Gamma(Z;F) &= \Gamma(Z;i_Z^{-1}F). \end{aligned}$$

If Z is open, these definitions agree with the previous ones. The morphism $F \to i_{Z*}i_Z^{-1}F$ defines the morphism $a_{X*}F \to a_{Z*}i_Z^{-1}F$, that is the morphism:

$$\Gamma(X;F) \to \Gamma(Z;F).$$

One denotes by $s|_Z$ the image of a section s of F on X by this morphism.

Replacing X by an open subset U containing Z, we get the natural morphism:

$$\varinjlim_{U \supset Z} \Gamma(U;F) \to \Gamma(Z;F). \tag{4.1}$$

This morphism is injective, since if a section $s \in \Gamma(U;F)$ is zero in $\Gamma(Z;F)$, this implies $s_x = 0$ for all $x \in Z$, hence $s = 0$ on an open neighborhood of Z. But one shall take care that this morphism is not an isomorphism in general. This is true in some particular situations (see Proposition 4.1.2).

Definition 4.1.1. (i) A subset Z of a topological space X is relatively Hausdorff if two distinct points in Z admit disjoint neighborhoods in X. If $Z = X$, one says that X is Hausdorff.

(ii) A paracompact space X is a Hausdorff space such that for each open covering $\{U_i\}_{i\in I}$ of X there exists an open refinement $\{V_j\}_{j\in J}$ (*i.e.*, for each $j \in J$ there exists $i \in I$ such that $V_j \subset U_i$) which is locally finite.

Recall that, by its definition, a compact set is in particular Hausdorff.

If X is paracompact and $\{U_i\}_i$ is a locally finite open covering, there exists an open refinement $\{V_i\}_i$ such that $\overline{V_i} \subset U_i$. Closed subspaces of paracompact spaces are paracompact. Locally compact spaces countable at infinity (*i.e.*, countable union of compact subspaces), are paracompact.

Proposition 4.1.2. *Assume one of the following conditions:*

(i) *Z is open,*

(ii) *Z is a relatively Hausdorff compact subset of X,*

(iii) *Z is closed and X is paracompact.*

Then the morphism (4.1) *is an isomorphism.*

Proof. (i) is obvious.
(ii) Let $s \in \Gamma(K; F)$. There exist a finite family of open subsets $\{U_i\}_{i=1}^n$ covering K and sections $s_i \in \Gamma(U_i; F)$ such that $s_i|_{K\cap U_i} = s|_{K\cap U_i}$. Moreover, we may find another family of open sets $\{V_i\}_{i=1}^n$ covering K such that $K\cap\bar{V}_i \subset U_i$. We shall glue together the sections s_i on a neighnorhood of K. For that purpose we may argue by induction on n and assume $n = 2$. Set $K_i = K \cap \bar{V}_i$. Then $s_1|_{K_1\cap K_2} = s_2|_{K_1\cap K_2}$. Let W be an open subset of X such that $s_1|_W = s_2|_W$ and let $W_i (i = 1, 2)$ be an open subset of U_i such that $W_i \supset K_i \setminus W$ and $W_1 \cap W_2 = \emptyset$. Such W_i's exist thanks to the hypotheses. Set ${U'}_i = W_i \cup W$, $(i = 1, 2)$. Then $s_1|_{{U'}_1\cap {U'}_2} = s_2|_{{U'}_1\cap {U'}_2}$. This defines $t \in \Gamma({U'}_1 \cup {U'}_2; F)$ with $t|_K = s$.
(iii) We shall not give the proof here and refer to [Go58]. q.e.d.

Let $f : X \to Y$ be a continuous map and let F be a sheaf on X. Let $y \in Y$. The natural morphism $\varinjlim_{V\ni y} \Gamma(f^{-1}V; F) \to \Gamma(f^{-1}(y); F|_{f^{-1}(y)})$ defines the morphism:

$$(f_*F)_y \to \Gamma(f^{-1}(y); F|_{f^{-1}(y)}). \tag{4.2}$$

This morphism is not an isomorphism in general.

Examples 4.1.3. (i) Assume f is an open inclusion $U \hookrightarrow X$ and choose $x \in \partial U$ $(= \bar{U} \setminus U)$. Then $f^{-1}(x) = \emptyset$ and $\Gamma(f^{-1}(x); G|_{f^{-1}(x)}) = 0$ but

$$(f_*G)_x = \varinjlim_V \Gamma(U \cap V; G),$$

where V ranges through the family of open neighborhoods of x in X, and this group is not zero in general.

(ii) Let $X = \mathbb{C}$ with coordinate $z = x + iy$, $Y = \mathbb{R}$, $f : X \to \mathbb{R}$ the map $f(x + iy) = y$. Then

$$(f_*\mathcal{O}_{\mathbb{C}})_0 = \varinjlim_{\varepsilon > 0} \Gamma(\{|y| < \varepsilon\}; \mathcal{O}_{\mathbb{C}})$$

and

$$\Gamma(f^{-1}(0); \mathcal{O}_{\mathbb{C}}|_{f^{-1}(0)}) = \varinjlim_U \Gamma(U; \mathcal{O}_{\mathbb{C}})$$

where U ranges through the (non countable) family of open neighborhoods of $\mathbb{R}$ in $\mathbb{C}$.

4.2 Sheaves associated with a locally closed subset

Let X be a topological space, U an open subset of X and $F \in \mathrm{Mod}(\mathbf{k}_X)$. Recall that $F_U = \mathrm{j}_U^{-1} \mathrm{j}_{U*} F \simeq i_{U!} i_U^{-1} F$.

Propositions 4.2.1, 4.2.3 and 4.2.4 below are easy exercises whose proof is left to the reader. Note that the result of Proposition 4.2.1 (i) and (ii) has already been proved in the more general setting of sheaves on sites.

Proposition 4.2.1. (i) *The functor* $(\bullet)_U \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$, $F \mapsto F_U$, *is exact and commutes with inductive limits.*

(ii) *One has* $F_U \simeq F \otimes \mathbf{k}_{XU}$.

(iii) *For* $x \in X$, $(F_U)_x \simeq F_x$ *or* $(F_U)_x \simeq 0$ *according whether* $x \in U$ *or not.*

(iv) *Let* U' *be another open subset. Then* $(F_U)_{U'} = F_{U \cap U'}$.

(v) *Let* U_1 *and* U_2 *be two open subsets of* X. *Then there is an exact sequence*

$$0 \to F_{U_1 \cap U_2} \xrightarrow{\alpha} F_{U_1} \oplus F_{U_2} \xrightarrow{\beta} F_{U_1 \cup U_2} \to 0. \tag{4.3}$$

Here $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = \beta_1 - \beta_2$ *are induced by the natural morphisms* $\alpha_i : F_{U_1 \cap U_2} \to F_{U_i}$ *and* $\beta_i : F_{U_i} \to F_{U_1 \cup U_2}$.

Now set $S := X \setminus U$. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, define the sheaf F_S by

$$F_S = i_{S*} i_S^{-1} F. \tag{4.4}$$

Notation 4.2.2. For a closed set $S \subset X$ one sets $\mathbf{k}_{XS} := (\mathbf{k}_X)_S$. If there is no risk of confusion, we also write $\mathbf{k}_S$ instead of $\mathbf{k}_{XS}$. This last notation is justified by Remark 4.2.5 bellow.

Proposition 4.2.3. (i) *There is a natural exact sequence* $0 \to F_U \to F \to F_S \to 0$*.*

(ii) *The functor* $(\bullet)_S \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$*,* $F \mapsto F_S$*, is exact.*

(iii) *One has* $F_S \simeq F \otimes \mathbf{k}_{XS}$*, where one sets* $\mathbf{k}_{XS} := (\mathbf{k}_X)_S$ *for short.*

(iv) *For* $x \in X$*,* $(F_S)_x \simeq F_x$ *or* $(F_S)_x \simeq 0$ *according whether* $x \in S$ *or not.*

(v) *Let* S' *be another closed subset. Then* $(F_S)_{S'} = F_{S \cap S'}$*.*

(vi) *Let* S_1 *and* S_2 *be two closed subsets of* X*. Then the sequence below is exact:*

$$0 \to F_{S_1 \cup S_2} \xrightarrow{\alpha} F_{S_1} \oplus F_{S_2} \xrightarrow{\beta} F_{S_1 \cap S_2} \to 0. \tag{4.5}$$

Here $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = \beta_1 - \beta_2$ *are induced by the natural morphisms* $\alpha_i : F_{S_1 \cup S_2} \to F_{S_i}$ *and* $\beta_i : F_{S_i} \to F_{S_1 \cap S_2}$*.*

(vii) *Setting* $\Gamma_S(F) = \mathcal{H}om\,(\mathbf{k}_{XS}, F)$ *and* $\Gamma_S(X; F) = \mathrm{Hom}\,(\mathbf{k}_{XS}, F)$*, one has* $\Gamma_S(X; F) \simeq \Gamma(X; \Gamma_S(F))$ *and*

$$\Gamma_S(X; F) = \{s \in \Gamma(X; F); \mathrm{supp}(s) \text{ is contained in } S\}.$$

Recall that a locally closed set Z is the (non unique) intersection of an open subset U and a closed subset S of X. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, one sets

$$F_Z := (F_U)_S. \tag{4.6}$$

Proposition 4.2.4. (i) *The functor* $(\bullet)_Z \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$*,* $F \mapsto F_Z$*, is well defined and satisfies the properties* (ii)–(v) *of* Proposition 4.2.3 (*with* S *replaced by* Z)*.*

(ii) *Let* Z *be as above and let* Z' *be a closed subset of* Z*. One has an exact sequence*

$$0 \to F_{Z \setminus Z'} \to F_Z \to F_{Z'} \to 0. \tag{4.7}$$

(iii) *Setting* $\Gamma_Z(F) = \mathcal{H}om\,(\mathbf{k}_{XZ}, F)$ *and* $\Gamma_Z(X;F) = \mathrm{Hom}\,(\mathbf{k}_{XZ}, F)$*, one has for* $Z = U \cap S$*:*

$$\Gamma_Z(X;F) = \{s \in \Gamma(U;F); \mathrm{supp}(s) \textit{ is contained in } Z\}.$$

Let U_1, U_2 be open subsets, S_1, S_2 closed subsets, Z a locally closed of X, Z' a closed subset of Z. Consider the exact sequences (4.3), (4.5) and (4.7). They give rise to distinguished triangles in the category $\mathrm{D}^+(\mathbf{k}_X)$:

$$\begin{aligned}
&F_{U_1\cap U_2} \to F_{U_1} \oplus F_{U_2} \to F_{U_1\cup U_2} \xrightarrow{+1},\\
&F_{S_1\cup S_2} \to F_{S_1} \oplus F_{S_2} \to F_{S_1\cap S_2} \xrightarrow{+1},\\
&F_{Z\setminus Z'} \to F_Z \to F_{Z'} \xrightarrow{+1} .
\end{aligned}$$

Choosing $F = \mathbf{k}_X$ and applying the functor $R\mathcal{H}om\,(\,\bullet\,, F)$, we get new distinguished triangles, called Mayer-Vietoris triangles :

$$\mathrm{R}\Gamma_{U_1\cup U_2}F \to \mathrm{R}\Gamma_{U_1}F \oplus \mathrm{R}\Gamma_{U_2}F \to \mathrm{R}\Gamma_{U_1\cap U_2}F \xrightarrow{+1}, \tag{4.8}$$

$$\mathrm{R}\Gamma_{S_1\cap S_2}F \to \mathrm{R}\Gamma_{S_1}F \oplus \mathrm{R}\Gamma_{S_2}F \to \mathrm{R}\Gamma_{S_1\cup S_2}F \xrightarrow{+1}, \tag{4.9}$$

$$\mathrm{R}\Gamma_{Z'}(F) \to \mathrm{R}\Gamma_Z(F) \to \mathrm{R}\Gamma_{Z\setminus Z'}(F) \xrightarrow{+1} . \tag{4.10}$$

When applying $\mathrm{R}\Gamma(X;\,\bullet\,)$, we find other distinguished triangles and taking the cohomology, we find long exact sequences, such as for example the Mayer-Vietoris long exact sequences :

$$\begin{aligned}
\cdots \to H^j(U_1 \cup U_2; F) &\to H^j(U_1;F) \oplus H^j(U_2;F)\\
&\to H^j(U_1 \cap U_2; F) \to H^{j+1}(U_1 \cup U_2; F) \to \cdots
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\cdots \to H^j(X; F_{S_1\cup S_2}) &\to H^j(X;F_{S_1}) \oplus H^j(X;F_{S_2})\\
&\to H^j(X; F_{S_1\cap S_2}) \to H^{j+1}(X; F_{S_1\cup S_2}) \to \cdots
\end{aligned} \tag{4.12}$$

Remark 4.2.5. Let S be a closed subset of X. Then

$$\mathrm{R}\Gamma(X;F_S) \simeq \mathrm{R}\Gamma(S;F|_S). \tag{4.13}$$

This follows from the isomorphism $F_S \simeq i_{S*}i_S^{-1}F$, the fact that i_{S*} is exact and the isomorphism $Ra_{X*} \circ i_{S*} \simeq R(a_X \circ i_S)_*$. Note that (4.13) would not remain true when replacing S with an open subset.

When Z is locally closed in X, one also sets

$$H^j_Z(F) = H^j(\mathrm{R}\Gamma_Z(\,\bullet\,))(F), \quad H^j_Z(X;F) = H^j(\mathrm{R}\Gamma_Z(X;\,\bullet\,))(F). \tag{4.14}$$

4.3 Čech complexes for closed coverings

We shall adapt the construction of §3.2 to the case of closed coverings.

As in § 3.2, we consider a total ordered set $I_{\rm ord}$ and denote by I the underlying set. For $J \subset I$, J finite, we denote by $|J|$ its cardinal. If J is endowed with the order induced by I, we write $J \subset I_{\rm ord}$. We denote by $\{e_i\}_{i\in I}$ the canonical basis of $\mathbb{Z}^{\oplus I}$ and for $J \subset I_{\rm ord}$, $J = \{i_0 < i_1 < \cdots < i_p\} \subset I_{\rm ord}$, we denote by e_J the element $e_{i_0} \wedge \cdots \wedge e_{i_p}$ of $\bigwedge^{p+1} \mathbb{Z}^{\oplus I}$.

