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AN INTRODUCTION TO DIFFERENTIAL GEOMETRY

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CONTENTS

CHAPTER 1

MANIFOLDS

1.1. Topological and differential manifolds

The fundamental objects of differential geometry are called *manifolds* and they should be thought of as topological spaces that locally look like $\mathbb{R}^n$. Let us start with a basic definition.

1.1.1 Definition. — Let M be a topological space. A *topological atlas* of dimension n is a family of couples (U_i, ϕ_i), indexed by $i \in I$, where $(U_i)_{i\in I}$ is an open covering of M and, for each i in I, ϕ_i is a homeomorphism between U_i and an open subset of $\mathbb{R}^n$.

The maps ϕ_i are called *charts* or *systems of coordinates* and the open sets U_i are the *domains* of of theses charts. The maps $\phi_i \circ \phi_j^{-1}$ are homeomorphisms between the open subsets $\phi_j(U_i \cap U_j)$ and $\phi_i(U_i \cap U_j)$ of $\mathbb{R}^n$; they are called *transition functions* or *changes of coordinates* of the atlas.

1.1.2 Definition. — A *topological manifold* of dimension n is a Hausdorff topological space endowed with a countable atlas of dimension n.

In this definition, instead of requiring the atlas to be countable, one could assume that the topological space is a countable union of compact subsets, or that it admits a countable basis ; these induce equivalent definitions. The reason for the Hausdorff and countability requirements in the definition lies the following proposition, giving some of the nice properties of topological manifolds. They will ensure the existence of the very useful partitions of unity (see Proposition 1.1.11 below). The proof is an exercise in topology.

1.1.3 Proposition. — *A topological manifold is locally compact, locally path-connected, locally contractible, separable, paracompact, metrizable.*

1.1.4 Remark. — Let M be a set. An abstract atlas of dimension n on M is a family of couples (U_i, ϕ_i), indexed by $i \in I$, satisfying the following properties.

– For each i in I, U_i is a subset of M and ϕ_i is a bijection between U_i and an open subset of $\mathbb{R}^n$.
– The union of all U_i's is the whole M.
– For all i and j in I, $\phi_i(U_i \cap U_j)$ is an open subset of $\mathbb{R}^n$ and $\phi_j\phi_i^{-1}$ is a homeomorphism between $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$.

Then M carries a unique topology for which the U_i's are open and the ϕ_i's are homeomorphisms : a subset U of M is open for this topology iff, for each i, $\phi_i(U \cap U_i)$ is open in $\mathbb{R}^n$. The family $((U_i, \phi_i))_{i\in I}$ is then a topological atlas for M, endowed with this topology. This is a way to turn the naked set M into a topological manifold, *provided* the topology is Hausdorff and the atlas is countable.

It turns out that topological manifolds are not nice enough for our purposes : there is no notion of differentiation on them. This is why we need to introduce *differential* manifolds.

1.1.5 Definition. — A smooth *atlas* is a topological atlas whose transition functions are C^∞ diffeomorphisms (between open sets of $\mathbb{R}^n$).

Two atlas are *equivalent* if their union is an atlas. Concretely this means that if ϕ_i and ψ_j are the charts of the first and second atlas, then the composites $\phi_i \circ \psi_j^{-1}$ are C^∞ on the open sets where they are defined.

1.1.6 Definition. — A *differential manifold* (or *smooth manifold*) is a topological manifold admitting a smooth atlas. A *smooth structure* is an equivalence class of smooth atlases.

One can also define C^k (resp. analytic) manifolds by asking that the transition functions be C^k (resp. analytic). When the dimension n of the manifold is even, say $n = 2m$, the charts take values in $\mathbb{C}^m = \mathbb{R}^{2m}$; if the transition functions are *biholomorphic*, then the manifold is called a *complex manifold.*

1.1.7 Remark. — A theorem of Whitney asserts that any C^1 manifold admits a compatible C^∞ atlas, unique up to equivalence : the equivalence class of C^1 atlases defining its C^1 structure contains a C^∞ atlas and any two such C^∞ atlases are equivalent. In dimension 1, 2 or 3, this is even true for topological manifolds. In dimension four, things get much more complicated : for instance, the (standard) topological manifold $\mathbb{R}^4$ admits infinitely many compatible smooth structures ! There are also topological manifolds admitting no differential structure.

In what follows, we will only deal with differential manifolds. So "manifold" will always mean "differential manifold". We will often write M^n to express that M is n-dimensional.

1.1.8. Examples. — By definition, any open set of $\mathbb{R}^n$ is an n-dimensional differential manifold. For instance, it follows that the group of invertible matrices $GL_p(\mathbb{R})$ is a manifold of dimension p^2.

The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is a beautiful example, owing to to the obvious geographic interpretation. In coordinates $(x^0, \dots, x^n)$, its North Pole N and South Pole S are the points $(\pm 1, 0, \dots, 0)$. Let us define two charts with values in $\mathbb{R}^n$, considered as the hyperplane $\{x^0 = 0\}$ in $\mathbb{R}^{n+1}$. For $x \in S^n - \{N\}$ define the stereographic projection $\phi_N(x)$ from the North Pole to be the point of $\mathbb{R}^n$ where the line passing through N and x meets $\mathbb{R}^n$; the stereographic projection ϕ_S from the South Pole is defined similarly. In formulas:

$$\phi_N(x^0, x^1, \dots, x^n) = \frac{(x^1, \dots, x^n)}{1 - x^0}, \quad \phi_S(x^0, x^1, \dots, x^n) = \frac{(x^1, \dots, x^n)}{1 + x^0}.$$

The transition function is the inversion

$$\phi_N \phi_S^{-1}(x^1, \dots, x^n) = \frac{(x^1, \dots, x^n)}{(x^1)^2 + \dots + (x^n)^2}.$$

This defines a (finite) smooth atlas for the sphere and explains why the sphere S^n is a smooth manifold of dimension n (it is of course Hausdorff).

By definition, any countable union and any finite product of differential manifolds is a differential manifold. In particular, the torus $\mathbb{T}^n = S^1 \times \dots \times S^1$ (n times) is a manifold of dimension n.

The projective space is the space $\mathbb{R}P^n$ of all real lines in $\mathbb{R}^{n+1}$. It can also be identified with the quotient $S^n/(\mathbb{Z}/2\mathbb{Z})$ of the sphere by the antipodal map ($x \mapsto -x$). A nonzero vector $(x^0, \dots, x^n) \in \mathbb{R}^n$ generates a line in $\mathbb{R}^n$, that is a point of $\mathbb{R}P^n$ which is denoted $[x^0 : \dots : x^n]$. Of course, if λ is any non vanishing number, one has $[x^0 : \dots : x^n] = [\lambda x^0 : \dots : \lambda x^n]$. The $[x^0 : \dots : x^n]$ are the *homogeneous coordinates* on $\mathbb{R}P^n$. Let us see $\mathbb{R}P^n$ as a manifold by giving an explicit atlas. Let $U_i \subset \mathbb{R}P^n$ the open set given by $U_i = \{[x^0 : \dots : x^n], x^i \neq 0\}$. On U_i we have the chart $\phi_i : U_i \to \mathbb{R}^n$ given by

$$\phi_i([x^0 : \dots : x^n]) = (\frac{x^0}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^n}{x^i}),$$

where the hat means that the corresponding term is omitted. One can check that transition functions are smooth and the topology is Hausdorff so that $\mathbb{R}P^n$ is a smooth n-dimensional manifold. The $\mathbb{R}P^n$ for different n's are related in the following way. The chart open set $U_n = \{x^n = 1\}$ is diffeomorphic to $\mathbb{R}^n$ by ϕ_n. The complement $\mathbb{R}P^n - U_n = \{[x^0 : \cdots : x^{n-1} : 0]\}$ identifies naturally with $\mathbb{R}P^{n-1}$. In this way one obtains the $\mathbb{R}P^n$ inductively: starting from $\mathbb{R}P^0$ which is reduced to a point,

- $\mathbb{R}P^1 = \mathbb{R} \cup \{\text{pt.}\}$ is a circle;
- $\mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$ is the union of the plane and the line at infinity;
- more generally, $\mathbb{R}P^n = \mathbb{R}^n \cup \mathbb{R}P^{n-1}$.

Observe also that all we have done has a meaning if we decide that the x^i are complex coordinates rather than real coordinates. In this way, one obtains the structure of a complex manifold on the space $\mathbb{C}P^m$ of complex lines in $\mathbb{C}^{m+1}$. We also see that $\mathbb{C}P^1 = \mathbb{C} \cup \{\text{pt}\}$ is homeomorphic to a 2-sphere.

1.1.9 Definition. — A map $f : M^n \to N^p$ between manifolds M and N is smooth (or C^∞) near a point $x \in M$ if there are charts $\phi : U \subset M \to \mathbb{R}^n$ and $\psi : V \subset N \to \mathbb{R}^p$, with $x \in U$ and $f(U) \subset V$, such that the map $\psi f \phi^{-1}$ is C^∞ on $\phi(U)$.

This definition does not depend on the choice of charts, because the transition between two charts is always a C^∞ diffeomorphism, by the very definition of a differential manifold.

Locally, thanks to the chosen charts, we can identify f with a function between $\mathbb{R}^n$ and $\mathbb{R}^p$, which we write $(f_1(x^1, \dots, x^n), \dots, f_p(x^1, \dots, x^n))$; f is smooth if and only if each f_i is C^∞.

A function that is smooth around every point is called smooth. If $f : M^n \to N^p$ is a smooth bijection such that f^{-1} is also smooth, we say that f is a C^∞ diffeomorphism. Of course this implies $n = p$. In that case we say that M and N are diffeomorphic. The notions of submersion and immersion are local hence extend to the setting of manifolds. More precisely, we say that a smooth map f between manifolds M and N is a submersion (resp. an immersion) near a point x of M if there are charts ϕ, ψ as in the definition above, such that $\psi f \phi^{-1}$ is a submersion (resp. an immersion). Again, one can choose any pair of charts in this definition because transition functions are diffeomorphisms.

1.1.10. Exercise. — Prove that the following maps are smooth:

- the quotient by the antipodal map $S^n \to \mathbb{R}P^n$;

– the map $S^3 \to \mathbb{C}P^1$ taking a vector $x \in S^3$ to the complex line that it generates in $\mathbb{C}^2$.

Prove that S^2 and $\mathbb{C}P^1$ are diffeomorphic.

Let us finally introduce an important technical tool. As a consequence of the countability assumption in their definition, manifolds carry partitions of the unity. They will be used to patch together local objects, typically defined on the domain of charts, into a global one, defined on the whole manifold.

1.1.11 Proposition. — *For every open covering $\mathscr{U} := (U_i)_{i\in I}$ of a manifold M, there is a partition of unity $(\chi_j)_{j\in J}$ subordinate to $\mathscr{U}$: each χ_j is a smooth nonnegative function with support in some U_i and, for every point x in M, there is a neighborhood of x on which all but a finite number of the χ_j's vanish and $\sum_{j\in J} \chi_j = 1$.*

The proof is left to the reader (the bibliography might help).

1.2. Submanifolds

A manifold is basically something that locally looks like a vector space $\mathbb{R}^n$. So a submanifold should be something that locally looks like a vector subspace of the $\mathbb{R}^n$ above.

1.2.1 Definition. — Let M be an N-dimensional manifold. A subset X of M^N is an n-dimensional *submanifold* of M^N if, for every point x of X, there is a chart ϕ of M, with domain U containing x and range $V \subset \mathbb{R}^N$, such that $\phi(X \cap U) = (\mathbb{R}^n \times \{0\}) \cap V \subset \mathbb{R}^n \times \mathbb{R}^{N-n} = \mathbb{R}^N$.

The charts mentioned in this definition are called submanifolds charts. A submanifold X inherits a manifold structure. First, it inherits a Hausdorff topology with a countable basis, as a topological subspace of a manifold. And the submanifold charts induce charts on X : if ϕ is a submanifold chart, then the corresponding chart for X is $\pi \circ \phi|_X$, where π is the projection $\mathbb{R}^n \times \mathbb{R}^{N-n} \to \mathbb{R}^n$.

For example, in a manifold M of dimension n, the submanifolds of dimension n (or codimension 0) are simply the open subsets of M. Most interesting examples are built from submersions and immersions.

1.2.2 Theorem. — *Let $f : M^N \longrightarrow N^p$ be a submersion. Then for any a in N, the set $f^{-1}(a)$ is either empty or an $(N-p)$-dimensional submanifold of M^N.*

It is useful to notice that in this statement we just need f to be a submersion along $f^{-1}(a)$ (in this case, a is called a regular value).

Proof. — Let x be a point of $f^{-1}(a)$. Picking charts φ near x and ψ near a, with $\varphi(x) = 0$ and $\psi(a) = 0$, we obtain a submersion $\psi f \varphi^{-1}$ defined between open sets $V \subset \mathbb{R}^N$ and $W \subset \mathbb{R}^p$, mapping $0 \in V$ to $0 \in W$. The constant rank theorem in calculus provides a local normal form for such a map : it ensures the existence of a diffeomorphism θ, defined in a smaller neighborhood of 0 in V such that

$$\psi f \varphi^{-1} \theta(x^1, \dots, x^N) = (x^{N-p+1}, \dots, x^N).$$

The promised submanifold chart ϕ is $\theta^{-1}\varphi$. □

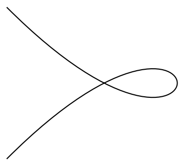
1.2.3. Examples. — 1) The curve $y^2 = x^3 - x$ is a smooth curve (i.e. a 1-dimensional submanifold) of $\mathbb{R}^2$. Indeed, consider $f(x, y) = y^2 - x^3 + x$, then $d_{(x,y)} f = (-3x^2 + 1, 2y)$ which vanishes only at the points $(\pm\frac{1}{\sqrt{3}}, 0)$. Since these two points are not in $f^{-1}(0)$, the result follows from the theorem applied to the map f on the open set $U = \mathbb{R}^2 - \{(\pm\frac{1}{\sqrt{3}}, 0)\}$.

2) The sphere $S^n = \{(x^0)^2 + \cdots + (x^n)^2 = 1\}$ and the hyperbolic space $H^n = \{x^0 > 0, (x^0)^2 - (x^1)^2 - \cdots (x^n)^2 = 1\}$ are submanifolds of $\mathbb{R}^{n+1}$.

3) (Exercise) The group $O(n)$ is a submanifold of $GL_n(\mathbb{R})$ (the space of $n \times n$ matrices). Apply the theorem to the map $f(A) = AA^T - 1$ from matrices to symmetric matrices. To prove that f is a submersion at each point $x \in O(n)$, use the invariance $f(Ax) = f(A)$ to reduce to the case $x = 1$.

4) (Exercise) If $f : U \subset \mathbb{R}^n \to \mathbb{R}^{N-n}$ is a smooth map defined on an open set U in $\mathbb{R}^n$, then the graph $M = \{(x, f(x)), x \in U\}$ is a n-dimensional submanifold of $\mathbb{R}^N$.

To understand the link between immersions and submanifolds, it is useful to look at some examples. A positive example is provided by the hyperbolic space : the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(x^1, \dots, x^n) \mapsto (1 + (x^1)^2 + \cdots + (x^n)^2, x^1, \dots, x^n)$ is an immersion and a bijection from $\mathbb{R}^n$ to its image $H^n \subset \mathbb{R}^{n+1}$, the hyperbolic space, which is already known to be a submanifold. But the two figures below show that the image of an immersion is not always so nice. They both represent immersions $\mathbb{R} \to \mathbb{R}^2$ whose image is not a submanifold: the first is not injective, it has a double point; the second one is injective but not proper.



An injective immersion f between manifolds is called an *embedding* if it is a homeomorphism onto its image.

1.2.4 Theorem. — *Let $f : N^n \longrightarrow M^N$ be an injective immersion between manifolds.*

- *If f is proper, then f is an embedding.*
- *If f is an embedding, then $f(N)$ is an n-dimensional submanifold of M.*

Proof. — The first statement is an exercise in topology (it amounts to prove that a proper map between locally compact spaces is a closed map). As for the second statement, observe that locally we may find charts φ and ψ such that $\psi f \varphi^{-1}$ is an immersion between open neighborhoods of 0 in $\mathbb{R}^n$ and $\mathbb{R}^N$, mapping 0 to 0. The constant rank theorem in calculus provides a local normal form for such a map : it ensures the existence of a diffeomorphism θ, defined in a neiborhood of 0 in $\mathbb{R}^N$, such that

$$\theta\psi f \varphi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

It follows that for any point x in N, there is an open neighborhood W of x and an open neighborhood V of 0 in $\mathbb{R}^N$ such that

$$\theta\psi(f(W)) = (\mathbb{R}^n \times \{0\}) \cap V \subset \mathbb{R}^N.$$

Since f is a homeomorphism onto its image, there is an open neighborhood U of $f(x)$ in M such that $f(W) = f(N) \cap U$ (and we may assume $\theta\psi$ is defined on U). Then $\phi := \theta\psi$ is a submanifold chart, with domain U. □

1.2.5 Remark. — In fact, any differential manifold is diffeomorphic to a submanifold of $\mathbb{R}^N$, for some N (this is an exercise when the manifold is compact). A theorem of Whitney even ensures that a manifold of dimension n can be embedded in $\mathbb{R}^{2n}$ (cf. the book of Hirsch in the bibliography).

1.3. Tangent vectors

1.3.1. The case of a submanifold of $\mathbb{R}^N$. — There is a natural notion of tangent vectors for submanifolds of a vector space. Let M^n be a submanifold of $\mathbb{R}^N$ and let x be a point of M. A vector $X \in \mathbb{R}^N$ is a *tangent vector to M at x* if there exists a smooth curve $c :]-\epsilon, \epsilon[\to M \subset \mathbb{R}^N$, such that $c(0) = x$ and $c'(0) = X$. The space T_xM of all tangent vectors to M at x is called the *tangent space of M at x*.

For instance, if M^n is an affine subspace of $\mathbb{R}^N$, namely $M = x_0 + V$ where V is a vector subspace of $\mathbb{R}^N$, then for all $x \in M$, one has $T_xM = V$.

Near a point x of a submanifold M^n of $\mathbb{R}^N$, we can pick a submanifold chart $\phi : U \to V \subset \mathbb{R}^N$, with $\phi(M \cap U) = (\mathbb{R}^n \times \{0\}) \cap V$. So ϕ turns a piece of M into a piece of the affine subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^N$. Since diffeomorphisms exchange smooth curves, we are back to the example above and we can see that $T_xM = (d_x\phi)^{-1}(\mathbb{R}^n \times \{0\})$. In particular, T_xM is always an n-dimensional vector subspace of $\mathbb{R}^N$.

Let us consider the following subset of $\mathbb{R}^N \times \mathbb{R}^N$:

$$TM = \{(x, X) \in M \times \mathbb{R}^N \quad / \quad X \in T_xM\}.$$

It is the set of all tangent vectors to the submanifold M^n of $\mathbb{R}^N$. We will call it the tangent bundle of M. It turns out that TM itself is a $2n$-dimensional submanifold of $\mathbb{R}^N \times \mathbb{R}^N$: if $\phi : U \subset \mathbb{R}^N \to \mathbb{R}^N$ is a submanifold chart for M, then the map $\Phi : U \times \mathbb{R}^N \longrightarrow \phi(U) \times \mathbb{R}^N$ defined by $\Phi(x, X) = (\phi(x), d_x\phi(X))$ is a submanifold chart for TM.

We now turn to the notion of a tangent vector at a point x in a manifold M. We will introduce an abstract notion of vector by saying that it is a an equivalence class of curves sharing the same Taylor expansion up to order 1 (in a chart). More precisely :

1.3.2 Definition. — Let M be a manifold and x a point of M.

1. Two paths $c_1, c_2 :]-\epsilon, \epsilon[\to M$ such that $c_1(0) = c_2(0) = x$ are called *equivalent* if, for any local chart ϕ at x, one has $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$.
2. A *tangent vector* at x to M is an equivalence class of paths for this relation.
3. The set of all tangent vectors at x to M is called the *tangent space* of M at x and denoted by T_xM.

Observe that in the first part of the definition, it is equivalent to ask the equality of the derivatives for one chart or for all charts.

Given a smooth map between two manifolds, $f : M^n \to N^p$ and a path c at $x \in M$, we can associate a path $f(c)$ at $f(x)$. It is easy to check that if c_1 and c_2 are equivalent, then so are $f(c_1)$ and $f(c_2)$. It follows that we obtain a well defined map

$$d_x f : T_x M \to T_{f(x)} N. \tag{1.1}$$

If f is a diffeomorphism, it is easy to check that $(d_x f)^{-1} = d_{f(x)}(f^{-1})$.

Let us apply this to a local chart ϕ at x, centered at x, which means that $\phi(x) = 0$: the map ϕ is a diffeomorphism between an open subset U of M and an open subset V of $\mathbb{R}^n$. We obtain an isomorphism $d_x\phi : T_x M \tilde{\longrightarrow} \mathbb{R}^n$ such that $d_x\phi([c]) = (\phi \circ c)'(0)$. We would like to deduce that $T_x M$ carries a natural structure of vector space, provided by this identification with $\mathbb{R}^n$. To ensure that this is true, we need to show that this structure of vector space does not depend on the choice of the chart ϕ. This is a good place to check this kind of statement, that we will use repeatedly. Given another chart ψ, centered at x, we obtain the following commutative diagram :

$$\begin{array}{ccc} & T_x M & \\ {\scriptstyle d_x\phi} \swarrow & & \searrow {\scriptstyle d_x\psi} \\ \mathbb{R}^n & \xrightarrow{d_0(\psi\phi^{-1})} & \mathbb{R}^n \end{array}$$

So the two different identifications of $T_x M$ with $\mathbb{R}^n$ coming from the charts ϕ and ψ differ by the linear isomorphism $d_0(\psi\phi^{-1})$, which preserves the vector space structure. So the vector space structures induced on $T_x M$ from $d_x\phi$ and $d_x\psi$ coincide.

Observe that if $f : M^n \to N^p$ is a smooth map and if x is a point of M, then $d_x f : T_x M \to T_{f(x)} N$ is a linear map (exercise : check it in charts).

1.3.3. Tangent bundle. — We now turn to the problem of constructing the *manifold* of all tangent vectors at all points of a manifold M. As a set, we define the *tangent bundle* of a manifold M^n by

$$TM = \amalg_{x \in M} T_x M = \{(x, X), x \in M, X \in T_x M\}.$$

It is endowed with a natural projection $\pi : TM \to M$ given by $\pi(x, X) = x$. So $\pi^{-1}(x) = T_x M$.

1.3.4 Proposition. — *The tangent bundle TM of the manifold M^n is a manifold of dimension $2n$ and π is a smooth surjective submersion.*

Proof. — To endow TM of a topological and differential structure, we resort to Remark 1.1.4, so we need to construct a convenient (abstract) atlas. Let

$((U_i,\phi_i))_{i\in I}$ be a countable atlas for the manifold M. For every i, we introduce the map $d\phi_i : \pi^{-1}(U_i) \subset TM \longrightarrow \mathbb{R}^{2n}$:

$$d\phi_i(x,X) = (\phi(x), d_x\phi(X)).$$

This is a bijection onto $\phi_i(U_i)\times\mathbb{R}^n$. Given i and j in I, $d\phi_i(U_i\cap U_j) = \phi_i(U_i\cap U_j)\times\mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n}$. Finally, the maps $d\phi_i$ and $d\phi_j$ are related by the following commutative diagram:

$$\begin{array}{ccc} & \pi^{-1}(U_i\cap U_j) & \\ {}^{d\phi_i}\swarrow & & \searrow^{d\phi_j} \\ \phi_i(U_i\cap U_j)\times\mathbb{R}^n & \xrightarrow{d(\phi_j\phi_i^{-1})} & \phi_j(U_i\cap U_j)\times\mathbb{R}^n \end{array} \tag{1.2}$$

where, if $y\in\phi_i(U_i\cap U_j)$ and $V\in\mathbb{R}^n$, then

$$d(\phi_j\phi_i^{-1})(y,V) = (\phi_j\phi_i^{-1}(y), d_y(\phi_j\phi_i^{-1})(V)).$$

This is clearly smooth, as well as its inverse $d(\phi_j\phi_i^{-1})$, and therefore a diffeomorphism. All in all, the family $((\pi^{-1}(U_i), d\phi_i))_i$ is a countable abstract atlas of dimension $2n$ in the sense of Remark 1.1.4, so it gives a topology to TM, as well as a countable differential atlas.

Observe that π is continuous for this topology : if V is an open subset of M, $d\phi_i(\pi^{-1}(V)\cap\pi^{-1}(U_i)) = \phi_i(V\cap U_i)\times\mathbb{R}^n$ is open, so $\pi^{-1}(V)$ is open (cf. Remark 1.1.4).

To see that TM is a manifold, the last thing that we have to check is that the topology induced by the atlas is Hausdorff. Let (x_1,X_1) and (x_2,X_2) be two distinct points of TM. If $x_1\neq x_2$, then there are disjoint open subsets $V_1\ni x_1$ and $V_2\ni x_2$, for M is Hausdorff. Then, since π is continuous, $\pi^{-1}(V_1)\ni(x_1,X_1)$ and $\pi^{-1}(V_2)\ni(x_2,X_2)$ are two disjoint open subsets of TM and we are done when $x_1\neq x_2$. Now assume $x_1=x_2=x$, while $X_1\neq X_2$: pick a chart (U_i,ϕ_i) around $x\in M$ and observe that the points (x,X_1) and (x,X_2) lie inside the domain $\pi^{-1}(U_i)$ of the chart $d\phi_i$, so that they can also be separated by open subsets (like in an open subset of $\mathbb{R}^{2n}$). So TM is Haudorff, hence, eventually, a manifold.

Finally, for any i in I, $\phi_i\pi d\phi_i^{-1} : \phi_i(U_i)\times\mathbb{R}^n \longrightarrow \phi_i(U_i)$ is simply the projection onto the first factor, so it is certainly a smooth surjective submersion. □

The tangent bundle of a submanifold of M is a submanifold of TM. For instance, if $f : M\longrightarrow N$ is a submersion and $a\in N$, then the tangent bundle to the submanifold $f^{-1}(a)$ of M is given by the kernel of the differential of f : if $f(x)=a$, then $T_x\left(f^{-1}(a)\right) = \ker d_xf$. If $f : N\longrightarrow M$ is an embedding, then the tangent bundle to the submanifold $f(N)$ is given by the range of the differential of f : $T_{f(x)}f(N) = \operatorname{ran} d_x f$.

Now, let us come back to a smooth map $f : M^n \to N^p$. Then the collection of the maps $d_x f : T_x M \to T_{f(x)} N$ gives a smooth *tangent map* $df : TM \to TN$ and we have the commutative diagram

$$\begin{array}{ccc} TM & \stackrel{df}{\longrightarrow} & TN \\ \downarrow & & \downarrow \\ M & \stackrel{f}{\longrightarrow} & N \end{array}$$

where for each x in M, the induced map $d_x f : T_x M \to T_{f(x)} N$ is a linear map. An important remark is that the usual chain rule

$$d(g \circ f) = dg \circ df$$

holds true (exercise : check it in charts).

1.4. Vector bundles

The atlas that we have introduced on the tangent bundle TM of a manifold M^n is of a special kind: it ensures that TM is locally diffeomorphic to the product of an open subset of M and of a vector space $\mathbb{R}^n$, in such a way that each fiber of the projection $\pi : TM \longrightarrow M$ gets locally identified to $\mathbb{R}^n$; moreover, the fibers of π carry a structure of vector space that is compatible with this local identification. It follows that we may see TM as a family of vector spaces, indexed by a manifold M in a "locally trivial way", in the sense that it is locally a product. This kind of structure is a vector bundle and it is defined as follows.

