VARIETIES WITH AMPLE COTANGENT BUNDLE

Paris, Juin 2004

Olivier Debarre

Introduction

Let E be a vector bundle on a variety X. The associated projective bundle

$$\mathbf{P}(E) = \mathbf{Proj}(\bigoplus_{m \geq 0} \mathbf{S}^m E) \to X$$

comes equipped with a line bundle $\mathscr{O}_{\mathbf{P}(E)}(1)$. We say that E is ample if $\mathscr{O}_{\mathbf{P}(E)}(1)$ is. This is equivalent to saying that for any coherent sheaf $\mathscr{F}$ on X, the sheaf $\mathbf{S}^m E \otimes \mathscr{F}$ is generated by global sections for all $m \gg 0$.

If E is globally generated, it is ample if and only if the composition

$$\mathbf{P}(E) \overset{\iota}{\hookrightarrow} \mathbf{P}(H^0(X,E)) \times X \xrightarrow{pr_1} \mathbf{P}(H^0(X,E))$$

is finite, where ι is the closed immersion induced by the evaluation map

$$H^0(X,E) \otimes \mathscr{O}_X \twoheadrightarrow E$$

This talk is about constructing smooth projective varieties with ample cotangent bundle.

Properties (1) Any subvariety of a smooth projective variety X with ample cotangent bundle is of general type. In particular, X contains no rational or elliptic curves, and no nonzero abelian varieties.

(2) Any smooth complex projective variety X with ample cotangent bundle is analytically hyperbolic: any holomorphic map $\mathbf{C} \to X$ is constant.

(3) It is conjectured that a smooth projective variety with ample cotangent bundle defined over a number field should have only finitely many rational points.

Examples (1) A smooth projective curve has ample cotangent bundle if and only if its genus is at least 2.

(2) The cotangent bundle of a product of two positive-dimensional smooth projective varieties is *not* ample.

(3) The cotangent bundle of a smooth hypersurface in $\mathbf{P}^n$ is never ample for $n \geq 3$.

(4) Any smooth complex projective variety that is uniformized by the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$ inherits from the Bergman metric a metric with negative sectional Riemannian curvature hence has ample cotangent bundle. A smooth projective surface has this property if and only if $c_1^2 = 3c_2$ (Borel–Hirzebruch, Yau, Miyaoka). Examples were constructed by:

- Borel in 1963 (compact quotients of $\mathbf{B}_2$ by a discontinuous group of analytic automorphisms);

• Mumford in 1979 (“fake projective plane” with $c_1^2 = 3$, $c_2 = 1$, and $b_1 = 0$);

• Hirzebruch in 1983 (minimal desingularizations of certain coverings of $\mathbf{P}^2$ branched along a union of lines).

(5) Mostow and Siu constructed in 1980 a compact Kähler surface not covered by $\mathbf{B}_2$, with negative sectional Riemannian curvature, hence ample cotangent bundle. No simply connected examples of such surfaces seem to be known.

(6) Deschamps constructed in 1984 smooth projective surfaces smoothly fibered over a curve with everywhere nonzero Kodaira–Spencer map and proved that their cotangent bundle is ample. Around the same time, Bogomolov produced many (simply connected) examples: if S is a smooth projective surface of general type with $c_1^2 > c_2$, a general linear section of S^m of dimension $\leq m/3$ has ample cotangent bundle.

Subvarieties of abelian varieties

The cotangent bundle to a smooth subvariety X of an abelian variety A is globally generated. It is therefore ample if and only if the map

$$\begin{array}{ccc} \mathbf{P}(\Omega_X) \overset{\iota}{\hookrightarrow} \mathbf{P}(\Omega_A)|_X \simeq \mathbf{P}(\Omega_{A,0}) \times X & \xrightarrow{pr_1} & \mathbf{P}(\Omega_{A,0}) \\ (x,\xi) \longrightarrow & & \xi \end{array}$$

is finite. In other words if, for any nonzero (constant) vector field ∂ on A, the set

$$\{x \in X \mid \partial(x) \in T_{X,x}\}$$

is finite. This is possible only if $\dim(X) \leq \dim(A)/2$, but many other things can prevent Ω_X from being ample. For example, if $X \supset X_1 + X_2$, where X_1 and X_2 are subvarieties of A of positive dimension, and x_1 is smooth on X_1, we have

$$\forall\ x_2 \in X_2 \qquad T_{X_1,x_1} \subset T_{X,x_1+x_2}$$

hence Ω_X is not ample.