Let $\mathcal{S} = \{S_i\}_{i\in I}$ be a family of closed subsets of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. For $J \subset I$ we set

$$S_J := \bigcap_{j\in J} S_j, \quad S_{-1} = \bigcup_{i\in I} S_i,$$
$$F^p_{\mathcal{S}} := \oplus_{|J|=p+1} F_{S_J}, \quad F^{-1}_{\mathcal{S}} = F_S.$$

For $J \subset I$ and $i \in I$, we denote by $\alpha_{(i,J)}\colon F_{S_J} \to F_{S_{J\cup\{i\}}}$ the natural restriction morphism. We set for $p \geq 0$

$$F^p_{\mathcal{S}} := \bigoplus_{J\subset I_{\rm ord}, |J|=p+1} F_{S_J} \otimes e_J$$

and we define the differential

$$d\colon F^p_{\mathcal{S}} \to F^{p+1}_{\mathcal{S}} \tag{4.15}$$

by setting for $s_J \in F_{S_J}$:

$$d(s_J \otimes e_J) = \sum_{i\in I} \alpha_{(i,J)}(s_J) \otimes e_i \wedge e_J.$$

Here we consider $J \cup \{i\}$ as a subset of $I_{\rm ord}$. One easily checks that

$$d \circ d = 0$$

and we obtain a complex

$$F^\bullet_{\mathcal{S}} := 0 \to F^0_{\mathcal{S}} \xrightarrow{d^0} F^1_{\mathcal{S}} \xrightarrow{d^1} \cdots . \tag{4.16}$$

We also consider the augmented complex

$$F^{\bullet,+}_{\mathcal{S}} := 0 \to F_S \xrightarrow{d^{-1}} F^0_{\mathcal{S}} \xrightarrow{d^0} F^1_{\mathcal{S}} \xrightarrow{d^1} \cdots . \tag{4.17}$$

Proposition 4.3.1. *Consider a family $\mathcal{S} = \{S_i\}_{i\in I}$ of closed subsets of X. Then the complex* (4.17) *is exact. Equivalently, $F_S \to F^\bullet_{\mathcal{S}}$ is a qis.*

Proof. Let $x \in X$ and denote by $M_x^\bullet := (F_{\mathcal{S}}^{\bullet,+})_x$ the stalk of the complex (4.17) at x. It is enough to check that this complex is exact. Let $K = \{i \in I; x \in S_i\}$. Replacing I with K, we may assume from the beginning that $x \in S_i$ for all $i \in I$. In this case, $M_x^\bullet$ is the Koszul complex associated with the module $M = F_x$ and the family of morphisms $\{\varphi_i\}_{i\in I}$ with $\varphi_i = \mathrm{id}_M$ for all $i \in I$. This last complex is clearly exact. q.e.d.

Example 4.3.2. Assume that $X = S_0 \cup S_1 \cup S_2$, where the S_i's are closed subsets. We get the exact complex of sheaves

$$0 \to F \xrightarrow{d^{-1}} F_{S_0} \oplus F_{S_1} \oplus F_{S_2} \xrightarrow{d^0} F_{S_{12}} \oplus F_{S_{02}} \oplus F_{S_{01}} \xrightarrow{d^1} F_{S_{012}} \to 0.$$

Let us denote by

$$s_i \colon F \to F_{S_i}, \quad s_{ij}^a \colon F_{S_a} \to F_{S_{ij}}, \quad s_{\widehat{\mathbf{k}}} \colon F_{S_{ij}} \to F_{S_{012}}\, (a,i,j,k) \in \{0,1,2\}),$$

the natural morphisms. Then

$$d^{-1} = \begin{pmatrix} s_0, \\ s_1, \\ s_2 \end{pmatrix}, \quad d^0 = \begin{pmatrix} 0 & -s_{12}^1 & s_{12}^2 \\ -s_{02}^0 & 0 & s_{02}^2 \\ -s_{01}^0 & s_{01}^1 & 0 \end{pmatrix} \quad d^1 = (s_{\widehat{2}}, -s_{\widehat{0}}, s_{\widehat{1}}).$$

4.4 Flabby sheaves

Definition 4.4.1. On a topological space X, an object $F \in \mathrm{Mod}(\mathbf{k}_X)$ is flabby if for any open subset U of X the restriction map $\Gamma(X;F) \to \Gamma(U;F)$ is surjective.

By applying the functor $\mathrm{Hom}\,(\,\bullet\,, F)$ to the epimorphism $\mathbf{k}_X \twoheadrightarrow \mathbf{k}_{XU}$, one sees that injective sheaves are flabby. The converse is true if $\mathbf{k}$ is a field (see Exercise 3.5).

Proposition 4.4.2. *Let F be a flabby sheaf on X.*

(i) *If U is open in X, $F|_U$ is flabby on U,*

(ii) *if $f\colon X \to Y$ is a continuous map, f_*F is flabby on Y,*

(iii) *if Z be a locally closed subset of X, $\Gamma_Z(F)$ is flabby.*

Proof. (i)–(ii) are obvious.
(iii) Let U be an open subset containing Z as a closed subset. Since

$$\begin{aligned} \Gamma(V;\Gamma_Z(F)) &\simeq \Gamma(U \cap V; \Gamma_Z(F)) \\ &\simeq \Gamma_{Z\cap V}(U \cap V; F), \end{aligned}$$

we may assume from the beginning (replacing X by U) that Z is closed in X. Let V be an open subset, and let $s \in \Gamma_{Z\cap V}(V;F)$. First, we extend s by 0 on $X \setminus Z$, thus defining $s' \in \Gamma_Z(X \setminus (Z \setminus V); F)$. Then one extends s' using the flabbiness of F. q.e.d.

Proposition 4.4.3. *Let* $0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$ *be an exact sequence of sheaves, and assume* F' *is flabby. Then the sequence*

$$0 \to \Gamma(X;F') \xrightarrow{\alpha} \Gamma(X;F) \xrightarrow{\beta} \Gamma(X;F'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma(X;F'')$ and let $\sigma = \{(U;s);\, U \text{ open in } X,\ s \in \Gamma(U;F),\ \beta(s) = s''|_U\}$. Then σ is naturally inductively ordered. Let $(U;s)$ be a maximal element, and assume $U \neq X$.

Let $x \in X \setminus U$, let V be an open neighborhood of x and let $t \in \Gamma(U;F)$ such that $\beta(t) = s''|_V$. Such a pair $(V;t)$ exists since $\beta : F_x \to F''_x$ is surjective. On $U \cap V$, $s - t \in \Gamma(U \cap V; F')$. Let $r \in \Gamma(X;F')$ which extends $s-t$. Then $s-(t+r) = 0$ on $U\cap V$, hence there exists a section $\tilde{s} \in \Gamma(U \cup V; F)$ with $\tilde{s}|_U = s$, $\tilde{s}|_V = t + r$, and $\beta(\tilde{s}) = s''$. This is a contradiction. q.e.d.

Proposition 4.4.4. *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence of sheaves. Assume* F' *and* F *are flabby. Then* F'' *is flabby.*

Proof. Let U be an open subset of X and consider the diagram:

$$\begin{array}{ccccc}
\Gamma(X;F) & \longrightarrow & \Gamma(X;F'') & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \\
\Gamma(U;F) & \xrightarrow{\beta} & \Gamma(U;F'') & \longrightarrow & 0
\end{array}$$

Then α is surjective since F is flabby and β is surjective since F' is flabby, in view of the preceding proposition. This implies γ is surjective, hence F'' is flabby. q.e.d.

Theorem 4.4.5. *The category of flabby sheaves is injective with respect to the functors* $\Gamma(X;\cdot)$, $\Gamma_Z(\cdot)$, f_*.

Proof. Since the category of sheaves has enough injectives, and injective sheaves are flabby, the result for $\Gamma(X;\cdot)$ follows from Propositions 4.4.3 and 4.4.4, and the other functors are similarly treated. q.e.d.

Proposition 4.4.6. *Let* $X = \bigcup_{i\in I} U_i$ *be an open covering of* X *and let* $F \in \mathrm{Mod}(\mathbf{k}_X)$. *Assume that* $F|_{U_i}$ *is flabby for all* $i \in I$. *Then* F *is flabby.*

In other words, flabbyness is a local property.

Proof. Let U be an open subset of X and let $s \in F(U)$. Let us prove that s extends to a global section of F. Let $\mathfrak{S}$ be the family of pairs (t, V) such that V is open and contains U and $t|_U = s$. We order $\mathfrak{S}$ as follows:$(t, V) \leq (t', V')$ if $V \subset V'$ and $t'|_V = t$. Then $\mathfrak{S}$ is inductively ordered. Therefore, there exists a maximal element (t, V). Let us show that $V = X$. Otherwise, there exists $x \in X \setminus V$ and an $i \in I$ such that $x \in U_i$. Then $t|_{U_i \cap V} \in F(U_i \cap V)$ extends to a section $t_i \in F(U_i)$. Since $t_i|_{U_i \cap V} = t|_{U_i \cap V}$, the section t extends to a section on $V \cup U_i$ which contredicts the fact that V is maximal. q.e.d.

4.5 Sheaves on the interval $[0,1]$

Lemma 4.5.1. *Let $I = [0,1]$ and let $F \in \mathrm{Mod}(\mathbf{k}_I)$. Then:*

(i) *For $j > 1$, one has $H^j(I; F) = 0$.*

(ii) *If $F(I) \to F_t$ is an epimorphism for all $t \in I$, then $H^1(I; F) = 0$.*

Proof. Let $j \geq 1$ and let $s \in H^j(I; F)$. For $0 \leq t_1 \leq t_2 \leq 1$, consider the morphism:

$$f_{t_1,t_2} : H^j(I; F) \to H^j([t_1, t_2]; F)$$

and let

$$J = \{t \in [0,1];\ f_{0,t}(s) = 0\}.$$

Since $H^j(\{0\}; F) = 0$ for $j \geq 1$, we have $0 \in J$. Since $f_{0,t}(s) = 0$ implies $f_{0,t'}(s) = 0$ for $0 \leq t' \leq t$, J is an interval. Since $H^j([0, t_0]; F) = \varinjlim_{t > t_0} H^j([0, t]; F)$ (see 4.8), this interval is open. It remains to prove that J is closed. For $0 \leq t \leq t_0$, consider the Mayer-Vietoris sequence (see (4.12) and (4.13)):

$$\cdots \to H^j([0, t_0]; F) \to H^j([0, t]; F) \oplus H^j([t, t_0]; F) \to H^j(\{t\}; F) \to \cdots$$

For $j > 1$, or else for $j = 1$ assuming $H^0(I; F) \to H^0(\{t\}; F)$ is surjective, we obtain:

$$H^j([0, t_0]; F) \simeq H^j([0, t]; F) \oplus H^j([t, t_0]; F). \tag{4.18}$$

Let $t_0 = \sup\ \{t; t \in J\}$. Then $f_{0,t}(s) = 0$, for all $t < t_0$. On the other hand,

$$\varinjlim_{t < t_0} H^j([t, t_0]; F) = 0.$$

Hence, there exists $t < t_0$ with $f_{t,t_0}(s) = 0$. By (4.18), this implies $f_{0,t_0}(s) = 0$. Hence $t_0 \in J$. q.e.d.

Lemma 4.5.2. *let $X = U_1 \cup U_2$ be a covering of X by two open sets. Let F be a sheaf on X and assume that:*

(i) *$U_{12} = U_1 \cap U_2$ is connected and non empty,*

(ii) *$F|_{U_i}$ $(i = 1, 2)$ is a constant sheaf.*

Then F is a constant sheaf.

Proof. It follows from the hypothesis that there is a set M and isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$. Let $\theta_{12} = \theta_1 \circ \theta_2^{-1} : (M_X)|_{U_1\cap U_2} \xrightarrow{\sim} (M_X)|_{U_1\cap U_2}$. Since $U_1 \cap U_2$ is connected and non empty, $\Gamma(U_1 \cap U_2; \mathcal{H}om\,(M_X, M_X)) \simeq \mathrm{Hom}\,(M, M)$ and θ_{12} defines an invertible element of $\mathrm{Hom}\,(M, M)$. Using the map $\mathrm{Hom}\,(M, M) \to \Gamma(X; \mathcal{H}om\,(M_X, M_X))$, we find that θ_{12} extends as an isomorphism $\theta : M_X \simeq M_X$ all over X. Now define the isomorphisms: $\alpha_i : F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$ by $\alpha_1 = \theta_1$ and $\alpha_2 = \theta|_{U_2} \circ \theta_2$. Then α_1 and α_2 will glue together to define an isomorphism $F \xrightarrow{\sim} M_X$. q.e.d.

Proposition 4.5.3. *Let I denote the interval* $[0, 1]$.

(i) *Let F be a locally constant sheaf on I. Then F is a constant sheaf.*

(ii) *In particular, if $t \in I$, the morphism $\Gamma(I; F) \to F_t$ is an isomorphism.*

(iii) *Moreover, if $F = M_I$ for a $\mathbf{k}$-module M, then the composition*

$$M \simeq F_0 \xleftarrow{\sim} \Gamma(I; M_I) \xrightarrow{\sim} F_1 \simeq M$$

is the identity of M.

Proof. (i) We may find a finite open covering $U_i, (i = 1, \dots, n)$ such that F is constant on U_i, $U_i \cap U_{i+1}$ $(1 \le i < n)$ is non empty and connected and $U_i \cap U_j = \emptyset$ for $|i - j| > 1$. By induction, we may assume that $n = 2$. Then the result follows from Lemma 4.5.2.
(ii)–(iii) are obvious. q.e.d.

4.6 Invariance by homotopy

In this section, we shall prove that the cohomology of locally constant sheaves is an homotopy invariant. First, we define what it means.

In the sequel, we denote by I the closed interval $I = [0, 1]$.

Definition 4.6.1. Let X and Y be two topological spaces.

(i) Let f_0 and f_1 be two continuous maps from X to Y. One says that f_0 and f_1 are homotopic if there exists a continuous map $h : I \times X \to Y$ such that $h(0, \cdot) = f_0$ and $h(1, \cdot) = f_1$.

(ii) Let $f : X \to Y$ be a continuous map. One says that f is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. In such a case one says that X and Y are homotopic.

(iii) One says that a topological space X is contractible if X is homotopic to a point $\{x_0\}$.

One checks easily that the relation "f_0 is homotopic to f_1" is an equivalence relation. If $f_0, f_1 : X \rightrightarrows Y$ are homotopic, one gets the diagram

$$\begin{array}{ccccc} & & \overset{f_t}{\frown} & & \\ X \simeq \{t\} \times X & \xhookrightarrow{i_t} & I \times X & \xrightarrow{h} & Y \\ & & \downarrow{\scriptstyle p} & & \\ & & X & & \end{array} \tag{4.19}$$

where $t \in I$, $i_t : X \simeq \{t\} \times X \hookrightarrow I \times X$ is the embedding, p is the projection and $f_t = h \circ i_t$.

A topological space is contractible if and only if there exist $g : \{x_0\} \to X$ and $f : X \to \{x_0\}$ such that $f \circ g$ is homotopic to id_X. Replacing x_0 with $g(x_0)$, this means that there exists $h : I \times X \to X$ such that $h(0, x) = \mathrm{id}_X$ and $h(1, x)$ is the map $x \mapsto x_0$. Note that contractible implies non empty.

Example 4.6.2. Let V be a real vector space. A non empty convex set in V as well as a closed cone are contractible sets.

Statement of the main theorem

Let $f : X \to Y$ be a continuous map and let $G \in \mathrm{Mod}(\mathbf{k}_Y)$. Remark that $a_X \simeq a_Y \circ f$. The morphism of functors $\mathrm{id} \to Rf_* \circ f^{-1}$ defines the morphism $Ra_{Y*} \to Ra_{Y*} \circ Rf_* \circ f^{-1} \simeq Ra_{X*} \circ f^{-1}$. We get the morphism:

$$f^\sharp : \mathrm{R}\Gamma(Y; G) \to \mathrm{R}\Gamma(X; f^{-1}G). \tag{4.20}$$

If $g : Y \to Z$ is another morphism, we have:

$$f^\sharp \circ g^\sharp = (g \circ f)^\sharp. \tag{4.21}$$

The aim of this section is to prove:

Theorem 4.6.3. (Invariance by homotopy Theorem.) *Let $f_0, f_1 : X \rightrightarrows Y$ be two homotopic maps, and let G be a locally constant sheaf on Y. Consider the two morphisms $f_t^\sharp : \mathrm{R}\Gamma(Y;G) \to \mathrm{R}\Gamma(X;f_t^{-1}G)$, for $t = 0,1$. Then there exists an isomorphism $\theta\colon \mathrm{R}\Gamma(X;f_0^{-1}G) \to \mathrm{R}\Gamma(X;f_1^{-1}G)$ such that $\theta\circ f_0^\sharp = f_1^\sharp$.*

If $G = M_Y$ for some $M \in \mathrm{Mod}(\mathbf{k})$, then, identifying $f_t^{-1}M_Y$ with M_X $(t = 0,1)$, we have $f_1^\sharp = f_0^\sharp$.