1.4.1 Definition. — A *vector bundle* of rank p over a manifold M^n is the data of a manifold E, of a smooth map $\pi : E \to M$ and of a structure of vector space on each fiber $E_x := \pi^{-1}(x)$, $x \in M$, such that, for every point x in M, there is an open neighborhood U of x and a diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^p$ with the following compatibility conditions:

– $pr_1 \circ \psi = \pi$;
– for every x in M, $pr_2 \circ \psi|_{E_x} : E_x \longrightarrow \mathbb{R}^p$ is a linear isomorphism.

The manifold E is the *total space* of the vector bundle, the manifold M is the *base* of the vector bundle and the maps ψ are called *local trivializations*.

The first compatibility condition means the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^p \\ & {}_{\pi}\searrow \quad \swarrow_{pr_1} & \\ & U & \end{array}$$

is commutative. Given two local trivializations (U_1, ψ_1) and (U_2, ψ_2), we may introduce the *transition function* $\psi_1\psi_2^{-1} : (U_1 \cap U_2) \times \mathbb{R}^p \longrightarrow (U_1 \cap U_2) \times \mathbb{R}^p$. The compatibility conditions imply that for x in $U_1 \cap U_2$ and ξ in $\mathbb{R}^p$, we have $\psi_i\psi_j^{-1}(x, \xi) = (x, u_{ij}(x)(\xi))$, where $u_{ij}(x)$ is a linear isomorphism of $\mathbb{R}^p$.

One can define similarly *complex* vector bundles (replace $\mathbb{R}^p$ by $\mathbb{C}^p$ in the definition above).

If (E, π, M) is a vector bundle and if N is a submanifold of M, then $(\pi^{-1}(N), \pi|_{\pi^{-1}(N)}, N)$ inherits a structure of vector bundle, often denoted by $E|_N$ (exercise).

1.4.2 Definition. — A *smooth section* s of a vector bundle (E, π, M) is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = id$. This just means that $s(x)$ belongs to E_x for every x in M. The set of smooth sections of E over M will be denoted by $\Gamma(M, E)$ or $\Gamma(E)$.

In a local trivialization $E|_{U_i} \simeq U_i \times \mathbb{R}^p$, a section is given by $p = \operatorname{rank} E$ coordinates $s_1(x), \ldots, s_p(x)$ that are smooth functions. In particular, there are many sections in $\Gamma(U_i, \pi^{-1}(U_i))$. Observe that partitions of unity (see Proposition 1.1.11) make it possible to patch them together into (many) global sections, i.e. elements of $\Gamma(M, E)$: given a family of U_i's covering M, a partition of unity (χ_i) subordinate to this covering and local sections $\sigma_i \in \Gamma(U_i, \pi^{-1}(U_i))$, we can construct an global section $\sigma \in \Gamma(M, E) : \sigma := \sum_i \chi_i \sigma_i$ (recall this is a locally finite sum).

Given a manifold M, the product $M \times \mathbb{R}^p$ is a vector bundle of rank p : it is *trivial* in that there is a global trivialization (namely, over $U = M$). It is useful to remark that a vector bundle of rank p is trivial if and and only if it admits p sections that are everywhere linearly independent. For instance, a line bundle (namely, a vector bundle of rank 1) is trivial if and only if it carries a section that is everywhere nonzero.

The Möbius band provides an example of a non-trivial line bundle. Its total space E is $[0, 1] \times \mathbb{R}/ \sim$, where $(0, t) \sim (1, -t)$. Its base is the circle S^1 (obtained by gluing 0 and 1 in $[0, 1]$) and its projection π is induced by the projection onto the first factor $[0, 1]$.

An interesting example is given by the tangent bundle TM of M, as we have seen above ; its rank is $\dim M$. To construct more examples from it, a simple remark is in order. Given one or several bundles, any algebraic operation on the underlying vector spaces can be done fiberwise to give rise to a new vector bundle. For example, if E and F are vector bundles over M, then there is a natural way to define $E \oplus F$, $E \otimes F$, E^* (and therefore $\mathrm{Hom}(E, F) = E^* \otimes F$) as vector bundles over M, whose fibers at $x \in M$ are $E_x \oplus F_x$, $E_x \otimes F_x$, E_x^*. We leave these constructions as an exercise.

In particular, any vector bundle E comes with a vector bundle $\mathrm{End}(E)$, whose fibers consist of the endomorphisms of the fibers of E. When E is a line bundle (i.e. has rank 1), $\mathrm{End}\, E$ is always trivial (why ?).

This gives rise to the *cotangent bundle* of a manifold M^n. This is a vector bundle, denoted by T^*M or $\Lambda^1 M$, whose fiber at $x \in M$ is the dual $T_x^*M = (T_xM)^*$ of the tangent space T_xM. If $f : M \to \mathbb{R}$ is a smooth function, its differential d_xf is a linear form on T_xM, so $d_xf \in T_x^*M$. It follows that df can be interpreted as a section of T^*M. Sections of T^*M are called 1-forms. Let us consider a chart $\phi : U \longrightarrow \mathbb{R}^n$ and denote by x^i the n components of ϕ (these are the *local coordinates*). Then a local basis of T^*M is given by the differentials dx^i of the coordinates : it means that any element α of T_y^*M with y in U can be written as $\alpha = \sum_i \alpha_i dx^i$, $\alpha_i \in \mathbb{R}$ (note a slight abuse : $dx^i = d_y x^i$), so that $\alpha \mapsto (\alpha_1, \ldots, \alpha_n)$ is a local trivialization. In particular, for any function f, we may write $df = \frac{\partial f}{\partial x^i} dx^i$. In this formula we dropped the sum symbol and used the **Einstein summation convention**: if you find in a formula the same i as an index and as an exponent, then you must understand that the result is just the sum on all possible i's ; an exponent in a denominator is considered as an index.

More generally, any manifold comes with a wealth of natural vector bundles built from the tangent bundle by taking duals, tensor products, sums, exterior and symmetric powers. These are called tensor bundles. We will study some of them.

1.5. Vector fields

1.5.1 Definition. — The smooth sections of the tangent bundle of a manifold are called *vector fields.*

Let X be a vector field on a manifold M. Let f be an element of the space $C^\infty(M)$ of smooth functions defined on M and with values in $\mathbb{R}$. Then df is a section of T^*M : at each point x of M, d_xf belongs to the dual of T_xM. Now, $X(x)$ belongs to T_xM, so we may introduce $(\mathscr{L}_X f)(x) := d_xf(X(x)) \in \mathbb{R}$. The *Lie derivative* of f along X is the function $\mathscr{L}_X f = df(X) \in C^\infty(M)$.

It is clear from the formula that $L_X f(x)$ depends only on $X(x)$ and not on the values of X outside x ; in particular, we may compute $L_X f$ on some open set U as soon as f is defined on U (and maybe not elsewhere).

Let us see what this means in a local trivialization of TM, coming from a chart ϕ defined on an open subset U of M. TU is identified with $U \times \mathbb{R}^n$ (through $d\phi$) and an element of $\Gamma(TU)$ can be viewed as a smooth map $U \to \mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ therefore induces n elements of $\Gamma(TU)$, which are constant in the chart :

$$e_1 = d\phi^{-1}((1,0,\dots,0)), \quad e_n = d\phi^{-1}((0,\dots,0,1)).$$

Any vector field X can then be written as $X = X^i e_i$ (remind the implicit summation convention). If f is a function on M, we can consider locally f as a function of the local coordinates x^i (recall these are the components of ϕ), $f(x^1,\dots,x^n)$. Then one can easily check that

$$\mathscr{L}_{e_i} f = \frac{\partial f}{\partial x^i}. \tag{1.3}$$

This is why the vector field e_i is usually identified with the corresponding derivation : $e_i = \frac{\partial}{\partial x^i}$. In the sequel we will use the notation $\frac{\partial}{\partial x^i}$ instead of e_i.

Let us generalize this identification. We introduce the space $D(M)$ of all derivations of $C^\infty(M)$, i.e. $\mathbb{R}$-linear endomorphisms D of $C^\infty(M)$ satisfying the Leibniz rule

$$D(fg) = (Df)g + fDg.$$

The reader may check that for any vector field X, the Lie derivative $\mathscr{L}_X$ is a derivation of $C^\infty(M)$. Indeed, a vector field is the same thing as a derivation:

1.5.2 Theorem. — *The map $X \mapsto \mathscr{L}_X$ is an isomorphism $\Gamma(TM) \to D(M)$.*

Proof. — The map is clearly linear. To prove the injectivity, we assume $\mathscr{L}_X = 0$ for some X. Given local coordinates $(x^1,\dots,x^n)$ around any point, we get $0 = \mathscr{L}_X x^i = dx_i(X)$ for each i, so that X vanishes in the domain of any chart, hence $X = 0$. So the map is one-to-one.

To prove that it is onto, we let D be a derivation of $C^\infty(M)$ and seek a vector field X such that $D = \mathscr{L}_X$. We proceed in three steps.

First step. We wish to prove that D is "local", in that, if U is an open set, then $f|_U = 0$ implie $(Df)|_U = 0$. To prove it, we pick a function χ with compact support in U. Then $D(\chi f) = \chi Df + (D\chi)f$. If $f|_U = 0$, then $\chi f = 0$, so $\chi Df = -(D\chi)f$ vanishes on U. Since χ is arbitrary, $(Df)|_U = 0$. So D is local, which implies that if f and g coincide on some U, then $D(f-g)|_U = 0$ so Df and Dg coincide on U: in particular $Df(x)$ depends only on the values of f on an arbitrary small neighborhood of x. It makes it possible to define Df for f defined on any open subset of M.

Second step. $D(1) = 0$: indeed, from Leibniz rule, we have $D(1) = D(1^2) = 1D(1) + D(1)1 = 2D(1)$, hence $D(1) = 0$.

Third step. Let us work in some local coordinates (x^i), defined on some open set U and pick a point p of U. Setting $p^i := x^i(p)$, we may write

$$f(x^1,\dots,x^n) = f(p) + (x^i - p^i)g_{i,p}(x^1,\dots,x^n)$$

for some smooth functions $g_{i,p}$ such that $g_{i,p}(p) = \frac{\partial f}{\partial x^i}(p)$. Thanks to the first step, we can apply D to these local functions: using Leibniz identity and the second step, we get

$$(Df)(p) = (Dx^i)(p)g_{i,p}(p) = (Dx^i)(p)\frac{\partial f}{\partial x^i}(p).$$

Defining $X_U = (Dx^i)\frac{\partial}{\partial x^i}$, we see from (1.3) that $Df = df(X_U) = L_{X_U}f$ on U. We can do this in any domain of chart U. From the (local) injectivity proved above, we see that, given two domains of charts U and V, $X_U = X_V$ on the overlap of their domains. So these local vector fields patch together into a global smooth section X of TM such that $D = \mathscr{L}_X$. □

It is easy to check that the commutator of two derivations is still a derivation. In view of the isomorphism above, we obtain a structure on $\Gamma(TM)$, a bracket.

1.5.3 Definition. — If X and Y are two vector fields on M, then their *bracket* $[X,Y]$ is the vector field corresponding to the derivation $[\mathscr{L}_X,\mathscr{L}_Y] = \mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X$. In other words, $\mathscr{L}_{[X,Y]} = [\mathscr{L}_X,\mathscr{L}_Y]$.

This rather abstract definition corresponds to a simple calculation: taking local coordinates (x^i), we write $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$, then

$$\begin{aligned}\mathscr{L}_X\mathscr{L}_Y f - \mathscr{L}_Y\mathscr{L}_X f &= X^j\frac{\partial}{\partial x^j}\left(Y^i\frac{\partial f}{\partial x^i}\right) - Y^j\frac{\partial}{\partial x^j}\left(X^i\frac{\partial f}{\partial x^i}\right)\\ &= \left(X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\right)\frac{\partial f}{\partial x^i}\end{aligned}$$

Therefore

$$[X,Y] = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \tag{1.4}$$

The abstract definition implies the so-called Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

while the local expression (1.4) immediately yields the formula

$$[X, fY] = f[X,Y] + (\mathscr{L}_X f)Y.$$

If $\phi : M \to N$ is a diffeomorphism and X a vector field on N, we can define the *pullback* $\phi^* X$ of X by ϕ : it is the vector field on M defined by

$$(\phi^* X)_x = (d_x \phi)^{-1} X_{\phi(x)}.$$

The bracket of vector fields behaves well under diffeomorphisms, it is "natural" : the reader may use the abstract definition of the bracket to see that

$$\phi^*[X,Y] = [\phi^* X, \phi^* Y].$$

Another consequence of (1.4) is the following property.

1.5.4 Proposition. — *Let N be a submanifold of a manifold M and let X, Y be vector fields on M. If the restrictions of X and Y to N lie inside $TN \subset TM|_N$, then $[X,Y]|_N$ is tangent to N and equals $[X|_N, Y|_N]$.*

1.5.5. First order ordinary differential equations. — Let $c : I \to M$ be a curve in a manifold, defined on an interval $I \subset \mathbb{R}$. By the very definition of a tangent vector, this defines a vector $\dot{c}(t) \in T_{c(t)}M$ for every $t \in I$ (the dot means derivation with respect to t). Given now a vector field X on the manifold M, we will look for solutions $c : I \to M$ of the equation

$$\dot{c}(t) = X(c(t)) \tag{1.5}$$

where t belongs to some open interval I of $\mathbb{R}$.

For example, if $M = \mathbb{R}^2$, a vector field is given by $X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$, a curve is $c(t) = (x(t), y(t))$ and the equation (1.5) is the system

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y). \end{cases}$$

More generally, in local coordinates (x^i), the equation (1.5) becomes

$$\dot{x}^i = X^i(x^1, \ldots, x^n).$$

So if we give $c(t_0) \in M$, near t_0 the path $c(t)$ will remain in the open set of coordinates and the equation (1.5) translates into a first order system of ordinary differential equations. Usual results in calculus then ensure that the equation has a unique solution in some interval containing t_0. It follows that if we give the initial condition $c(t_0) = x \in M$, there is a unique solution defined on a maximal interval $I \ni t_0$. We shall denote this solution $c_x(t)$.

1.5.6 Definition. — A vector field X on M is *complete* if, for any initial condition x, the solution c_x the equation (1.5) is defined on $\mathbb{R}$.

1.5.7 Lemma. — *If a vector field X has compact support, then it is complete.*

Proof. — The only way a solution can exist only on a bounded interval is that $c(t)$ gets out of any compact of M. But this is impossible since $X = 0$ outside a compact set K so that solutions starting from outside K are constant. □

Now change the perspective: we consider t as fixed and we vary the initial condition x : we define $\phi_t(x) = c_x(t)$. So ϕ_t consists in following the solution of $\dot{c} = X(c)$ from the initial condition x during a time t. It is the *flow* at time t. Unicity of solutions yields that whenever the flow is defined, we have

$$\phi_t \circ \phi_{t'} = \phi_{t+t'}.$$

In particular, $\phi_t \circ \phi_{-t} = id$. The following result follows.

1.5.8 Proposition. — *If X is a complete vector field on M, then its flow $(\phi_t)_{t\in\mathbb{R}}$ is a 1-parameter group of diffeomorphisms of M.*

1.5.9 Example. — 1) Check that the radial vector field $X = x^i \frac{\partial}{\partial x^i}$ generates an homothety ϕ_t of ratio e^t. 2) Check that the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is a vector field on $S^2 \subset \mathbb{R}^3$ that generates a rotation of angle t around the z axis.

1.5.10. Flow and bracket. — The Lie derivative of a function f along a vector field X measures the variation of f along the flow ϕ_t of X : a slight reformulation of the definition of the Lie derivative is

$$\mathscr{L}_X f(x) = \frac{d}{dt}\Big|_{t=0} f \circ \phi_t(x). \tag{1.6}$$

We will see that the bracket of two vector fields admits a similar interpretation.

1.5.11 Lemma. — *Let X and Y be two vector fields. We denote by ϕ_t and ψ_u their respective flows. Then the following identities hold true:*

$$\frac{d}{du}\,(\phi_{-t}\;\psi_u\;\phi_t) = (\phi_t^* Y) \circ (\phi_{-t}\;\psi_u\;\phi_t) \qquad \textit{and} \qquad \frac{d}{dt}\phi_t^* Y = \phi_t^*[X,Y].$$

Proof. — To prove the first identity, we use the chain rule

$$\frac{d}{du}\left(\phi_{-t}\psi_u\phi_t(x)\right) = d_{\psi_u\phi_t(x)}\phi_{-t}\left(\frac{d}{du}\psi_u \circ \phi_t(x)\right)$$

and then basic properties of flows:

$$\frac{d}{du}\left(\phi_{-t}\psi_u\phi_t(x)\right) = \left(d_{\phi_{-t}\psi_u\phi_t(x)}\phi_t\right)^{-1}\left(Y(\psi_u \circ \phi_t(x))\right) = \left(\phi_t^*Y\right)\left(\phi_{-t}\psi_u\phi_t(x)\right).$$

Hence the first formula. As a consequence, we find :

$$\frac{d}{ds}\Big|_{s=0}\phi_s^*Y(x) = \frac{d}{ds}\Big|_{s=0}\frac{d}{du}\Big|_{u=0}\left(\phi_{-s}\psi_u\phi_s(x)\right) =: Z(x).$$

To understand the vector field Z hidden behind the right hand side, we look at the corresponding derivation $\mathscr{L}_Z$. Given any smooth function f, we compute

$$\begin{aligned}
\mathscr{L}_Z f(x) &= d_x f\left(\frac{d}{ds}\Big|_{s=0}\frac{d}{du}\Big|_{u=0}\left(\phi_{-s}\psi_u\phi_s(x)\right)\right)\\
&= \frac{d}{ds}\Big|_{s=0} d_x f\left(\frac{d}{du}\Big|_{u=0}\left(\phi_{-s}\psi_u\phi_s(x)\right)\right)\\
&= \frac{d}{ds}\Big|_{s=0}\frac{d}{du}\Big|_{u=0} f\left(\phi_{-s}\psi_u\phi_s(x)\right)\\
&= -\frac{d}{du}\Big|_{u=0} d_{\psi_u(x)}f\left(X(\psi_u(x))\right) + \frac{d}{du}\Big|_{u=0}\left(d_x(f\psi_u)(X(x))\right)\\
&= -\frac{d}{du}\Big|_{u=0}\mathscr{L}_X f\left(\psi_u(x)\right) + \frac{d}{du}\Big|_{u=0}\mathscr{L}_X(f\psi_u)(x)\\
&= -\mathscr{L}_Y\mathscr{L}_X f(x) + \mathscr{L}_X\mathscr{L}_Y f(x).
\end{aligned}$$

Hence $\frac{d}{ds}\Big|_{s=0}\phi_s^*Y = Z = [X,Y]$. Next apply ϕ_t^* and use the group property :

$$\phi_t^*[X,Y] = \phi_t^*\frac{d}{ds}\Big|_{s=0}\phi_s^*Y = \frac{d}{ds}\Big|_{s=0}\phi_t^*\phi_s^*Y = \frac{d}{ds}\Big|_{s=0}\phi_{s+t}^*Y = \frac{d}{dt}\phi_t^*Y.$$

□

Note that the second identity, for $X = Y$, implies that $\phi_t^*X = X$: a vector field is preserved by its own flow. It also provides an alternative and more geometric definition for the bracket : $[X,Y] = \frac{d}{dt}\Big|_{t=0}\phi_t^*Y$. In analogy with (1.6), we may introduce the Lie derivative of a vector field Y along X :

$$\mathscr{L}_X Y := \frac{d}{dt}\Big|_{t=0}\phi_t^*Y = [X,Y].$$

Let us point out another formula, which stems from the computations in the proof of Lemma 1.5.11:

$$[X,Y] = \frac{d}{ds}\Big|_{s=0}\frac{d}{du}\Big|_{u=0}\left(\psi_{-u}\phi_{-s}\psi_u\phi_s\right)$$

The following corollary explains what it means for two vectors to commute.

1.5.12 Corollary. — *The following statements are equivalent.*

1. $[X, Y] = 0$.
2. *The flow* ϕ_t *of* X *preserves* Y : $\phi_t^* Y = Y$.
3. *The flow* ψ_u *of* Y *preserves* X : $\psi_u^* X = X$.
4. *The flows generated by two vector fields* X *and* Y *commute.*

Proof. — If the flows commute then the first identity of lemma 1.5.11 implies $Y = \phi_t^* Y$, and then, from the second identity of lemma 1.5.11, $[X, Y] = 0$. This proves $4 \Rightarrow 2 \Rightarrow 1$. Conversely, if $[X, Y] = 0$, then, from the second formula of lemma 1.5.11, $\phi_t^* Y$ is constant, so $\phi_t^* Y = Y$, and then the first equation of lemma 1.5.11 says that, for any t, the flow generated by Y and parameterized by u) is $(\phi_t \psi_u \phi_t^{-1})$. But this flow is (ψ_u) by definition, whence $\phi_t \psi_u \phi_t^{-1} = \psi_u$: the flows commute. So $1 \Rightarrow 4$. It follows that 1, 2, 4, and also 3 (by symmetry of the roles of X and Y), are equivalent. □

The typical example of two vector fields with $[X, Y] = 0$ is $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$. The corresponding flows translate by t along the x^i and x^j variables, so they clearly commute. Somehow this is a very general example, in view of Frobenius theorem.

1.6. Frobenius theorem

1.6.1 Definition. — A p-dimensional *distribution* in a manifold M^n is a subbundle of rank p of the vector bundle TM, namely it is the data at each point $x \in M$ of a p-dimensional subspace $D_x \subset T_x M$ depending smoothly on x in the following sense : for any point x_0, one can find p smooth vector fields $X_1, \ldots, X_p$ defined in a neighborhood U of x_0 and such that D_x is the vector space generated by $X_1(x), \ldots, X_p(x)$.

For instance, any non-vanishing vector field X defines a 1-dimensional distribution $\mathbb{R}X$ on a manifold M. In this example, the distribution appears as the tangent bundle to the trajectories of the vector field, that is the solutions $c : I \subset \mathbb{R} \longrightarrow M$ of $\dot{c} = X(c)$. One says that these trajectories are *integral* curves for the distribution. It is natural to ask for a higher dimensional analogue of this phenomenon : for instance, does a 2-dimensional distribution induce some surfaces ? In general, for a p-dimensional distribution, the convenient replacement for the curves c will be immersions from a p-dimensional manifold to M.

1.6.2 Definition. — An *integral submanifold* for a distribution D on M is an immersion $i : X \longrightarrow M$ such that, for each point x of X, $d_x i(T_x X) = D_{i(x)}$. A distribution on M is called *integrable* if every point of M belongs to the image of an integral submanifold.

Beware the image of an integral submanifold is *not* necessarily a submanifold ! Nonetheless, every x in X admits a neighborhood $U \subset X$ such that $i(U)$ is a submanifold (from the local normal form for immersions). The problems arise when you look at maximal integral submanifolds. An example to keep in mind is the torus $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$: given an irrational number α, the constant vector field $(1, \alpha)$ on $\mathbb{R}^2$ is invariant under the action of $\mathbb{Z}^2$ so that it induces a non-vanishing vector field on $\mathbb{T}^2$, hence a distribution. Its integral curves are dense in the torus so they are not submanifolds.

As noticed above, every 1-dimensional is integrable, but in general distributions need not be integrable. A simple criterion will be provided by Frobenius theorem below.

1.6.3 Definition. — A distribution D is called *involutive* if for any vector fields X and Y lying in D, the vector field $[X, Y]$ also lies in D.

It is equivalent to ask that, around any point, the distribution D is generated by vector fields $X_1, \dots, X_p$ such that $[X_i, X_j]$ lies in D for every (i, j).

1.6.4 Lemma. — *If D is a p-dimensional involutive distribution of M^n, then, around any point, there are local coordinates $x^1, \dots, x^n$ such that D is generated by the vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}$.*

This means that, locally, we may identify M^n with $\mathbb{R}^n$ in such a way that the distribution admits the submanifolds $\mathbb{R}^p \times \{y\} \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-p}$, as integral submanifolds. Such a structure is called a *foliation.*

Proof. — The first step consists in producing vector fields $X_1, \dots, X_p$ spanning D near x such that $[X_i, X_j] = 0$. Choose local coordinates $(x^1, \dots, x^n)$ near x, x corresponding to 0, and local vector fields $Y_1, \dots, Y_p$ generating D near x. Up to composing by a linear isomorphism of $\mathbb{R}^n$, we can assume that at the point 0, $Y_i = \frac{\partial}{\partial x^i}$, $1 \leqslant i \leqslant p$. Then, near 0, write $Y_i = \sum_{j=1}^n a_i^j \frac{\partial}{\partial x^j}$ and observe that the matrix $(a_i^j)_{1 \leqslant i,j \leqslant p}$ is close to the identity hence invertible, with inverse (b_i^j). We then define a new local basis of D, consisting of the vectors $X_i = \sum_{j=1}^p b_i^j Y_j$, $1 \leqslant i \leqslant p$. One can check that

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=p+1}^n f_i^j \frac{\partial}{\partial x^j},$$

where f_i^j denotes a smooth function vanishing at 0. Then we calculate

$$[X_i, X_j] = \sum_{k=p+1}^n (\mathscr{L}_{X_i} f_j^k - \mathscr{L}_{X_j} f_i^k) \frac{\partial}{\partial x^k}.$$

But since D is involutive $[X_i, X_j] \in D$. From the form of the basis (X_i) of D we see that this implies $[X_i, X_j] = 0$.

The second step in the proof consists in considering the flows $\phi^1, \ldots, \phi^p$ generated by the vector fields $X_1, \ldots, X_p$. We let $Y^{n-p} \subset M^n$ be the local submanifold corresponding to $\{0\} \times \mathbb{R}^{n-p}$ in the local coordinates above and consider the map

$$\begin{array}{rccl} f: & \mathbb{R}^p \times Y & \longrightarrow & M \\ & (x^1, \ldots, x^p, y) & \longmapsto & \phi^1_{x^1} \cdots \phi^p_{x^p}(y) \end{array}.$$

The differential at $(0, y)$ of this map is

$$(x^1, \ldots, x^p, W) \longmapsto x^1 X_1 + \cdots + x^p X_p + W,$$

which is an isomorphism $\mathbb{R}^p \times T_y Y \to T_y M$ (since $T_y Y \oplus D_y = T_y M$), so f is a local diffeomorphism. Since the X_i's have vanishing brackets, the $\phi^i_{x^i}$'s commute. It follows that

$$d_{(x,y)} f \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} \phi^1_{x^1} \cdots \phi^p_{x^p}(y) = \frac{\partial}{\partial x^i} \phi^i_{x^i} \phi^1_{x^1} \cdots \widehat{\phi^i_{x^i}} \cdots \phi^p_{x^p}(y),$$

where the hat indicates an omitted factor. We deduce :

$$d_{(x,y)} f \left(\frac{\partial}{\partial x^i} \right) = X_i \left(\phi^i_{x^i} \phi^1_{x^1} \cdots \widehat{\phi^i_{x^i}} \cdots \phi^p_{x^p}(y) \right) = X_i \left(f(x, y) \right),$$

namely $f^* X_i = \frac{\partial}{\partial x^i}$. The promised coordinates on M are obtained by applying f^{-1} and taking coordinates on Y. □

***1.6.5 Theorem* (Frobenius)**. — *A distribution is integrable if and only if it is involutive.*

Proof. — The implication "involutive ⇒ integrable" is clear from lemma 1.6.4. Now assume the distribution D on M is integrable. Pick a point y in M and an integral submanifold $i : X \longrightarrow M$ such that $y = i(x_0)$ for some x_0 in X. As noticed above, we can always shrink X and assume $i(X)$ is a submanifold. Then if A and B are two vector fields on M, which lie in D, their restriction to $i(X)$ are vector fields on the submanifold $i(X)$, so that the restriction of $[A, B]$ on $i(X)$ lies in D (from Proposition 1.5.4). So $[A, B](y)$ is in D_y, for all y in M : D is involutive. □

1.6.6. Exercise. — Let x, y, z denote the standard coordinates on $\mathbb{R}^3$. We consider the distribution $D = \mathrm{Ker}(dz - ydx) = \mathrm{Vect}(\partial_y, \partial_x + y\partial_z)$ on $\mathbb{R}^3$. 1) Check that this is not integrable. 2) Compute the flows ϕ_t (resp. ψ_u) of ∂_y (resp. $\partial_x + y\partial_z$) and then the commutator $\psi_{-u}\phi_{-t}\psi_u\phi_t$. 3) Deduce that any two points in $\mathbb{R}^3$ can be connected by a (piecewise smooth) path that remains tangent to D. Compare this phenomenon with what happens on a foliation (namely, an integrable distribution).