Theorem 1 *Let L be a very ample line bundle on a simple abelian variety A of dimension n. Let $e > n$ and $c \geq n/2$. For $H_1, \ldots, H_c$ general in $|L^e|$, the cotangent bundle of $H_1 \cap \cdots \cap H_c$ is ample.*

Proof. We prove that the fibers of the map

$$\begin{array}{ccc} \mathbf{P}(\Omega_X) & \longrightarrow & \mathbf{P}(\Omega_{A,0}) \\ (x, \xi) & \longmapsto & \xi \end{array}$$

all have dimension $n - 2c$ for $c \leq n/2$. This means that for $H_1, \ldots, H_c$ general in $|L^e|$ and *any* nonzero (constant) vector field ∂ on A,

$$\dim\{x \in H_1 \cap \cdots \cap H_c \mid \partial(x) \in T_{X,x}\} = n - 2c$$

In other words,

$$\operatorname{codim}(H_1 \cap \partial H_1 \cap \cdots \cap H_c \cap \partial H_c) = 2c$$

We proceed by induction on c, assuming that for all $\partial \neq 0$, the variety $Y_\partial = H_1 \cap \partial H_1 \cap \cdots \cap H_{c-1} \cap \partial H_{c-1}$, with irreducible components $Y_{\partial,1}, \ldots, Y_{\partial,q}$, has codimension $2c - 2$ in A.

Set $Y_i = (Y_{\partial,i})_{\mathrm{red}}$ and let

$$\mathscr{U}_e(Y_\partial) \;=\; \{H \in |L^e| \mid Y_i \cap H \text{ is integral of codimension 1 in } Y_i \text{ for all } i\}$$

If $H \in \mathscr{U}_e(Y_\partial)$, I claim that $Y_\partial \cap H \cap \partial H$ has codimension 2 in Y_∂: if $s \in H^0(A, L^e)$ defines H, we have a commutative diagram

$$\begin{array}{ccc}
H^0(H, L^e|_H) & \longrightarrow & H^1(A, \mathscr{O}_A) \\
\big\downarrow & \partial s \longmapsto \partial \smile c_1(L^e) & \big\downarrow \rho \\
H^0(Y_i \cap H, L^e|_{Y_i \cap H}) & \longrightarrow & H^1(Y_i, \mathscr{O}_{Y_i})
\end{array}$$

The restriction ρ is injective because Y_i generates A, hence ∂s does not vanish identically on the integral scheme $Y_i \cap H$.

It follows that for $H \in \mathscr{U}_e(Y_\partial)$, the scheme $Y_\partial \cap H \cap \partial H$ has codimension $2c$ in A. Thus, for $H_c \in \bigcap_{[\partial] \in \mathbf{P}(\Omega_{A,0})} \mathscr{U}_e(Y_\partial)$, the intersection

$$H_1 \cap \partial H_1 \cap \cdots \cap H_c \cap \partial H_c$$

has codimension $2c$ in A for *all* $\partial \neq 0$.

An elementary geometric lemma shows that the complement of $\mathscr{U}_e(Y_\partial)$ in $|L^e|$ has codimension at least $e-1$. For $e > n$, the above intersection is nonempty and the theorem follows. $\square$

Remarks 2 (1) The assumption that A be simple is unnecessary, although the proof becomes quite complicated without it.

(2) Izadi and I have recently proved that on a *general* Jacobian fourfold, the intersection of a theta divisor with a translate by a general point is a smooth surface with ample cotangent bundle.

Cohomology of symmetric tensors

Ampleness has cohomological consequences: under the hypotheses of the theorem,

$$\forall\ q > \max\{n-2c, 0\} \quad H^q(X, \mathbf{S}^r\Omega_X) = 0$$

for $r \gg 0$.