This is visualized by the diagram

$$\begin{array}{ccc} & \mathrm{R}\Gamma(Y;G) & \\ {\scriptstyle f_0^\sharp}\swarrow & & \searrow{\scriptstyle f_1^\sharp} \\ \mathrm{R}\Gamma(X;f_0^{-1}G) & \xrightarrow[\sim]{\theta} & \mathrm{R}\Gamma(X;f_1^{-1}G). \end{array}$$

Proof of the main theorem

In order to prove Theorem 4.6.3, we need several lemmas.

Recall that the maps $p\colon I \times X \to X$ and $i_t\colon X \to I \times X$ are defined in (4.19). We also introduce the notation $I_x := I \times \{x\}$.

Lemma 4.6.4. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then*

(i) $F \xrightarrow{\sim} Rp_*p^{-1}F$,

(ii) *the morphism $p^\sharp : \mathrm{R}\Gamma(X;F) \to \mathrm{R}\Gamma(I \times X;p^{-1}F)$ is an isomorphism,*

(iii) *the morphisms $i_t^\sharp : \mathrm{R}\Gamma(I \times X;p^{-1}F) \to \mathrm{R}\Gamma(X;F)$ are isomorphisms and do not depend on $t \in I$.*

Proof. (i) Let $x \in X$ and let $t \in I$. Using Theorem 5.3.5, one gets the isomorphism $((Rp_*)p^{-1}F)_x \simeq \mathrm{R}\Gamma(I_x;p^{-1}F|_{I_x})$. This complex is concentrated in degree 0 and isomorphic to F_x by Lemma 4.5.1.
(ii) We have $Ra_{X*}Rp_*p^{-1}F \simeq Ra_{X*}F$ by (i). Hence $p^\sharp$ is an isomorphism.
(iii) By (4.21), $i_t^\sharp \circ p^\sharp$ is the identity, and $p^\sharp$ is an isomorphism by (i). Hence, $i_t^\sharp$ which is the inverse of $p^\sharp$ does not depend on t. q.e.d.

Lemma 4.6.5. *Let $H \in \mathrm{Mod}(\mathbf{k}_{I\times X})$ be a locally constant sheaf. Then*

(i) *the natural morphism $p^{-1}Rp_*H \to H$ is an isomorphism,*

(ii) *for each $t \in I$, the morphism $i_t^\sharp : \mathrm{R}\Gamma(I \times X;H) \to \mathrm{R}\Gamma(X;i_t^{-1}H)$ is an isomorphism.*

(iii) *If $H = M_{I\times X}$ for some $M \in \mathrm{Mod}(\mathbf{k})$, the isomorphism $i_t^\sharp : \mathrm{R}\Gamma(I \times X;M_{I\times X}) \to \mathrm{R}\Gamma(X;M_X)$ does not depend on t.*

Proof. (i) One has

$$\begin{aligned}(p^{-1}Rp_*H)_{(t,x)} &\simeq (Rp_*H)_x \\ &\simeq \mathrm{R}\Gamma(I_x; H|_{I_x}) \simeq H_{(t,x)}.\end{aligned}$$

Here the last isomorphism follows from Lemmas 4.5.3 and 4.5.1.
(ii) Consider the commutative diagram

$$\begin{array}{ccc}\mathrm{R}\Gamma(I\times X;H) & \xleftarrow{\sim} & \mathrm{R}\Gamma(I\times X;p^{-1}Rp_*H) \\ {\scriptstyle i_t^\sharp}\downarrow & & {\scriptstyle i_t^\sharp}\downarrow \\ \mathrm{R}\Gamma(X;i_t^{-1}H) & \xleftarrow{\sim} & \mathrm{R}\Gamma(X;i_t^{-1}p^{-1}Rp_*H)\end{array}$$

The horizontal arrows are isomorphisms by (i) and the vertical arrow on the vertical arrow on the right is an isomorphism by Lemma 4.6.4 (ii).
(iii) follows from Lemma 4.6.4 (iii). q.e.d.

End of the proof of Theorem 4.6.3. Set $H = h^{-1}G$. Then $F_t = i_t^{-1}H$ and the results follow from Lemma 4.6.4 (ii)–(iii). q.e.d.

Corollary 4.6.6. *Assume $f : X \to Y$ is a homotopy equivalence and let G be a locally constant sheaf on Y. Then $\mathrm{R}\Gamma(X, f^{-1}G) \simeq \mathrm{R}\Gamma(Y;G)$.*

In other words, the cohomology of locally constant sheaves on topological spaces is a homotopy invariant.

Proof. Let $g : Y \to X$ be a map such that $f\circ g$ and $g \circ f$ are homotopic to the identity of Y and X, respectively. Consider $f^\sharp : \mathrm{R}\Gamma(Y;G) \to \mathrm{R}\Gamma(X;f^{-1}G)$ and $g^\sharp : \mathrm{R}\Gamma(X;f^{-1}G) \to \mathrm{R}\Gamma(Y;G)$. Then: $(f\circ g)^\sharp = g^\sharp \circ f^\sharp \simeq \mathrm{id}_X^\sharp = \mathrm{id}$ and $(g\circ f)^\sharp = f^{\sharp j} \circ g^\sharp \simeq \mathrm{id}_Y^\sharp = \mathrm{id}$. q.e.d.

Corollary 4.6.7. *If X is contractible and $M \in \mathrm{Mod}(\mathbf{k})$, then $\mathrm{R}\Gamma(X;M_X) \simeq M$.*

We shall apply Theorem 4.6.3 to calculate the cohomology of various spaces.

Theorem 4.6.8. *Let $X = \bigcup_{i\in I} Z_i$ be a finite covering of X by closed subsets satisfying the condition*

(4.22) *for each non empty subset $J \subset I$, Z_J is contractible or empty.*

Let F be a locally constant sheaf on X. Then $H^j(X;F)$ is isomorphic to the j-th cohomology object of the complex

$$\Gamma(X;F_{\mathcal{Z}}^\bullet) := 0 \to \Gamma(X;F_{\mathcal{Z}}^0) \xrightarrow{d} \Gamma(X;F_{\mathcal{Z}}^1) \to \cdots$$

Proof. Recall that if Z is closed in X, then $\Gamma(X;F_Z) \simeq \Gamma(Z;F|_Z)$. Therefore the sheaves $F_{\mathcal{Z}}^p$ $(p \geq 0)$ are acyclic with respect to the functor $\Gamma(X;\bullet)$, by Corollary 4.6.7. Applying Proposition 4.3.1, the result follows. q.e.d.

4.7 Cohomology of some classical manifolds

Here, $\mathbf{k}$ denotes as usual a commutative unitary ring and M denotes a $\mathbf{k}$-module.

1-sphere

Let X be the circle $\mathbb{S}^1$ and let Z_j's be a closed covering by intervals such that the Z_{ij}'s are single points and $Z_{012} = \emptyset$. Applying Theorem 4.6.8, we find that if F is a locally constant sheaf on X, the cohomology groups $H^j(X;F)$ are the cohomology objects of the complex:

$$0 \to F_{Z_0} \oplus F_{Z_1} \oplus F_{Z_2} \xrightarrow{d} F_{Z_{12}} \oplus F_{Z_{20}} \oplus F_{Z_{01}} \to 0.$$

Recall Example 3.3.10: $\mathbb{S}^1 = U_1 \cup U_2$, $U_1 \cap U_2$ has two connected components U_{12}^+ and U_{12}^-, $\mathbf{k}$ is a field, $\alpha \in \mathbf{k}^\times$ and L_α denotes the locally constant sheaf of rank one over $\mathbf{k}$ obtained by glueing $\mathbf{k}_{U_1}$ and $\mathbf{k}_{U_2}$ by the identity on U_{12}^+ and by multiplication by $\alpha \in \mathbf{k}^\times$ on U_{12}^-.

Then for $j = 0$ (resp. for $j = 1$), $H^j(\mathbb{S}^1; L_\alpha)$ is the kernel (resp. the cokernel) of the matrix $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -\alpha \\ -1 & 1 & 0 \end{pmatrix}$ acting on $\mathbf{k}^3$. (See Example 4.3.2.) Note that these kernel and cokernel are zero except in case of $\alpha = 1$ which corresponds to the constant sheaf $\mathbf{k}_X$.

It follows that if M is a $\mathbf{k}$-module, then $\mathrm{R}\Gamma(\mathbb{S}^1; M_{\mathbb{S}^1}) \simeq M \oplus M[-1]$.

n-sphere

Consider the topological n-sphere $\mathbb{S}^n$. Recall that it can be defined as follows. Let $\mathbb{E}$ be an $\mathbb{R}$-vector space of dimension $n+1$ and denote by $\dot{\mathbb{E}}$ the set $\mathbb{E} \setminus \{0\}$. Then

$$\mathbb{S}^n \simeq \dot{\mathbb{E}}/\mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers and $\mathbb{S}^n$ is endowed with the quotient topology. In other words, $\mathbb{S}^n$ is the set of all half-lines in $\mathbb{E}$. If one chooses an Euclidian norm on $\mathbb{E}$, then one may identify $\mathbb{S}^n$ with the unit sphere in $\mathbb{E}$.

We have $\mathbb{S}^n = \bar{D}^+ \cup \bar{D}^-$, where $\bar{D}^+$ and $\bar{D}^-$ denote the closed hemispheres, and $\bar{D}^+ \cap \bar{D}^- \simeq \mathbb{S}^{n-1}$. Let us prove that:

$$\mathrm{R}\Gamma(\mathbb{S}^n; \mathbf{k}_{\mathbb{S}^n}) = \mathbf{k} \oplus \mathbf{k}[-n]. \tag{4.23}$$

Consider the Mayer-Vietoris long exact sequence

$$\begin{aligned}(4.24)\qquad \to H^j(\bar{D}^+;\mathbf{k}_{\bar{D}^+})\oplus H^j(\bar{D}^-;\mathbf{k}_{\bar{D}^-})\to H^j(\mathbb{S}^{n-1};\mathbf{k}_{\mathbb{S}^{n-1}})\\ \to H^{j+1}(\mathbb{S}^n;\mathbf{k}_{\mathbb{S}^n})\to\cdots\end{aligned}$$

The closed hemispheres being contractible, their cohomology is concentrated in degree 0. Then we find by induction on n that the cohomology of $\mathbb{S}^n$ is concentrated in degree 0 and n and isomorphic to $\mathbf{k}$ in these degrees. To conclude that $\mathrm{R}\Gamma(\mathbb{S}^n;\mathbf{k}_{\mathbb{S}^n})$ is the direct sum of its cohomology objects, use the fact that $\mathrm{Ext}^n_{\mathrm{D}^b(\mathbf{k})}(\mathbf{k},\mathbf{k})\simeq 0$ for $n\neq 0$ and Exercise 8.5 of [Sc02].

Let $\mathbb{E}$ be a real vector space of dimension $n+1$, and let $X=\mathbb{E}\setminus\{0\}$. Assume $\mathbb{E}$ is endowed with a norm $|\cdot|$. The map $x\mapsto x((1-t)+t/|x|)$ defines an homotopy of X with the sphere $\mathbb{S}^n$. Hence the cohomology of a constant sheaf with stalk M on $V\setminus\{0\}$ is the same as the cohomology of the sheaf $M_{\mathbb{S}^n}$

As an application, one obtains that the dimension of a finite dimensional vector space is a topological invariant. In other words, if V and W are two real finite dimensional vector spaces and are topologically isomorphic, they have the same dimension. In fact, if V has dimension n, then $V\setminus\{0\}$ is homotopic to $\mathbb{S}^{n-1}$.

Notice that $\mathbb{S}^n$ is not contractible, although one can prove that any locally constant sheaf on $\mathbb{S}^n$ for $n\geq 2$ is constant.

Denote by a the antipodal map on $\mathbb{S}^n$ (the map deduced from $x\mapsto -x$) and denote by $a^{\sharp n}$ the action of a on $H^n(\mathbb{S}^n;M_{\mathbb{S}^n})$. Using (4.24), one deduces the commutative diagram:

$$(4.25)\qquad \begin{array}{ccc} H^{n-1}(\mathbb{S}^{n-1};M_{\mathbb{S}^{n-1}}) & \xrightarrow{u} & H^n(\mathbb{S}^n;M_{\mathbb{S}^n}) \\ \downarrow{\scriptstyle a^{\sharp n-1}} & & \downarrow{\scriptstyle a^{\sharp n}} \\ H^{n-1}(\mathbb{S}^{n-1};M_{\mathbb{S}^{n-1}}) & \xrightarrow{-u} & H^n(\mathbb{S}^n;M_{\mathbb{S}^n}) \end{array}$$

For $n=1$, the map a is homotopic to the identity (in fact, it is the same as a rotation of angle π). By (4.25), we deduce:

$$(4.26)\qquad a^{\sharp n}\text{ acting on }H^n(\mathbb{S}^n;M_{\mathbb{S}^n})\text{ is }(-)^{n+1}.$$

n-torus

The Künneth formula will be proved in the next chapter (see Corollary 5.3.8). It allows us to calculate the cohomology of the n-torus $\mathbb{T}^n:=(\mathbb{S}^1)^n$. Applying (4.23) for $n=1$ we get:

$$(4.27)\qquad \mathrm{R}\Gamma(\mathbb{T}^n;\mathbf{k}_{\mathbb{T}^n})\simeq(\mathbf{k}\oplus\mathbf{k}[-1])^{\otimes n}.$$

For example, for $n = 2$, we find

$$\begin{aligned}(\mathbf{k} \oplus \mathbf{k}[-1]) \otimes (\mathbf{k} \oplus \mathbf{k}[-1]) &\simeq \mathbf{k} \otimes \mathbf{k} \oplus \mathbf{k} \otimes \mathbf{k}[-1] \oplus \mathbf{k}[-1] \otimes \mathbf{k} \oplus \mathbf{k}[-1] \otimes \mathbf{k}[-1] \\ &\simeq \mathbf{k} \oplus \mathbf{k}^{\oplus 2}[-1] \oplus \mathbf{k}[-2].\end{aligned}$$

Action of groups

Let G be a fgroup (with unit denoted e). We identify G with the category with one object, the morphisms of this object being G. Consider the category

$$G\text{-Mod}(\mathbf{k}) := \mathrm{Fct}(G, \mathrm{Mod}(\mathbf{k})).$$

An object of G-Mod($\mathbf{k}$) is thus a $\mathbf{k}$-module endowed with a left action of G. One defines the functor

$$\begin{aligned}&I^G\colon G\text{-Mod}(\mathbf{k}) \to \mathrm{Mod}(\mathbf{k}), \\ &I^G(M) = \{m \in M; g \cdot m = m \text{ for all } g \in G\}.\end{aligned}$$