1.6.7. Example. — Many problems can be expressed in terms of the integrability of a distribution and are thus solved by Frobenius theorem. Here is an example: we explain how the problem of finding a function with given differential can be expressed in these terms. Of course the result is a well-known basic fact, but it will serve for us as a very simple illustration of the use of the theorem.

So suppose we have a 1-form α on a manifold M^n and we want to understand conditions on α in order to find a function f such that $df = \alpha$ locally. We consider the manifold $X^{n+1} = M \times \mathbb{R}$, with the n-dimensional distribution

$$D_{(x,t)} = \{(\xi, \alpha_x(\xi)), \xi \in T_xM\}.$$

An integral submanifold of D is locally a submanifold $Y^n \subset M^n \times \mathbb{R}$ tangent to D. Since D is always transverse to the $\mathbb{R}$ factor of $T_xX = T_xM \oplus \mathbb{R}$ (meaning that $T_xX = D_x \oplus \mathbb{R}$), such a Y can be seen as the graph of a function $f : M \to \mathbb{R}$. Then $T_{(x,f(x))}Y = \{(\xi, d_xf(\xi)), \xi \in T_xM\}$ so Y is an integral submanifold of D if and only if $df = \alpha$.

So we see that the problem of finding locally f such that $df = \alpha$ is equivalent to finding an integral submanifold of D. By Frobenius theorem, this is possible if and only if D is involutive. Let us write down the condition in local coordinates (x^i) on M: we have $\alpha = \alpha_i dx^i$ an the distribution D is generated by the vector fields $X_i = \frac{\partial}{\partial x^i} + \alpha_i \frac{\partial}{\partial t}$. Now :

$$[X_i, X_j] = (\frac{\partial \alpha_j}{\partial x^i} - \frac{\partial \alpha_i}{\partial x^j})\frac{\partial}{\partial t}.$$

This belongs to D only if it vanishes (because D is transverse to ∂_t), and we recover in this way the fact that α is locally a df if and only if the first derivatives of α are symmetric : $\frac{\partial \alpha_j}{\partial x^i} - \frac{\partial \alpha_i}{\partial x^j} = 0$.

1.7. Differential forms

1.7.1. Linear algebra. — If E is an n dimensional real vector space, we define $\Lambda^k E^*$ as the space of alternate k-linear forms on E, also called exterior forms of degree k. Note $\Lambda^k E^* \subset \otimes^k E^*$. The degree k of the form $\alpha \in \Lambda^k E^*$ is denoted by $|\alpha|$ or $\deg \alpha$. Observe that $\Lambda^0 E^* = \mathbb{R}$, $\Lambda^1 E^* = E^*$ and $\Lambda^k E^* = \{0\}$ for $k > n$. Sometimes one considers all k-forms together: $\Lambda^\bullet E = \oplus_{k=0}^n \Lambda^k E$.

Concretely, if $(e_i)_{i=1,\dots,n}$ is a basis of E, and (e^i) denotes the dual basis of E^*, then a basis of $\Lambda^k E^*$ consists of $(e^{i_1} \wedge \cdots \wedge e^{i_k})_{i_1<\cdots<i_k}$, where the exterior product $\alpha_1 \wedge \cdots \wedge \alpha_k$ of k one-forms is defined by

$$\alpha_1 \wedge \cdots \wedge \alpha_k(x_1, \dots, x_k) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \alpha_1(x_{\sigma(1)}) \cdots \alpha_k(x_{\sigma(k)}).$$

In particular, the dimension of $\Lambda^k E^*$ is $\binom{n}{p}$.

The exterior product (or wedge product) extends to all forms to define an associative product mapping $\Lambda^k \otimes \Lambda^l$ to Λ^{k+l} and satisfying the commutation rule

$$\beta \wedge \alpha = (-1)^{|\alpha||\beta|} \alpha \wedge \beta.$$

1.7.2. Differential forms on a manifold. — The algebraic constructions described above can be implemented on each fiber of the tangent bundle of a manifold M. For every nonnegative integer k, this provides a vector bundle $\Lambda^k T^*M$ or simply $\Lambda^k M$ over M, whose fiber $\Lambda^k_x M$ at a point x is $\Lambda^k (T_x M)^*$ (see section 1.4). This is the bundle of exterior forms of degree k.

1.7.3 Definition. — A *k-differential form* on a manifold M is a smooth section of the vector bundle $\Lambda^k M$.

The set of differential forms of degree k is $\Gamma(\Lambda^k M) =: \Omega^k(M)$. For instance, $\Omega^0(M) = C^\infty(M)$, the set of smooth functions on M with values in $\mathbb{R}$, and $\Omega^1(M) = \Gamma(T^*M)$. We also set $\Omega^\bullet(M) := \oplus_k \Omega^k(M)$.

In local coordinates, we have a basis (dx^i) of 1-forms, and a k-differential form ω is a linear combination

$$\sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

1.7.4. Exercise. — Check that the form $4\frac{dx\wedge dy}{(1+x^2+y^2)^2}$ defined on $S^2 - \{N\}$ in the coordinates (x, y) given by stereographic projection extends to a global 2-form on S^2. (As we will see later, this is the volume form of the sphere, and its integral gives the volume of the sphere, that is 4π).

If $f : M \to N$ is a smooth map and α is a k-form on N, then one can define the *pull-back* of α by f on M, defined at the point $x \in M$, on vectors $X_1, \dots, X_k \in T_xM$, by

$$(f^*\alpha)_x(X_1, \dots, X_k) = \alpha_{f(x)}(d_xf(X_1), \dots, d_xf(X_k)).$$

The pull-back satisfies $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

Finally, a k-form ω on M defines an alternate $C^\infty(M)$-linear form on the $C^\infty(M)$-module $\Gamma(TM)$ of vector fields on M: $(X_1, \dots, X_p) \mapsto \omega(X_1, \dots, X_p)$. Conversely:

1.7.5 Lemma. — *Any $C^\infty(M)$-linear alternate k-form α on $\Gamma(TM)$ is induced by some smooth k-differential form.*

One says that the form α is *tensorial*, namely it comes from a section of a tensor bundle (in this case : $\Lambda^k M \subset \otimes^k T^*M$).

Proof. — One first prove that such a $C^\infty(M)$-linear form α is local, as in the proof of theorem 1.5.2. Then one is reduced to consider only local vector fields, and one can use local coordinates (x^i): if $X_j = X_j^i \frac{\partial}{\partial x^i}$, then by $C^\infty(M)$-linearity

$$\alpha(X_1, \dots, X_k) = \sum_{(i_1, \dots, i_k)} X_1^{i_1} \cdots X_k^{i_k} \alpha\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)$$

which is induced by the k-differential form

$$\sum_{i_1 < \cdots < i_k} \alpha\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

□

Let us make the following more general statement. It is often useful in differential geometry and its proof is the same as the argument above (for Lemma 1.7.5). The details are left to the reader.

1.7.6 Lemma. — *Let E and F be two vector bundles over the manifold M. Assume $P : \Gamma(E) \to \Gamma(F)$ is a $C^\infty(M)$-linear map. Then P can be identified with an element of $\Gamma(E^* \otimes F)$: it is a section of the bundle $E^* \otimes F$ whose fiber over x consists of linear maps from E_x to F_x.*

Differential forms come with an extra structure.

1.7.7 Lemma and definition. — *The* exterior differential *d on M is the unique linear map $d : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$ with the following properties.*

1. *d maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$.*

2. *$d : \Omega^0(M) \longrightarrow \Omega^1(M)$ coincides with the differential of smooth functions.*
3. *d is an odd derivation of $\Omega^\bullet M$: for any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.*
4. *$d \circ d = 0$.*

Proof. — Uniqueness. We first prove that such a map d is necessarily local : if α vanishes on some open set U, then $d\alpha$ vanishes on U. The proof consists in observing that for any point x of U, we may pick a smooth function χ vanishing on a neighborhood of x and constant to 1 outside U. Then $\alpha = \chi\alpha$, hence $d\alpha = d\chi \wedge \alpha + \chi d\alpha$, which implies $d\alpha(x) = 0$, hence the locality property. It follows that, even if β is defined only on some open set V, we may define $d\beta$, on V. Now, given local coordinates, a differential k-form α can always be written locally as

$$\alpha = f_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

It follows from axioms 3 and 4 that

$$d\alpha = df_{i_1,\dots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{1.7}$$

Since the right-hand side only uses d on functions, it is well defined by axiom 2, hence the local uniqueness and then the global uniqueness.

Existence. Given a differential form α, formula 1.7 defines $d\alpha$ in any chart. From the local uniqueness property (applied on the overlap of any two charts), this defines globally $d\alpha$ on the whole M. Axioms 1 and 2 are clearly satisfied. To prove 3, it is sufficient to consider

$$\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \text{and} \quad \beta = g dx^{j_1} \wedge \cdots \wedge dx^{j_l}.$$

Then we compute from formula 1.7 and Leibniz rule for the differential on functions:

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(fg) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= (gdf + fdg) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge g dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&\quad + (-1)^k f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dg \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.
\end{aligned}$$

To prove 4, we use formula 1.7 twice to get

$$\begin{aligned} d^2\alpha &= d(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d\left(\frac{\partial f}{\partial x^a} dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= d\left(\frac{\partial f}{\partial x^a}\right) \wedge dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{\partial^2 f}{\partial x^b \partial x^a} dx^b \wedge dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

From Schwarz theorem, $\frac{\partial^2 f}{\partial x^b \partial x^a}$ is symmetric in a and b, while $dx^b \wedge dx^a$ is skewsymmetric. So the sum over a and b vanishes : $d^2 = 0$. □

As a consequence of unicity, the reader may check that, for any smooth map $f : M \to N$ and $\omega \in \Omega^\bullet(N)$, one has $f^* d\omega = d(f^*\omega)$.

It is important to be able to calculate the exterior differential from the point of view of linear forms on vector fields. The following formula shows that the exterior differential is somehow a counterpart of the Lie bracket.

1.7.8 Lemma. — *For a differential k-form α and vector fields X_i,*

$$\begin{aligned} d\alpha(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i \mathscr{L}_{X_i}\big(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) \\ &+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k). \end{aligned}$$

When $k = 1$, this formula reduces to

$$d\alpha(X, Y) = \mathscr{L}_X\big(\alpha(Y)\big) - \mathscr{L}_Y\big(\alpha(X)\big) - \alpha([X, Y]).$$

Proof. — One checks that the right hand side of the formula is $C^\infty(M)$-linear in X_0, X_1,..., X_k, and is alternate, so it actually defines a $(k+1)$-differential form. To determine it, it suffices to take the X_j among a local basis of vector fields $(\frac{\partial}{\partial x^i})$. Then the calculation becomes very simple because all the brackets vanish. □

As an application of this formula, we may give a reformulation of the Frobenius theorem. Let D be a distribution of rank p on M^n and $I(D)$ denote the ideal of differential forms β that vanish along D. To understand whether D is integrable or not, we just need to work *locally.* So we work with a trivialization of TM by vector fields $X_1, \ldots, X_n$ and assume $X_1, \ldots X_p$ generate D. We let $(\alpha_1, \ldots, \alpha_n)$ denote the dual basis, so that D can be seen as the intersection of the kernels of $\alpha_{p+1}, \ldots, \alpha_n$ and $I(D)$ appears as the ideal generated by $\alpha_{p+1}, \ldots, \alpha_n$. Then D is involutive if and only if $[X_i, X_j] \in \Gamma(D)$

for every $i, j \leqslant p$, which in view of Lemma 1.7.8 means $d\alpha_k \in I(D)$ for very $k > p$. In other words D is involutive, and therefore integrable, if and only if $dI(D) \subset I(D)$.

For instance, if $D = \operatorname{Ker} \alpha$ is a rank two distribution on M^3, for some non-vanishing one-form α, then $I(D) = \alpha \wedge \Omega^{\cdot}(M)$ and the Frobenius integrability criterion is exactly $\alpha \wedge d\alpha = 0$. If on the contrary $\alpha \wedge d\alpha$ is non-zero everywhere, one says D is a *contact* distribution. A theorem, known as the Darboux theorem, asserts that a contact distribution is always locally isomorphic to the distribution described in 1.6.6. See also 1.9.3.

1.7.9. Lie derivative. — The *Lie derivative* of a differential form ω along a vector field X with flow ϕ_t is defined by

$$\mathscr{L}_X \omega = \frac{d}{dt}\Big|_{t=0} \phi_t^* \omega. \tag{1.8}$$

Unsurprisingly, it combines nicely with the exterior product and differential :

$$\mathscr{L}_X(\alpha \wedge \beta) = (\mathscr{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathscr{L}_X \beta), \qquad \mathscr{L}_X d\alpha = d\mathscr{L}_X \alpha.$$

Given a vector field X on M, we can introduce the interior product $\iota_X : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$, defined by the formula

$$(\iota_X \alpha)(X_1, \dots, X_k) = \alpha(X, X_1, \dots, X_k).$$

This is an odd derivation of the algebra $\Omega^\bullet(M)$. The Lie derivative on differential forms can be easily computed as follows.

***1.7.10 Lemma* (Cartan's magic formula).** — $\mathscr{L}_X = d \circ \iota_X + \iota_X \circ d$.

Proof. — First, observe that $d \circ \iota_X + \iota_X \circ d$ is a derivation of the algebra $\Omega^\bullet(M)$ (because d and ι_X are both odd derivations), as well as $\mathscr{L}_X$. In particular, both operators are local, so we only need to check that they coincide locally, in a chart, on k-forms $\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. From the derivation property again, we only need to check that the operators agree on the function f and one-forms dx^i's. Indeed, since both operators commute with d, we are left to check that they coincide on functions (f and x^i) and this is obvious : they both map a function f to $df(X)$. □

1.8. Orientation and integration

Notice that $\Lambda^n \mathbb{R}^n = \mathbb{R}$: every alternate n-form is proportional to $dx^1 \wedge \cdots \wedge dx^n$, the determinant.

On a manifold M^n, $dx^1 \wedge \cdots \wedge dx^n$ is well defined in local coordinates, but of course does not extend in general to the whole manifold. If we change coordinates, $(x^1, \dots, x^n) = \psi(y^1, \dots, y^n)$, for some diffeomorphism ψ of $\mathbb{R}^n$, then, from the usual behaviour of determinants with respect to a change of basis, we have

$$dx^1 \wedge \cdots \wedge dx^n = J(\psi) dy^1 \wedge \cdots \wedge dy^n \tag{1.9}$$

where $J(\psi) = det(d\psi)$ is the Jacobian determinant of ψ.

1.8.1 Definition. — A manifold M is called *orientable* if it admits an atlas whose transition functions ψ have positive Jacobian determinant : $J(\psi) > 0$.

Such an atlas is an *oriented atlas.* Two oriented atlases are called equivalent if their union is still an oriented atlas. An *orientation* is an equivalence class of oriented atlases. A manifold endowed with an orientation is called *oriented.*

1.8.2 Lemma. — *Any oriented manifold M^n carries a* volume form*, namely an element ω of $\Omega^n(M)$ that is positive in the following sense : in any chart of an oriented atlas, $\omega = f\ dx^1 \wedge \cdots \wedge dx^n$ for some positive function f.*

Proof. — Let us pick an oriented atlas $((U_i, \phi_i))_i$. On each U_i, we may define an n-form $\omega_i = dx^1 \wedge \cdots \wedge dx^n$. Now, we choose a partition of unity (χ_i) subordinate to the covering (U_i) and set $\omega = \sum_i \chi_i \omega_i$. It is easy to check in coordinates that this is positive (since, at any point, one of the χ_i's is nonzero). □

1.8.3 Proposition. — *Let M^n be a manifold. The following statements are equivalent.*

1. *M^n is orientable.*
2. *M carries a differential n-form that is everywhere nonzero.*
3. *The line bundle $\Lambda^n M$ is trivial.*

Proof. — "2 ⇔ 3" is clear. "1 ⇒ 2" follows from the lemma above. Let us prove "2 ⇒ 1". Let us pick any atlas $((U_i, \phi_i))_i$ of M. Let ω be a nonvanishing element of $\Lambda^n M$. On each U_i we may write $\omega = f_i dx_i^1 \wedge \cdots \wedge dx_i^n$ in the local coordinates corresponding to ϕ_i. Then f_i does not vanish on U_i so it is either positive or negative. In the case where $f_i < 0$, we change ϕ_i by composing it by a reflection of $\mathbb{R}^n$; in the new chart ϕ_i, f_i is changed into $-f_i$ so it is positive. So we assume that f_i is positive for all i. Now on each $U_i \cap U_j$, we have $f_i dx_i^1 \wedge \cdots \wedge dx_i^n = \omega = f_j dx_j^1 \wedge \cdots \wedge dx_j^n$ so that the Jacobian determinants of the atlas are given by the quotient $\frac{f_i}{f_j} > 0$, so the atlas is oriented. □

If M^n is orientable, it carries a nowhere vanishing n-form ω and then the set of nowhere vanishing n-forms splits into the two disjoint subsets $\{f\omega/f \in C^\infty(M), f > 0\}$ and $\{f\omega/f \in C^\infty(M), f < 0\}$. The choice of an orientation is just the choice of one these subsets as the set of volume forms.

When M^n is not orientable, there is a canonical way to find a better behaved manifold: the quotient $\left(\Lambda^n M \backslash \{\text{zero section}\}\right)/\mathbb{R}_+^*$ (quotient by positive scalings in each fiber) is a non-trivial double cover of M and an orientable manifold.

1.8.4. Examples. —

1. $\mathbb{R}^n$ is orientable.
2. The sphere S^n is oriented, with volume form $\iota_{\vec{n}}(dx^1 \wedge \cdots \wedge dx^{n+1})$, where $\vec{n} = x^i \frac{\partial}{\partial x^i}$ is the outward normal vector to S^n. This means that at each point, a direct basis of S^n is given by $(e_1, \dots, e_n)$ so that $(\vec{n}, e_1, \dots, e_n)$ is a direct basis of $\mathbb{R}^{n+1}$.
3. The projective space $\mathbb{R}P^n$ is orientable iff n is odd. Indeed consider the map $\pi : S^n \to \mathbb{R}P^n$. This is a 2:1 local diffeomorphism, given by quotient by the antipodal map a. If ω is a volume form on $\mathbb{R}P^n$, then $\pi^*\omega$ is a nowhere vanishing n-form on S^n (since π is a local diffeomorphism), satisfying $a^*\pi^*\omega = \pi^*\omega$ (since $\pi \circ a = \pi$). This implies that a preserves the orientation of S^n. Now remark that $a^*\vec{n} = \vec{n}$, so
$$a^*(\iota_{\vec{n}}(dx^1 \wedge \cdots \wedge dx^{n+1})) = (-1)^{n+1}\iota_{\vec{n}}(dx^1 \wedge \cdots \wedge dx^{n+1}),$$
so a preserves the orientation of S^n if and only n is odd. So if $\mathbb{R}P^n$ is orientable then n is odd. Conversely if n is odd, then the standard volume form of S^n is invariant under a, so it induces a well-defined volume form on $\mathbb{R}P^n$.
4. The tangent bundle of any manifold is orientable.
5. Any complex manifold is an oriented (real) manifold.
6. The Möbius band is not orientable.

Given a volume form ω on a manifold M, we may define the *divergence* of a vector field X as the function $\operatorname{div}_\omega(X)$ defined by $\mathscr{L}_X\omega = \operatorname{div}_\omega(X)\omega$. In view of Cartan's magic formula, this means $\operatorname{div}_\omega(X) = d(\iota_X\omega)/\omega$ (exercise: check this definition is consistent with the usual one on $\mathbb{R}^n$). Basicallly, a vector field has vanishing divergence if and only if its flow preserves the volume form ($\phi_t^*\omega = \omega$).

1.8.5. Exercise. — Let $X^{2n} = T^*M$ denote the cotangent bundle of a manifold M^n. We denote by π the natural projection from $X = T^*M$ to M. For any point (x, ξ) of X (namely $x \in M$ and $\xi \in T_x^*M$), we set $\alpha_{(x,\xi)} := \xi \circ d_{(x,\xi)}\pi$. Check that this defines a (canonical !) one-form on X. It is known as the Liouville form. The choice of some local coordinates x^i on M gives rise to some local coordinates x^i, p_i on $T^*M = X$. What is α in these coordinates ? Prove that the closed two-form $\Omega := d\alpha$ is non-degenerate, i.e. $V \mapsto \Omega(V, .)$ is a linear isomorphism between T_zX and T_z^*X for every $V \in T_zX$, $z \in X$. As a consequence, $\Omega^n = \Omega \wedge \cdots \wedge \Omega$ (n times) is a *canonical* volume form on the cotangent bundle.

1.8.6. Exercise. — The data of a non-degenerate closed two-form Ω on a manifold M^{2n} (necessarily of even dimension) is known as a *symplectic* structure. Pick a function H on M (H for Hamiltonian). The non-degeneracy of Ω makes it possible to define a vector field X_H by the relation $\iota_{X_H}\Omega = dH$. Check that the flow of X_H automatically preserves the symplectic for Ω, as well as the function H. In particular, its divergence with respect to the volume form Ω^n vanishes.

We refer to the book of Marsden and Ratiu (for instance) for more symplectic geometry and applications to mechanics. Let us just point out that, like in contact geometry (cf. 1.6.6, 1.7.2), there is a Darboux theorem, asserting that a symplectic structure is always locally isomorphic to the canonical symplectic structure of the cotangent bundle described above, in that there are coordinates where it is given by the same expression (cf. exercise 1.9.3).

1.8.7. Integration. — Let M^n be an oriented manifold. We are now going to define the integral $\int_M \omega$ of any compactly supported n-form ω on M. First suppose that ω has his support contained in the domain of a chart, with coordinates x^i. Then $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ where f has compact support, and we can define

$$\int_M \omega := \int f(x)dx^1 \cdots dx^n.$$

The right hand side is just an integral over an open set of $\mathbb{R}^n$, for the Lebesgue measure. Now, pick other coordinates y^j, with $(x^1, \ldots, x^n) = \psi(y^1, \ldots, y^n)$. On the one hand, formula (1.9) yields :

$$\omega = f(y)J(\psi)(y)dy^1 \wedge \cdots \wedge dy^n.$$

On the other hand, the formula for the change of variables, in $\mathbb{R}^n$, yields :

$$\int f(x)dx^1 \cdots dx^n = \int f(y)|J(\psi)|(y)dy^1 \cdots dy^n.$$

If we work with an oriented atlas, we have $J(\psi) > 0$, so that our definition of $\int_M \omega$ does not depend on the choice of coordinates.

The definition of $\int_M \omega$ is then extended to any ω by a partition of unity (χ_i) subordinate to a covering of M by coordinate charts: $\omega = \sum \chi_i \omega$ and $\int_M \omega = \sum \int_M (\chi_i \omega)$.

The resulting linear form $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ does not depend on the partition of unity chosen, it only depends on the orientation chosen on M : if M_+ and M_- denote the same manifold M with its two different orientations, then $\int_{M_+} = -\int_{M_-}$.

The fundamental formula of calculus

$$f(b) - f(a) = \int_a^b f'(x)dx \tag{1.10}$$

has a powerful extension to the setting of manifolds, known as Stokes formula. It requires the notion of manifold with boundary.

1.8.8 Definition. — Let M be a Hausdorff topological space. M is a *differential manifold with boundary* of dimension n if there is a countable family $((U_i, \phi_i))_{i \in I}$ with the following properties.

- $(U_i)_{i \in I}$ is an open covering of M.
- For every i in I, $\phi_i : U_i \longrightarrow \Omega_i$ is a homeomorphism onto an open subset Ω_i of $(-\infty, 0] \times \mathbb{R}^{n-1}$.
- For every i and j in I, $\phi_i \circ \phi_j^{-1}$ is smooth.

The boundary ∂M of M is the set of points $x \in M$ that are mapped into $\{0\} \times \mathbb{R}^{n-1}$ by some ϕ_i. It follows that $M \backslash \partial M$ is a manifold (without boundary) of dimension n, while ∂M is a manifold (without boundary) of dimension $n-1$ (charts are obtained by restriction). A simple example is the closed unit ball in $\mathbb{R}^n$, whose boundary is the unit sphere.

There is a natural notion of differential form on a manifold with boundary M. If ω is an element of $\Omega^k(M)$, it induces a k-form on ∂M : $i^*\omega$, where $i : \partial M \longrightarrow M$ is the inclusion. This k form $i^*\omega$ is usually denoted by ω.

A manifold with boundary M^n is said to be oriented if its transition functions have positive Jacobian determinants. We can define the integral of n-forms with compact support in M as in the case of manifolds without boundary. Note also that an orientation on M induces an orientation on ∂M : locally, this is the orientation of $\mathbb{R}^{n-1} \cong \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ (if $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$, $(e_2, \dots, e_n)$ is an oriented basis of $\{0\} \times \mathbb{R}^{n-1}$).

***1.8.9 Theorem* (Stokes).** — *If M^n is an oriented manifold with boundary and ω is a compactly supported $(n-1)$-form on M, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, if M^n is a manifold without boundary, then $\int_M d\omega = 0$ for any compactly supported $(n-1)$-form.

Proof. — Using a partition of unity, it is sufficient to check the case where the support of ω is contained in the domain of some coordinates x^i, namely, we may assume $M = \{x^1 \leqslant 0\} \subset \mathbb{R}^n$. Then $\omega = \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$, so

$$d\omega = \left(\sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n,$$

and

$$\int_M d\omega = \sum_i (-1)^{i-1} \int_{x^1 \leqslant 0} \frac{\partial \omega_i}{\partial x^i} dx^1 \cdots dx^n.$$

Using formula (1.10), we see that all the terms with $i > 1$ vanish (because ω has compact support), while the remaining term, with $i = 1$, yields

$$\int_M d\omega = \int_{x^1=0} \omega_1 dx^2 \cdots dx^n = \int_{\partial M} \omega.$$

□

This theorem contains as special cases some classical identities, well known to physicists, such as the Gauss-Ostrogradsky formula (involving the divergence of a vector field in $\mathbb{R}^3$) or the Kelvin-Stokes formula (involving the curl operator) or the Green-Riemann formula.