Claim: for any smooth subvariety X of A, the restriction

$$H^q(A, \mathbf{S}^r\Omega_A) \to H^q(X, \mathbf{S}^r\Omega_X) \text{ is } \begin{cases} \text{bijective} & \text{for } q < n-2c \\ \text{injective} & \text{for } q = n-2c \end{cases}$$

Let $r \gg 0$. For $c < n/2$, the only nonzero cohomology groups of $\mathbf{S}^r\Omega_X$ are therefore

$$\begin{array}{rcl} H^0(X, \mathbf{S}^r\Omega_X) & \simeq & H^0(A, \mathbf{S}^r\Omega_A) \\ & \vdots & \\ H^{n-2c-1}(X, \mathbf{S}^r\Omega_X) & \simeq & H^{n-2c-1}(A, \mathbf{S}^r\Omega_A) \\ H^{n-2c}(X, \mathbf{S}^r\Omega_X) & \supset & H^{n-2c}(A, \mathbf{S}^r\Omega_A) \end{array}$$

For $c \geq n/2$, the only nonzero cohomology group is $H^0(X, \mathbf{S}^r\Omega_X)$.

Proof of the claim. The symmetric powers of the exact sequence $0 \to N^*_{X/A} \to \Omega_A|_X \to \Omega_X \to 0$ yield, for each $r > 0$, a long exact sequence

$$0 \to \wedge^c N^*_{X/A} \otimes \mathbf{S}^{r-c}\Omega_A \to \cdots$$
$$\cdots \to N^*_{X/A} \otimes \mathbf{S}^{r-1}\Omega_A \to$$
$$\to \mathbf{S}^r\Omega_A|_X \to \mathbf{S}^r\Omega_X \to 0$$

Since $N_{X/A}$ is ample, $H^q(X, \wedge^i N^*_{X/A}) = 0$ for $n-c-q > c-i$ and $i > 0$ by Le Potier's vanishing theorem. Since Ω_A is trivial, an elementary homological algebra argument yields

$$\forall\, q \leq n-2c \quad H^q(X, \mathrm{Ker}(\mathbf{S}^r\Omega_A|_X \to \mathbf{S}^r\Omega_X)) = 0$$

The claim follows from the fact that the restriction $H^q(A, \mathscr{O}_A) \to H^q(X, \mathscr{O}_X)$, hence also the restriction $H^q(A, \mathbf{S}^r\Omega_A) \to H^q(X, \mathbf{S}^r\Omega_A|_X)$, is bijective for $q \leq n - 2c$ (Sommese). $\square$

Subvarieties of the projective space

Conjecture 3 *The cotangent bundle of the intersection in $\mathbf{P}^n$ of at least $n/2$ general hypersurfaces of sufficiently high degrees is ample.*

This conjecture has a more general cohomological formulation.

Conjecture 4 *Let X be the intersection in $\mathbf{P}^n$ of c general hypersurfaces of sufficiently high degrees and let m be an integer. For $r \gg 0$,*

$$\forall\ q \neq \max\{n-2c, 0\} \quad H^q(X, \mathbf{S}^r\Omega_X(m)) = 0$$

For $q < 2n - c$, this was proved by Schneider (for all $r > 0$). The conjecture holds for $c \leq 1$ by results of Bogomolov and Demailly that use the existence on X of (some kind of) a Kähler–Einstein metric.

Using the same techniques as above, one can get a weak positivity result in this direction. If X is a smooth subvariety of $\mathbf{P}^n$, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N^*_{X/\mathbf{P}^n}(1) & = & N^*_{X/\mathbf{P}^n}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega_{\mathbf{P}^n}(1)|_X & \to & \mathscr{O}_X^{n+1} & \to & \mathscr{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \Omega_X(1) & \to & \mathscr{E} & \to & \mathscr{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathscr{E}$ is the dual of the pull-back by the Gauss map $X \to G(\dim X, \mathbf{P}^n)$ of the universal subbundle on the Grassmannian. It is globally generated and we have a map

$$\begin{array}{ccc}
\mathbf{P}(\mathscr{E}) \hookrightarrow \mathbf{P}^n \times X & \xrightarrow{pr_1} & \mathbf{P}^n \\
(x, \xi) & \longrightarrow & \xi
\end{array}$$

whose image is the tangential variety of X. The vector bundle $\mathscr{E}$ is ample if and only if no point of $\mathbf{P}^n$ is on infinitely many tangent spaces to X.

Theorem 5 *For the complete intersection in $\mathbf{P}^n$ of at least $n/2$ general hypersurfaces of degree $\geq n+2$, the vector bundle $\mathscr{E}$ is ample.*

This can be shown to imply that the vector bundle $\Omega_X(1)$ is big.