The module $I^G(M)$ (also denoted M^G in the literature) is thus the submodule of G-invariants of M. One checks easily that the functor I^G is left exact. If $M \in G$-Mod($\mathbf{k}$), one sets

$$H^p(G; M) = H^p(RI^G(M)).$$

Assume now that G is endowed with the discrete topology and acts on a topological space X, that is, we have a continuous map

$$\mu\colon G \times X \to X$$

satisfying $\mu(e, x) = x$, $\mu(g_1 \cdot g_2, x) = \mu(g_2, \mu(g_1, x))$. On X, the relation $x \sim y$ if there exists $g \in G$ such that $\mu(g, x) = y$ is an equivalence relation and one denotes by X/G the quotient space. One sets for short $Y = X/G$, one endows Y with the quotient topology and one denotes by

$$\rho\colon X \to Y = X/G$$

the quotient map. For $g \in G$, we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ & \searrow^{\rho} & \downarrow{\scriptstyle\rho} \\ & & Y. \end{array}$$

For $g \in G$, we deduce a morphism of functors (see (4.21)):

$$g^\sharp\colon \mathrm{id}_Y \to \rho_* \circ \rho^{-1}, \tag{4.28}$$

as the composition

$$\mathrm{id}_Y \to \rho_* \circ g_* \circ g^{-1} \circ \rho^{-1} \simeq \rho_* \circ \rho^{-1}.$$

Applying $\rho_* \circ \rho^{-1}$, we get a morphism

$$g^\sharp \colon \rho_* \circ \rho^{-1} \to \rho_* \circ \rho^{-1}. \tag{4.29}$$

For $F \in \mathrm{Mod}(\mathbf{k}_Y)$ define F^g by the exact sequence

$$0 \to F^g \to \rho_* \circ \rho^{-1} F \xrightarrow{\mathrm{id} - g^\sharp} \rho_* \circ \rho^{-1} F$$

and set

$$I^G(F) = \bigcap_{g \in G} F^g.$$

Of course, there is a natural morphism $F \to I^G(F)$. The functor $\Gamma(Y; \bullet)$ being left exact, we get:

Lemma 4.7.1. *Let $F \in \mathrm{Mod}(\mathbf{k}_Y)$. Then*

$$I^G(\Gamma(Y; F)) \simeq \Gamma(Y; I^G(F)).$$

Lemma 4.7.2. *Assume that X is Hausdorff and G is finite and acts freely on X. Then one has the isomorphism $F \xrightarrow{\sim} F^G$.*

Proof. Notice first that the fibers of ρ are finite. Let $y \in Y$. The module F_y is isomorphic to the submodule of $\Gamma(\rho^{-1}(y); \rho^{-1}F)$ on which G acts trivially, that is, $F_y \simeq I^G((\rho_*\rho^{-1}F)_y)$. Since $I^G((\rho_*\rho^{-1}F)_y) \simeq (F^G)_y$, we get the result. q.e.d.

Recall that for a space Z, one denotes by a_Z the map $Z \to \mathrm{pt}$. Applying the functor a_{Y*} to (4.29), we deduce that there is a well-defined functor

$$a_{X*} \circ \rho^{-1} \colon \mathrm{Mod}(\mathbf{k}_Y) \to G\text{-}\mathrm{Mod}(\mathbf{k}).$$

Lemma 4.7.3. *Assume that X is Hausdorff and G is finite and acts freely on X. Then one has the isomorphism of functors $RI^G \circ Ra_{X*} \circ \rho^{-1} \simeq Ra_{Y*}$.*

Proof. (i) The isomorphism $I^G \circ a_{X*} \circ \rho^{-1} \simeq a_{Y*}$. follows from Lemmas 4.7.2 and 4.7.1 .

(ii) It follows from the hypotheses that ρ^{-1} sends injective sheaves to injective sheaves (apply the result of Exercise 4.7), and one knows that a_{X*} sends injective sheaves to injective sheaves. Therefore, the derived functor of the composition is the composition of the derived functors. q.e.d.

Application: real projective spaces [1]

by $\mathbb{P}^n$ the real projective space of dimension n. It can be obtained as the quotient of $\mathbb{S}^n$ by the antipodal map, that is, the quotient of $\mathbb{S}^n$ by the group $\mathbb{Z}/2\mathbb{Z}$. Hence, we may apply Lemma 4.7.3 with $X = \mathbb{S}^n$, $Y = \mathbb{P}^n$ and $G = \mathbb{Z}/2\mathbb{Z}$. We choose $\mathbf{k} = \mathbb{Z}$.

We have a distinguished triangle

$$\tau^{<n} Ra_{X*}\mathbb{Z}_X \to Ra_{X*}\mathbb{Z}_X \to H^n(Ra_{X*}\mathbb{Z}_X) \xrightarrow{+1} .$$

Since $X = \mathbb{S}^n$, this triangle reduces to

$$\mathbb{Z} \to Ra_{X*}\mathbb{Z}_X \to \widetilde{\mathbb{Z}}\,[-n] \xrightarrow{+1}$$

where $\widetilde{\mathbb{Z}} = \mathbb{Z}$, the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $\mathbb{Z}$, is trivial on $\widetilde{\mathbb{Z}}$ if n is odd and this action on $\widetilde{\mathbb{Z}}$ is the multiplication by -1 if n is even (we apply (4.26)).

Using resolution, one easily obtains:

$$H^p(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p \text{ is even}, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, if n is odd one proves

$$H^p(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, n, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p = 2, 4, \dots, n-1, \\ 0 \text{ otherwise.} \end{cases}$$

When n is even, one find

$$H^p(\mathbb{Z}/2\mathbb{Z};\widetilde{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } p \text{ is odd}, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, if n is even we find

$$H^p(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p = 2, 4, \dots, n, \\ 0 \text{ otherwise.} \end{cases}$$

[1] The classical proofs calculating the cohomology of the real projective space use spectral sequences. The proof proposed here, using truncation functors instead, is much shorter. It is due to Tony Yue Yu who did it when he was a Master 2 student at UPMC around 2011.

Exercises to Chapter 4

Exercise 4.1. Let X be a topological space, M a closed subspace and F a sheaf on X. Assume there is an $n > 0$ such that $H^j_M(F) \simeq 0$ for $j < n$. Prove that the presheaf $U \mapsto H^n_{M\cap U}(U;F)$ is a sheaf and is isomorphic to the sheaf $H^n_M(F)$. (See Notation 4.14.)

Exercise 4.2. Let X be a locally compact space, M a closed subspace and F a sheaf on X. Assume there is an $n > 0$ such that for any compact $K \subset M$, $H^j_K(X;F) = 0$ for all $j < n$ and that for each pair $K_1 \subset K_2$ of compact subsets of M, the natural morphism $H^n_{K_1}(X;F) \to H^n_{K_2}(X;F)$ is injective.
(i) Prove that for each open subset ω of M, $H^j_\omega(X;F) = 0$ for all $j < n$, and the presheaf $\omega \mapsto H^n_\omega(X;F)$ is the sheaf $H^n_M(F)$.
(ii) Prove that if $K \subset M$ is compact, $\Gamma_K(M;H^n_M(F)) \simeq H^n_K(X;F)$.
(iii) Assume moreover that $H^j_K(X;F) = 0$ for all compact subsets of M and all $j > n$. Prove that the sheaf $H^n_M(F)$ is flabby.
(Remark: when M is a real analytic manifold of dimension n, X a complexification, and $F = \mathcal{O}_X$, all hypotheses are satisfied. The sheaf $H^n_M(\mathcal{O}_X) \otimes or_M$ is called the sheaf of Sato's hyperfunctions.)

Exercise 4.3. Let $X = \mathbb{N}$ endowed with the topology for which the open subsets are the intervals $[0,\dots,n], n \geq -1$ and $\mathbb{N}$.
(i) Prove that a presheaf F of $\mathbf{k}$-modules on X is nothing but a projective system $(F_n, \rho_{m,n})$ indexed by $\mathbb{N}$ and that this presheaf is a sheaf if and only if $F(X) \simeq \varprojlim_n F_n$.
(ii) Prove that if $F_{n+1} \to F_n$ is onto, then the sheaf F is flabby.
(iii) Deduce that if $0 \to M'_n \to M_n \to M''_n \to 0$ is an exact sequence of projective systems of $\mathbf{k}$-modules and the morphisms $M'_{n+1} \to M'_n$ are onto, then the sequence $0 \to \varprojlim_n M'_n \to \varprojlim_n M_n \to \varprojlim_n M''_n \to 0$ is exact.
(iv) Prove that for any sheaf F on X there exists an exact sequence $0 \to F \to F_0 \to F_1 \to 0$, with F_0 and F_1 flabby.
(v) Denote by π the left exact functor

$$\pi\colon (\mathrm{Mod}(\mathbf{k}))^{\mathbb{N}} \to \mathrm{Mod}(\mathbf{k}), \quad \{M_n\}_n \mapsto \varprojlim_n M_n. \tag{4.30}$$

Prove that $R^j\pi \simeq 0$ for $j > 1$.

Exercise 4.4. let X be a real n-dimensional vector space and let U be an open convex subset, $j : U \hookrightarrow X$ the embedding. Calculate $Rj_*\mathbf{k}_U$.

Exercise 4.5. Assume that $\mathbf{k}$ is a field. Prove that a sheaf of $\mathbf{k}_X$-modules is injective if and only if it is flabby.

Exercise 4.6. By considering the space $X = \mathbb{S}^1$ and the map $a_X : X \to \mathrm{pt}$, prove that the isomorphism $Rf_* \circ f^{-1} \simeq R(f_* \circ f^{-1})$ does not hold in general. (Here, $R(f_* \circ f^{-1})$ denotes the right derived functor of the left exact functor $f_* \circ f^{-1}$.)

Exercise 4.7. Let X be a topological space and les $F, G \in \mathrm{Mod}(\mathbf{k}_X)$.
(i) Prove that if F is injective, then $\mathcal{H}om\,(G, F)$ is flabby.
(ii) Deduce from (i) and Proposition 4.4.6 that to be injective is a local property.

Exercise 4.8. Let K be a compact relatively Hausdorff subset of a topological space X. Prove the isomorphism $H^j(K; F) \simeq \varinjlim_U H^j(U; F)$, where U ranges through the family of open neighborhoods of K.

Exercise 4.9. Let $X = \mathbb{S}^n \times \mathbb{S}^n$. Calculate $\mathrm{R}\Gamma(X; \mathbf{k}_X)$.

Exercise 4.10. Assume $\mathbf{k}$ is a field, and for $\alpha \in \mathbf{k}^\times$ let L_α be the locally free sheaf of rank one on $\mathbb{S}^1$ constructed in Example 3.3.10. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Calculate $\mathrm{R}\Gamma(X; L_\alpha \boxtimes L_\beta)$ for $\alpha, \beta \in \mathbf{k}^\times$.

Exercise 4.11. Let $Y = [0,1] \times]0,1[$ and let X denote the manifold obtained by identifying $(0,t)$ and $(1, 1-t)$. Let S denote the hypersurface of X, the image of the diagonal of Y. Calculate $\Gamma(X; or_{S/X})$.

Exercise 4.12. Let $\bar{D}$ denote the closed disc in $\mathbb{R}^2$ with boundary $\mathbb{S}^1$. Let $\iota : \mathbb{S}^1 \hookrightarrow \bar{D}$ denote the embedding. Prove that there exists no continuous map $f : \bar{D} \to \mathbb{S}^1$ such that the composition $f \circ \iota$ is the identity.

Exercise 4.13. Let Y and Y' be two topological spaces, S and S' two closed subsets of Y and Y' respectively, $f : S \simeq S'$ a topological isomorphism. Define the topological space $X := Y \sqcup_S Y'$ as the quotient $Y \sqcup Y' / \sim$ where $\sim$ is the equivalence relation which identifies $x \in Y$ and $y \in Y'$ for $x \in S$, $y \in S'$ and $f(x) = y$.

Let $\mathbb{S}^n$ be the unit sphere of thee Euclidian space $\mathbb{R}^{n+1}$, Z the intersection of $\mathbb{S}^n$ with an open ball of radius ε $(0 < \varepsilon << 1)$ centered in some point of $\mathbb{S}^n$ and let Σ denote its boundary in $\mathbb{S}^n$. Set $Y = \mathbb{S}^n \setminus Z$, $S = \Sigma$ denote by Y' and S' another copy of Y and S.
(i) Calculate $\mathrm{R}\Gamma(Y \sqcup_S Y'; \mathbf{k}_{Y \sqcup_S Y'})$.
(ii) Same question when replacing the sphere $\mathbb{S}^n$ by the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$.

Chapter 5

Duality on locally compact spaces

In this chapter all sites X, Y, etc. are locally compact topological spaces.

Recall that we assume that $\mathbf{k}$ has finite global dimension.

5.1 Proper direct images

Proper maps

Definition 5.1.1. A continuous map $f\colon X \to Y$ is proper if f is closed (i.e. the image of any closed subset in X is closed in Y) and its fibers are relatively Hausdorff and compact.

If X and Y are locally compact, f is proper if and only if the inverse image of a compact subset of Y is compact in X. If $Y = \mathrm{pt}$, f is proper if and only if X is compact.

Proposition 5.1.2. *Assume that $f\colon X \to Y$ is proper. Then the morphism*

$$(f_*F)_y \to \Gamma(f^{-1}(y); F|_{f^{-1}(y)}). \tag{5.1}$$

is an isomorphism.

Proof. When V ranges over the family of open neighborhoods of y, $f^{-1}(V)$ ranges over a neighborhood system of $f^{-1}(y)$. Hence $\varinjlim_{V \ni y} \Gamma(f^{-1}V; F) \xrightarrow{\sim} \Gamma(f^{-1}(y); F|_{f^{-1}(y)})$ by Proposition 4.1.2. q.e.d.

One says that a map f is finite if it is proper and moreover the inverse image of a finite set is finite.

Corollary 5.1.3. *If f is finite, the functor f_* is exact.*

Lemma 5.1.4. *Let K be a relatively Hausdorff compact subset of X and let $\{F_i\}_{i\in I}$ be a small filtrant inductive system of sheaves. Then the natural morphism $\varinjlim_i \Gamma(K;F_i) \to \Gamma(K;\varinjlim_i F_i)$ is an isomorphism.*

Proof. The proof is left as an exercise. q.e.d.

Lemma 5.1.5. *Let $f\colon X \to Y$ be a morphism of locally compact spaces and let $\{F_i\}_{i\in I}$ be an inductive system of sheaves on X with I small and filtrant. Let $Z \subset X$ be a closed subset and assume that the map $f|_Z$ is proper. Then the natural morphism $\varinjlim_i f_*(F_i)_Z \to f_*(\varinjlim_i (F_i)_Z)$ is an isomorphism.*

Proof. We shall apply Lemma 5.1.5. Let K be a compact subset of Y. One has

$$\begin{aligned}\Gamma(K;f_*(\varinjlim_i (F_i)_Z)) &\simeq \Gamma(f^{-1}K\cap Z;\varinjlim_i (F_i)_Z)\\ &\simeq \varinjlim_i \Gamma(f^{-1}K\cap Z;(F_i)_Z)\\ &\simeq \varinjlim_i \Gamma(K;f_*((F_i)_Z)).\end{aligned}$$

By choosing for K a fundamental neighborhood system of $y \in Y$ we get that the natural morphism of the statement induces an isomorphism on the talks at each $y \in Y$. q.e.d.

Proper direct images

Definition 5.1.6. Let $f\colon X \to Y$ be a morphism of locally compact spaces and let $F \in \mathrm{Mod}(\mathbf{k}_X)$.

(a) One defines the functor $f_!\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y)$ by setting for $F \in \mathrm{Mod}(\mathbf{k}_X)$:

$$f_!F = \varinjlim_{U\subset\subset X} f_*(F_U)$$

where U ranges over the family of relatively compact open subsets of X.

(b) One sets $\Gamma_c(X;\,\bullet\,) = a_{X!}$, where $a_X\colon X \to \mathrm{pt}$.