1.9. De Rham cohomology

Let us briefly mention a few facts about de Rham cohomology. Proofs for what we state, and much more, can be found in the references, in particular in the book by Bott and Tu. Let M^n be a differential manifold. The vector space of *closed* k-forms is

$$Z^k(M) := \{\alpha \in \Gamma(\Omega^k M), d\alpha = 0\},$$

while the vector space of *exact* k-forms is

$$B^k(M) := \{d\beta, \beta \in \Gamma(\Omega^{k-1} M)\}.$$

Then, since $d \circ d = 0$, $B^k(M)$ is a subspace of $Z^k(M)$. The k-th group of de Rham cohomology is defined as the quotient :

$$H^k(M) = Z^k(M)/B^k(M).$$

For instance, it is clear that $H^0(M)$ consists of locally constant functions on M so, if M has N connected components, then $H^0(M) = \mathbb{R}^N$.

Locally, on any manifold, if $d\alpha = 0$ then there exists β such that $d\beta = \alpha$ (this is Poincaré Lemma, cf. exercise 1.9.3), so the cohomology does not depend on local properties of M. It turns out that $H^k(M)$ is a topological invariant of M (it depends on the class of M modulo homeomorphisms, and even modulo homotopy equivalences) ; in fact, it coincides with singular cohomology. If M is compact, then $H^k(M)$ is finite dimensional and its dimension $b_k(M) = \dim H^k(M)$ is called the k-th Betti number of M. At the end of this course, we will discuss Hodge theorem, which makes it possible to see a few general properties of de Rham cohomology at once. Let us just point out an immediate application of Stokes' theorem: on a compact oriented manifold M^n (without boundary), the integration of n-forms gives rise to a well-defined and surjective map $H^n_{DR}(M) \to \mathbb{R}$; so $b_n(M)$ is nonzero. This map is in fact an isomorphism, cf. 3.10.1.

There are some techniques that make it possible to compute the de Rham cohomology. The cohomology of $\mathbb{R}^n$ vanishes except in degree 0. For the sphere S^n, the cohomology vanishes in every degree, except in degrees 0 and n, and $H^0(S^n) = H^n(S^n) = \mathbb{R}$. For the complex projective space $\mathbb{C}P^n$, the cohomology vanishes in odd degrees, and in even degree $2k$ for $k = 0, \dots, n$ one has $H^{2k}(\mathbb{C}P^n) = \mathbb{R}$.

1.9.1. Exercise. — Let M^{2n} be a compact manifold (without boundary), endowed with a symplectic form Ω (the definition is given in 1.8.6). Prove that $b_{2k}(M)$ is non-zero for $k = 0, \dots, n$. This fact explains the origin of the cohomology of $\mathbb{C}P^n$, which is a symplectic manifold, owing to the following exercise.

1.9.2. Exercise. — Let us see S^{2n+1} as the unit sphere in the Hermitian space $(\mathbb{C}^{n+1}, \langle ., . \rangle)$. For every point z of S^{2n+1}, we may define a unit vector $T_z := iz$ tangent to S^{2n+1} and an element η_z of $T_z^* S^{2n+1}$, by $\eta_z(V) = \langle iz, V \rangle$ for all vectors $V \in T_z S^{2n+1}$. What is the flow of the vector field T ? Prove that $L_T \eta$ and $\iota_T d\eta$ vanish. Let π denote the natural projection $S^{2n+1} \to \mathbb{C}P^n$. Prove that there is a unique two-form Ω on $\mathbb{C}P^n$ such that $d\eta = \pi^*\Omega$. Prove that Ω is symplectic (for instance, prove that $\eta \wedge d\eta^n$ does not vanish). Ω is known as the *Fubini-Study form.*

1.9.3. Exercise. — 1) Let M be a manifold and $(\psi_t)_{0\leqslant t\leqslant 1}$ denote a smooth one-parameter family of diffeomorphisms ($(t,x)\mapsto \psi_t(x)$ is smooth). Prove that for any form α, $\frac{d}{dt}\psi_t^*\alpha = \psi_t^* L_{X_t}\alpha$, where X_t is the time dependent vector field defined by $X_t = \frac{d}{dt}\psi_t \circ (\psi_t)^{-1}$. Prove that if α is closed, then

$$\psi_1^*\alpha - \psi_0^*\alpha = d\left(\int_0^1 \psi_t^*(\iota_{X_t}\alpha)dt\right)$$

2) Prove the Poincaré lemma: any closed form of positive degree is locally exact. Hint: use 1) on a small ball of $\mathbb{R}^n$, with $\psi_t(x) = tx$ and check that the apparent problem at $t = 0$ is irrelevant.

3) Prove the Hairy Ball theorem: there is no nonvanishing vector field on the sphere S^{2n}. Hint: if such a vector field X existed, you could associate to any point x of S^{2n} the point $\psi_t(x)$ obtained by rotating x of an angle πt in the plane containing x and $X(x)$, in the direction of $X(x)$; deduce that the standard volume form of S^{2n} would be preserved by the antipodal map. (Observe the previous exercise yields a nonvanishing vector field on the sphere S^{2n+1}.)

4) Prove the symplectic Darboux theorem: any symplectic form Ω is locally isomorphic to the standard symplectic form $\Omega_0 = \sum_{i=1}^n dx^i \wedge dx^{i+n}$ on $\mathbb{R}^{2n}$, namely $\Omega = \psi^*\Omega_0$ for some local diffeomorphism ψ. To prove this, you may work in coordinates, on a small ball around 0 in $\mathbb{R}^{2n}$, assume Ω and Ω_0 coincide at the origin and consider $\Omega_t = t\Omega + (1-t)\Omega_0$. The trick (known as Moser's trick) consists in finding a family of diffeomorphisms ψ_t such that ψ_0 is the identity and $\psi_t^*\Omega_t$ is constant in time, so that ψ_1 will do the job. In order to build this family of diffeomorphisms, you will first find the vector fields X_t by computing the derivative of $\psi_t^*\Omega_t$ and then integrate the time-dependent ordinary differential equation $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

5) A distribution D of rank $2n$ on a manifold M^{2n+1} is called a contact distribution if it can be written locally as $D = \operatorname{Ker}\alpha$ for some one-form α such that $\alpha \wedge d\alpha^n$ does not vanish; it is equivalent to require that the restriction of $d\alpha$ on D is non-degenerate. Prove the contact Darboux theorem: any contact distribution D on M^{2n+1} is locally isomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$, given by the kernel of $\alpha_0 = dx^{2n+1} - \sum_{i=1}^n x^i dx^{i+n}$. Hint: Moser's trick.

1.10. Bibliography

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CHAPTER 2

CONNECTIONS

2.1. Connections as covariant derivatives

Here, we address the following problem: find a way to take derivatives of sections of vector bundles. More precisely, suppose E is a vector bundle over the manifold M. Pick a section s of E and a vector $X \in T_xM$. What we want to define is a derivative of s along X at x, denoted by $(\nabla_X s)_x \in E_x$. This should be linear in $X \in T_xM$, so at the point x the object $(\nabla s)_x$ should belong to $\mathrm{Hom}(T_xM, E_x) = T_x^*M \otimes E_x$. This means that ∇s should be a section of the bundle $T^*M \otimes E$ and justifies the following definition.

2.1.1 Definition. — A *connection* ∇ on a vector bundle E over the manifold M is a $\mathbb{R}$ -linear operator

$$\nabla : \Gamma(M, E) \longrightarrow \Gamma(M, T^*M \otimes E)$$

satisfying the following Leibniz rule: if $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$, then

$$\nabla(fs) = df \otimes s + f\nabla s.$$

There is an obvious variant on complex vector bundles, requiring the connection to be $\mathbb{C}$-linear. Note also that connections are sometimes called *covariant derivatives*.

If E is the trivial line bundle over M (namely, $E = M \times \mathbb{R}$), then $\Gamma(E) = C^\infty(M)$ and an obvious example of connection on E is given by the exterior differential $d : C^\infty(M) \longrightarrow \Omega^1(M)$. Letting d act componentwise, we therefore obtain a map $d : C^\infty(M, \mathbb{R}^p) \longrightarrow \Omega^1(M, \mathbb{R}^p)$. This yields a connection on the trivial bundle of rank p ($E = M \times \mathbb{R}^p$). Such a connection is said to be *trivial*.

As we have already seen in other contexts, the Leibniz rule implies immediately that any connection ∇ is a local operator: if U is an open set, $(\nabla s)|_U$ depends only on $s|_U$. In other words, it induces an operator $\Gamma(U, E|_U) \to \Gamma(U, T^*U \otimes E|_U)$.

By definition of a vector bundle E over M, we can always cover M by open sets U_i such that $E|_{U_i}$ can be identified with a trivial vector bundle $U_i \times \mathbb{R}^p$ over U_i. On every such $E|_{U_i}$, we have seen above that there is a connection ∇^i (induced by the differential d on U_i). Using a partition of unity (χ_i) subordinate to (U_i), we may then build a connection ∇ on E :

$$\nabla_X s = \sum_i \chi_i \nabla^i_{X|_{U_i}} s|_{U_i}.$$

The set of connections is therefore always non-empty. It turns out that it carries a natural affine structure.

2.1.2 Lemma. — *The space of connections on a vector bundle E over the manifold M is an affine space with direction* $\Gamma(M, T^*M \otimes \operatorname{End} E)$. □

Proof. — First, if ∇ is a connection and $a \in \Gamma(M, T^*M \otimes \operatorname{End} E)$, it is easy to check that $\nabla + a$ is again a connection. Then, using Leibniz rule for two connections ∇ and ∇', we obtain that for any vector field X,

$$(\nabla_X - \nabla'_X)(fs) = f(\nabla_X - \nabla'_X)s,$$

which means $\nabla_X - \nabla'_X$ is a $C^\infty(M)$-linear endomorphism of $\Gamma(E)$, hence a *section* of the vector bundle $E^* \otimes E = \operatorname{End} E$, the vector bundle whose fiber over x consists of endomorphisms of E_x (Lemma 1.7.6). Since it is also $C^\infty(M)$-linear with respect to the vector field X (by the definition of connections), it follows that it defines a section of $T^*M \otimes \operatorname{End} E$. □

2.1.3. The local point of view. — Let E be a vector bundle of rank p over M^n. We choose a chart of M and a local trivialization of E around some point of M : this yields local coordinates x^i, $i = 1, \dots, n$, and p linearly independent local sections $e_1, \dots, e_p$ of E. Define the *Christoffel symbols* Γ^b_{ia} of a connection ∇ by

$$\nabla e_a = \Gamma^b_{ia} dx^i \otimes e_b.$$

A general section of E reads $s = s^a e_a$ and we can apply Leibniz rule to get

$$\nabla s = ds^a \otimes e_a + s^a \nabla e_a = (\frac{\partial s^a}{\partial x^i} + \Gamma^a_{ib} s^b) dx^i \otimes e_a$$

or, equivalently,

$$\nabla_{\frac{\partial}{\partial x^i}} s = (\frac{\partial s^a}{\partial x^i} + \Gamma^a_{ib} s^b) e_a.$$

Therefore we shall write

$$\nabla s = ds + dx^i \otimes \Gamma_i s,$$

where $\Gamma_i = (\Gamma^b_{ia})$ is a matrix, i.e. an endomorphism of E : the connection ∇ is locally given by a 1-form with values in $\operatorname{End} E$, $\Gamma = dx^i \otimes \Gamma_i$, which is called the *connection 1-form*. Beware this is only a *local* section of $T^*M \otimes \operatorname{End} E$, depending on the chart and the local trivialization.

Let us see what happens when we change of trivialization. Given a new local basis (f_b) of E, such that $e_a = u^b_a f_b$, then a section $s = s^b f_b$ has components $u^{-1}s$ in the basis (e_a) and therefore in this basis ∇s writes $d(u^{-1}s) + dx^i \Gamma_i u^{-1} s$. Coming back to the basis (f_b), we obtain

$$\nabla s = u\big(d(u^{-1}s) + dx^i \Gamma_i u^{-1} s\big) = ds + \big(- du u^{-1} + dx^i u \Gamma_i u^{-1}\big) s.$$

In particular, we see that the matrices (Γ'_i) in the basis (f_b) can be expressed as

$$\Gamma'_i = -\frac{\partial u}{\partial x^i} u^{-1} + u \Gamma_i u^{-1}. \tag{2.1}$$

2.1.4. Example : the tautological bundle $\mathscr{O}(-1)$ over $\mathbb{C}P^1$. — This is the complex line bundle whose fiber over a point $x \in \mathbb{C}P^1$ is the complex line $x \subset \mathbb{C}^2$. Recall that homogeneous coordinates $[z^1 : z^2]$ on $\mathbb{C}P^1$ yield two charts. On $U_1 = \{z^1 \neq 0\}$, we have one complex coordinate $\zeta = \frac{z^2}{z^1}$ (namely, ζ corresponds to the point $[1 : \zeta]$), while on $U_2 = \{z^2 \neq 0\}$, $\zeta' = \frac{z^1}{z^2}$ is a complex coordinate. On U_1, $\mathscr{O}(-1)$ is trivialized by the non-vanishing section $s_1(\zeta) = (1, \zeta)$; on U_2, $s_2(\zeta') = (\zeta', 1)$ is a non-vanishing section. On $U_1 \cap U_2$, we have $\zeta' = \zeta^{-1}$ and $s_2(\zeta') = \zeta^{-1} s_1(\zeta)$ (this ζ^{-1} is somehow the reason for the odd notation $\mathscr{O}(-1)$). The transition function $u(\zeta)$, in the sense above (cf. (2.1)), is the multiplication by ζ.

Now we define a connection on $\mathscr{O}(-1)$ in the following way: locally we can consider a section as a map $s : \mathbb{C}P^1 \to \mathbb{C}^2$ such that $s(x) \in x$, and we define

$$\nabla_X s = \pi_x(d_x s(X)), \tag{2.2}$$

where π_x is the orthogonal projection on x (in $\mathbb{C}^2$). It follows that on U_1,

$$\nabla_X s_1 = \pi_{(1,\zeta)}(0, X) = \frac{X\bar{\zeta}}{1 + |\zeta|^2} s_1.$$

Similarly, on $U_1 \cap U_2$, we have

$$\nabla_X s_2 = -\frac{X}{\zeta(1+|\zeta|^2)} s_2.$$

So in the two charts we have the Christoffel symbols $\Gamma = \frac{\bar\zeta d\zeta}{1+|\zeta|^2}$ and $\Gamma' = -\frac{d\zeta}{\zeta(1+|\zeta|^2)}$. In particular, we get $\Gamma' = \Gamma - \frac{d\zeta}{\zeta}$, which coincides with (2.1) since $u(\zeta) = \zeta$.

2.1.5. Submanifolds of a Euclidean space. — Note that for any submanifold M^n of $\mathbb{R}^N$, one can define a connection on TM much as in the example above : consider at each point the tangent space T_xM as a subspace of $\mathbb{R}^N$ and denote by π_{T_xM} the orthogonal projection $\mathbb{R}^N \to T_xM$, so that we can set

$$\nabla^M_X s = \pi_{T_xM}(\nabla^{\mathbb{R}^n}_X s), \quad X \in T_xM. \tag{2.3}$$

It is easy to check that it is indeed a connection on TM.

2.1.6. Induced connections. — Given a connection ∇^E on a vector bundle E over M, the dual vector bundle E^* is automatically endowed with a connection ∇^{E^*}: for $s \in \Gamma(E)$, $t \in \Gamma(E^*)$ and $X \in \Gamma(TM)$, we require

$$\mathscr{L}_X \langle t, s\rangle = \langle \nabla^{E^*}_X t, s\rangle + \langle t, \nabla^E_X s\rangle. \tag{2.4}$$

If (e_a) is a local basis of sections of E, then the dual basis (e^a) is a local basis for E^* and the duality bracket reads, for $s = s^a e_a$ and $t = t_b e^b$, $\langle t, s\rangle = t_a s^a$. The equation (2.4) then gives immediately

$$\nabla_{\frac{\partial}{\partial x^i}} t = (\frac{\partial t_a}{\partial x^i} - \Gamma^b_{ia} t_b) e^a = \frac{\partial t}{dx^i} - {}^t\Gamma_i t.$$

Therefore the (local) connection 1-form for E^* is $-{}^t\Gamma$.

In the same spirit, suppose we have connections ∇^E and ∇^F on the vector bundles E and F over M. Then there is a naturally induced connection on $G = \mathrm{Hom}(E,F) = E^* \otimes F$, defined similarly: we require that if $s \in \Gamma(E)$ and $u \in \mathrm{Hom}(E,F)$, then

$$\nabla^F_X(u(s)) = (\nabla^G_X u)(s) + u(\nabla^E_X s). \tag{2.5}$$

From this it follows quickly that

$$\nabla^G_{\frac{\partial}{\partial x^i}} u = \frac{\partial u}{\partial x^i} + \Gamma^F_i \circ u - u \circ \Gamma^E_i.$$

(Remark that for $F = \mathbb{R}$ we recover the previous case $G = E^*$).

More generally, by asking that the Leibniz rule (like in 2.5) is true for algebraic operations, one easily extends a connection on E to all associated bundles (tensor products, exterior products).

2.2. Parallel transport

If we have a trivial vector bundle $E = M \times \mathbb{R}^k$, then all fibers of the bundle can be identified with a fixed vector space $\mathbb{R}^k$. But for a general vector bundle E over M, there is no canonical way to identify the fibers of E. We will see that a connection provides a way to identify the fibers *along curves* on the base.

2.2.1 Lemma. — *Let E be a vector bundle over M with a connection ∇. We consider a curve $c : I \longrightarrow M$ and a section s of E. Then, for t in I, the quantity $\left(\nabla_{\dot c(t)} s\right)_{c(t)}$ depends only on the values of s* along *c and in a neighborhood of $c(t)$.*

Proof. — In a local trivialization over a coordinate chart, we have $c(t) = (x^1(t), \dots, x^n(t))$ and s has values in $\mathbb{R}^p$, hence the formula

$$\nabla_{\dot c} s = \dot x^i \left(\frac{\partial s}{\partial x^i} + \Gamma_i s\right) = \dot s + \Gamma_{\dot c} s \tag{2.6}$$

which justifies the statement. □

It follows that, given a curve $c : I \longrightarrow M$, we may look for smooth maps $s : I \longrightarrow E$ such that $s(t) \in E_{c(t)}$ for every t in I and s satisfies the equation

$$\nabla_{\dot c} s = 0, \tag{2.7}$$

meaning that (2.6) holds locally. Since it is locally a first order linear ordinary differential equation on s, given some initial condition $s(0)$, one can construct a unique solution of (2.7) along c. This leads to the following definition:

2.2.2 Definition. — Let (E, ∇) be a bundle with connection over M. If $(c(t))_{t\in[a,b]}$ is a path in M, then the *parallel transport* along c is the application $E_{c(a)} \to E_{c(b)}$, $s(a) \mapsto s(b)$ obtained by solving the equation (2.7) along c.

The parallel transport $E_{c(a)} \to E_{c(b)}$ is always a linear isomorphism, since the inverse is obtained by parallel transport along c in the reverse direction.

The connection ∇ can be computed from its parallel transport. Let X be a tangent vector at $x \in M$ and s a section. We consider a curve c defined on some neighborhood of 0 with $c(0) = x$ and $\dot c(0) = X$. Then for any small

t we can define the parallel transport of $s_{c(t)}$ along c from $c(t)$ to $c(0) = x$, resulting in an element $\tilde{s}(t)$ of E_x. Then :

$$(\nabla_X s)_x = \frac{d}{dt}\Big|_{t=0} \tilde{s}(t).$$

2.2.3. Example. — Let E be the quotient of $\mathbb{R} \times \mathbb{R}^p$ by $(t, v) \sim (t-1, Av)$, where $A \in GL(\mathbb{R}^p)$. This is the total space of a vector bundle over the circle S^1, with a projection π induced by the projection onto the first factor (t). A section is then a simply a map $\sigma : \mathbb{R} \to \mathbb{R}^p$ such that $\sigma(t+1) = A^{-1}\sigma(t)$. Let us define a connection ∇ by setting $\nabla_{\frac{\partial}{\partial t}} \sigma = \frac{d\sigma}{dt}$ (check it makes sense !). Then for any path running k times around S^1, the parallel transport is given by A^k. The reader may convince himself that any vector bundle over the circle arises as such a quotient and is therefore either trivial or the sum of a Moebius plane bundle and a trivial vector bundle.

2.2.4. Orientation. — Let us make a bunch of rather trivial remarks about the natural notion of orientation for vector bundles. Each fiber is a vector space so it carries two different orientations. A vector bundle of rank p is said to be *orientable* when it can be endowed with local trivializations such that any of its transition function $(x, \xi) \mapsto (x, u(x)\xi)$ takes its values in $GL_+(\mathbb{R}^p)$ (the component of the identity in $GL(\mathbb{R}^p)$) in that $u(x) \in GL_+(\mathbb{R}^p)$ for every point x. It is then possible to choose an orientation for all fibers in a continuous manner: locally, the fibers can all be identified with $\mathbb{R}^p$ thanks to the local trivializations and these identifications yields a consistent notion of positive basis owing to the orientability. Such a choice makes the vector bundle into an *oriented* vector bundle. For instance: the tangent bundle TM of a manifold M is an orientable vector bundle if and only if M is an orientable manifold; a line bundle is orientable if and only if is trivial; the Moebius plane bundle is not orientable. What is the interplay between this notion of orientation and connections ? It is easy to see that the parallel transport with respect to any connection preserves the orientation of an oriented vector bundle. And conversely, any connection on a vector bundle (E, π, B) can detect its eventual non-orientability: if (E, π, B) is not orientable, then there is a closed path $\gamma : [0, 1] \to B$, with $\gamma(0) = \gamma(1) = x \in B$, such that the corresponding parallel transport is an element of $GL(E_x) \backslash GL_+(E_x)$. In fact, it even shows that the restriction of E over the range of γ contains a Moebius plane bundle, as a subbundle... The Moebius band is the universal source of non-orientability.

2.3. Metric connections

Given a vector bundle E over M, we may introduce the vector bundle $\mathrm{Sym}^2(E^*)$ over M whose fiber $\mathrm{Sym}^2(E^*)_x$ consists of the symmetric bilinear forms over E_x.

2.3.1 Definition. — A *(Euclidean) metric* g on the vector bundle E is a smooth section g of $\mathrm{Sym}^2(E^*)$ that is positive definite in each fiber. In other words, it is the smooth data of a positive definite bilinear symmetric form g_x on each fiber E_x.

If E is complex vector bundle, the relevant notion is that of a *Hermitian metric* g, which is the smooth data of a positive definite Hermitian form g_x on each fiber E_x .

For instance, if M is a submanifold of $\mathbb{R}^N$, the Euclidean inner product on $\mathbb{R}^N$ restricts as a metric on the tangent space TM : each fiber $T_xM \subset \mathbb{R}^N$ is endowed with the restriction of the scalar product of $\mathbb{R}^N$. Another example is the bundle $\mathscr{O}(-1)$ of example 2.1.4: each fiber is naturally a complex line of $\mathbb{C}^2$ and so inherits a Hermitian metric from that of $\mathbb{C}^2$. If M is a manifold, a metric on TM is called a Riemannian metric on M. It is the subject of the third chapter.

Again, algebraic operations (sum, tensor product,...) can be used to produce new metrics. For instance, given a metric g on E, there is an induced metric on $\Lambda^k E^*$: if $(e_1, \dots, e_p)$ is a g-orthonormal basis of E_x, with dual basis $(e^1, \dots, e^p)$, then we decide that the set of $e^{i_1} \wedge \dots \wedge e^{i_k}$, for $i_1 < \dots i_k$, is an orthonormal basis of $(\Lambda^k E^*)_x$.

2.3.2 Definition. — If the bundle E has a metric g, we say that a connection ∇ on E is a *metric connection* or preserves g if for any sections s, t of E and any vector field X:

$$\mathscr{L}_X\big(g(s,t)\big) = g(\nabla_X s, t) + g(s, \nabla_X t).$$

Note that a connection is metric if and only if its parallel transport is an isometry between fibers. Another equivalent definition is that the induced connection ∇^{Sym} on $\mathrm{Sym}^2(E^*)$ kills g: $\nabla^{\mathrm{Sym}} g = 0$ (exercises).

If E is a complex vector bundle, a connection preserving a Hermitian metric in the sense above is called a *unitary connection.*

Using a partition of unity, it is easy to prove that any vector bundle admits a metric and that any vector bundle with a metric carries a metric connection.

What does "metric" mean on the Christoffel symbols ? Suppose that (e_a) is a local *orthonormal* basis of E (it always exists, by Gram-Schmidt), then for all a, b we must have $g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) = 0$, whence $\Gamma_{ia}^b = -\Gamma_{ib}^a$ (resp. $\Gamma_{ia}^b = -\overline{\Gamma_{ib}^a}$ in the complex case). This condition characterizes the metric connections. It means that the matrices Γ_i take values in antisymmetric or anti-Hermitian endomorphisms of E. We shall denote the bundle of antisymmetric endomorphisms by $\mathfrak{so}(E)$ and, in the complex case, the bundle of anti-Hermitian endomorphisms $\mathfrak{u}(E)$. Then we have proved the following version of lemma 2.1.2: the space of metric connections of (E, g) is an affine space with direction $\Gamma(T^*M \otimes \mathfrak{so}(E))$ in the real case, $\Gamma(T^*M \otimes \mathfrak{u}(E))$ in the complex case.

2.3.3. Examples. — 1) The standard connection d on $M \times \mathbb{R}^p$ is metric with respect to the Euclidean metric induced by the scalar product of $\mathbb{R}^p$. 2) The connection we defined on $\mathscr{O}(-1)$ is a metric connection. 3) The connection induced on the tangent bundle of a submanifold of $\mathbb{R}^N$ is a metric connection for the metric induced by the scalar product of $\mathbb{R}^N$.

2.3.4. Exercise. — Consider the two-sphere, endowed with its standard connection, coming from the embedding in $\mathbb{R}^3$. Observe that parallel transport along a great circle preserves the velocity vector of any parametrization at constant speed and deduce the parallel transport of any vector along a great circle. Describe the parallel transport of any path obtained by starting from the North Pole, going southward along a meridian, following the Equator for some time, and finally coming back to the North Pole along a meridian.

2.4. Curvature

Schwarz theorem ensures that partial derivatives of functions on $\mathbb{R}^n$ commute : $\frac{\partial^2 f}{\partial x^i x^j} = \frac{\partial^2 f}{\partial x^j x^j}$. It turns out that this symmetry property is not true for a general connection ∇ on a vector bundle : in general, $\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} s$ is different from $\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} s$. The curvature F^∇ of ∇ will measure the defect of commutation : $F^\nabla \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j} \right) = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} s - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} s$. More precisely, the definition is the following.

2.4.1 Definition. — Let E be a vector bundle over the manifold M, with a connection ∇. The *curvature* of ∇ is the section F^∇ of $\Lambda^2 M \otimes \operatorname{End} E$ defined by

$$F^\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s,$$

where X and Y are vector fields on M and s is a section of E.

There is something to check : it is not at all obvious that the formula above defines a section of $\Lambda^2 M \otimes \operatorname{End} E$ for the right hand side seems to depend on the derivatives of X, Y and s. Using the very definition of a connection, the reader may check that this expression is indeed tensorial (i.e. $C^\infty(M)$-linear) with respect to X, Y and s, which justifies our claim (Lemma 1.7.6).

In local trivialization and chart, we may write $\nabla = d + dx^i \otimes \Gamma_i$ and also $F^\nabla = dx^i \otimes dx^j \otimes F_{ij} = \frac{1}{2} dx^i \wedge dx^j \otimes F_{ij}$. A direct computation yields

$$F_{ij} = \frac{\partial \Gamma_j}{\partial x^i} - \frac{\partial \Gamma_i}{\partial x^j} + [\Gamma_i, \Gamma_j]. \tag{2.8}$$

2.4.2 Definition. — A connection is said to be *flat* if its curvature vanishes.

For instance, the connection induced by d on any trivial vector bundle is flat, because of Schwarz theorem.