Proposition 5.1.7. (i) *In the situation of Definition 5.1.6, one has for V an open subset of Y:*

$$\Gamma(V; f_! F) \simeq \varinjlim_{Z} \Gamma_Z(f^{-1}(V); F)$$

where Z ranges through the family of closed subsets of $f^{-1}(V)$ such that $f|_Z : Z \to V$ is proper. In particular, $\Gamma_c(X; F) \simeq \varinjlim_{K} \Gamma_K(X; F)$, where K ranges through the family of compact subsets of X.

(ii) *If f is proper on* supp(F)*, then $f_! F \xrightarrow{\sim} f_* F$. In particular, if f is proper, then $f_! \xrightarrow{\sim} f_*$.*

(iii) *The functor $f_!$ is left exact and commutes with small filtrant inductive limits.*

(iv) *Let $g \colon Y \to Z$ be a continuous map of locally compact spaces. Then $f_! \circ g_! = (f \circ g)_!$.*

(v) *Let $i_U \colon U \hookrightarrow X$ be an open embedding. Then the functor $i_{U!}$, as given by Definition 5.1.6, coincides with the functor j_U^{-1} (see Notation 2.7.2).*

Proof. (i) We shall apply Lemma 5.1.5. For any W open and relatively compact subset of V

$$\begin{aligned}\Gamma(\overline{W}; \varinjlim_{U} f_* F_U) &\simeq \varinjlim_{U} \Gamma(\overline{W}; f_* F_U) \\ &\simeq \varinjlim_{U,W'} \Gamma(f^{-1}(W'); F_U)\end{aligned}$$

where U ranges over the family of relatively compact open subsets of X and W' over the family of open neighbourhoods of $\overline{W}$.

Let $s \in \Gamma(V; f_* F)$.Then $s \in \Gamma(V; f_! F)$ if and only if for any $W \subset V$ open and relatively compact in V, there exists $U \subset X$ open and relatively compact such that supp$(s) \cap f^{-1}W$ is contained in some U relatively compact in X. This is equivalent to saying that the support of s is proper over Y.
(ii) is obvious.
(iii) The functor $F \mapsto F_U$ is exact, the functor f_* is left exact and the functor $\varinjlim$ over small filtrant categories is exact. Hence, $f_!$ is left exact. It commutes with small filtrant inductive limits by Lemma 5.1.5.
(iv) In the sequel, U ranges over the family of relatively compact open subsets of X, and similarly with V in Y.

By Proposition 4.2.1), the functor $F \mapsto F_U$ commutes with inductive limits and by Lemma 5.1.5 the functor $g_*((\bullet)_V)$ commutes with filtrant inductive limits. Therefore:

$$\begin{aligned} g_! f_! F &\simeq \varinjlim_V g_*((\varinjlim_U f_* F_U)_V) \simeq \varinjlim_V g_*(\varinjlim_U (f_* F_U)_V) \\ &\simeq \varinjlim_V (\varinjlim_U g_*(f_* F_U)_V) \simeq \varinjlim_U g_* f_* F_U \simeq (g \circ f)_* F. \end{aligned}$$

(v) Applying Notation 2.7.2, we find

$$\begin{aligned} i_{U!} F &= j_U^{-1} F \\ &\simeq j_U^{-1} \varinjlim_V F_V \simeq \varinjlim_V j_U^{-1} F_V \end{aligned}$$

where V ranges over the family of relatively compact open subsets of U. Hence, to recover Definition 5.1.6, it is enough to check that for such a V,

$$j_U^{-1} F_V \simeq i_{U*} F_V.$$

This is left as an exercise. q.e.d.

Base change formula (non derived)

Consider a *Cartesian square* of locally compact topological spaces:

(5.2)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

This means that $g \circ f' = f \circ g'$ and X' is isomorphic (as a topological space) to the fiber product:

$$X \times_Y Y' = \{(x, y') \in X \times Y';\ f(x) = g(y')\}.$$

Note that for any compact $K \subset Y'$, g' induces a topological isomorphism $f'^{-1}(K) \xrightarrow{\sim} f^{-1}(g(K))$.

Also note that choosing $y \in Y$ and setting $X' = f^{-1}(y)$, we get the Cartesian square:

(5.3)
$$\begin{array}{ccc} f^{-1}(y) & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow{g} & Y. \end{array}$$

Proposition 5.1.8. *Consider the Cartesian square* (5.2)*. There is a natural morphism of functors*

$$g^{-1} \circ f_* \to f'_* \circ g'^{-1} \tag{5.4}$$

Moreover, if $F \in \mathrm{Mod}(\mathbf{k}_X)$ *and* f *is proper on* $\mathrm{supp}\, F$*, then* f' *is proper on* $\mathrm{supp}\, g'^{-1}F$ *and the morphism* (5.4) *induces an isomorphism* $g^{-1}f_*F \xrightarrow{\sim} f'_*g'^{-1}F$*. In particular, if* f *is proper, then* (5.4) *is an isomorphism.*

Proof. (i) The isomorphism $f_* \circ g'_* \simeq g_* \circ f'_*$ defines by adjunction the morphism $g^{-1} \circ f_* \circ g'_* \to f'_*$, hence the morphisms

$$\begin{aligned} g^{-1} \circ f_* &\to g^{-1} \circ f_* \circ g'_* \circ g'^{-1} \\ &\to f'_* \circ g'^{-1}. \end{aligned}$$

(ii) Let $y' \in Y'$ and set $y = g(y')$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. We have

$$\begin{aligned} (g^{-1}f_*F)_{y'} &\simeq (f_*F)_y \\ &\simeq \Gamma(f^{-1}(y); F) \end{aligned}$$

and

$$(f'_*g'^{-1}F)_{y'} \simeq \Gamma(f'^{-1}(y'); g'^{-1}F).$$

Since g' induces a topological isomorphism $f'^{-1}(y') \xrightarrow{\sim} f^{-1}(g(y'))$, the result follows. q.e.d.

Theorem 5.1.9. *Consider the Cartesian square* (5.2)*. Then there is a natural isomorphism of functors:*

$$f'_! \circ g'^{-1} \xrightarrow{\sim} g^{-1} \circ f_!.$$

In particular, setting $Y' = \{y\}$ for $y \in Y$, one gets for $F \in \mathrm{Mod}(\mathbf{k}_X)$ the isomorphism:

$$(f_!F)_y \simeq \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}). \tag{5.5}$$

Proof. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. We have the isomorphisms below in which U ranges over the family of relatively compact open subsets of X and similarly with U' in X':

$$\begin{aligned} g^{-1}f_!F &\simeq g^{-1} \varinjlim_U f_*F_U \simeq \varinjlim_U g^{-1}f_*F_U \\ &\simeq \varinjlim_U f'_*(g'^{-1}(F_U)) \simeq \varinjlim_U f'_*((g'^{-1}F)_{g'^{-1}U}) \end{aligned}$$

and

$$f'_{!}g'^{-1}F \simeq \varinjlim_{U'} f'_{*}(g'^{-1}F)_{U'}.$$

Let K be a compact subset of Y'. The family $\{f'^{-1}K \cap U'\}_{U'}$ and the family $\{f'^{-1}K \cap g^{-1}U\}_{U}$ are cofinal. Therefore, the morphism

$$\Gamma(K; f'_{!}g'^{-1}F) \to \Gamma(K; g^{-1}f_{!}F)$$

is an isomorphism. q.e.d.

Projection formula (non derived)

Lemma 5.1.10. *Let X be a locally compact space and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Let M be a flat $\mathbf{k}$-module. Then the natural morphism:*

$$\Gamma_c(X; F) \otimes M \to \Gamma_c(X; F \otimes M_X)$$

is an isomorphism.

Proof. Since $\Gamma_c(X; F) \simeq \varinjlim_{K} \Gamma(X; F_K)$, we may assume from the beginning that X is compact. Let $K = \cup_j K_j$ be a finite covering by compact subsets and set $K_{ij} = K_i \cap K_j$. Consider the diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \Gamma(X; F) \otimes M & \xrightarrow{\lambda} & \oplus_i \Gamma(K_i; F) \otimes M & \xrightarrow{\mu} & \oplus_{ij} \Gamma(K_{ij}; F) \otimes M \\
 & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
0 \longrightarrow & \Gamma(X; F \otimes M_X) & \xrightarrow{\lambda'} & \oplus_i \Gamma(K_i; F \otimes M_X) & \xrightarrow{\mu'} & \oplus_{ij} \Gamma(K_{ij}; F \otimes M_X)
\end{array}$$

Notice first the isomorphism

$$\varinjlim_{U} (\Gamma(U; F) \otimes M) \xrightarrow{\sim} \varinjlim_{U} \Gamma(U; F \otimes M_X), \tag{5.6}$$

where U ranges thourgh the family of open neighborhoods of $x \in X$. In fact, both sides are isomorphic to $F_x \otimes M$.

(i) α is injective. Let $s \in \Gamma(X; F) \otimes M$, with $\alpha(s) = 0$. By (5.6) there exists a covering such that $\lambda(s) = 0$. Hence, $s = 0$. The same argument shows that β and γ are injective.

(ii) α is surjective. Let $t \in \Gamma(X; F \otimes M_X)$. By (5.6) there exists a finite covering such that $\lambda'(t)$ is in the image of β. Then the result follows, using the injectivity of γ. q.e.d.

Now we consider a continuous map $f\colon X \to Y$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$. There are natural morphisms :

$$\begin{aligned} f^{-1}(f_*F \otimes G) &\simeq f^{-1}f_*F \otimes f^{-1}G \\ &\to F \otimes f^{-1}G \end{aligned}$$

which defines by adjunction: $f_*F\otimes \to f_*(F \otimes f^{-1}G)$. This last morphism induces:

$$f_!F \otimes G \to f_!(F \otimes f^{-1}G). \tag{5.7}$$

Proposition 5.1.11. *Let $f\colon X \to Y$ be a morphism of locally compact spaces. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$. Assume that G is a flat $\mathbf{k}_Y$-module. Then the natural morphism* (5.7) *is an isomorphism.*

Proof. It is enough to check the isomorphism at each $y \in Y$. Denote by $g : \{y\} \hookrightarrow Y$ the embedding and consider the Cartesian square (5.3). Applying the base change formula, we get

$$\begin{aligned} (f_!(F \otimes f^{-1}G))_y &\simeq g^{-1}f_!(F \otimes f^{-1}G) \\ &\simeq f'_!g'^{-1}(F \otimes f^{-1}G) \\ &\simeq f'_!(g'^{-1}F \otimes g'^{-1}f^{-1}G). \end{aligned}$$

Applying Lemma 5.1.10 with F replaced by $g'^{-1}F$ and M replaced by $f^{-1}G = G_y$, we get

$$\begin{aligned} f'_!(g'^{-1}F \otimes g'^{-1}f^{-1}G) &\simeq f'_!g'^{-1}F \otimes G_y \\ &\simeq (f_!F)_y \otimes G_y \\ &\simeq (f_!F \otimes G)_y. \end{aligned}$$

q.e.d.

5.2 c-soft sheaves

Definition 5.2.1. Assume X is locally compact. A sheaf F is *c*-soft if for any compact subset K of X, the map $\Gamma(X; F) \to \Gamma(K; F)$ is onto.

Lemma 5.2.2. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. The conditions bellow are equivalent.*

(i) *the sheaf F is c-soft*

(ii) *for any locally closed subset Z of X, the restriction map $\Gamma_c(X; F) \to \Gamma_c(Z; F|_Z)$ is surjective,*

(iii) *for any compact subset K of X, the restriction map $\Gamma_c(X;F) \to \Gamma(K;F)$ is surjective,*

Proof. (i) For K compact:, we have $\Gamma(K;F) = \Gamma_c(K;F|_K)$. Therefore, $(ii) \Rightarrow (iii) \Rightarrow (i)$ is clear.
(ii) Assume F is c-soft. Let $s \in \Gamma_c(Z;F|_Z)$ with support in K and let U be a relatively compact open neighborhood of K in X. Define $\tilde{s} \in \Gamma(\partial U \cup (Z \cap \bar{U});F)$ by setting $\tilde{s}|_{Z\cap\bar{U}} = s$, $\tilde{s}|_{\partial U} = 0$. Then $\tilde{s} \in \Gamma(\partial U \cup Z \cap \bar{U};F)$ extends to a section of $\Gamma(X;F)$, and since $\tilde{s}|_{\partial U} = 0$, we may assume t is supported by $\bar{U}$. q.e.d.

Lemma 5.2.3. *A small filtrant inductive limit of c-soft sheaves is c-soft. In particular, a small direct sum of c-soft sheaves is c-soft.*

Proof. Apply Proposition 5.1.7 and Lemma 5.2.2 after remarking that a small direct sum is a filtrant inductive limit of finite direct sums. q.e.d.

Proposition 5.2.4. *Assume F is c-soft on X.*

(i) *If $i_Z : Z \hookrightarrow X$ is the embedding of a locally closed subset in X, then $i_Z^{-1}F$ is c-soft,*

(ii) *If $f : X \to Y$ is continuous, then $f_!F$ is c-soft on Y,*

(iii) *for Z as in* (i)*, F_Z is c-soft.*

Proof. (i) If Z is open, this is clear and if Z is closed, this follows from Lemma 5.2.2.
(ii) Let K be a compact subset of X. Consider the diagram:

$$\begin{array}{ccc}
\Gamma_c(X;F) & \longrightarrow & \Gamma_c(f^{-1}(K);F) \\
\downarrow & & \downarrow \\
\Gamma_c(Y;f_!F) & \longrightarrow & \Gamma_c(K;f_!F)
\end{array}$$

The first horizontal arrow is surjective by Lemma 5.2.2 and the vertical arrows are isomorphisms.
(iii) follows from (i) and (ii) since $F_Z \simeq i_{Z!}i_Z^{-1}F$. q.e.d.

Proposition 5.2.5. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume F' is c-soft. Then the sequence*

$$0 \to \Gamma_c(X;F') \xrightarrow{\alpha} \Gamma_c(X;F) \xrightarrow{\beta} \Gamma_c(X;F'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma_c(X; F'')$ and let U be an open neighborhood of $\mathrm{supp}(s'')$, U being relatively compact. In order to prove that s is in the image of $\Gamma_c(X; F) \to \Gamma_c(X; F'')$, we may replace F', F, F'' by F'_U, F_U, F''_U. Then we may replace X by $\bar{U}$, hence we may assume from the beginning that X is compact.

Let $\{K_i\}_{i=1}^n$ be a finite covering of X by compact subsets and let $s_i \in \Gamma(K_i; F)$ such that $\beta(s_i) = s''|_{K_i}$. We argue by induction on n, and reduce the proof to the case $n = 2$. Then $s_1|_{K_1 \cap K_2} - s_2|_{K_1 \cap K_2}$ belongs to $\Gamma(K_1 \cap K_2; F')$. We extend this element to $s' \in \Gamma(X; F')$ and replace s_2 by $s_2 + s'$. Hence there exists $t \in \Gamma(K_1 \cup K_2; F)$ with $\beta(t) = s''$ and the induction proceeds. q.e.d.

Proposition 5.2.6. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves, and assume F' and F are c-soft. Then F'' is soft.*

The proof is similar to that of Proposition 4.4.4.

Proposition 5.2.7. *Let S be a closed subset and K a compact subset of X. The category of c-soft sheaves is injective with respect to the functors $\Gamma_c(X; \bullet)$, $\Gamma_c(S; \bullet|_S)$, $f_!$ and $\Gamma(K; \bullet)$.*

The proof is left as an exercise.