2.4.3. Example. — A non-flat example is given by the line bundle $\mathscr{O}(-1)$ over $\mathbb{C}P^1$, with the connection $\Gamma = \frac{\bar z dz}{1+|z|^2}$. The brackets vanish since we are on a line bundle, and we get

$$F^{\mathscr{O}(-1)} = -\frac{dz\wedge d\bar z}{(1+|z|^2)^2} = \frac{2i dx\wedge dy}{(1+x^2+y^2)^2},$$

which is a 2-form with values in $i\mathbb{R} = \mathfrak{u}_1$ (recall the endomorphism bundle of a line bundle is trivial).

2.4.4. Exercise. — Let L be a complex line bundle over some (real) manifold M. Since $\operatorname{End} L$ is trivial, any two connections ∇ and ∇' on L differ by some complex-valued one-form α (an element of $\Omega(M) \otimes_{\mathbb{R}} \mathbb{C}$). Prove that $F^\nabla - F^{\nabla'} = d\alpha$. Prove that for any unitary connection ∇ on L, $\frac{i}{2\pi} F^\nabla$ is a closed real-valued two-form, whose de Rham cohomology class does not depend on ∇, but only on L. This class is known as the first Chern class of L, $c_1(L)$. Check that $c_1(L^*) = -c_1(L)$, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ (where L_1 and L_2 are two line bundles over M). When M is a compact oriented surface, the integral of the first Chern class is a well-defined number (owing to Stokes theorem). Prove that this 'Chern' number is -1 for $\mathscr{O}(-1)$) and k for the line bundle $\mathscr{O}(k) := (\mathscr{O}(-1)^*)^{\otimes k}$, whose fiber at some point x of $\mathbb{C}P^1$ consists of the k-linear forms on x. Prove that the Chern number is always an integer, equal to the index of any section σ of L with isolated zeros, defined as follows:

for every x such that $\sigma(x) = 0$, pick a chart and a local trivialization, so that σ can be seen as a map from a neighborhood of 0 in $\mathbb{R}^2$ to a neighborhood of 0 in $\mathbb{R}^2$; on a small circle around 0, $\frac{\sigma}{|\sigma|}$ then defines a map from S^1 to S^1 and we let $k_x \in \mathbb{Z}$ denote its winding number (or degree), which depends only on σ and x; the index of σ is defined as the (finite) sum of all k_x. Hint : make the connection trivial in the trivialization of L induced by σ away from its zeros. This is a version of Gauss-Bonnet formula, due to S.S. Chern. This construction can be widely generalized: this is the Chern-Weil theory of characteristic classes.

2.5. The horizontal distribution

Let $\pi : E \to M$ be a vector bundle over M with a connection ∇, and $x \in M$. We have seen in section 2.2 that if we have a path c in M with $c(0) = x$ and an initial value $s_0 \in E_x$, then c can be *lifted* to a path s in E, the parallel transport of s_0 along c, such that $s(0) = s_0$, $\pi \circ s = c$ and $\nabla_{\dot{c}} s = 0$.

There is an infinitesimal version of this process: we define the *horizontal lift* of $X = \dot{c}(0) \in T_x M$ at $s_0 \in E_x$ to be

$$\tilde{X} = \frac{d}{dt}\Big|_{t=0} s(t) \ \in T_{s_0} E. \tag{2.9}$$

We claim that $\tilde{X}$ does not depend on the choice of c, but only on its initial speed vector X. To see this, we calculate $\tilde{X}$ in a local trivialization $(e_1, \ldots, e_r)$ of E, over the domain U of a chart of M, with coordinates (x^i). Therefore, locally,

$$E|_U \simeq U \times \mathbb{R}^r$$

with coordinates $(x^i, s^a)_{i=1,\ldots,n,\, a=1,\ldots,r}$, and the corresponding vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial s^a}$. Observe that the latter vectors are tangent to the fibers of E. The connection then locally reads $\nabla = d + \Gamma$, where Γ is a local 1-form with values in $\operatorname{End} E$. From equation (2.6) we obtain $\dot{s}(0) = -\Gamma_X s_0$ and therefore

$$\tilde{X} = (X, -\Gamma_X s_0). \tag{2.10}$$

For example for $X = \frac{\partial}{\partial x^i}$, noting $s_0 = s^a e_a$, we obtain

$$\tilde{X} = (\frac{\partial}{\partial x^i}, -\Gamma_i s_0) = \frac{\partial}{\partial x^i} - \Gamma^a_{ib} s^b \frac{\partial}{\partial s^a}.$$

.

2.5.1 Definition. — The *horizontal distribution* H^∇ of (E, ∇) is the distribution on E defined at each point $s_0 \in E$ as the vector space of horizontal lifts : $H^\nabla_{s_0} := \{\tilde{X}, X \in T_{\pi(s_0)}M\} \subset T_{s_0}E$.

By definition, the parallel transport of s_0 along c can be interpreted as the unique curve in E that goes through s_0, projects onto c and is everywhere tangent to the horizontal distribution. For instance, on the trivial vector bundle $M \times \mathbb{R}^p$, the horizontal distribution is the one that admits the submanifolds $M \times \{s\}$, $s \in \mathbb{R}^p$, as integral submanifolds and the parallel transport along any path between $x \in M$ and $y \in M$ is just $(x, s) \mapsto (y, s)$.

Note that there is a natural notion of *vertical distribution* V on E, independent of any connection : it is simply given by the kernel of $d\pi$, namely at $s_0 \in E$, we set $V_{s_0} := \ker d\pi_{s_0}$. In other words, it is the tangent bundle to the fibers : $V_{s_0} = E_{\pi(s_0)} = T_{s_0}(E_{\pi(x_0)}) \subset T_{s_0}E$. If E has rank p and M has dimension n, the vertical distribution has dimension p, while the horizontal one has dimension n. It is easy to see that the horizontal and vertical distributions are supplementary :

$$T_{s_0}E = V_{s_0} \oplus H^\nabla_{s_0}. \tag{2.11}$$

It means $d\pi_{s_0}$ is an isomorphism between $H^\nabla_{s_0}$ and $T_{\pi(s_0)}M$.

2.5.2 Proposition. — *The horizontal distribution of a connection ∇ is integrable if and only if the vector bundle admits local trivializations in which ∇ is trivial.*

An immediate consequence of the proof below is that, if ∇ is a metric connection, then the local trivializations can be supposed to be orthonormal (i.e. isometric on each fiber).

Proof. — The implication $\Leftarrow$ has been settled above so we tackle the other one. Suppose H^∇ is integrable. Then for any given point $(x, 0)$ in the zero section of E, there is an open neighborhood $V \subset E$ of $(x, 0)$ and a diffeomorphism $\psi : V \longrightarrow L \times W$ such that $\psi(x, 0) = (0, 0)$, $L \subset \mathbb{R}^n$, $W \subset \mathbb{R}^p$ and the integral submanifolds of H^∇ are given by the $\psi^{-1}(L \times \{w\})$, $w \in W$. Now consider the application $f : L \times W \to M \times W$ defined by

$$f(l, w) = (\pi(\psi^{-1}(l, w)), w).$$

The differential of f at $(0, 0)$ is an isomorphism, so the inverse function theorem ensures f is a diffeomorphism near $(0, 0)$. It follows that we may shrink $V \in E$ and $W \subset \mathbb{R}^p$ so as to obtain a diffeomorphism $\phi : V \longrightarrow U \times W$, for some open (connected) neighborhood U of x in M, such that

– the integral submanifolds of H^∇ are still given by $\phi^{-1}(U \times \{w\})$, $w \in W$,
– and we furthermore have $\pi \circ \phi^{-1}(y,w) = y$.

In this picture, we see from the structure of integral submanifolds that parallel transport is given by the identity of W : the parallel transport P_{xy} from x to y, along any path between x and y in some neighborhood of x, maps $\phi^{-1}(x,w)$ to $\phi^{-1}(y,w)$.

$$\begin{array}{ccc} E_x \cap V & \xrightarrow{\phi} & \{x\} \times W \cong W \\ \big\downarrow P_{xy} & & \big\downarrow id \\ E_y \cap V & \xrightarrow{\phi} & \{y\} \times W \cong W \end{array}$$

Now recall that parallel transport acts by linear isomorphisms between fibers. It stems from the commutative diagram that if we identify W with the open subset $E_x \cap V$ of the vector space E_x through ϕ, then every diffeomorphism $\phi : E_y \cap V \longrightarrow W$ is indeed linear, hence extends uniquely as a linear isomorphism $\phi : E_y \longrightarrow E_x$. The map ϕ therefore extends as a diffeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times E_x \cong U \times \mathbb{R}^p$ such that $\pi\phi^{-1}(y,w) = y$ and linear on each fiber of π : this is local trivialization of E as a vector bundle. Then equation (2.10) gives immediately $\Gamma = 0$ everywhere in this trivialization, so ∇ is the trivial connection $\nabla = d$. □

2.5.3 Proposition. — *The horizontal distribution of a connection is integrable if and only if the connection is flat.*

Proof. — The implication ⇒ is obvious in view of Proposition 2.5.2. For the other one, Frobenius theorem 1.6.5 says that integrable means involutive so we need to understand what it means for H^∇ to be involutive. We pick a local trivialization and local coordinates on the base to compute like in paragraph 2.1.3 at some point with coordinates (x,s) in E :

$$\begin{aligned} \left[\widetilde{\frac{\partial}{\partial x^i}}, \widetilde{\frac{\partial}{\partial x^j}}\right] &= \left[\frac{\partial}{\partial x^i} - \Gamma^a_{ib} s^b \frac{\partial}{\partial s^a}, \frac{\partial}{\partial x^j} - \Gamma^a_{jb} s^b \frac{\partial}{\partial s^a}\right] \\ &= \left(-\frac{\partial \Gamma^a_{jb}}{\partial x^i} + \frac{\partial \Gamma^a_{ib}}{\partial x^j} + \Gamma^a_{jc}\Gamma^c_{ib} - \Gamma^a_{ic}\Gamma^c_{jb}\right) s^b \frac{\partial}{\partial s^a} \\ &= -\left(\frac{\partial \Gamma_j}{\partial x^i} - \frac{\partial \Gamma_i}{\partial x^j} + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right)(s), \end{aligned}$$

which is exactly $-F^\nabla_{ij}(s)$, see formula (2.8). It follows that if $F^\nabla = 0$, then H^∇ is involutive hence integrable. □

Propositions 2.5.2 and 2.5.3 say that a flat connection is locally trivial : it is locally completely understood. Given a connection, the basic question is : is it trivial ? Locally, it amounts to find good coordinates in which one sees the connection as a mere differential. This problem is a priori very hard, but Frobenius theorem reduces it to the computation of a single quantity: the curvature. It is the only obstruction, so we just need to compute it and see whether it is zero or not ! This kind of idea is central in differential geometry, we will see one of its striking occurrences in the chapter on Riemannian geometry.

In the course of the proof of Proposition 2.5.3, we basically proved that if $[X,Y]=0$, then

$$F^{\nabla}(X,Y) = -[\widetilde{X},\widetilde{Y}],$$

which yields a nice interpretation of curvature : it measures the commutation of the lifts of commuting vector fields. To rephrase this, pick a point x and look at the path σ obtained by starting from x, following the flow of X for some time t, then Y for some time u, then $-X$ for some time t and finally $-Y$ for some time u. Since X and Y have vanishing bracket, σ ends up in x, so the parallel transport along σ gives an endomorphism $\tau(t,u)$ of E_x. Then the curvature is given by

$$F^{\nabla}(X,Y) = -\frac{d}{dt}\Big|_{t=0}\frac{d}{du}\Big|_{u=0}\tau(t,u).$$

Finally, let us describe all flat vector bundles. Let ∇ be a connection on a vector bundle E over a connected base M. Given any point x in M, we consider the *holonomy group* $Hol(\nabla,x)$, defined as the subgroup of $GL(E_x)$ consisting of the parallel transports of all loops in M based at x. If ∇ is flat, the parallel transport along a loop only depends on the homotopy class of the loop, so ∇ gives rise to representation ρ of the fundamental group $\pi_1(M,x)$ in $GL(E_x)$. Up to conjugacy, this does not depend on the choice of x. Conversely, given a representation ρ of $\pi_1(M,x)$ on $\mathbb{R}^p$, we may build a flat vector bundle: letting $\tilde{M}$ denote the universal cover of M, we define E as the quotient $M\times\mathbb{R}^p/\sim$, where $(z,v)\sim(\gamma.z,\rho(\gamma)v)$ for any γ in $\pi_1(M)$, z in $\tilde{M}$ and v in $\mathbb{R}^p$. In this way, we see that the data of a flat vector bundle of rank p over M is the same as the data of a representation of the fundamental group of M on $\mathbb{R}^p$.

2.6. Bibliography

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– *Einstein Manifolds*, Besse.

CHAPTER 3

RIEMANNIAN GEOMETRY

3.1. Riemannian metrics

3.1.1 Definition. — Let M be a manifold. A *Riemannian metric* on M is a metric g on the vector bundle TM : it is the data for each point $x \in M$ of a positive definite quadratic form g_x on T_xM, depending smoothly on the point x. A *Riemannian manifold* (M, g) is manifold M endowed with a Riemannian metric g.

A Riemannian metric measures the length of tangent vectors, as well as angles between them. In local coordinates (x^i), it is given by a positive definite matrix $(g_{ij}(x)) = \left(g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\right)$, where $g_{ij}(.)$ are smooth local functions. This is often written in the following way:

$$g = g_{ij} dx^i dx^j.$$

If we have other coordinates (y^j), then it is easy to see that

$$g = g_{ij} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} dy^k dy^l.$$

Sometimes, we will write $\langle X, Y \rangle$ for $g(X, Y)$ and $|X|$ for $\sqrt{g(X, X)}$.

Recall that any manifold can be endowed with a Riemannian metric, because one can patch local Riemannian metrics together thanks to a partition of unity. This fact relies on the convexity of the set of positive definite symmetric matrices. A natural generalization of Riemannian metrics consists in considering non-degenerate quadratic forms with arbitrary signature. For instance, the smooth data of a quadratic form of signature $(1, n)$ on a manifold M^{n+1} is known as a Lorentz metric; this is the basic object in relativistic physics, cf. B. O'Neill's book in the bibliography. A major difference with the

Riemannian case is that some manifolds cannot carry Lorentz metrics. The partition of unity argument fails owing to a lack of convexity. Exercise: prove that the sphere S^2 does not carry any Lorentz metric.

3.1.2. Examples. — 1) The Euclidean metric $g_{\mathbb{R}^n} = (dx^1)^2 + \cdots + (dx^n)^2$ on $\mathbb{R}^n$. At each point, the tangent space identifies to $\mathbb{R}^n$ and the metric is the standard metric of $\mathbb{R}^n$.

2) The circle S^1 is parametrized by an angle $\theta \in [0, 2\pi[$ and a Riemannian metric is given by $g_{S^1} = d\theta^2$. The Euclidean metric of $\mathbb{R}^2$ in polar coordinates (r, θ) reads $g_{\mathbb{R}^2} = dr^2 + r^2 d\theta^2$, namely the basis $(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{d\theta})$ is orthonormal.

3) More generally, the Euclidean metric of $\mathbb{R}^n - \{0\} =]0, +\infty[\times S^{n-1}$ can be written as $g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$ where $g_{S^{n-1}}$ is a Riemannian metric on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. In the coordinates y^i given by the stereographic projection (cf. 1.1.8), one calculates

$$g_{S^{n-1}} = 4\frac{\sum (dy^i)^2}{(1+|y|^2)^2}. \tag{3.1}$$

The proof consists in using the coordinates y^i to express the coordinates of the corresponding point in $\mathbb{R}^n$, namely $\frac{1}{1+|y|^2}(|y|^2 - 1, 2y^1, \ldots, 2y^{n-1})$, and therefore

$$g = d\left(\frac{|y|^2-1}{1+|y|^2}\right)^2 + d\left(\frac{2y^1}{1+|y|^2}\right)^2 + \cdots + d\left(\frac{2y^{n-1}}{1+|y|^2}\right)^2.$$

Expand and simplify to get the formula above.

4) Any submanifold of a Riemannian manifold inherits a Riemannian metric by restricting the metric of the manifold to the tangent bundle of the submanifold.

5) Any product of Riemannian manifolds inherits a Riemannian metric obtained by summing the metrics on each factor. For instance, the torus $T^n = S^1 \times \cdots \times S^1$ is therefore endowed with a metric g_{T^n}, induced by the metric on each factor S^1 given by 2).

6) A surface of revolution in $\mathbb{R}^3$, say around the z axis. We take polar coordinates (r, θ) in the xy plane. The surface is given by an equation of the type $r = f(z)$, but it is more convenient to parameterize it in a different way: the intersection with the xz plane is a curve, which we parameterize by the length u. Then the metric of the surface is $g = du^2 + r(u)^2 d\theta^2$.

7) The hyperbolic space H^n. We consider $\mathbb{R}^{n+1}$ endowed with the quadratic form $h = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$, of signature $(1, n)$: it is a Lorentz metric and the space we describe is known as Minkowski space $\mathbb{R}^{1,n}$. The

Riemannian manifold (H^n, g_{H^n}) is defined as the component of $\{-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1\}$ with $x^0 > 0$, endowed with the restriction of h to the tangent bundle of this submanifold. The reader may check that the restriction of h to each tangent space of this hypersurface is indeed positive definite.

3.1.3 Remark. — A deep theorem of Nash (Nobel prize in Economics, main character of the movie *A beautiful mind*) says that any Riemannian manifold can be seen as a submanifold of some $\mathbb{R}^N$, with the induced metric.

A Riemannian metric gives many structures, many useful objects. For instance, it induces a metric on the vector bundle $\Lambda^\bullet M$, which can be used to distinguish a volume form on oriented manifolds.

3.1.4 Definition. — Assume (M^n, g) is an oriented Riemannian manifold. Its *volume form* $d\,\mathrm{vol}^g$ is the unique positive n-form of norm 1.

Beware the (standard) notation $d\,\mathrm{vol}^g$ is misleading : this is not an exact form in general. If (e_i) is an orthonormal basis of T_xM, then $d\,\mathrm{vol}^g = e^1 \wedge \cdots \wedge e^n$ In local coordinates, it is given by the formula

$$d\,\mathrm{vol}^g = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

Let N be a hypersurface of an oriented Riemannian manifold M. We assume there is a unit normal vector field $\nu \in \Gamma(TM|_N)$ (it exists iff N is orientable). Then N is oriented. Moreover, it can be endowed with the induced Riemannian metric and the corresponding volume form is given by $\iota_\nu dvol_M$. .

The (possibly infinite) volume of an oriented Riemannian M is defined by $\mathrm{vol}^g(M) = \int_M d\,\mathrm{vol}^g$. For instance, the volumes of the spheres are given by

$$\mathrm{vol}(S^{2n}) = (4\pi)^n \frac{(n-1)!}{(2n-1)!}, \quad \mathrm{vol}(S^{2n+1}) = 2\frac{\pi^{n+1}}{n!}.$$

3.1.5 Remark. — Orientation is necessary to integrate n-forms on a manifold of dimension n and to get Stokes theorem. It also makes it possible to integrate a function f, by integrating the n-form $f\,dvol$, but this is not the best way to do it. Let (M^n, g) be a Riemannian manifold, possibly non-orientable. Then $\Lambda^n M$ is a line bundle (trivial iff M is orientable). If we identify ω with $-\omega$ in each fiber of $\Lambda^n M$, we obtain a “half-line” bundle DM (the bundle of “densities”), which is always trivial ! Indeed, locally, one can always find sections of $\Lambda^n M$ with constant norm 1 : there are two possible choices, ω and $-\omega$. Now these two choices yield the same local section in DM ($|\omega|$). This uniqueness makes it possible to obtain a well-defined global section, that we may call $|dvol|$ (usually, simply $dvol$). Then one can define the integral of a function f, $\int_M f|dvol|$, exactly like how we defined the integral of n-forms. The key is the formula for the change of variables in $\mathbb{R}^n$, which involves the absolute value of a Jacobian determinant ; the reader will easily guess how this absolute value relates to the bundle of densities... In particular, the volume of any Riemannian manifold is well-defined ; for instance, the volume of $\mathbb{R}P^2$ (for the metric induced by that of S^2, see below) is 2π, i.e. half the volume of S^2.

3.2. Isometries

Given a smooth map ϕ between manifolds, one can define the pullback of a metric as in the case of differential forms :

$$(\phi^* g)_x(X,Y)) = g_{\phi(x)}(d_x\phi(X), d_x\phi(Y)).$$

3.2.1 Definition. — A diffeomorphism $\phi : (M,g) \to (N,h)$ is an *isometry* if $\phi^* h = g$.

The definition means that ϕ is a diffeomorphism such that for every point x, $d_x\phi$ is a linear isometry between T_xM and $T_{\phi(x)}N$. The *isometry group* $\mathrm{Isom}(M,g)$ of (M,g) is the set of diffeomorphisms of M that are g-isometric. A local diffeomorphism ϕ with $\phi^* h = g$ is a *local isometry.*

3.2.2. Examples. — 1) The antipodal map $x \to -x$ on S^n is an isometry. As a consequence, since $\mathbb{R}P^n$ is the quotient of S^n by this isometry, the metric of S^n induces a metric on $\mathbb{R}P^n$ (see below for more details on quotients).

2) The isometries of $\mathbb{R}^n$ are obtained from orthogonal transformations and translations: $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \ltimes O(n)$.

3) $\mathrm{Isom}(S^n) = O(n+1)$, and $\mathrm{Isom}(H^n) = O_o(1,n)$, where the index means that we take the subgroup preserving the component of $\{(x^0)^2 - (x^1)^2 - \cdots (x^n)^2 = 1\}$ with $x^0 > 0$. If we write SO instead of O in these examples, we obtain the orientation-preserving isometries. These two spaces are homogeneous spaces, that is the isometry group acts transitively. Therefore they are quotient of the isometry group by the isotropy group of a point:

$$S^n = O(n+1)/_{O(n)}, \quad H^n = O_0(1,n)/_{O(n)}.$$

Note that for $M = \mathbb{R}^n$, S^n or H^n, we have given a group that is clearly a group of isometries, but we have not proved that there is no other isometry. Nevertheless it is easy to see that these groups have a stronger property than being just homogeneous: actually, for any points x et y and any isometry $u : T_xM \to T_yM$, there exists an element ϕ of the group such that $\phi(x) = y$ and $d_x\phi = u$ (this is because the stabilizer of a point is each time $O(n)$). We will see later from the study of the exponential map that for a complete connected Riemannian manifold, there is at most one isometry with given (x, y, u), cf Lemma 3.6.6, so this proves that there is no possible other isometry.

3.2.3 Example. — A Lie group G is by definition a manifold endowed with smooth group operations. Every element γ of G therefore defines a diffeomorphism $L_\gamma : x \mapsto \gamma x$, by multiplication on the left. It turns out that every Lie group G can be endowed with metrics g that are *left-invariant* in that every map L_γ is an isometry of g. The recipe is the following. Pick any scalar product g_e on the tangent space T_eG at the identity element e and define g at any point $x \in G$ by

$$g_x := g_e\left((d_e(L_x))^{-1}(.), (d_e(L_x))^{-1}(.)\right).$$

The corresponding volume form is of course also invariant under left translations and it gives rise to a *Haar measure* on G. For instance, a left-invariant metric on the Lie group $GL_n(\mathbb{R})$ is given by $g_A(B, C) = \mathrm{Tr}((A^{-1}B)^T A^{-1}C)$, for $A \in GL_n(\mathbb{R})$ and B, C in $T_A GL_n(\mathbb{R}) = M_n(\mathbb{R})$.

In presence of isometries, it is natural to consider quotients. Let us state an important result on smooth actions of Lie groups G on manifolds M. Recall that the action of G on M is *free* if any non-trivial element of G has no fixed point in M ($\exists x \in M, \gamma x = x \Rightarrow \gamma = e$) and *proper* if, for any compact subset K of M, the closure of $\{\gamma \in G / \gamma K \cap K \neq \emptyset\}$ is a compact subset of G. Note that 'proper' is automatic if G is compact and, when G is non-compact, it roughly means that any compact portion of M is really moved away by the action of most elements of G, those outside a compact part. When G is a discrete group, it implies that any two points x and y which are not in the same orbit can be separated by two open subsets U and V of M such that GU does not intersect V, i.e. the topological quotient space M/G is Hausdorff! 'Free' means we avoid for instance a quotient like $\mathbb{R}^2/\pm id$ which has a conical singularity at the origin.

3.2.4 Proposition. — *Let G be a Lie group acting smoothly, freely and properly on a manifold M. Then M/G carries a unique structure of smooth manifold such that the projection $\pi : M \to M/G$ is a smooth submersion. When G is a discrete group, π is in fact a covering map.*

When G acts freely and properly by *isometries* on a Riemannian manifold (M, g), then M/G inherits a well-defined Riemannian metric $\breve{g}$ such that for all vectors v and w in $(\operatorname{Ker} d_x\pi)^\perp \subset T_xM$,

$$\breve{g}_{\pi(x)}(d_x\pi(v), d_x\pi(w)) = g_x(v, w).$$

For instance, the standard metrics on the Euclidean space and the spheres therefore induce metrics on $\mathbb{T}^n = \mathbb{R}^/\mathbb{Z}^n$, $\mathbb{R}P^n = S^n/\pm id$ and $\mathbb{C}P^n = S^n/S^1$.

3.3. The Levi-Civita connection

Our aim here is to prove that any Riemannian metric comes with a connection on the tangent bundle. First, we need a general definition.

3.3.1 Definition. — If ∇ is a connection on the tangent bundle TM of a manifold M, the *torsion* T^∇ of ∇ is given by TM, called the *torsion* of M.given by

$$T^\nabla_{X,Y} := \nabla_X Y - \nabla_Y X - [X, Y],$$

where X and Y are vector fields on M.

The reader may check that T^∇ is a section of $\Lambda^2 M \otimes TM$, given locally by the formula

$$T^\nabla\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = (\Gamma^k_{ij} - \Gamma^k_{ji})\frac{\partial}{\partial x^k}. \tag{3.2}$$

In particular, a *torsion-free connection*, i.e. a connection with zero torsion, is a connection whose Christoffel symbols satisfy the symmetry property

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{3.3}$$

A geometric interpretation of the torsion will be given in 3.4.3.

3.3.2 Theorem and definition. — *The tangent bundle of any Riemannian manifold (M, g) carries a unique torsion-free g-metric connection : this is the* Levi-Civita connection *of (M, g).*

Proof. — Let us deal with the uniqueness. Suppose we have a convenient connection ∇. Since it is g-metric, we can write

$$\mathscr{L}_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

as well as similar formulas for $\mathscr{L}_Y\langle X,Z\rangle$ and $\mathscr{L}_Z\langle X,Y\rangle$, and then use these identities to compute $\mathscr{L}_X\langle Y,Z\rangle+\mathscr{L}_Y\langle X,Z\rangle-\mathscr{L}_Z\langle X,Y\rangle$. In view of $T^\nabla=0$, the result of this computation reads:

$$2\langle\nabla_X Y,Z\rangle=\mathscr{L}_X\langle Y,Z\rangle+\mathscr{L}_Y\langle X,Z\rangle-\mathscr{L}_Z\langle X,Y\rangle \\ -\langle X,[Y,Z]\rangle-\langle Y,[X,Z]\rangle+\langle Z,[X,Y]\rangle. \tag{3.4}$$

Since $g=\langle .,.\rangle$ is non-degenerate on each fiber, this equality determines ∇ in terms of g, hence the uniqueness.