Proposition 5.2.8. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then F is c-soft if and only if $H^j_c(U; F) \simeq 0$ for any U open in X and any $j > 0$.*

Proof. It follows from Proposition 5.2.7 that the condition is necessary. Let us prove the converse. Assume that $H^j_c(U; F) \simeq 0$ for any U open in X and any $j > 0$. Let K be a compact subset. Applying Proposition 4.2.4, we have an exact sequence

$$0 \to F_{X \setminus K} \to F \to F_K \to 0.$$

Applying the functor $\Gamma_c(X; \bullet)$ to this exact sequence, the result follows since $H^1_c(X \setminus K; F) \simeq 0$ by the hypothesis. q.e.d.

Proposition 5.2.9. *Assume X is locally compact and countable at infinity. Then the category of c-soft sheaves is injective with respect to the functor $\Gamma(X; \bullet)$.*

Proof. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves, with F' c-soft. Let $\{K_n\}_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of X, with $X = \cup_n K_n$.

The sequences

$$0 \to \Gamma(K_n; F') \to \Gamma(K_n; F) \to \Gamma(K_n; F'') \to 0$$

are all exact, and the morphisms $\Gamma(K_{n+1}; F') \to \Gamma(K_n; F')$ are all surjective. Hence the sequence obtained by taking the projective limit will remain exact by the Mittag-Leffler property. (See Exercise 4.3.) q.e.d.

Proposition 5.2.10. *Assume X is locally compact and countable at infinity. Let $X = \bigcup_{i\in I} U_i$ be an open covering of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Assume that $F|_{U_i}$ is soft for all $i \in I$. Then F is soft.*

In other words, to be soft is a local property.

Proof. The proof is similar to that of Proposition 4.4.6. q.e.d.

Example 5.2.11. (i) On a locally compact space X, any sheaf of C^0_X-modules is soft.
(ii) Let X be a real manifold of class C^∞, let K be a compact subset of X and U an open neighborhood of K in X. By the existence of "partition of unity", there exists a real $\mathcal{C}^\infty$-function φ with compact support contained in U and which is identically 1 in a neighborhood of K. It follows that any sheaf of C^∞_X-modules is soft.
(iii) Flabby sheaves are soft.

5.3 Derived proper direct images

Consider a morphism $f: X \to Y$ of locally compact spaces. One denotes by $Rf_!$ its right derived functor:

$$Rf_!: \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y).$$

By Proposition 5.2.7, if $F \in \mathrm{Mod}(\mathbf{k}_X)$, then $Rf_!F \simeq f_!F^\bullet$, where $F^\bullet$ is a c-soft resolution of F. Moreover, if $g: Y \to Z$ is another morphism of locally compact spaces, then, by Proposition 5.2.4,

$$R(g \circ f)_! \simeq Rg_! \circ Rf_!. \tag{5.8}$$

In the sequel, we shall always make Hypothesis 5.3.2 below.

Definition 5.3.1. Let $d \in \mathbb{N}$. One says that f has c-soft dimension $\leq d$ if $H^j(Rf_!F) = 0$ for all $j > d$ and all $F \in \mathrm{Mod}(\mathbf{k}_X)$. One says that f has finite c-soft dimension if there exists $d \geq 0$ such that f has c-soft dimension $\leq d$.

Hypothesis 5.3.2. The map f has finite c-soft dimension.

Remark 5.3.3. It follows from Theorem 5.1.2 that f has c-soft dimension $\leq d$ only if, for any $y \in Y$, the restriction $f|_{f(y)^{-1}}$ has c-soft dimension $\leq d$.

Note that assuming Hypothesis 5.3.2, the functor $Rf_!$ induces a functor:

$$Rf_! \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y).$$

Projection formula

First, we derive the isomorphism in Proposition 5.1.11.

Theorem 5.3.4. (Projection formula.) *Let $f \colon X \to Y$ be a morphism of locally compact spaces. Let $F \in \mathrm{D}^+(\mathbf{k}_X)$ and $G \in \mathrm{D}^+(\mathbf{k}_Y)$. Then there is a natural isomorphism*

$$Rf_! F \overset{\mathrm{L}}{\otimes} G \simeq Rf_!(F \overset{\mathrm{L}}{\otimes} f^{-1}G).$$

Proof. Let $F^\bullet$ be a c-soft resolution of F in $K^+(\mathrm{Mod}(\mathbf{k}_X))$ and let $G^\bullet$ be a flat resolution of G in $K^-(\mathrm{Mod}(\mathbf{k}_Y))$. By the hypothesis on the Tor-dimension of $\mathbf{k}$, we may assume that $G^\bullet \in K^b(\mathrm{Mod}(\mathbf{k}_Y))$.

Notice that if F^i is c-soft and G^j is a flat sheaf, then $F^i \otimes f^{-1}G^j$ is acyclic for the functor $f_!$. It follows that $Rf_!(F \overset{\mathrm{L}}{\otimes} f^{-1}G)$ is represented by the complex $f_!(F^\bullet \otimes f^{-1}G^\bullet)$. On the other hand, $Rf_! F \overset{\mathrm{L}}{\otimes} G$ is represented by the complex $f_! F^\bullet \otimes G^\bullet$. Hence, the result follows from Proposition 5.1.11. q.e.d.

Base change formula

Next, we derive the isomorphism in Theorem 5.1.9.

Theorem 5.3.5. (Base change formula.) *Consider the Cartesian square (5.2). Then there is an isomorphism in $\mathrm{D}^+(\mathbf{k}_{Y'})$, functorial in $F \in \mathrm{D}^+(\mathbf{k}_X)$:*

$$g^{-1} Rf_! F \simeq Rf'_! g'^{-1} F.$$

In particular, for $y \in Y$, we have the isomorphism

$$(Rf_! F)_y \simeq \mathrm{R}\Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}). \tag{5.9}$$

Proof. It is enough to prove that $g^{-1} \circ Rf_!$ is the derived functor of $g^{-1} \circ f_!$, which is obvious and $Rf_! \circ g'^{-1}$ is the derived functor of $f'_! \circ g'^{-1}$.

Denote by $\mathcal{I}_X$ the subcategory of $\mathrm{Mod}(\mathbf{k}_X)$ consisting of sheaves F such that for all $y \in Y$, $F|_{f^{-1}(y)}$ is c-soft, and define similarly $\mathcal{I}_{X'}$. Then $\mathcal{I}_X$ is injective with respect to g'^{-1} and g'^{-1} sends $\mathcal{I}_X$ into $\mathcal{I}_{X'}$. Moreover, $\mathcal{I}_{X'}$ is injective with respect to $f'_!$. q.e.d.

Let us give two important corollaries which are particularly important in Algebraic Topology. The first one tells us that the cohomology with compact support of a topological space X, with values in a commutative group M, *i.e.,* the cohomology of the constant sheaf M_X, is known as soon as it is known over $\mathbb{Z}$. The second one tells us how to calculate the cohomology of a product.

Universal coefficients formula

Corollary 5.3.6. (Universal coefficients formula.) *Let* $M \in \mathrm{Mod}(\mathbf{k})$.

(i) *One has the isomorphism* $\mathrm{R}\Gamma_c(X; M_X) \simeq \mathrm{R}\Gamma_c(X; \mathbf{k}_X) \overset{\mathrm{L}}{\otimes} M$.

(ii) *Assume* $\mathbf{k} = \mathbb{Z}$. *Then*

$$\begin{aligned} \mathrm{R}\Gamma_c(X; M_X) &\simeq \bigoplus_j H_c^j(X; M_X)\,[-j] \\ &\simeq \bigoplus_j \Big(H_c^j(X; \mathbb{Z}_X) \otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{Z}(H_c^{j+1}(X; \mathbb{Z}_X), M) \Big)\,[-j]. \end{aligned}$$

Proof. (i) One has $M_X = a_X^{-1} M_{\mathrm{pt}} \overset{\mathrm{L}}{\otimes} \mathbf{k}_X$. By the projection formula, we get:

$$Ra_{X!}(a_X^{-1} M_{\mathrm{pt}} \overset{\mathrm{L}}{\otimes} \mathbf{k}_X) \simeq Ra_{X!}\mathbf{k}_X \overset{\mathrm{L}}{\otimes} M.$$

(ii) Since the homological dimension of the ring $\mathbb{Z}$ is one, we have for $N \in \mathsf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{Z}))$ and $M \in \mathrm{Mod}(\mathbb{Z})$:

$$\begin{aligned} &N \simeq \oplus_j H^j(N)[-j], \\ &N \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} M \simeq \bigoplus_j \big(H^j(N) \otimes M \oplus Tor_1^{\mathbb{Z}}(H^{j+1}(N), M) \big)\,[-j]. \end{aligned}$$

q.e.d.

Notation 5.3.7. Let X and Y be two topological spaces. One sets:

$$F \overset{\mathrm{L}}{\boxtimes} G = q_1^{-1} F \overset{\mathrm{L}}{\otimes} q_2^{-1} G.$$

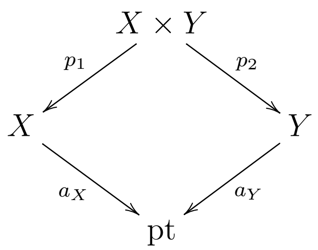
Künneth formula

Corollary 5.3.8. (Künneth formula.) *Let X and Y be two locally compact spaces. Let $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G \in \mathrm{D}^+(\mathbf{k}_Y)$. Then:*

$$\mathrm{R}\Gamma_c(X \times Y; F \overset{\mathrm{L}}{\boxtimes} G) \simeq \mathrm{R}\Gamma_c(X; F) \overset{\mathrm{L}}{\otimes} \mathrm{R}\Gamma_c(Y; G). \tag{5.10}$$

Proof. Consider the diagram:

(5.11)



Then:

$$\begin{aligned} Ra_{X\times Y!}(F \overset{\mathrm{L}}{\boxtimes} G) &\simeq Ra_{Y!}Rp_{2!}(p_1^{-1}F \overset{\mathrm{L}}{\otimes} p_2^{-1}G) \\ &\simeq Ra_{Y!}((Rp_{2!}p_1^{-1}F) \overset{\mathrm{L}}{\otimes} G) \\ &\simeq Ra_{Y!}(a_Y^{-1}Ra_{X!}F \overset{\mathrm{L}}{\otimes} G) \\ &\simeq Ra_{X!}F \overset{\mathrm{L}}{\otimes} Ra_{Y!}G. \end{aligned}$$

q.e.d.

5.4 The functor $f^!$

All over this section, we shall assume that all morphisms of locally compact spaces have finite c-soft dimension (see hypothesis 5.3.2).

Let $f\colon X \to Y$ be a continuous map of locally compact topological spaces. Applying Theorem 1.1.9 and Lemma 5.2.3, we get:

Theorem 5.4.1. *The functor $Rf_! : \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y)$ admits a right adjoint.*

One denotes by $f^!$ this adjoint.

In other words, for $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G \in \mathrm{D}^+(\mathbf{k}_Y)$, we have an isomorphism functorial with respect to F and G:

$$\mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_Y)}(Rf_!F, G) \simeq \mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_X)}(F, f^!G).$$

Notice that the functor $f^!\colon \mathrm{D}^+(\mathbf{k}_Y) \to \mathrm{D}^+(\mathbf{k}_X)$ is not the derived functor of any functor in general.

For a direct proof not using the Brown representability theorem, we refer to[GM96], [KS90].

We discuss its applications. First, notice that we get natural morphisms:

$$Rf_!f^!G \to G, \quad F \to f^!Rf_!F.$$

Proposition 5.4.2. *Let $g : Y \to Z$ be a continuous map satisfing Hypothesis 5.3.2. Then $g \circ f$ satisfies Hypothesis 5.3.2 and*

$$(g \circ f)^! \simeq f^! \circ g^!.$$

Proof. Both results immediately follow from (5.8). q.e.d.

Proposition 5.4.3. *Consider the Cartesian square* (5.2) *Assume f satisfies Hypothesis 5.3.2. Then f' satisfies Hypothesis 5.3.2 and there is a natural isomorphism of functors from* $\mathrm{D}^+(\mathbf{k}_{X'})$ *to* $\mathrm{D}^+(\mathbf{k}_Y)$*:*

$$f^! \circ Rg_* \simeq Rg'_* \circ f'^!. \tag{5.12}$$

Proof. The results follow from Remark 5.3.3. and Theorem 5.3.5 by adjunction. q.e.d.

Proposition 5.4.4. *In the situation of Theorem 5.4.1, one has:*

(i) $\mathrm{RHom}\,(Rf_!F, G) \simeq \mathrm{RHom}\,(F, f^!G)$

(ii) $R\mathcal{H}om\,(Rf_!F, G) \simeq Rf_*R\mathcal{H}om\,(F, f^!G)$.

Proof. (i) follows from (ii) by applying $\mathrm{R}\Gamma(Y; \bullet)$.
(ii) Consider:

$$\begin{aligned} Rf_*R\mathcal{H}om\,(F, f^!G) &\to R\mathcal{H}om\,(Rf_!F, Rf_!f^!G) \\ &\to R\mathcal{H}om\,(Rf_!F, G). \end{aligned}$$

Let us prove that the composite of these two morphisms is an isomorphism by applying $H^j\mathrm{R}\Gamma(V; \bullet)$ to both terms for V open in Y. We get :

$$\begin{aligned} H^j\mathrm{R}\Gamma(V; Rf_*R\mathcal{H}om\,(F, f^!G)) &\simeq \mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_{f^{-1}(V)})}(F|_{f^{-1}(V)}, f^!G)[j]) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_V)}(Rf_!F|_V, G[j]) \\ &\simeq H^j\mathrm{R}\Gamma(V; R\mathcal{H}om\,(Rf_!F, G). \end{aligned}$$

q.e.d.

Proposition 5.4.5. *Let $G_1, G_2 \in \mathrm{D}^+(\mathbf{k}_Y)$. There is a natural morphism*

$$f^! G_1 \otimes f^{-1} G_2 \to f^!(G_1 \overset{\mathrm{L}}{\otimes} G_2).$$

Proof. Consider the chain of morphisms:

$$\begin{aligned} \mathrm{Hom}\,(G_1 \otimes G_2, H) &\to \mathrm{Hom}\,(Rf_! f^! G_1 \otimes G_2, H) \\ &\simeq \mathrm{Hom}\,(Rf_!(f^! G_1 \otimes f^{-1} G_2), H) \\ &\simeq \mathrm{Hom}\,(f^! G_1 \otimes f^{-1} G_2, f^! H). \end{aligned}$$

Choosing $H = G_1 \otimes G_2$, we get the result. q.e.d.

Given a map $f\colon X \to Y$, we may decompose it by its graph:

$$f\colon X \hookrightarrow X \times Y \to Y.$$

In view of Proposition 5.4.2, in order to calculate $f^!$ it is thus enough to do it when f is an isomorphism on a closed subset and when f is a projection.

Proposition 5.4.6. *Assume that $f\colon X \to Y$ is a closed embeddding, that is, induces an isomorphism from X onto a closed subset Z of Y. Then*

$$f^!(\,\bullet\,) \simeq f^{-1} \circ \mathrm{R}\Gamma_Z(\,\bullet\,).$$

Proof. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)$.

$$\begin{aligned} \mathrm{Hom}\,(Rf_! F, G) &\simeq \mathrm{Hom}\,(Rf_! F \otimes \mathbf{k}_Z, G) \simeq \mathrm{Hom}\,(Rf_! F, \mathrm{R}\Gamma_Z G) \\ &\simeq \mathrm{Hom}\,(f^{-1} Rf_! F, f^{-1}\mathrm{R}\Gamma_Z G) \simeq \mathrm{Hom}\,(F, f^{-1}\mathrm{R}\Gamma_Z G). \end{aligned}$$

q.e.d.