As for the existence, we merely define ∇ by formula (3.4). The reader may check that (i) the expression really yields a connection (just use basic properties of Lie derivatives and brackets to see how the formula behaves when you multiply X, Y or Z by some function f), (ii) it is metric (straightforward) and (iii) torsion-free (obvious on coordinate vector fields). □

Formula (3.4) is known as the *Koszul formula.* It is very important because it makes it possible to compute the Levi-Civita connection in any reasonable context; in particular, it yields an expression for Christoffel symbols in local coordinates :

$$\Gamma^l_{ij}=\frac{1}{2}g^{kl}\Big(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\Big), \tag{3.5}$$

where (g^{kl}) is the inverse matrix of (g_{kl}).

The Levi-Civita connection of $g_{\mathbb{R}^n}$ is of course the trivial connection. Any submanifold of $\mathbb{R}^N$ inherits a Riemannian metric by restriction of $g_{\mathbb{R}^n}$. The Levi-Civita connection is then induced by the trivial connection on $\mathbb{R}^N$ (cf. 2.1.5). In particular, this yields the Levi-Civita connection of $S^n\subset\mathbb{R}^{n+1}$ or $T^n=S^1\times\cdots\times S^1\subset\mathbb{R}^{2n}$. The Levi-Civita connection of the hyperbolic space H^n is also induced by the trivial connection on $\mathbb{R}^{n+1}$.

3.3.3. Exercise. — Use Koszul's formula to compute the Levi-Civita connection of a surface of revolution in $\mathbb{R}^3$, with the metric $g=du^2+r(u)^2d\theta^2$ (cf. 3.1.2).

The immediate interest of the Levi-Civita connection is two-fold : firstly, it yields a notion of geodesics generalizing "straight lines" from affine geometry to any Riemannian manifold ; secondly, it involves a notion of curvature, which is a fundamental invariant, generalizing the Gaussian curvature of surfaces.

3.4. Geodesics

Straight lines in the affine space $\mathbb{R}^n$ can be seen as curves c with zero acceleration : $\ddot{c} = 0$. Given any connection ∇ on the tangent bundle of a manifold M, we can define the acceleration vector of a curve $c : I \to M$. This is the derivative of the velocity vector $\dot{c}$ along the curve, namely along $\dot{c}$: the acceleration vector is $\nabla_{\dot{c}}\dot{c}$. In what follows, we will stick to the case of the Levi-Civita connection ∇^g of a metric g.

3.4.1 Definition. — A curve c on a Riemannian manifold (M, g) is called a *geodesic* if $\nabla^g_{\dot{c}}\dot{c} = 0$.

In other words, the velocity vector is obtained by parallel transport of the initial velocity vector along the curve. Since the Levi-Civita connection $\nabla^g = \nabla$ is g-metric, the definition implies $\frac{d}{dt}|\dot{c}|^2 = 2\langle \dot{c}, \nabla_{\dot{c}}\dot{c}\rangle = 0$, so the velocity vector $\dot{c}$ of a geodesic has always constant norm.

In local coordinates (x^i), we can compute the geodesic equation : for a curve $c(t) = (x^i(t))$,

$$\nabla_{\dot{c}}\dot{c} = \dot{x}^j\Big(\frac{\partial \dot{x}^i}{\partial x^j} + \Gamma^i_{jk}\dot{x}^k\Big)\frac{\partial}{\partial x^i} = (\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k)\frac{\partial}{\partial x^i},$$

so the equations for a geodesic are :

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k, \qquad 1 \leqslant i \leqslant n. \tag{3.6}$$

This is a nonlinear second order differential equation on $(x^i(t))$. It has a unique solution on some maximal interval as soon as the initial position $c(0)$ and initial velocity vector $\dot{c}(0)$ are given.

3.4.2. Examples. — 1) On $\mathbb{R}^n$, geodesics are of course straight lines, parameterized at constant speed.

2) On $S^n \subset \mathbb{R}^{n+1}$ the Levi-Civita connection is the projection of the Levi-Civita connection of $\mathbb{R}^{n+1}$ so the geodesics with unit speed can be seen as curves c in $\mathbb{R}^{n+1}$ with $|c| = 1$, $|c'| = 1$ and whose acceleration vector c'' in $\mathbb{R}^{n+1}$ is normal to the sphere, i.e. $c'' = \lambda c$. It follows that $\lambda = \langle c'', c\rangle = -\langle c', c'\rangle = -1$ so that $c'' + c = 0$. The geodesics of the sphere are therefore the great circles.

3) The geodesics of $T^n = S^1 \times \cdots \times S^1$ are the curves that turn around each circle S^1 at constant speed. This torus can also be seen as $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and then the geodesics are simply the projections of straight lines in $\mathbb{R}^n$.

3.4.3 Remark. — The notion of geodesics extends immediately if ∇ is any connection on the tangent bundle TM and it allows a geometric interpretation of its torsion. Let X and Y denote two vectors in T_pM, $p \in M$. For small positive t and u, we let $\tau_{XY}(t,u)$ denote the point obtained in following way: start from p, follow the geodesic γ_X with initial velocity vector X for some time t and then follow for some time u the geodesic with initial velocity vector $\tilde{Y}(t)$, where $\tilde{Y}(t)$ is the parallel transport of Y from p to $\gamma_X(t)$, along γ_X. Another point $\tau_{YX}(t,u)$ is obtained by switching the roles of X and Y: let γ_Y be the geodesic starting from p with velocity Y, let $\tilde{X}(u)$ denote the parallel transport of X from p to $\gamma_Y(u)$ along γ_Y, and define $\tau_{YX}(t,u)$ as the point at time t along the geodesic starting from $\gamma_Y(u)$ with velocity $\tilde{X}(u)$. It is instructive to compare the Taylor expansions of $\tau_{XY}(t,u)$ and $\tau_{YX}(t,u)$ in a chart. Using the equation of geodesics 3.6 and the equation of parallel transport 2.6, one finds

$$\begin{aligned}\tau_{XY}(t,u) = \quad & p + tX + uY - tu\Gamma(X)Y \\ - \quad & \frac{t^2}{2}\Gamma(X)X - \frac{u^2}{2}\Gamma(Y)Y + \text{terms of order three.}\end{aligned}$$

The expansion of $\tau_{YX}(t,u)$ follows at once and the outcome is:

$$\tau_{YX}(t,u) - \tau_{XY}(t,u) = tu\,T^{\nabla}(X,Y) + \text{terms of order three.}$$

So the torsion measures the infinitesimal defect of τ_{XY} and τ_{YX} to coincide.

In the Euclidean space $\mathbb{R}^n$, straight lines yield the shortest paths between any two points. We will see that a similar property holds true for geodesics on a Riemannian manifold. First, observe that a Riemannian metric g on M makes it possible to define the *length* of a path $c : [a,b] \to M$: it is

$$L(c) = \int_a^b \sqrt{g(\dot{c}(t), \dot{c}(t))}dt = \int_a^b |\dot{c}|.$$

This is independent of the parameterization of c and one can always change the parameterization so that c is parameterized by arc length: $|\dot{c}| = 1$ (as in $\mathbb{R}^n$). We want to analyze the paths realizing the minimum distance from x to y (which we call *minimizing paths*), and for this we will find the critical points of L. We consider a family of paths $c_s : [a,b] \to M$ depending on $s \in]-\epsilon,\epsilon[$, namely a smooth map $c : [a,b] \times]-\epsilon,\epsilon[\to M$ and we wish to calculate $\frac{d}{ds}L(c_s)$ at $s = 0$.

Shortening $\frac{\partial c}{\partial t}$ into $\partial_t c$ and using the fact that ∇ is a metric connection, we find

$$\frac{d}{ds}L(c_s) = \frac{d}{ds}\int_a^b \sqrt{g(\partial_t c(t,s), \partial_t c(t,s))}dt = \int_a^b \frac{g(\partial_t c, \nabla_{\partial_s c}\partial_t c)}{|\partial_t c|}dt.$$

Observe that in local coordinates we may write :

$$\nabla_{\partial_s c}\partial_t c = \frac{\partial^2 c}{\partial s \partial t} + \Gamma_{\partial_s c}\partial_t c.$$

In this expression, the first term is symmetric in s and t, because of Schwarz theorem, and the second one also, because ∇ is torsion free, cf. (3.3). Then $\nabla_{\partial_s c}\partial_t c = \nabla_{\partial_t c}\partial_s c$ and we can further compute

$$\frac{d}{ds}L(c_s) = \int_a^b \frac{g(\partial_t c, \nabla_{\partial_t c}\partial_s c)}{|\partial_t c|}dt = \int_a^b \frac{\frac{d}{dt}g(\partial_t c, \partial_s c) - g(\nabla_{\partial_t c}\partial_t c, \partial_s c)}{|\partial_t c|}dt.$$

Let us assume $c_0 = c(.,0)$ is parameterized at constant speed, that is $|\dot{c_0}| = v_0 := \frac{L(c_0)}{b-a}$, and define a vector field along c_0 by $N_0(t) = \frac{\partial}{\partial s}|_{s=0}\ c(t,s)$. Then we obtain the formula for the variation of length:

(3.7)

$$\frac{d}{ds}\Big|_{s=0} L(c_s) = \frac{1}{v_0}\left(g(\dot{c_0}, N_0)|_{t=b} - g(\dot{c_0}, N_0)|_{t=a} - \int_a^b g(\nabla_{\dot{c_0}}\dot{c_0}, N_0)dt \right).$$

We are interested in critical points of L among paths between two given given points x to y. These are paths $c_0 : [a,b] \longrightarrow M$ parameterized with constant speed (recall one can always reparameterize to get this) with $c_0(a) = x$ and $c_0(b) = y$ such that for any deformation c_s of c_0 with $c_s(a) = x$ and $c_s(b) = y$, the derivative of $L(c_s)$ at $s = 0$ vanishes. This means that the right hand side of (3.7) must vanish for any vector field N_0 along c_0 such that $N_0(a) = 0$ and $N_0(b) = 0$, namely $\nabla_{\dot{c_0}}\dot{c_0} = 0$:

3.4.4 Proposition. — *A path $c : [a,b] \to M$ parameterized with constant speed is a critical point of the length among paths from $c(a)$ to $c(b)$ if and only if it is a geodesic.*

In particular, up to reparameterization, minimizing paths are geodesics. The converse cannot be true in full generality. For instance, the geodesics of the sphere are the great circles, which we can see as paths from some point x to itself ; these are certainly not minimizing paths between x and x, since they are beaten by the constant path ! What we will show later (cf. the study of the exponential map, corollary 3.6.4) is that geodesics are exactly the locally minimizing paths : given any two close enough points of a geodesic, the geodesic is the unique minimizing path between them.

3.4.5. Exercise. — Prove that the critical points of the *energy*

$$E(c) = \int_a^b |\dot{c}|^2 dt \tag{3.8}$$

among paths from $c(a)$ to $c(b)$ are also the geodesics.

3.5. Symmetries and geodesics

Basically, an isometry preserves the metric, so it preserves every natural object determined by the metric, such as the volume form, the Levi-Civita connection, the geodesics... The presence of isometries can indeed be used to understand the geodesics. For instance, let us explain why every geodesics of the sphere S^2 is a great circle, namely the intersection of the sphere with some plane through 0. We pick a point x and a tangent vector v to the sphere at this point. Let H be the plane generated by x in v in $\mathbb{R}^3$. If the geodesic starting at x with velocity v did not remain inside H, the reflection in H (which is an isometry) would yield another geodesic starting at x with velocity v and that is not possible (because of Cauchy-Lipschitz theorem) : so the geodesic stays in H, which is what we wanted to prove. The same argument extends to any dimension and also shows that the geodesics of the hyperbolic space consist of intersections of H^n with planes.

Now, let us introduce the infinitesimal analogue of an isometry.

3.5.1 Definition. — On a Riemannian manifold, a *Killing field* is a vector field whose flow consists of isometries.

Observe there is a natural notion of a *Lie derivative of a metric* g along a vector field X (with flow ϕ_t):

$$\mathscr{L}_X g = \frac{d}{dt}\Big|_{t=0} \phi_t^* g. \tag{3.9}$$

As for differential forms, we have

$$(\mathscr{L}_X g)(Y,Z) = \mathscr{L}_X(g(Y,Z)) - g(\mathscr{L}_X Y, Z) - g(Y, \mathscr{L}_X Z). \tag{3.10}$$

Using $\mathscr{L}_X Y = [X,Y]$ and the properties of the Levi-Civita connection, one can then see that

$$(\mathscr{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

In words, $(\mathscr{L}_X g)_x$ is twice the symmetric part of the bilinear form $g(\nabla_. X, .)_x$ on T_xM. The reader may check that twice the skew-symmetric part of $g(\nabla_. X, .)$ is the exterior derivative $d(X^\sharp)$ of the one-form $X^\sharp = g(X, .)$ that is dual to X

with respect to the metric g (since g is non-degenerate on each fiber, it yields an identification between the tangent and the cotangent bundle).

3.5.2 Proposition. — *A vector field X on a Riemannian manifold (M, g) is Killing iff $\mathscr{L}_X g = 0$, i.e. $g(\nabla_. X, .)$ is skew-symmetric.*

Proof. — The group property of the flow gives $\frac{d}{dt}\phi_t^* g = \phi_t^* \mathscr{L}_X g$. The result then follows immediately from the remarks above. □

3.5.3 Lemma. — *If X is a Killing vector field and c a geodesic, then $\langle \dot{c}, X\rangle$ is constant along c.*

Proof. — One has $\mathscr{L}_{\dot{c}}\langle \dot{c}, X\rangle = \langle \dot{c}, \nabla_{\dot{c}} X\rangle = 0$ (the first equality by the geodesic equation, the second by the Killing condition). □

The quantity $\langle \dot{c}, X\rangle$ is preserved along a geodesic, it is a *first integral* of the geodesic equation. This is useful for finding the solutions of the geodesic equation when the metric has symmetries, and we shall now give an example.

3.5.4. Example. — Suppose we have a surface of revolution, with metric $g = du^2 + r(u)^2 d\theta^2$ (see example 3.1.2). The rotation vector $X = \frac{\partial}{\partial\theta}$ generates the flow of rotations of the surface, and is therefore a Killing field. Then our first integral says immediately that along a geodesic c, the quantity $r^2\dot{\theta}$ is a constant, say C; it is known as the Clairaut invariant of the geodesic. If we assume c is parametrized by arc length and denote by α the angle between $\dot{c}$ and $\frac{\partial}{\partial\theta}$, then $\cos\alpha = \frac{\dot{\theta}}{r}$ and the Clairaut invariant can be expressed as $r\cos\alpha = C$.

On the other hand, requiring that $\dot{c}$ has unit length yields $\dot{u}^2 + r^2\dot{\theta}^2 = 1$. Therefore, up to reversing time, we obtain the system

$$\dot{\theta} = \frac{C}{r^2}, \quad \dot{u} = \sqrt{1 - \frac{C^2}{r^2}}. \tag{3.11}$$

This system of first order differential equations can be solved to provide potential geodesics. Two special kinds of solutions are interesting:

- $\theta =$ constant, $C = 0$ and $u(t) = t$: these are the meridians and one can check that they have zero acceleration and are therefore geodesics;
- $u(t) =$ constant $= u_0$, $C = r(u_0)$ and $\theta(t) = \frac{t}{r(u_0)}$: these are parallels (i.e. horizontal circles) and they are geodesics if and only if $r(u_0)$ is a critical value, i.e. $\frac{dr}{du}|_{u=u_0} = 0$ (check it).

Observe that, in fact, any other solution of 3.11 (i.e. neither a parallel or a meridian) yields a geodesic. The behaviour of these geodesics c is ruled by their Clairaut invariant. Owing to $r\cos\alpha = C$, we have $r \geqslant C$. Equations 3.11 allow the following phenomenons. If $c(0)$ lies between two consecutive parallels with radius $r = C$ non-critical, then c will keep bouncing between these two parallels forever. If the surface is (complete and) non-compact and $c(0)$ is not sandwiched like in the previous case, c will escape to infinity, maybe after bouncing on one parallel with $r = C$ non-critical. If by chance c points toward ($\dot{u}$ is monotone) a parallel with radius $r = C$ critical, then it will accumulate on it.

3.5.5. Exercise. — On the 2-sphere S^2 we consider the metric of revolution

$$g = \frac{(1+f(z))^2}{1-z^2}dz^2 + (1-z^2)d\theta^2,$$

where $f : [-1,1] \to]-1,1[$ is any smooth function with $f(-1) = f(1) = 0$. Show that if f is an odd function ($f(-z) = -f(z)$), then all geodesics of g are circles of length 2π (Zoll, 1903). Hint: deduce the behaviour of geodesices from the discussion above and compute the periods of the coordinates $z(t)$ and $\theta(t)$ of a geodesic.

3.6. The exponential map

Let (M^n, g) be a Riemannian manifold. Given $x \in M$ and $X \in T_xM$, let γ_X be the geodesic such that $\gamma_X(0) = x$ and $\dot{\gamma_X}(0) = X$. If γ_X is defined up to time 1, we set :

$$\exp_x(X) := \gamma_X(1). \tag{3.12}$$

This defines a map $\exp_x$ from some subset of T_xM to M. This is the *exponential map* at x.

Now, take any vector X. If r is a small enough real number, then $\exp_x$ is defined on rX and

$$\exp_x(rX) = \gamma_X(r). \tag{3.13}$$

It follows that $\exp_x$ is defined on some open star-shaped neighborhood of the origin in T_xM (star-shaped : if X is inside and $0 \leqslant r \leqslant 1$, then rX is inside).

Taking the derivative of (3.13) with respect to r at $r = 0$, we see that $d_0 \exp_x(X) = X$ (with the canonical identification between $T_0(T_xM)$ and

T_xM). Therefore :

$$d_0 \exp_x = id_{T_xM}. \tag{3.14}$$

Therefore $\exp_x$ is a diffeomorphism between a neighborhood U of 0 in T_xM and a neighborhood V of x in M. Then consider

$$\exp_x^{-1} : V \longrightarrow U \subset T_xM \simeq \mathbb{R}^n \tag{3.15}$$

is a smooth chart near x. Moreover, T_xM carries a natural structure of Euclidean vector space, given by g_x, so we may identify T_xM with the Euclidean vector space $\mathbb{R}^n$. What we are saying is that, given a metric g, there is a canonical chart around any point, well defined up to the action of $O(n)$ on $\mathbb{R}^n$. The corresponding coordinates are called *normal coordinates* or *geodesic coordinates* or *exponential coordinates*.

An important notion in Riemannian geometry is the *injectivity radius*, which measures the size of the domain of the exponential map: the injectivity radius at x, denoted by inj_x, is the largest number r such that $\exp_x$ is a diffeomorphism on the ball of radius r and centered in $0 \in T_xM$; the injectivity radius $\mathrm{inj}(M)$ of M is the infimum of all inj_x, $x \in M$.

The exponential chart is expected to have special properties, so let us write the metric in normal coordinates: $g = g_{ij}dx^i dx^j$ (in other words, we look at the pullback metric $\exp_x^* g$). Since $d_0 \exp_x$ is the identity and we identify T_xM with the Euclidean $\mathbb{R}^n$, we certainly have

$$g_{ij}(0) = \delta_{ij}. \tag{3.16}$$

In view of (3.13), unit speed geodesics starting from x are given by $r \mapsto \exp_x(rX)$ for some unit vector X. In normal coordinates, these geodesics are thus described by straight rays from the origin : letting r be the radial coordinate, we have

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0. \tag{3.17}$$

Using this equation at the origin together with $T^\nabla = 0$, we find $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(0) = 0$ for all indices so that Christoffel symbols vanish at the origin:

$$\Gamma_{ij}^k(0) = 0. \tag{3.18}$$

Finally, since ∇ is g-metric, we have

$$\frac{\partial g_{ij}}{\partial x^k} = \langle \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle + \langle \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \rangle$$

which vanishes at the origin, hence the Taylor expansion :

$$g_{ij} = \delta_{ij} + O(r^2). \tag{3.19}$$

This means that in normal coordinates, the metric is approximated up to second order by the Euclidean metric $\sum(dx^i)^2$. As we shall see later, it is not possible in general to obtain a better approximation, because the second derivatives of the coefficients g_{ij} can be interpreted as the curvature of the Levi-Civita connection.

3.6.1. Examples. — 1) In $\mathbb{R}^n$, one has $\exp_x(X) = x+X$ since the geodesics are the straight lines. The injectivity radius at every point is $+\infty$.

2) In $\mathbb{R}^n \times S^1$, one has $\exp_{(x,z)}(X, t\partial_\theta) = (x+X, ze^{it})$. The injectivity radius at every point is π.

3) In S^n, in the stereographic projection from the north pole, the geodesics issued from the south pole become straight lines, but the velocity in the coordinates is not constant, see formula (3.1). To obtain the normal coordinates, it is therefore sufficient to re-parameterize each ray by arc length: this gives the change of coordinates $\rho = 2\arctan r$ (so $\rho < \pi$), and the formula

$$g = d\rho^2 + \sin^2(\rho) g_{s^{n-1}}. \tag{3.20}$$

The injectivity radius at every point is π.

4) Similarly prove that the hyperbolic metric can be written in normal coordinates as

$$g = d\rho^2 + \sinh^2(\rho) g_{S^{n-1}}. \tag{3.21}$$

The injectivity radius at every point is $+\infty$.

3.6.2. Exercise. — 1) Prove that any compact Lie group carries a *bi-invariant* Riemannian metric, i.e. invariant under both left translations ($L_\gamma : x \mapsto \gamma x$, $\gamma \in G$) and right translations ($R_\gamma : x \mapsto x\gamma$).

2) Let X be a left-invariant vector field on a Lie group G. (i.e. $L_\gamma^* X = X$ for every $\gamma \in G$). Let ϕ_t denote the flow of X and $\gamma(t) := \phi_t(e)$. Prove the formula

$$\phi_t(x) = x\gamma(t) = L_x(\gamma(t)) = R_{\gamma(t)}(x).$$

3) Let G be a Lie group endowed with a bi-invariant metric $\langle\rangle$, with levi-Civita connection ∇. Prove that for any left-invariant vector fields X and Y,

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{3.22}$$

Hint: use 2) and 1.5.11 to prove that $\langle [X, .], .\rangle$ is skewsymmetric. Why does it completely determine ∇? Deduce that the exponential map is given by the flow of left-invariant vector fields, or in other words $\exp_e(tX_e) = \gamma(t)$, where γ is like in 2).

4) Let us work out the example of $SO(n)$. We endow $T_{I_n}SO(n)$ (the space of skewsymmetric matrices) with the scalar product $\langle A, B\rangle = \operatorname{Tr} A^T B = -\operatorname{Tr} AB$ and extend it into a left-invariant metric on $SO(n)$. a) Prove that it is a bi-invariant metric. b) Write down in terms of matrices what left invariant vector fields look like, and what their flow and Lie bracket are. c) Prove that $\exp_{I_n}$ is the usual exponential map on matrices :

$$\exp_{I_n} A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

d) Prove that the injectivity radius at any point of $SO(n)$ is $\sqrt{2}\pi$.

3.6.3 Gauss Lemma. — *Let γ be a geodesic issued from $\gamma(0) = x$ with inital velocity vector $X = \dot\gamma(0) \in T_xM$. If $Y \in T_xM$ is orthogonal to X then $d_X \exp_x(Y)$ is also orthogonal to $d_X \exp_x(X) = \dot\gamma(1)$.*

We already know that $\exp_x$ maps straight rays to geodesics issued from x. What Gauss Lemma says is that any vector orthogonal to a straight ray is mapped to a vector orthogonal to the corresponding geodesic. In other words, the image of any sphere in T_xM is orthogonal to all the geodesics issued from the point x.

Proof. — Let $\sigma(s)$ be any path in T_xM such that $\sigma(0) = X$, $\sigma'(0) = Y$ and $|\sigma(s)| = |X|$: σ starts at X, orthogonally to X and wanders on the sphere of radius $|X|$. We consider $c(t, s) = \exp_x(t\sigma(s))$. This is a deformation of the geodesic γ with fixed starting point $c(0, s) = x$. Moreover, every path $c(., s)$ has length $|X|$ (because $|\sigma(s)| = |X|$ and $0 \leqslant t \leqslant 1$). The formula (3.7) for the variation of the length then reads

$$0 = \frac{1}{|X|}\langle \partial_t c(1, 0), \partial_s c(1, 0)\rangle.$$

Now, $\partial_t c(1, 0)$ is the derivative at $t = 1$ of $\exp_x(tX) = \gamma(t)$, so we have $d_X \exp_x(X) = \dot\gamma(1)$, and $\partial_s c(1, 0)$ is the derivative at $s = 0$ of $\exp_x(\sigma(s))$, which is certainly $d_X \exp_x(Y)$, hence the result. □

This lemma has several important consequences. First it tells us that in normal coordinates, the rays from the origin are orthogonal to the concentric spheres, which implies that we have polar coordinates in the following sense :

$$g = dr^2 + g_r \tag{3.23}$$

where g_r denotes metrics on the sphere S^{n-1}, depending smoothly on r. Besides, from the Taylor expansion of g at 0, we get

$$g_r = r^2 g_{S^{n-1}} + O(r^4),$$

For the Euclidean metric $g_r = r^2 g_{S^{n-1}}$, for the sphere and the hyperbolic space, see the formulas (3.20) and (3.21).

3.6.4 Corollary. — *Let x and y be two point in M such that y belongs to the image of the ball $B(0, \mathrm{inj}_x)$ by exp_x. Then there is a unique shortest path from x to y. It is given by the geodesic $t \mapsto \exp_x(tX)$, where $|X| < \mathrm{inj}_x$ and $\exp_x(X) = y$.*

In particular, any small part of a geodesic is length minimizing : *geodesics are locally minimizing.*

Proof. — The geodesic $t \mapsto \exp_x(tX)$ has length $|X|$ by definition of the exponential map. So we must show that any other path γ from x to y has length greater than or equal to $|X|$. Let T be the first time t such that $|\gamma(t)| = |X|$ and let $\bar\gamma$ be the restriction of γ to $[0, T]$: $\bar\gamma$ is shorter than γ and remains in $B(0, \mathrm{inj}_x)$. Gauss Lemma yields

$$L(\bar\gamma) = \int_0^T |dr(\dot{\bar\gamma})| dt + \int_0^T \sqrt{g_r(\dot{\bar\gamma}, \dot{\bar\gamma})} dt.$$

It follows that

$$L(\bar\gamma) \geqslant \int_0^T dr(\dot{\bar\gamma}) dt = r(\bar\gamma(T)) - r(\bar\gamma(0)) = |X|.$$

So the length of γ is at least $|X|$. □

Our definition of the length of a path does not require much regularity on the path : we may consider paths that are only piecewise C^1 for instance. A consequence of Gauss Lemma (cf. the proof above) is that minimizing paths are bound to be smooth because they must be geodesics.

On any connected Riemannian manifold (M, g) there is a natural distance : the distance $d_g(x, y)$ between x and y is defined as the infimum of the lengths of paths from x to y. The corollary above tells us that the image of any small ball $B_{g_x}(0, t)$ of T_xM by $\exp_x$ is the ball $B_{d_g}(x, t)$. Note that the topology induced by this distance is the same as the topology of the manifold.

We say that a connected Riemannian manifold (M, g) is *complete* if the metric space (M, d_g) is complete. Hopf-Rinow theorem explains what it means to be complete.

3.6.5 Hopf-Rinow theorem. — *On a connected Riemannian manifold (M, g), the following statements are equivalent:*

1. (M, g) *is complete;*
2. *for any $x \in M$, $\exp_x$ is defined on T_xM;*
3. *there exists $x \in M$, such that $\exp_x$ is defined on T_xM ;*
4. *closed balls are compact.*

If (M, g) is complete, then any two points of M can be joined by a minimizing geodesic.