Proposition 5.4.7. *Let $G_1, G_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)$. Then:*

$$f^! R\mathcal{H}om\,(G_2, G_1) \simeq R\mathcal{H}om\,(f^{-1} G_2, f^! G_1).$$

Proof. For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, one has:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)}(F, f^! R\mathcal{H}om\,(G_2, G_1)) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)}(Rf_! F, R\mathcal{H}om\,(G_2, G_1)) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)}(Rf_! F \overset{\mathrm{L}}{\otimes} G_2, G_1) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)}(Rf_!(F \overset{\mathrm{L}}{\otimes} f^{-1} G_2), G_1) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)}(F \overset{\mathrm{L}}{\otimes} f^{-1} G_2, f^! G_1) \\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)}(F, R\mathcal{H}om\,(f^{-1} G_2, f^! G_1)). \end{aligned}$$

Since these isomorphisms hold for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, the result follows. q.e.d.

Consider the diagram, were as usual δ_X denotes the diagonal embedding:

$$\begin{array}{ccccc} & \Delta_X & \overset{\delta}{\hookrightarrow} & X \times X & \\ & & \swarrow^{q_1} & & \searrow^{q_2} \\ & X & & & X \end{array}$$

Corollary 5.4.8. *Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then, identifying Δ_X with X by q_1,*

$$R\mathcal{H}om\,(F_2, F_1) \simeq \delta^! R\mathcal{H}om\,(q_2^{-1}F_2, q_1^! F_1).$$

Proof.

$$\begin{aligned} \delta^! R\mathcal{H}om\,(q_2^{-1}F_2, q_1^! F_1) &\simeq R\mathcal{H}om\,(\delta^{-1}q_2^{-1}F_2, \delta^! q_1^! F_1) \\ &\simeq R\mathcal{H}om\,(F_2, F_1). \end{aligned}$$

q.e.d.

The next proposition is analogous to the Künneth formula, replacing the functor $q_2^{-1}(\,\bullet\,)\overset{\mathrm{L}}{\otimes} q_1^!(\,\bullet\,)$ with the functor $\mathrm{RHom}\,(q_2^{-1}(\,\bullet\,), q_1^!(\,\bullet\,))$.

Proposition 5.4.9. *Let X and Y be topological spaces with finite c-soft dimension. Then for $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)$, $F \in \mathsf{D}^{+}(\mathbf{k}_X)$, one has:*

$$\mathrm{RHom}\,(q_2^{-1}G, q_1^! F)) \simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(Y; G), \mathrm{R}\Gamma(X; F)).$$

Proof. Consider Diagram 5.11. Then:

$$\begin{aligned} \mathrm{R}\Gamma(X \times Y; R\mathcal{H}om\,(q_2^{-1}G, q_1^! F)) &\simeq Ra_{X*}Rq_{1*}R\mathcal{H}om\,(q_2^{-1}G, q_1^! F) \\ &\simeq Ra_{X*}R\mathcal{H}om\,(Rq_{1!}q_2^{-1}G, F) \\ &\simeq Ra_{X*}R\mathcal{H}om\,(a_X^{-1}Ra_{Y!}G, F) \\ &\simeq \mathrm{RHom}\,(Ra_{Y!}G, Ra_{X*}F). \end{aligned}$$

q.e.d.

Definition 5.4.10. Assume that $f : X \to Y$ satisfies Hypothsis 5.3.2. One sets:

$$\omega_{X/Y} = f^! \mathbf{k}_Y$$

and calls $\omega_{X/Y}$ the relative dualizing complex.

If X has finite c-soft dimension, one sets:

$$\omega_X = \omega_{X/\mathrm{pt}} = f^! \mathbf{k}_{\mathrm{pt}},$$

and calls ω_X the dualizing complex on X.

Note that by applying Proposition 5.4.5 with $F_1 = \mathbf{k}_X$, we get a natural morphism:

$$f^{-1}G \otimes \omega_{X/Y} \to f^!G.$$

For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, one defines the two dual objects to F:

$$\begin{aligned} \mathrm{D}'F &= R\mathcal{H}om\,(F, \mathbf{k}_X), \\ \mathrm{D}F &= R\mathcal{H}om\,(F, \omega_X). \end{aligned}$$

The object $\mathrm{D}F$ is often called "the Verdier dual" of F. We denote by * the duality functor on $\mathrm{Mod}(\mathbf{k})$:

$$(5.13) \qquad {}^* = \mathrm{RHom}_{\mathbf{k}}(\,\bullet\,, \mathbf{k}),\ \mathsf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbf{k}))^{\mathrm{op}} \to \mathrm{D}^{+}(\mathrm{Mod}(\mathbf{k})).$$

Using the adjunction $(Ra_{X!}, a_X^!)$, we get :

$$\begin{aligned} \mathrm{RHom}\,(F, \omega_X) &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(X; F), \mathbf{k}) \\ &= (\mathrm{R}\Gamma_c(X, F))^*. \end{aligned}$$

Choosing $F := \mathbf{k}_X$, we find:

Corollary 5.4.11. *Assume that X has finite c-soft dimension. Then*

$$(\mathrm{R}\Gamma_c(X; \mathbf{k}_X))^* \simeq \mathrm{R}\Gamma(X; \omega_X).$$

When X is a topological n-dimensional manifold of class $\mathcal{C}^\infty$, we shall see that that ω_X is the orientation sheaf shifted by n, and Corollary 5.4.11 is a formulation of the classical Poincaré duality theorem.

5.5 Orientation and duality on $\mathcal{C}^0$-manifolds

A $\mathcal{C}^0$-manifold X is a Hausdorff, locally compact, countable at infinity topological space which is locally isomorphic to a real finite dimensional vector space. Recall that the dimension of such a vector space is a topological invariant, hence the dimension of X is a well-defined locally constant function on X that we denote by d_X.

Lemma 5.5.1. *Let V be a real vector space of dimension n and let F be a sheaf on V. Then $H^j_c(V; F) = 0$ for $j > n$.*

Proof. (i) Assume $n = 1$. We may replace V by the open interval $]0, 1[$. Denote by j the embedding $]0, 1[\hookrightarrow [0, 1]$. Then $j_!$ is exact and we deduce that $H^j_c(]0, 1[; F) \simeq H^j([0, 1]; j_!F)$. Then, the result follows from Lemma 4.5.1.

(ii) Assume the result if proved for linear spaces of dimension less than n. Let $p : V \to V'$ be a surjective linear map with $\dim V' = n-1$. By the above result and the base change formula, $R^j p_! = 0$ for $j \neq 0, 1$. Hence have a d.t. $R^0 p_! F \to R p_! F \to R^1 p_! F[-1] \xrightarrow{+1}$, which gives a long exact sequence:

$$\cdots \to H_c^j(V'; R^0 p_! F) \to H_c^j(V; F) \to H_c^{j-1}(V'; R^1 p_! F) \to \cdots$$

Then the result follows by induction. q.e.d.

Proposition 5.5.2. *Let X be a $\mathcal{C}^0$-manifold of constant dimension n and let F be a sheaf on X. Then:*

(i) $H^j(X; F) = 0$ *for* $j > n$,

(ii) $H_c^j(X; F) = 0$ *for* $j > n$,

(iii) *the c-soft dimension of X is n.*

Proof. (i)–(ii) Let $0 \to F \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots$ be an injective resolution of F, and let $G^n := \operatorname{Ker} d^n$. It is enough to prove that G^n is c-soft. This is a local problem, and we may assume $X = V$ is a real vector space. Let U be an open subset of V. Since $H_c^j(U; F) \simeq H_c^j(V; F_U)$, these groups vanish for $j > n$ by Lemma 5.5.1 and the result follows from Proposition 5.2.8.
(iii) By (ii) the c-soft dimension of X is $\leq n$. The result follows since $H_c^n(X; \mathbf{k}_X) \neq 0$ when $X = \mathbb{R}^n$. q.e.d.

Lemma 5.5.3. *Let X be a topological manifold of dimension n. Then $H^k(\omega_X) = 0$ for $k \neq -n$, and the sheaf $H^{-n}(\omega_X)$ is locally isomorphic to $\mathbf{k}_X$.*

Proof. We may assume $X = \mathbb{R}^n$. Then for U open in X, one has the isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma(U; \omega_X) &\simeq \mathrm{RHom}\,(\mathbf{k}_U, a_X^!(\mathbf{k})) \\ &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U; \mathbf{k}_X), \mathbf{k}) \\ &= (\mathrm{R}\Gamma_c(U; \mathbf{k}_X))^*. \end{aligned}$$

If U is convex and non empty, one already knows that $\mathrm{R}\Gamma_c(U; \mathbf{k}_U)$ is isomorphic to $\mathbf{k}\,[-n]$. Hence $H^k(\omega_X) = 0$ for $k \neq -n$ and the restriction morphisms $\Gamma(X; H^{-n}(\omega_X)) \to \Gamma(U; H^{-n}(\omega_X))$ are isomorphisms for U convex and non empty. q.e.d.

Definition 5.5.4. Let X be a $\mathcal{C}^0$-manifold of dimension d_X. One sets:

$$\mathrm{or}_X^{\mathbf{k}} = H^{-d_X}(\omega_X)$$

and calls this sheaf the orientation sheaf on X. If there is no risk of confusion, we write or_X instead of $\mathrm{or}_X^{\mathbf{k}}$.

Note that

$$\omega_X \simeq \mathrm{or}_X^{\mathbf{k}}[d_X], \quad \mathrm{or}_X^{\mathbf{k}} \simeq \mathrm{or}_X^{\mathbb{Z}} \otimes_{\mathbb{Z}_X} \mathbf{k}_X.$$

Proposition 5.5.5. *Let X be a $\mathcal{C}^0$-manifold of dimension d_X.*

(i) or_X *is the sheaf associated to the presheaf:* $U \mapsto \mathrm{Hom}_k(H_c^{d_X}(U;\mathbf{k}_X)),\mathbf{k})$,

(ii) or_X *is locally free of rank one over* $\mathbf{k}_X$*, and* $\mathrm{or}_{X,x} \simeq (H^{d_X}_{\{x\}}(\mathbf{k}_X))^*$,

(iii) $\mathrm{or}_X \otimes \mathrm{or}_X \simeq \mathbf{k}_X$*, and* $\mathcal{H}om\,(\mathrm{or}_X,\mathbf{k}_X) \simeq \mathrm{or}_X$,

(iv) *if X is of class $\mathcal{C}^1$, then* or_X *coincides with the orientation sheaf defined in* Example 3.3.10.

Assertions (i) to (iii) are easily deduced from the previous discussion. We refer to [KS90] for a proof of (iv).

Applying Corollary 5.4.11, we obtain the Poincaré duality theorem with coefficients in $\mathbf{k}$:

Corollary 5.5.6. (Poincaré duality.) *Let X be $\mathcal{C}^0$-manifold of dimension d_X. Then*

$$(\mathrm{R}\Gamma_c(X;\mathbf{k}_X))^* \simeq \mathrm{R}\Gamma(X;\mathrm{or}_X)\,[d_X].$$

Definition 5.5.7. Let $f\colon X \to Y$ be a continuous map of topological spaces. One says that f is a topological submersion of relative dimension d if, locally on X, there exists an isomorphism $X \simeq Y \times \mathbb{R}^d$ and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & Y \times \mathbb{R}^d \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & \\ Y & & \end{array}$$

such that p is the projection.

Proposition 5.5.8. *Assume that $f : X \to Y$ is a topological submersion of relative dimension d. Let $G \in \mathrm{D}^+(\mathbf{k}_Y)$. Then there is a natural isomorphism* $f^{-1}G\overset{\mathrm{L}}{\otimes}\omega_{X/Y} \xrightarrow{\sim} f^!G$.

Proof. The natural morphism $f^{-1}G\overset{\mathrm{L}}{\otimes}\omega_{X/Y} \to f^!G$ is given by Proposition 5.4.5. To check it is an isomorphism, we may assume $X = Y \times T$ and f is the projection. We may assume $Y = U$ is a non empty open convex subset of a real vector space of dimension d. Then

$$\begin{aligned}
\mathrm{R}\Gamma(U \times V; f^!G) &\simeq \mathrm{RHom}\,(\mathbf{k}_{U\times V}, f^!G)\\
&\simeq \mathrm{RHom}\,(Rf_!\mathbf{k}_{U\times V}, G)\\
&\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbf{k}_U)\overset{\mathrm{L}}{\otimes}\mathbf{k}_V, G)\\
&\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbf{k}_U),\mathbf{k}) \otimes \mathrm{RHom}\,(\mathbf{k}_V, G)\\
&\simeq \mathrm{RHom}\,(\mathbf{k}_V, G)[d].
\end{aligned}$$

Here, we use the fact that the cohomology of $\mathrm{R}\Gamma_c(U;\mathbf{k}_U)$ is isomorphic to $\mathbf{k}[-d]$. On the other hand, since $\omega_{X/Y}$ is locally isomorphic to $\mathbf{k}_X[d]$, it remains to remark that

$$\mathrm{R}\Gamma(U \times V; f^{-1}G) \simeq \mathrm{R}\Gamma(V;G).$$

q.e.d.

5.6 Cohomology of real and complex manifolds

De Rham cohomology

Let X be a real $\mathcal{C}^\infty$-manifold of dimension n (this implies in particular that X is locally compact and countable at infinity). If $n > 0$, the sheaf $\mathbb{C}_X$ is not acyclic for the functor $\Gamma(X;\bullet)$ in general. In fact consider two connected open subsets U_1 and U_2 such that $U_1 \cap U_2$ has two connected components, V_1 and V_2. The sequence:

$$0 \to \Gamma(U_1 \cup U_2;\mathbb{C}_X) \to \Gamma(U_1;\mathbb{C}_X) \oplus \Gamma(U_2;\mathbb{C}_X) \to \Gamma(U_1 \cap U_2;\mathbb{C}_X) \to 0$$

is not exact since the locally constant function $\varphi = 1$ on V_1, $\varphi = 2$ on V_2 may not be decomposed as $\varphi = \varphi_1 - \varphi_2$, with φ_j constant on U_j. By the Mayer-Vietoris long exact sequence, this implies:

$$H^1(U_1 \cup U_2;\mathbb{C}_X) \neq 0.$$

On the other hand, for K a compact subset in X and U an open neighborhood of K in X, there exists a real $\mathcal{C}^\infty$-function φ with compact support contained in U and which is identically 1 in a neighborhood of K (existence of "partition of unity"). This implies that the sheaf $\mathcal{C}^\infty_X$ is c-soft, as well as any sheaf of $\mathcal{C}^\infty_X$-modules. In particular, the sheaves $\mathcal{C}^{\infty,(p)}_X$ or $\mathcal{D}b^{(p)}_X$ of differential forms with $\mathcal{C}^\infty_X$ or distributions coefficients are c-soft and in particular $\Gamma(X;\cdot)$ and $\Gamma_c(X;\cdot)$ acyclic.

Recall that by its definition, the space $\Gamma_c(X;\mathcal{D}b_X)$ of distributions with compact support is the topological dual of the space $\Gamma(X;\mathcal{C}^{\infty,(n)}_X \otimes \mathrm{or}_X)$ of $\mathcal{C}^\infty$-densities. Integration over X defines the embedding of $\Gamma_c(X;\mathcal{C}^\infty_X)$ in $\Gamma_c(X;\mathcal{D}b_X)$, hence defines $\mathcal{C}^\infty_X$ as a subsheaf of $\mathcal{D}b_X$.