In words, a Riemannian manifold is complete if all its geodesics are defined on $\mathbb{R}$, they are not allowed to reach "infinity" in finite time.

The examples mentionned previously are complete but it is easy to build non-complete Riemannian manifolds, like this one : $\mathbb{R}^n - \{0\}$ is not complete since a geodesic running towards the origin must stop in finite time.

Proof. — 1. $\Rightarrow$ 2. Let γ be a geodesic defined on $[0, T[$ with $T < +\infty$. Then $|\dot{\gamma}|$ is constant so that $\gamma(t)$ satisfies Cauchy criterion near T. By completeness, γ can therefore be extended continuously to $[0, T]$. Now, pick (normal) coordinates around $\gamma(T)$ and look at the geodesic equation $\ddot{\gamma}^k = -\Gamma_{ij}^k(\gamma)\dot{\gamma}^i\dot{\gamma}^j$. The right-hand side is bounded near $t = T$ so $\ddot{\gamma}$ is bounded and $\dot{\gamma}$ also extends continuously at $t = T$. We can then solve the geodesic equation on a neighborhood of T to extend γ beyond T. So γ is defined up to $+\infty$.

2. $\Rightarrow$ 3. Trivial.

3. $\Rightarrow$ (every point of M can be joined to x by a minimizing geodesic). Let $y \in M$. We shall construct a geodesic from x to y of length $d(x, y)$. As a first step, we pick a small number $\delta < \mathrm{inj}_x$ and observe that there exists a point z in the sphere $\partial B(x, \delta)$ such that

$$d(x, z) = \delta \quad \text{and} \quad d(x, y) = d(x, z) + d(z, y).$$

Indeed, the sphere $\partial B(x, \delta)$ is compact (it is the image of a standard sphere by $\exp_x$) so there is a point z of $\partial B(x, \delta)$ such that $d(z, y) = d(\partial B(x, \delta), y)$. Then, any path σ from x to y goes through $\partial B(x, \delta)$, say for the first time at z_σ, which implies $L(\sigma) \geqslant \delta + d(z_\sigma, y) \geqslant \delta + d(z, y)$. This is true for any σ so $d(x, y) \geqslant d(x, z) + d(z, y)$. And this is indeed an equality by triangle inequality.

The points x and z can be connected by a unique unit speed geodesic γ thanks to the choice of δ and we know by assumption that γ is defined on $\mathbb{R}$. The second step consists in proving that $\gamma(d(x,y)) = y$. We consider the set

$$I = \{t \in [0, d(x,y)] \;/\; d(x,\gamma(t)) = t \quad \text{and} \quad d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y)\}.$$

Then I is an interval containing $[0,\delta]$. Let $t \in I$ with $t < d(x,y)$. As in the first step, for any small enough $\epsilon > 0$, there exists $w \in \partial B(\gamma(t),\epsilon)$ such that

$$d(\gamma(t),y) = d(\gamma(t),w) + d(w,y).$$

With $t \in I$, it follows that

$$\begin{aligned} d(x,w) &\geqslant d(x,y) - d(y,w) = d(x,\gamma(t)) + d(\gamma(t),y) - d(w,y) \\ &= d(x,\gamma(t)) + d(\gamma(t),w) \geqslant d(x,w). \end{aligned}$$

This is therefore an equality, which implies that the path $\gamma_{|[0,t]}$ followed by the unique geodesic from $\gamma(t)$ to w has length $d(x,\gamma(t)) + d(\gamma(t),w) = d(x,w)$, hence is a minimizing path and thus a smooth geodesic: it must be $\gamma_{|[0,t+\epsilon]}$. Then $w = \gamma(t+\epsilon)$ and $t+\epsilon \in I$. So I is open in the interval $[0,d(x,y)]$. It is also closed and non-empty : $I = [0,d(x,y)]$. And $d(x,y) \in I$ means that γ is a minimizing geodesic from x to y.

3. $\Rightarrow$ 4. The previous statement ensures that for any point y of M, $y = \exp_x Y$ with $|Y| = d(x,y)$, so closed balls are in the image through $\exp_x$ of closed balls in T_xM, which are compact.

4. $\Rightarrow$ 1. A Cauchy sequence is bounded, so from 4. one can extract a converging subsequence. But then the whole sequence converges, for it is Cauchy. □

Completeness makes it possible to play with geodesics. For instance, any local isometry $f : M \longrightarrow N$ maps geodesics to geodesics, hence

$$f\left(\exp_x^M X\right) = \exp_{f(x)}^N \left(d_x f(X)\right), \tag{3.24}$$

provided that the geodesic $t \mapsto \exp_x^M(tX)$ is well defined on $[0,1]$. If M is complete, $\exp_x^M$ is well defined on T_xM and onto M, so the left-hand side with x fixed determines f. It follows that the right-hand side, for some fixed x, determines f, hence

3.6.6 Lemma. — *Let f_1 and f_2 be two local isometries between complete connected Riemannian manifolds M and N. Assume there is a point $x \in M$ such that $f_1(x) = f_2(x)$ and $d_x f_1 = d_x f_2$. Then $f_1 = f_2$.*

Another useful application of completeness is the following result.

3.6.7 Lemma. — *If M and N are complete connected Riemannian manifolds, then any local isometry $f : M \longrightarrow N$ is a covering map.*

Proof. — Firstly, observe that f is onto owing to completeness and (3.24). Secondly, for any y in N, if $r > 0$ is small (smaller than inj_y), then $f^{-1}(B(y,r))$ is the disjoint union of the balls $B(x_i, r)$, $x_i \in f^{-1}(y)$. Details are left to the reader (hint : push geodesics downstairs or lift them upstairs, by f). □

3.7. Riemannian curvature

The curvature of the Levi-Civita connection of a Riemannian metric g is a crucial invariant. It is often called the *Riemann curvature tensor*. In this text, we will denote it by R. A Riemannian manifold is said to be flat if its Levi-Civita connection is flat, i.e. $R = 0$.

Beware the two sign conventions for the curvature tensor are widely used in the literature ! The convention chosen here is the most natural from the viewpoint of general connections but it will force us to discreetly slip an undesired minus in the appropriate formulas, basically because *you know* that the sphere has positive curvature.

Since the Levi-Civita connection is very specific, the Riemann curvature tensor benefits from several special symmetries.

3.7.1 Proposition. — *The Riemann curvature tensor satisfies the following algebraic properties.*

1. $R_{X,Y} = -R_{Y,X}$.
2. $\langle R_{X,Y}Z, T\rangle = -\langle R_{X,Y}T, Z\rangle$.
3. $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0$ *(algebraic Bianchi identity).*
4. $\langle R_{X,Y}Z, T\rangle = \langle R_{Z,T}X, Y\rangle$.

The identities 1., 2. and 4. can be summed up by saying that the curvature tensor is a section of $\mathrm{Sym}^2(\Lambda^2 M)$.

Proof. — 1. is very general : the curvature of any connection on TM is in $\Lambda^2 M \otimes \mathrm{End}\, TM$.

2. is due to the fact that the Levi-Civita connection is metric.

3. comes from the fact the Levi-Civita connection is torsion-free. Indeed, the left-hand side is

$$\underline{\nabla_X\nabla_Y Z} - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X \underline{-\nabla_{[Y,Z]}X} \\ + \nabla_Z\nabla_X Y \underline{-\nabla_X\nabla_Z Y} - \nabla_{[Z,X]}Y$$

The three underlined terms give $[X,[Y,Z]]$. Gathering the other terms similarly, we get $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]$, which vanishes in view of Jacobi identity.

4. is a mess. The algebraic Bianchi identity yields

$$\langle R_{X,Y}Z,T\rangle+\langle R_{Y,Z}X,T\rangle+\langle R_{Z,X}Y,T\rangle=0.$$

Now rotate the roles of the four vector fields to obtain four identities and add them up ! With 1., 2. and patience, you will find the formula. □

From these symmetries, we see for example that in dimension 2, the curvature is determined by only one of its coefficients, $K=\langle R_{e_1,e_2}e_2,e_1\rangle$, in any orthonormal basis (e_1,e_2) ; this is the Gauss curvature of the surface. In higher dimension, one defines analogous 2-dimensional curvatures in the following way:

3.7.2 Definition. — Let (M,g) be a Riemannian manifold, and $P\subset T_xM$ a 2-plane. The *sectional curvature* of the plane P is given by

$$K(P)=K(e_1\wedge e_2):=\langle R_{e_1,e_2}e_2,e_1\rangle$$

for any orthonormal basis (e_1,e_2) of P.

The reader can check that this definition makes sense. Owing to the symmetries of the curvature tensor, the sectional curvatures of all 2-planes in T_xM completely determines the curvature tensor at the point x.

3.7.3. Examples. — 1) The curvature of the flat $\mathbb{R}^n$ vanishes and therefore all the sectional curvatures vanish.

2) For the sphere S^n, we first observe that the isometry group $SO(n+1)$ is transitive on 2-planes: indeed it is transitive on the points of S^n, and the isotropy group of a point is $SO(n)$ which acts transitively on 2-planes of $\mathbb{R}^n$. Since the curvature and the sectional curvatures are canonically defined from the metric, they are preserved by isometries and it follows that all the sectional curvatures of S^n equal a fixed constant (+1, as we shall see later).

3) Similarly the hyperbolic space H^n has constant sectional curvature.

4) The sectional curvatures of $\lambda^2 g$, where λ is a positive number, are related to that of g by the relation

$$K^{\lambda^2 g}=\frac{1}{\lambda^2}K^g. \tag{3.25}$$

This comes immediately from the fact that g and $\lambda^2 g$ have the same Levi-Civita connection (check it !) and therefore the same curvature tensor R. The formula corresponds to the idea that if we make a sphere very big (λ big), then its curvature becomes small, that is it becomes almost flat. Indeed the Earth looks locally very flat !

The following theorem is fundamental in Riemannian geometry. In fact, it was outlined in the first text on Riemannian geometry by B. Riemann, in 1854. It says that a Riemannian manifold has vanishing curvature iff it is locally isometric to the Euclidean space.

3.7.4 Theorem. — *A Riemannian manifold (M, g) is flat if and only if near any point there exist local coordinates x^i such that $g = \sum (dx^i)^2$.*

Proof. — If the Levi-Civita connection ∇ on TM is flat, then Propositions 2.5.2 and 2.5.3 ensure that near each point we have parallel vector fields $X_1, \dots, X_n$ (parallel means $\nabla X_i = 0$) forming an orthonormal basis. In particular, since ∇ is torsion-free, we have $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$. From the proof of lemma 1.6.4, it follows that there exist local coordinates such that $X_i = \frac{\partial}{\partial x^i}$. The other implication is clear. $\square$

3.8. Second fundamental form

Suppose that (M^n, g) is an oriented Riemannian manifold, and $N^{n-1} \subset M$ is a Riemannian submanifold oriented by the unit normal vector $\vec{n}$. Similarly to the case of submanifolds of $\mathbb{R}^n$, the Levi-Civita connection of N is

$$\nabla^N_X Y = \pi \left(\nabla^M_X Y \right), \tag{3.26}$$

where for every x in N, $\pi : T_x M \to T_x N$ is the orthogonal projection. Therefore, for two vector fields X, Y on N, the covariant derivative $\nabla^M_X Y$ decomposes as

$$\nabla^M_X Y = \nabla^N_X Y + \mathbb{I}(X, Y) \vec{n}. \tag{3.27}$$

Then II is tensorial in its variables and, developing the torsion-free condition $\nabla^M_X Y - \nabla^M_Y X = [X, Y]$ with (3.27), we obtain the symmetry condition

$$\mathbb{I}(X, Y) = \mathbb{I}(Y, X). \tag{3.28}$$

3.8.1 Definition. — The formula $\mathbb{I}(X, Y) = \langle \nabla^M_X Y, \vec{n} \rangle$ defines a section of $\mathrm{Sym}^2 T^* N$, called the *second fundamental form* of N.

Directly from the definition, using the properties of ∇^M, one also gets:

$$\mathbb{I}(X, Y) = -\langle \nabla^M_X \vec{n}, Y \rangle. \tag{3.29}$$

This gives another formula for the second fundamental form: $\mathbb{I} = -\nabla^M \vec{n}$.

In order to obtain a more geometric interpretation, let us consider the application $\phi : \mathbb{R} \times N \to M$, defined by

$$\phi(r,x) = \exp_x(r\vec{n}).$$

(If M is not complete, then ϕ may be defined only on an open subset of $\mathbb{R}\times N$). This means that from each point $x \in N$ we draw the geodesic from x which is orthogonal to N, and we parameterize it by its arc length r. The differential of ϕ at a point $(0,x)$ is $d_{(0,x)}\phi(r,X) = r\vec{n} + X$, so it is an isomorphism $\mathbb{R} \times T_xN \to T_xM$. It follows that ϕ is a diffeomorphism from a neighborhood of $\{0\} \times N \subset \mathbb{R} \times N$ onto a neighborhood of $N \subset M$. There is an analogue of Gauss lemma 3.6.3 in this situation, with a similar proof.

3.8.2 Lemma. — *The geodesics normal to N are orthogonal to the hypersurfaces $\phi(\{r\} \times N)$.*

It follows that on the open set where ϕ is a diffeomorphism, one has

$$\phi^* g = dr^2 + g_r, \qquad g_r \text{ metric on } N. \tag{3.30}$$

The normal vector $\vec{n}$ can then be extended to a neighborhood of N as $\phi_* \frac{\partial}{\partial r}$. If X is a vector field on N, one can extend it to a neighborhood of N as being independent of the $\mathbb{R}$ variable in the product $\mathbb{R} \times N$; equivalently, this is the unique extension so that $[\vec{n}, X] = 0$. Choose two vector fields X, Y on N and extend them in this way: then from (3.29) one deduces

$$\mathbb{I}(X,Y) = -\langle \nabla^M_{\vec{n}} X, Y\rangle + \langle [\vec{n}, X], Y\rangle = -\langle \nabla^M_{\vec{n}} X, Y\rangle$$

and by symmetry we get

$$\mathbb{I}(X,Y) = -\frac{1}{2}\big(\langle \nabla^M_{\vec{n}} X, Y\rangle + \langle \nabla^M_{\vec{n}} Y, X\rangle\big) = -\frac{1}{2}\mathscr{L}_{\vec{n}}\langle X, Y\rangle.$$

This proves the formula:

$$\mathbb{I} = -\frac{1}{2}\frac{\partial g_r}{\partial r}\Big|_{r=0}, \tag{3.31}$$

which gives an interesting way to calculate $\mathbb{I}$, as well as a geometric interpretation : any Riemannian submanifold comes with a natural deformation, generated by the (normal) exponential map, and the second fundamental form is simply the derivative of the deformed metric in the normal direction, up to a factor $-\frac{1}{2}$.

Often we will need to consider $\mathbb{I}$ as a g-symmetric endomorphism of TN instead of a quadratic form: therefore we define the *Weingarten endomorphism*

A by the formula

$$\mathbb{I}(X,Y) = g(A(X),Y), \qquad A(X) = -\nabla^M_X \vec{n}. \tag{3.32}$$

Using the notation of Definition 3.7.2, which relates the sectional curvatures of the submanifold N and the sectional curvature of the ambient manifold M, through the second fundamental form.

3.8.3 Lemma. — *If X and Y are two* unit *and* orthogonal *vectors in T_xN, then*

$$K^M(X\wedge Y) = K^N(X\wedge Y) + \mathbb{I}(X,Y)^2 - \mathbb{I}(X,X)\mathbb{I}(Y,Y); \tag{3.33}$$

$$K^M(X\wedge \vec{n}) = \langle (\mathscr{L}_{\vec{n}}A - A^2)X, X\rangle. \tag{3.34}$$

The Lie derivative $\mathscr{L}_{\vec{n}}A$ is defined as usual by computing the derivative of A along the flow of $\vec{n}$ or equivalently by the Leibniz rule $(\mathscr{L}_{\vec{n}}A)X = \mathscr{L}_{\vec{n}}(AX) - A(\mathscr{L}_{\vec{n}}X)$. The reader may check that $\mathscr{L}_{\vec{n}}A = \nabla_{\vec{n}}A$.

Proof. — To prove the lemma, we first extend X and Y in a neighborhood of x in M, in such a way that X and Y are tangent to N along N and $[X,Y] = [\vec{n},X] = [\vec{n},Y] = 0$ (pick coordinates on N and the normal coordinate r given by ϕ). In this proof, we will write ∇ for ∇^M and R for R^M. Then for any other vector fields Z, T tangent to N along N, we have at x :

$$\begin{aligned}
\langle \nabla_X\nabla_Y Z, T\rangle &= \langle \nabla_X(\nabla^N_Y Z + \mathbb{I}(Y,Z)\vec{n}), T\rangle \\
&= \langle \nabla^N_X\nabla^N_Y Z + \mathbb{I}(Y,Z)\nabla_X\vec{n}, T\rangle \\
&= \langle \nabla^N_X\nabla^N_Y Z, T\rangle - \mathbb{I}(Y,Z)\mathbb{I}(X,T).
\end{aligned}$$

Therefore

$$\langle R_{X,Y}Z,T\rangle = \langle R^N_{X,Y}Z,T\rangle - \mathbb{I}(Y,Z)\mathbb{I}(X,T) + \mathbb{I}(X,Z)\mathbb{I}(Y,T). \tag{3.35}$$

The first formula follows, with $Z = Y$ and $T = X$.

Now let us prove the second formula. Since $\nabla_{\vec{n}}\vec{n} = 0$ ($\vec{n}$ is the velocity vector of the geodesics normal to N) and $\nabla_{\vec{n}}Y = \nabla_Y\vec{n} = -A(Y)$, we have

$$\begin{aligned}
\langle R_{\vec{n},X}Y,\vec{n}\rangle &= \langle (\nabla_{\vec{n}}\nabla_X - \nabla_X\nabla_{\vec{n}})Y,\vec{n}\rangle \\
&= \mathscr{L}_{\vec{n}}\langle \nabla_X Y,\vec{n}\rangle + \langle \nabla_X(A(Y)),\vec{n}\rangle \\
&= \mathscr{L}_{\vec{n}}\langle A(X),Y\rangle + \langle A(X),A(Y)\rangle.
\end{aligned}$$

With $\mathscr{L}_{\vec{n}}X = \mathscr{L}_{\vec{n}}Y = 0$ and $\mathscr{L}_{\vec{n}}g = -2\mathbb{I} = -2\langle A.,.\rangle$ (cf. (3.31)), we have $\mathscr{L}_{\vec{n}}\langle A(X),Y\rangle = -2\langle A^2X,Y\rangle + \langle(\mathscr{L}_{\vec{n}}A)X,Y\rangle$. Since A is g-symmetric, we finally obtain

$$\langle R_{\vec{n},X}Y, \vec{n}\rangle = \langle(\mathscr{L}_{\vec{n}}A - A^2)X, Y\rangle. \tag{3.36}$$

The second formula follows, when $X = Y$. □

3.8.4. Examples. — 1) $g_{\mathbb{R}^{n+1}} = dr^2 + r^2 g_{S^n}$ is flat, the second fundamental form of the sphere of radius r is $\mathbb{I}_{S^n(r)} = -\frac{1}{2}\frac{d}{dr}r^2 g_{S^n} = -r g_{S^n}$ (hence $A(r) = -\frac{1}{r}$). Put $r = 1$ and use the lemma to find the sectional curvature of the unit sphere $S^n : 0 = K^{S^n} - 1$, hence $K^{S^n} = 1$.

2) There is a similar way to compute the curvature of H^n. Unfortunately, it requires to write down the analogue of the lemma in a pseudo-Riemannian setting (namely, by allowing non-positive-definite metrics), which we do not want to do here. Just note (at least heuristically) that $g_{\mathbb{R}^{1,n}} = dr^2 - r^2 g_{H^n}$ is flat so that the computation formally yields $K(-g_{H^n}) = 1$. Since the formula of the sectional curvature changes sign when g is replaced by $-g$, we obtain $K^{H^n} = -1$. Let us give a purely Riemannian proof. We already know, from the structure of isometries, that H^n has constant sectional curvature so we only need to compute the sectional curvature of one plane. Besides, we know that in normal coordinates, $g_{H^n} = dr^2 + (\sinh r)^2 g_{S^{n-1}}$. So S^{n-1} can be seen as a Riemannian submanifold of H^n, defined by $\sinh r = 1$. Its second fundamental form is given by $-\frac{1}{2}\frac{d}{dr}|_{\sinh r=1}(\sinh r)^2 g_{S^{n-1}} = -\sqrt{2} g_{S^{n-1}}$, so the lemma gives $K^{H^n} = K^{S^{n-1}} - (\sqrt{2})^2 = -1$.

3) If we have a surface $S \subset \mathbb{R}^3$, then the two eigenvalues λ_1 and λ_2 of $\mathbb{I}$ are called the *principal curvatures* of S. The first equation gives us the well-known formula for the Gauss curvature:

$$K^S = \lambda_1 \lambda_2.$$

The principal curvatures depend on the embedding $S \subset \mathbb{R}^3$ but the product depends only on the intrinsic geometry of S, which is known as Theorema Egregium, by Gauss. Also

$$H = \lambda_1 + \lambda_2$$

is called the *mean curvature*. Surfaces with $H = 0$ are called *minimal surfaces*: this is the equation satisfied by soap bubbles.

Finally, if the surface S is given by an equation $z = f(x, y)$, then the reader will check the following explicit formulas: the metric on S is given by

$$g_{11} = 1 + (\partial_x f)^2, \quad g_{12} = \partial_x f \partial_y f, \quad g_{22} = 1 + (\partial_y f)^2,$$

the normal vector is

$$\vec{n} = \frac{(-\partial_x f, -\partial_y f, 1)}{\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}},$$

from which one deduces the second fundamental form:

$$\mathbb{I}_{11} = \frac{\partial^2_{xx} f}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}, \quad \mathbb{I}_{12} = \frac{\partial^2_{xy} f}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}, \quad \mathbb{I}_{22} = \frac{\partial^2_{yy} f}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}.$$

It follows that the curvature of S is given by

$$K = \frac{\det(\mathbb{I}_{ij})}{\det(g_{ij})} = \frac{\partial^2_{xx} f \partial^2_{yy} f - (\partial^2_{xy} f)^2}{1 + (\partial_x f)^2 + (\partial_y f)^2}$$

and this formula allows a geometric interpretation. If $K > 0$, the surface is strictly convex while if $K < 0$, it is saddle-shaped.

3.8.5. A geometric interpretation of the curvature. — The sectional curvatures of a Riemannian metric g can be seen as coefficients in the Taylor expansion of the metric in normal coordinates. As a consequence of Gauss Lemma, we know that the pullback metric $\exp_x^* g$ on T_xM (namely, g in normal coordinates) has the following Taylor expansion :

$$\exp_x^* g = dr^2 + g_r = dr^2 + r^2(g_{S^{n-1}} + r^2\gamma + \cdots).$$

Here γ is a section of $\mathrm{Sym}^2 T^*S^{n-1}$ and we introduce the corresponding symmetric endomorphism : $\gamma = g_{S^{n-1}}(G.,.)$. Then we can expand the second fundamental form of the spheres centered in 0 :

$$\mathbb{I} = -\frac{1}{2}\frac{\partial g}{\partial r} \approx -rg_{S^{n-1}} - 2r^3\gamma = -rg_{S^{n-1}}(1 + 2r^2 G).$$

Since $g_r \approx r^2 g_{S^{n-1}}(1 + r^2 G, .)$, we have $g_{S^{n-1}} \approx r^{-2} g_r(1 - r^2 G, .)$, so that $\mathbb{I} \approx -\frac{1}{r} g_r((1 - r^2 G)(1 + 2r^2 G).,.)$ and

$$A \approx -\frac{1}{r} - rG.$$

Now, take two orthogonal unit vectors X and Y in (T_xM, g_x). We can use Lemma 3.8.3 to compute the sectional curvature of the plane generated by X and $Y = \partial_r$ at rY : this is

$$\frac{g_r(\partial_r A - A^2)X, X)}{g_r(X,X)} \approx g_r(-3GX, X) = -3r^2\gamma(X,X).$$

This gives γ in terms of the sectional curvature, hence

$$(\exp_x^* g)_{rY}(X,X) = 1 - \frac{1}{3}K^M(X \wedge Y)r^2 \tag{3.37}$$

Then for any X and Y in T_xM :

$$(\exp_x^* g)_Z(X,Y) = \langle X,Y\rangle + \frac{1}{3}\langle R(X,Z)Y,Z\rangle + o(|Z|^2) \tag{3.38}$$

So the information contained in the Riemann curvature tensor at some point x is exactly the Taylor expansion of g in normal coordinates centered at x, up to order two. The local behaviour of geodesics can then be compared with that of the Euclidean space ($K = 0$), whose geodesics are straight lines. Typically, if you look at two unit speed geodesics making an angle α at x, in a surface with sectional curvature K at x , then after a small time r, their mutual distance along the circle of radius r is about $\epsilon r\sqrt{1-\frac{Kr^2}{3}}$:

- when $K > 0$, the geodesics look like the geodesics of a sphere (the great circles), they get closer to each other ;
- when $K < 0$, the geodesics look like in the hyperbolic space, they get far away from each other.

3.9. Comparison theorems

3.9.1 Lemma. — *If (M^n, g) has constant sectional curvature, then in normal coordinates* $\exp^* g$ *coincides with the metric of* $\mathbb{R}^n$, S^n *or* H^n *(up to a multiplicative constant).*

In other words, it is locally isometric to such a *space form.*

Proof. — We use normal coordinates around a point x: then $g = dr^2 + g_r$, with g_r a metric on S^{n-1}. The second formula of lemma 3.8.3 yields

$$\partial_r A - A^2 = k,$$

where k is the (constant) sectional curvature. When $r \to 0$ we have the asymptotic behavior $A \sim -\frac{1}{r}$. In particular, A is invertible near 0 (but not defined at 0). We consider $B = A^{-1}$ and observe that B extends at 0, with with

$$\partial_r B = -1 - kB^2, \quad B(0) = 0.$$

It is thus easy to find B and therefore A ; g_r is then deduced through (3.31).

- If $k = 0$, then $A = -\frac{1}{r}$ and $g_r = r^2$;
- If $k > 0$, then $A = -\frac{\cot(\sqrt{k}r)}{\sqrt{k}}$ and $g_r = \frac{\sin^2(\sqrt{k}r)}{k}$;
- If $k < 0$, then $A = -\frac{\coth(\sqrt{-k}r)}{\sqrt{-k}}$ and $g_r = \frac{\sinh^2(\sqrt{-k}r)}{\sqrt{-k}}$.