Therefore, the sheaves $\mathcal{C}^{\infty,(j)}_X$ are naturally embedded into the sheaves $\mathcal{D}b^{(j)}_X$ of differential forms with distributions as coefficients and the differential on $\mathcal{D}b^{(j)}_X$ induces the differential on $\mathcal{C}^{\infty,(j)}_X$.

Notation 5.6.1. consider the complexes

$$\mathcal{C}^{\infty,(\bullet)}_X := 0 \to \mathcal{C}^{\infty,(0)}_X \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}_X \to 0, \tag{5.14}$$

$$\mathcal{D}b^{(\bullet)}_X := 0 \to \mathcal{D}b^{(0)}_X \xrightarrow{d} \cdots \to \mathcal{D}b^{(n)}_X \to 0. \tag{5.15}$$

We call them the De Rham complexes on X with $\mathcal{C}^\infty$ and distributions coefficients, respectively.

Lemma 5.6.2. (The Poincaré lemma.) *Let $I = (]0,1[)^n$ be the unit open cube in $\mathbb{R}^n$. The complexes below are exact.*

$$0 \to \mathbb{C} \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0,$$
$$0 \to \mathbb{C} \to \mathcal{D}b^{(0)}(I) \xrightarrow{d} \cdots \to \mathcal{D}b^{(n)}(I) \to 0.$$

Proof. We shall only treat the case of $\mathcal{C}^\infty(I)$. Consider the Koszul complex $K^\bullet(M,\varphi)$ over the ring $\mathbb{C}$, where $M = \mathcal{C}^\infty(I)$ and $\varphi = (\partial_1,\dots,\partial_n)$ (with $\partial_j = \frac{\partial}{\partial x_j}$). This complex is nothing but the complex:

$$0 \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0.$$

Clearly $H^0(K^\bullet(M,\varphi)) \simeq \mathbb{C}$, and it is enough to prove that the sequence $(\partial_1,\dots,\partial_n)$ is coregular. Let $M_{j+1} = \mathrm{Ker}(\partial_1) \cap \cdots \cap \mathrm{Ker}(\partial_j)$. This is the space of $\mathcal{C}^\infty$-functions on I constant with respect to the variables $x_1,\dots,x_j$. Clearly, ∂_{j+1} is surjective on this space. q.e.d.

The Poincaré lemma may be formulated intrinsically as:

Lemma 5.6.3. (The de Rham complex.) *Let X be a $\mathcal{C}^\infty$-manifold of dimension n. Then the natural morphisms $\mathbb{C}_X \to \mathcal{C}^{\infty,(\bullet)}_X$ and $\mathbb{C}_X \to \mathcal{D}b^{(\bullet)}_X$ are quasi-isomorphisms.*

We shall prove a finiteness and duality theorem for the cohomology of a compact manifold when the base ring **k** is the field $\mathbb{C}$. The duality result gives in this case an alternative proof of Corollary 5.5.6.

Theorem 5.6.4. (Poincaré duality on smooth manifolds.) *Assume X is compact. Then the $\mathbb{C}$-vector spaces $H^j(X;\mathbb{C}_X)$ and $H^{n-j}(X;or_X^{\mathbb{C}})$ are finite dimensional and dual one to each other.*

Proof. We shall make use of some results of functional analysis (refer to [Ko69]).

The vector spaces $\Gamma(X;\mathcal{C}_X^{\infty,(p)})$ are naturally endowed with a structure of Fréchet-Schwartz spaces (spaces of type FS), and the spaces $\Gamma(X;\mathcal{D}b_X^{(p)})$ are naturally endowed with a structure of dual of Fréchet-Schwartz spaces (spaces of type DFS). Set

$$\begin{aligned} E^\bullet &:= \Gamma(X;\mathcal{C}_X^{\infty,(\bullet)}), \\ F^\bullet &:= \Gamma(X;\mathcal{D}b_X^{(\bullet)}), \\ G^\bullet &:= \Gamma(X;\mathcal{D}b_X^{(\bullet)}\otimes \mathrm{or}_X). \end{aligned}$$

(i) Finiteness. The embedding $\mathcal{C}_X^{\infty,(j)} \hookrightarrow \mathcal{D}b_X^{(j)}$ defines the morphism of complexes $E^\bullet \to F^\bullet$. This morphism is continuous for the topologies of spaces FS and DFS and induces an isomorphism on the cohomology. This implies the finiteness of the vector spaces $H^j(E^\bullet)$.
(ii) Duality. Since the sheaf $\mathrm{or}_X^{\mathbb{C}}$ is locally isomorphic to $\mathbb{C}_X$, one gets the isomorphism

$$\mathrm{R}\Gamma(X;\mathrm{or}_X^{\mathbb{C}}) \xrightarrow{\sim} \Gamma(X;\mathcal{D}b_X^{(\bullet)}\otimes \mathrm{or}_X). \tag{5.16}$$

The topological vector spaces $\Gamma(X;C_X^{\infty,(p)})$ and $\Gamma(X;\mathcal{D}b_X^{(n-p)}\otimes \mathrm{or}_X)$ are naturally dual to each other, the pairing being defined by

$$(\varphi,u)\mapsto \int_X \varphi\cdot u.$$

This pairing is compatible to the differential:

$$(\varphi,du)=(d\varphi,u)$$

In other words, the two complexes $E^\bullet$ and $G^\bullet$ endowed with their topologies of vector spaces of type FS and DFS respectively are dual to each other. Since they have finite dimensional cohomology objects, this implies that the spaces $H^j(E^\bullet)$ and $H^{n-j}(G^\bullet)$ are dual to each other. q.e.d.

Corollary 5.6.5. *Let X be a real compact connected manifold of dimension n. Then $H^n(X;\mathbb{C}_X)$ has dimension 0 or 1, and X is orientable if and only if this dimension is one.*

Proof. One has $H^0(X;\mathrm{or}_X)\neq 0$ if and only if or_X has a non identically zero global section, and if such a section exists, it will define a global isomorphism of or_X with $\mathbb{C}_X$. By the duality theorem, $H^0(X;\mathrm{or}_X)$ is the dual space to $H^n(X;\mathbb{C}_X)$. q.e.d.

Cohomology of complex manifolds

Assume now that X is a complex manifold of complex dimension n, and let $X^{\mathbb{R}}$ be the real underlying manifold. The real differential d splits as $\partial+\bar\partial$, and one denotes by $\mathcal{C}_X^{\infty,(p,q)}$ the sheaf of $\mathcal{C}^\infty$ forms of type (p,q) with respect to $\partial,\bar\partial$. Consider the complexes

$$\begin{aligned}\mathcal{C}_X^{\infty,(p,\bullet)} &:= 0\to\mathcal{C}_X^{\infty(p,0)}\xrightarrow{\bar\partial}\cdots\to\mathcal{C}_X^{\infty,(p,n)},\\ \mathcal{D}b_X^{(p,\bullet)} &:= 0\to\mathcal{D}b^{(p,0)}\xrightarrow{\bar\partial}\cdots\to\mathcal{D}b_X^{(p,n)}.\end{aligned}$$

Denote by Ω_X^p the sheaf of holomorphic p-forms. One usually sets

$$\Omega_X=\Omega_X^n.$$

The Dolbeault-Grothendieck lemma is formulated as:

Lemma 5.6.6. *Let X be a complex manifold. Then the natural morphisms $\Omega_X^p\to\mathcal{C}_X^{\infty,(p,\bullet)}$ and $\Omega_X^p\to\mathcal{D}b_X^{(p,\bullet)}$ are quasi-isomorphisms.*

Since the sheaves $\mathcal{C}_X^{\infty,(p,q)}$ and $\mathcal{D}b_X^{(p,q)}$ are c-soft, it follows that

$$\mathrm{R}\Gamma(X;\Omega_X^p)\xrightarrow{\sim}\Gamma(X;\mathcal{C}_X^{\infty,(p,\bullet)})\xrightarrow{\sim}\Gamma(X;\mathcal{D}b_X^{(p,\bullet)}). \tag{5.17}$$

(5.18)

Theorem 5.6.7. (The Cartan-Serre finiteness and duality theorems.) *Let X be a compact manifold of complex dimension n. Then the $\mathbb{C}$-vector spaces $H^j(X;\Omega_X^p)$ and $H^{n-j}(X;\Omega_X^{n-p})$ are finite dimensional and dual one to each other.*

The proof goes as in the real case, recalling that a complex manifold is naturally oriented.

The Leray-Grothendieck integration morphism

Let $f : X \to Y$ be a morphism of complex manifolds. Denote by d_X (resp. d_Y) the complex dimension of X (resp. Y), and set for short;

$$l = d_X - d_Y, \quad \text{(hence } l \in \mathbb{Z}.\text{)}$$

For $p, q \in \mathbb{Z}$ we have a natural morphism (inverse image of differential forms):

$$f^{-1}\mathcal{C}_Y^{\infty,(p,q)} \to \mathcal{C}_X^{\infty,(p,q)}$$

which commutes with $\bar{\partial}$ and defines by duality (recall that the complex manifolds X and Y are naturally oriented):

$$\int_f : f_!\mathcal{D}b_X^{(p,q)} \to \mathcal{D}b_Y^{(p-l,q-l)}. \tag{5.19}$$

These morphisms commute to $\bar{\partial}$ and define a morphism of complexes:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & f_!\mathcal{D}b_X^{(p,q)} & \xrightarrow{\bar{\partial}} & \mathcal{D}b_X^{(p,q+1)} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle\int_f} & & \downarrow{\scriptstyle\int_f} & \\
\cdots \longrightarrow & \mathcal{D}b_Y^{(p-l,q-l)} & \xrightarrow{\bar{\partial}} & \mathcal{D}b_Y^{(p-l,q-l+1)} & \longrightarrow \cdots
\end{array}$$

If one decides that $f_*\mathcal{D}b_X^{(p,d_X)}$ is in degree zero (hence, $\mathcal{D}b_Y^{(p-l,d_Y)}$ will also be in degree zero), the first line is quasi-isomorphic to $Rf_!\Omega_X^p\,[d_X]$ and the second line to $\Omega_Y^{p-l}\,[d_Y]$. Therefore we have constructed a morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Y)$:

$$Rf_!\Omega_X^{p+d_X}\,[d_X] \to \Omega_Y^{p+d_Y}\,[d_Y].$$

In particular

Theorem 5.6.8. The residue morphism. *To each morphism $f : X \to Y$ of complex manifolds, the construction above defines functorially a morphism:*

$$\int_f : Rf_!\Omega_X\,[d_X] \to \Omega_Y\,[d_Y].$$

By "functorially", we mean that $\int_{\mathrm{id}_X} = \mathrm{id}$ and $\int_g \circ \int_f = \int_{g\circ f}$. In the absolute case we have thus obtained a map:

$$\int_X : H_c^{d_X}(X;\Omega_X) \to \mathbb{C}. \tag{5.20}$$

A cohomology class $u \in H_c^{d_X}(X;\Omega_X)$ may be represented by a distribution $v \in \Gamma_c(X;\mathcal{D}b_X^{(d_X,d_X)})$ modulo $\bar\partial w$ with $w \in \Gamma_c(X;\mathcal{D}b_X^{(d_X,d_X-1)})$. Since $\bar\partial w = (\bar\partial+\partial)w$, we get that $\int_X \bar\partial w = 0$ and $\int_X u$ is well defined. This is the required morphism.

If $X = \mathbb{C}$, we get in particular an integration map: $H^1_{\{0\}}(\mathbb{C};\Omega_{\mathbb{C}}) \to H^1_c(\mathbb{C};\Omega_{\mathbb{C}}) \to \mathbb{C}$, and one checks easily that, representing $H^1_{\{0\}}(\mathbb{C};\Omega_{\mathbb{C}})$ by $\Gamma(D\setminus\{0\};\Omega_{\mathbb{C}})/\Gamma(D;\Omega_{\mathbb{C}})$, where D is a disc centered at 0, the integral coincides, up to a non-zero factor, with the residue morphism.

Exercises to Chapter 5

Exercise 5.1. Let U be a convex open subset of $\mathbb{R}^d$. Prove that $\mathrm{R}\Gamma_c(U;\mathbf{k}_U)$ is concentrated in degree d and $H^d(U;\mathbf{k}_U) \simeq \mathbf{k}$.

Exercise 5.2. Let X be a locally compact space. Prove the isomorphisms $H^j_c(X;F) \simeq \varinjlim_K H^j_K(X;F)$, where K ranges over the family of compact subsets of X.

Exercise 5.3. (i) Let $I = [0,1[$. Show that $\mathrm{R}\Gamma_c(I;\mathbf{k}_I) = 0$.
(ii) Let s denote the map $\mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x+y$. Let $D \subset \mathbb{R}^2; D =]-1,1[\times[-1,1]$. Calculate $Rs_!(\mathbf{k}_D)$.

Exercise 5.4. Let $Y = [0,1]\times]0,1[$ and let X denote the manifold obtained by identifying $(0,t)$ and $(1,1-t)$. Let S denote the hypersurface of X, the image of the diagonal of Y. Calculate $\Gamma(X;or_{S/X})$.

Exercise 5.5. Let X be a locally compact space and let $\{F_i\}_{i\in I}$ be an inductive system of c-soft sheaves on X, with I filtrant. Prove that $\varinjlim_i F_i$ is c-soft.

Exercise 5.6. (i) Let t be an indeterminate, and denote by $F[t]$ the sheaf $F \otimes (\mathbf{k}[t])_X$. Prove that on $X = \mathbb{C}$, the sequence of sheaves $0 \to \mathcal{O}_X[t] \to \mathcal{C}^\infty_X[t] \xrightarrow{\bar\partial} \mathcal{C}^\infty_X[t] \to 0$ is exact.

(ii) Using the fact that there are $\mathcal{C}^\infty$-functions φ with compact support such that the support of any solution of the equation $\bar\partial\psi = \varphi$ is the whole set X, deduce that $H^1(\mathbb{C};\mathcal{O}_{\mathbb{C}}[t]) \neq 0$.

Exercise 5.7. Recall that $f : Z \to X$ is a trivial covering if there exists a non empty set S, a topological isomorphism $h : Z \xrightarrow{\sim} X \times S$ where S is endowed with the discrete topology, such that $f = p\circ h$ where $p : X\times S \to X$ is the projection. Also recall that $f : Z \to X$ is a locally trivial covering if f is surjective and any $x \in X$ has an open neighborhood U such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a trivial covering.

Prove that is $f : Z \to X$ is a locally trivial covering, then the functor f^{-1} is right adjoint to $f_!$.

Exercise 5.8. Assume $f\colon X \to Y$ is a covering. Prove that $f^! \simeq f^{-1}$. (Hint: the functor $f_!$ is a left adjoint to the exact functor f^{-1} and we get the isomorphism $\mathrm{Hom}\,(F, f^{-1}G) \simeq \mathrm{Hom}\,(F, f^!G)$ for all c-soft sheaf F.)

Exercise 5.9. Let $\mathbb{S}^n$ denote the real n-dimensional sphere, $\mathbb{P}^n$ the real n-dimensional projective space, $\gamma : \mathbb{S}^n \to \mathbb{P}^n$ the natural projection. Prove that γ is a 2-covering and deduce that for $n \geq 2$ there are at least two different locally constant sheaves of rank one on $\mathbb{P}^n$.

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