□

It is natural to try and extend this kind of result so as to obtain global results. We basically need to improve our understanding of the exponential map. In section 3.6 we have seen the notion of injectivity radius—the supremum of the $r > 0$ such that $\exp_x$ is a diffeomorphism on the ball of radius r. Here we will use another notion, the *conjugacy radius*, that is the supremum of the $r > 0$ such that $\exp_x$ is a local diffeomorphism on the ball of radius r. This is equivalent to $\exp_x^* g$ being a metric on the ball of radius r, so we can define alternatively the conjugacy radius at x by

$$\rho_{conj}(x) = \inf\{r > 0, \det(\exp_x^* g) \text{ vanishes at some point of } S(r)\}. \tag{3.39}$$

As in the proof of lemma 3.9.1, by Gauss lemma we have $\exp_x^* g = dr^2 + g_r$ and on the ball of radius $\rho_{conj}(x)$, one has, for $|X \models 1$,

$$\langle(\partial_r A - A^2)X, X\rangle = K(\partial_r \wedge X). \tag{3.40}$$

In particular, when $K \leqslant 0$, we have $\partial_r A - A^2 \leqslant 0$ and, like in the proof of Lemma 3.9.1, we deduce $A \leqslant -\frac{1}{r}$ and then $g_r \geqslant r^2 g_{S^{n-1}}$ from (3.31). It follows that $\det(\exp_x^* g)$ can never vanish and we obtain the first part of:

3.9.2 Proposition. — *If $K \leqslant 0$, then $\rho_{conj} = +\infty$. If $K \leqslant k$ with $k > 0$, then $\rho_{conj} \geqslant \frac{\pi}{\sqrt{k}}$.*

Proof. — It remains to deal with $K \leqslant k$: the same argument gives us $g_r \geqslant \frac{\sin^2(\sqrt{k}r)}{k}$ and therefore $\det(\exp_x^* g)$ cannot vanish for $r < \frac{\pi}{\sqrt{k}}$. □

3.9.3 Cartan-Hadamard theorem. — *If (M^n, g) is a complete connected Riemannian manifold with $K \leqslant 0$, then $\exp_x : T_xM \to M$ is a covering. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{R}^n$.*

Proof. — We have just seen that $\exp_x$ is a local diffeomorphism, so that $\exp_x : (T_xM, \exp_x^* g) \to (M, g)$ is a local isometry between complete Riemannian manifolds (for the completeness of T_xM, observe that its exponential at 0 is clearly defined on the whole T_xM ; it consists of straight rays). So it is a covering map from Lemma 3.6.7. □

3.9.4 Theorem. — *A connected and simply connected Riemannian manifold with constant curvature is isometric to $\mathbb{R}^n$, S^n or H^n, up to a constant.*

The universal cover $\tilde{M}$ of any Riemannian manifold (M, g) with constant curvature carries a metric with constant curvature $\tilde{g}$ (the pullback metric). The Theorem says that $(\tilde{M}, \tilde{g})$ is $\mathbb{R}^n$, S^n or H^n up to a constant. (M, g) is therefore obtained as a quotient of one of these models by a discrete group of isometries.

Proof. — In the case of negative or zero curvature, this is just the Cartan-Hadamard theorem, and the fact that we have an explicit formula for a constant curvature metric in normal coordinates.

In the case of positive curvature, we can suppose that $K = 1$. Then by corollary 3.9.2, the map $\exp_x$ is a local diffeomorphism on the ball $B(0,\pi) \subset T_xM$, and since $K = 1$,

$$\exp_x^* g = dr^2 + \sin^2(r) ds^2_{S^{n-1}} \quad \text{on } B(0,\pi).$$

Now if we consider S^n, we know that the exponential from the North pole $\exp_N$ is a diffeomorphism $B(0,\pi) \to S^n \backslash \{-N\}$ ($-N$ is the antipodal point, the South pole if you like) such that $\exp_N^* g_{S^n}$ has exactly the same expression as $\exp_x^* g$ in $B(0,1)$. So we deduce a local isometry $f : S^n\backslash\{-N\} \to M$ from the composition

$$S^n\backslash\{-N\} \xrightarrow{\exp_N^{-1}} (B(0,\pi), dr^2 + \sin^2(r) ds^2_{S^{n-1}}) \xrightarrow{\exp_x} M.$$

Now, if we choose a point p in S^n, outside the North and South poles, we obtain a local diffeomorphism $\tilde{f} : S^n\backslash\{-p\} \longrightarrow M$ by setting

$$\tilde{f} = \exp_{f(p)} \circ d_p f \circ (\exp_p)^{-1}.$$

But from (3.24 at p), we find $\tilde{f} = f$ outside a circle arc between $-N$ and $-p$. By continuity, they therefore coincide outside $-N$ and $-p$. Since $\tilde{f}$ is smooth at $-N$, we can thus smoothly extend f on the whole S^n and therefore obtain a local isometry $f : S^n \to M$, which is therefore a covering map (Lemma 3.6.7). □

To conclude this section, let us explain a comparison theorem about the diameter. The diameter $\mathrm{diam}(M,g)$ of a connected Riemannian manifold (M,g) is defined as the supremum of the distances between points of M. The most striking feature of the following result is its topological consequences : very Riemannian assumptions force the manifold we work with to be compact and have finite fundamental group.

3.9.5 Bonnet-Myers Theorem. — *If (M^n,g) is a complete connected Riemannian manifold, satisfying (Bonnet)*

$$K \geqslant k > 0,$$

or the weaker hypothesis (Myers)

$$\mathrm{Ric} \geqslant (n-1)k > 0,$$

then M is compact and $\pi_1(M)$ is finite. Moreover, the diameter of M satisfies $\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{k}}$.

Here Ric denotes the *Ricci tensor*, defined by

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}(Z \mapsto R_{Z,X}Y). \tag{3.41}$$

It is a section of $\mathrm{Sym}^2 T^*M$. If $|X| = 1$, we can complete X into an orthonormal basis $(X = e_1, \dots, e_n)$, and

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \langle R_{e_i,X}X, e_i \rangle = \sum_{i=2}^{n} K(X \wedge e_i). \tag{3.42}$$

Then it is clear that the first hypothesis of the theorem is stronger than the second one.

Before giving a proof of the theorem, we outline another approach, more in the spirit of what we have just done. At a point x, we have $\exp_x^* g = dr^2 + g_r$. As above, we consider the equation (3.40) and here take its trace

$$\frac{\partial \,\mathrm{Tr}(A)}{\partial r} - \frac{\mathrm{Tr}(A)^2}{n-1} \geqslant \mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \geqslant (n-1)k > 0,$$

which implies $\mathrm{Tr}(A) \geqslant -(n-1)\sqrt{k}\cot(\sqrt{k}r)$ and $\det(g_r) \leqslant (\frac{\sin(\sqrt{k}r)}{\sqrt{k}})^{n-1}$. On each ray from the origin, we see that $\det(g_r)$ must vanish at a radius $r \leqslant R$ with $R = \frac{\pi}{\sqrt{k}}$, that is $d\exp_x$ has a kernel on each ray at most at distance R. One says that x has a *conjugate point* on every geodesic from x at distance at most R. But it is known that a geodesic cannot be minimizing after a conjugate point, and it follows that all points of M are at most at distance R from x. The proof of this last fact requires the theory of Jacobi fields that will not be developed in these notes.

Proof. — The proof relies on the second variation of arc length: if $(c_s(t))$ is a family of paths defined on $[a,b]$, with fixed endpoints, and c_0 is a geodesic, then

$$\frac{d^2 L(c_s)}{ds}|_{s=0} = \int_a^b (|\nabla_{\dot{c}_0} \tilde{N}|^2 - \langle R_{\dot{c}_0,\tilde{N}}\tilde{N}, \dot{c}_0 \rangle) dt \tag{3.43}$$

where $N = \frac{\partial c}{\partial s}$ and $\tilde{N}$ is the projection of N orthogonally to $\dot{c}_0$. The proof of this formula is similar to that of the first variation of arc length (3.7) and is left to the reader.

Pick two points $x, y \in M$. By Hopf-Rinow theorem 3.6.5, there exists a minimizing geodesic c from x to y, of length $L = d(x, y)$. Now choose vectors $E_1, \ldots, E_{n-1}$ along c_0 such that $(\dot{c}, E_1, \ldots, E_{n-1})$ is a parallel basis of orthonormal vectors along c. For $i = 1, \ldots, n-1$ choose

$$N_i = \sin\left(\pi \frac{t}{L}\right) E_i.$$

These vectors vanish at the endpoints of $[0, L]$. Fix i and choose a variation (c_s) of c with fixed endpoints, such that $\frac{\partial c}{\partial s}|_{s=0} = N_i$. Since c is a minimizing geodesic, we have $\frac{d^2 L(c_s)}{ds^2}|_{s=0} \geqslant 0$, and therefore

$$\int_0^L |\nabla_{\dot{c}} N_i|^2 - \langle R_{\dot{c}, N_i} N_i, \dot{c}\rangle \geqslant 0.$$

But $\nabla_{\dot{c}} N_i = \frac{\pi}{L}\cos(\frac{\pi t}{L}) E_i$ so after summation over i we obtain

$$(n-1)\frac{\pi^2}{L^2}\int_0^L \cos(\tfrac{\pi t}{L})^2 \geqslant \int_0^L \sin(\tfrac{\pi t}{L})^2 \sum_1^{n-1} K(\dot{c}, E_i) = \int_0^L \sin(\tfrac{\pi t}{L})^2 \operatorname{Ric}(\dot{c}, \dot{c}).$$

The hypothesis gives $(n-1)\frac{\pi^2}{L^2} \geqslant (n-1)k$ so $L^2 \leqslant \frac{\pi^2}{k}$, hence the bound on the diameter.

Since the diameter is finite and (M, g) is complete, Hopf Rinow theorem implies immediately that M is compact. Observe that one can pull back the metric of M to its universal cover $\tilde{M}$, so that $\tilde{M}$ itself inherits a metric satisfying the assumptions of the theorem : $\tilde{M}$ is therefore compact, which implies that $\pi_1(M)$ is finite. □

Note that formula (3.43) can be rewritten as

$$\frac{d^2 L(c_s)}{ds}|_{s=0} = -\int_a^b \langle \nabla_{\dot{c}}\nabla_{\dot{c}}\tilde{N} + R_{\tilde{N},\dot{c}}\dot{c}, \tilde{N}\rangle dt \tag{3.44}$$

The vector fields J satisfying the second order linear ODE $\nabla_{\dot{c}}\nabla_{\dot{c}} J + R_{J,\dot{c}}\dot{c} = 0$ are the Jacobi fields alluded to above. They conceal a lot of geometric information, cf. the literature.

3.9.6. Exercise. — 1) Let G denote a Lie group endowed with a bi-invariant metric $\langle , \rangle$. Using 3.6.2, prove that for any left-invariant vector fields X, Y, Z, T:

$$\langle R_{X,Y} Z, T\rangle = -\frac{1}{4}\langle [X, Y], [Z, T]\rangle,$$

so that in particular G has non-negative sectional curvature!

2) Prove that $SL_n(\mathbb{R})$ does not carry any bi-invariant metric.

3.9.7 Remark. — To get a feeling about Ricci curvature Ric, it is useful to observe that a consequence of (3.38) is the following Taylor expansion of the volume form in normal coordinates :

$$(\exp_x^* dvol)_Z = \left(1 - \frac{1}{6}\operatorname{Ric}(Z,Z) + o(|Z|^2)\right) dx^1 \wedge \cdots \wedge dx^n.$$

So the Ricci curvature controls the local behaviour of the Riemannian volume. Nonnegative Ricci curvature means that the volume of balls of radius r will be smaller than the volume of a ball of radius r in $\mathbb{R}^n$. The Ricci tensor appears in many interesting contexts. For instance, Riemannian manifolds satisfying $\operatorname{Ric}_g = \lambda g$ for some constant λ are called *Einstein manifolds* (because their Lorentz analogues are the models for empty universes in Einstein's theory of General Relativity) and Besse's book (cf. the bibliography) might convince you that they are really nice objects to look at. The so-called *Ricci flow*, briefly mentioned below, is one of the most exciting current research subject.

3.10. The Hodge Theorem

In this last section, we wish to introduce another facet of Riemannian geometry : geometric analysis. Any Riemannian metric comes with a wealth of natural differential operators. The study of their properties, for instance of their kernel, is bound to depend on the topological and geometrical features of the manifold. So solving the corresponding Partial Differential Equation requires some geometric knowledge. And conversely, if by chance you know something about the PDE, this might help you understand the manifold. It turns out that it is a very powerful way to investigate Riemannian manifolds and more generally differential manifolds, once a Riemannian metric is chosen. A striking exemple is the proof of the Poincaré conjecture given by G. Perelman in 2002. He used PDE and Riemannian techniques to prove that the sphere is the unique compact connected and simply-connected topological manifold of dimension 3 (recall that in dimension 3, differential and topological manifolds coincide). This is a tremendous result, based on the study of the so-called Ricci flow, a parabolic PDE whose unknown is a Riemannian metric $g(t)$ on a fixed differential manifold :

$$\frac{\partial g(t)}{\partial t} = -2\operatorname{Ric}_{g(t)}\,.$$

This is a hard subject so, in this section, we will be more modest. The theorem we wish to describe gives an interpretation of de Rham cohomology in terms of a geometric PDE.

3.10.1. The Hodge Laplacian. — We consider a *compact* oriented[1] manifold M^n, endowed with a Riemannian metric g. Recall that $g = \langle , \rangle$ defines a metric on the bundle $\Lambda^\bullet M$, which we also denote by $\langle , \rangle$. This also determines a scalar product on the vector space $\Omega^\bullet(M) = \Gamma(\Lambda^\bullet M)$: for α and β in $\Omega^\bullet(M)$,

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle dvol_g.$$

In view of the orientation, we may write the scalar product in another way. Given an orthonormal basis $(e_1, \dots, e_n)$ of $T_x M$, the volume form reads $dvol_x = e^1 \wedge \dots e^n$. We define the *Hodge star* $* : \Lambda^k_x M \longrightarrow \Lambda^{n-k}_x M$ by

$$*(e^{i_1} \wedge \dots e^{i_k}) = e^{j_1} \wedge \dots e^{j_{n-k}} \quad \text{if} \quad e^{i_1} \wedge \dots e^{i_k} \wedge e^{j_1} \wedge \dots e^{j_{n-k}} = dvol.$$

By definition, $*1 = dvol$. It is characterized by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dvol,$$

so that the scalar product on $\Omega^\bullet(M)$ reads

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \alpha \wedge *\beta. \tag{3.45}$$

It is useful to note that $** = (-1)^{k(n-k)}$ on forms of degree k.

We wish to introduce natural differential operators. The most basic is $d : \Omega^k M \longrightarrow \Omega^{k+1} M$, depending only on the differential structure. The Riemannian structure makes it possible to consider a (formal) adjoint for this operator, the *codifferential* $d^* : \Omega^{k+1} M \longrightarrow \Omega^k M$. It is given on $(k+1)$-forms by

$$d^* = (-1)^{nk+1} * d *$$

and satisfies

$$\langle\langle d\alpha, \beta \rangle\rangle = \langle\langle \alpha, d^*\beta \rangle\rangle. \tag{3.46}$$

To prove this, just use (3.45), Leibniz rule and Stokes theorem. In particular, since $d^2 = 0$, we have

$$(d^*)^2 = 0.$$

Observe that $d^* : \Omega^1 M \longrightarrow \Omega^0 M$ determines by duality a map div that associates to any vector field X a function $\operatorname{div} X$, called the *divergence* of X. Exercise : $\mathscr{L}_X dvol = \operatorname{div}(X) dvol$, which means $\operatorname{div} X$ measures the variation of volume along the flow of X. Similarly, d induces a map grad from functions to vectors fields, the (Riemannian) *gradient*.

[1]The orientability assumption is not necessary for the Hodge theorem to hold, but it makes life a bit more simple.

We then introduce the *Hodge Laplacian* :

$$\Delta = (d + d^*)^2 = dd^* + d^*d.$$

It maps a form of degree k to a form of degree k and satisfies

$$\langle\langle \Delta\alpha, \beta\rangle\rangle = \langle\langle d\alpha, d\beta\rangle\rangle + \langle\langle d^*\alpha, d\beta\rangle\rangle. \tag{3.47}$$

In particular, it is (formally) self-adjoint. In degree 0, i.e. on functions, we have $\Delta = d^*d = \operatorname{div}\operatorname{grad}$. Beware this formula means that for $g = g_{\mathbb{R}^n}$, $\Delta f = -\sum_i \partial_{x^ix^i} f$ (and $\operatorname{div} X = -\sum_i \partial_{x^i} X^i$). Our sign convention is standard in geometric papers and books, while PDE specialists tend to prefer the opposite sign.

A differential form of degree k is called *harmonic* if it is in the kernel $\mathscr{H}^k$ of the Hodge Laplacian. With (3.47), we see that

$$\mathscr{H}^k = \{\alpha \in \Omega^k(M) \ / \ d\alpha = 0 \text{ and } d^*\alpha = 0\}.$$

Using (3.46) together with $d^2 = (d^*)^2 = 0$, we then obtain :

$$\Omega^k(M) \supset d\Omega^{k-1}(M) \overset{\perp}{\oplus} d^*\Omega^{k+1}(M) \overset{\perp}{\oplus} \mathscr{H}^k.$$

3.10.2 Hodge decomposition theorem. — *If (M, g) is a compact oriented Riemannian manifold, then*

$$\Omega^k(M) = d\Omega^{k-1}(M) \overset{\perp}{\oplus} d^*\Omega^{k+1}(M) \overset{\perp}{\oplus} \mathscr{H}^k.$$

In this decomposition, the space $Z^k(M)$ of closed forms of degree k is exactly $d\Omega^{k-1}(M) \overset{\perp}{\oplus} \mathscr{H}^k$ (just play with 3.46 to see it), so the quotient of $Z^k(M)$ by the subspace of exact forms identifies with the space of harmonic forms.

3.10.3 Corollary. — *Let M be a compact oriented manifold. For any Riemannian metric g on M,*

$$H^\bullet_{DR}(M) \cong \mathscr{H}^\bullet(M, g).$$

The isomorphism is given by orthogonal projection : in each de Rham class, there is a unique harmonic form and it minimizes the L^2-norm.

This isomorphism is a miracle ! The Hodge Laplacian strongly depends on g and it is not at all obvious that its kernel only depends on the topology of M^n (recall de Rham cohomology is a topological invariant). Now, you can play with this isomorphism. For instance, the proof of the Hodge theorem will ensure the Hodge Laplacian has a finite dimensional kernel, hence the finiteness of Betti numbers on a compact manifold. Observe also that the

Hodge star clearly commutes with the Hodge Laplacian, so that it induces an isomorphism $\mathscr{H}^{n-k} \cong \mathscr{H}^k$. Hence

3.10.4 Poincaré duality. — *For any compact oriented manifold M^n,*

$$H^k_{DR}(M) \cong H^{n-k}_{DR}(M).$$

For instance, the case $k = 0$ means $H^n_{DR}(M) \cong \mathbb{R}$, with an isomorphism given by integration. We refer to the exercises below for more applications.

Now, let us explain how to prove Hodge theorem. Basically, all we need to do is to prove that for any $\beta \in \Omega^k(M)$, if β is orthogonal to $\mathscr{H}^k$, then there is a form $\alpha \in \Omega^k(M)$ such that $\Delta\alpha = \beta$. This is based on quite standard functional analysis, which you can look up in the bibliography (cf. Brezis, Taylor).

We need Sobolev spaces $H^p = H^p(\Lambda^k(M))$, $p \in \mathbb{Z}$, on the compact Riemannian manifold (M, g). To define them, we choose a finite atlas, with a subordinate partition of unity (χ_i) , and we define the H^p-norm of a smooth form α to be the sum of the H^p norms of $\chi_i\alpha$ (that is the sum of the L^2 norms of the derivatives up to order p, computed in $\mathbb{R}^n$ thanks to the ith chart). The Hilbert space H^p is then defined as the completion of $\Omega^k(M)$ for this norm ; in fact, it does not depend on the choice of atlas. We identify H^0 with its dual and set $H^{-p} = (H^p)^*$. Then formula (3.47) implies that Δ determines a continuous linear operator $H^1 \to H^{-1}$.

3.10.5 Lemma. — *There are positive constants A and B such that for any $\alpha \in \Omega^k(M)$,*

$$\langle\langle \Delta\alpha, \alpha \rangle\rangle \geqslant A \left\|\alpha\right\|^2_{H^1} - B \left\|\alpha\right\|^2_{H^0} .$$

Proof. — It is sufficient to prove this locally so we work in local coordinates. The reader may check that the Laplacian reads $\Delta = g^{ij}\partial_{x^ix^j} + E_i\partial_{x^i} + F$, where (g^{ij}) is the inverse of the matrix of g and is therefore bounded from below. It then follows from integration by parts that there are positive constants A', B', C' such that

$$\langle\langle \Delta\alpha, \alpha \rangle\rangle \geqslant A' \left\|\alpha\right\|^2_{H^1} - B' \left\|\alpha\right\|^2_{H^0} - C' \left\|\alpha\right\|_{H^1} \left\|\alpha\right\|_{H^0} .$$

Then write $\left\|\alpha\right\|_{H^1} \left\|\alpha\right\|_{H^0} \leqslant \epsilon \left\|\alpha\right\|^2_{H^1} + \frac{1}{4\epsilon} \left\|\alpha\right\|^2_{H^0}$ and pick small enough ϵ, so that $A' - C'\epsilon > 0$. □

3.10.6 Lemma. — *There exists a Hilbert basis $(\psi_i)_{i\in\mathbb{N}}$ of H^0 consisting of elements ψ_i of H^1 such that*

$$\Delta\psi_i = \lambda_i\psi_i,$$

for some non-decreasing sequence of numbers $\lambda_i \geqslant 0$ *going to* $+\infty$.

The sequence of eigenvalues $(\lambda_i)_i$ (in degree $k = 0$) is called the *spectrum* of the compact Riemannian manifold (M, g). Note that the eigenvectors ψ_i are in fact smooth.

Proof. — As a consequence of Lemma 3.10.5, for any α in H^1,

$$\|(\Delta + B)\alpha\|_{H^{-1}} \geqslant A \|\alpha\|^2_{H^1}$$

so that $\Delta + B : H^1 \longrightarrow H^{-1}$ is injective and has closed range. Its range is moreover dense for its orthogonal (in $(H^{-1})^* = H^1$) coincides with the kernel of $\Delta+B$, which is $\{0\}$. In conclusion, $\Delta+B : H^1 \longrightarrow H^{-1}$ is an isomorphism. Then we look at the composition

$$P : H^0 \hookrightarrow H^{-1} \stackrel{(\Delta+B)^{-1}}{\longrightarrow} H^1 \hookrightarrow H^0.$$

Since M is compact, the inclusion $H^1 \hookrightarrow H^0$ is a compact operator (this is Rellich theorem, the Sobolev analogue of Ascoli theorem). P is thus compact, self-adjoint and positive. The spectral theorem for such operators ensures the existence of a Hilbert basis $(\psi_i)_{i\in\mathbb{N}}$ of H^0 such that $P\psi_i = \mu_i\psi_i$, for some non-increasing sequence of positive numbers μ_i going to 0. By construction of P, ψ_i lies in H^1 and $\Delta\psi_i = \lambda_i\psi_i$ with $\lambda_i = \frac{1}{\mu_i} - B$. □

In particular, this lemma says that the kernel of the Hodge Laplacian on k-forms has finite dimension, denoted by D. A k-form β is then orthogonal to the kernel of Δ iff we can decompose β into

$$\beta = \sum_{i=D}^{\infty} \beta_i \phi_i.$$

For $i \geqslant D$, we have $\lambda_i \geqslant \lambda_D > 0$, so we can set

$$\alpha := \sum_{i=D}^{\infty} \frac{\beta_i}{\lambda_i} \phi_i.$$

Then $\Delta\alpha = \beta$. Our construction does not a priori ensure that α is smooth. But the following inequality can be proved as in Lemma 3.10.5 : for any p, there are constants A_p and B_p such that

$$\|\alpha\|_{H^{p+2}} \leqslant A_p \|\alpha\|_{H^0} + B_p \|\Delta\alpha\|_{H^p}.$$

Since $\Delta\alpha = \beta$ is smooth and α is in H^0 by construction, α is automatically in H^p for every p, which implies that it is smooth (as in $\mathbb{R}^n$). This completes the proof of Hodge decomposition theorem.

Note the compactness assumption on M is crucial in the proof, because of Rellich theorem. Hodge theorem is generally false on non-compact manifolds.

3.10.7. Exercises. — 1) Compute the Hodge Laplacian on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ in the standard trivialisation of $\Lambda^\cdot \mathbb{T}^n$ induced by the coordinates of $\mathbb{R}^n$. Deduce the cohomology of the torus.

2) Prove that if N is a finite cover of a compact oriented manifold M, then $b_k(N) \geqslant b_k(M)$ for very degree k.

3) Prove that orientation-preserving isometries map harmonic forms to harmonic forms. Deduce that a compact non-orientable manifold M^n has $H^n(M) = 0$.

4) Let G be a connected Lie group acting by isometries on a compact oriented Riemnnian manifold M. Prove that for every element γ of G and every harmonic form α on M, $\gamma^*\alpha = \alpha$. Hint : use question 1 in exercise 1.9.3. Prove that $SO(n)$ does not fix any non-trivial exterior k-form on $\mathbb{R}^n$ for $0 < k < n$ and deduce the cohomology of S^n.

5) Consider the Riemannian product of two compact oriented Riemannian manifolds M and N. Use the Hilbert bases given by Lemma 3.10.6 on M and N to produce a Hilbert basis of the space of L^2 forms on $M \times N$ consisting of eigenvectors of the Laplacian. Deduce the Kunneth formula for the cohomology of the product of two (compact oriented) manifolds.

3.11. One last exercise.

To sum up, we wish to study a nice family of Riemannian metrics on the sphere S^3. Let us identify $\mathbb{R}^4$, with coordinates x_1, y_1, x_2, y_2, and $\mathbb{C}^2$, with complex coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. The following formulas yield three vector fields on $\mathbb{R}^4 = \mathbb{C}^2$:

$$\begin{aligned}
X_1 &= \begin{pmatrix} iz_1 \\ iz_2 \end{pmatrix} = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2},\\
X_2 &= \begin{pmatrix} -\bar{z_2} \\ \bar{z_1} \end{pmatrix} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2},\\
X_3 &= \begin{pmatrix} -i\bar{z_2} \\ i\bar{z_1} \end{pmatrix} = x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial y_1}.
\end{aligned}$$

1. Prove that the restrictions of these vector fields on S^3 are tangent to S^3 and provide a global trivialization of TS^3.

2. Prove that the Lie brackets obey $[X_1, X_2] = -2X_3$, $[X_2, X_3] = -2X_1$ et $[X_3, X_1] = -2X_2$. For every point $x \in S^3$, we set $H_x := \mathrm{Vect}(X_2, X_3)$. Prove that H is not integrable and is therefore a contact distribution on S^3.

3. Prove that S^3 is diffeomorphic to the Lie group $SU(2)$ in such a way that X_1, X_2 and X_3 are left-invariant. Compute their Lie brackets again.

Let $(\alpha_1, \alpha_2, \alpha_3)$ denote the dual coframe of (X_1, X_2, X_3) on each tangent space of S^3 and let λ be a positive number. We consider the Riemannian metric $g_\lambda = \lambda^2\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ on S^3. The Riemannian manifolds (S^3, g_λ) are known as 'Berger spheres', after the great French geometer Marcel Berger. Note that g_1 is the standard round metric on S^3.

4. In view of 2., prove that the metrics g_λ are left-invariant. Are they bi-invariant?

5. Prove that the volume of (S^3, g_λ) goes to zero as λ goes to zero.

6. What is the flow ϕ_t of X_1? Prove that it is isometric with respect to every metric g_λ.

Let π denote the standard projection $\S^3 \to \mathbb{C}P^1$. The Fubini-Study metric $g_{\mathbb{C}P^1}$ on $\mathbb{C}P^1$ is the (quotient metric) induced by g_1.

7. Prove that all metrics g_λ induce the same quotient metric, $g_{\mathbb{C}P^1}$.

8. Given $p \in \mathbb{C}P^1$, compute the length of the circle $\pi^{-1}(\{p\})$ with respect to g_λ.

Let ∇^λ denote the Levi-Civita connection of g_λ.

9. Let $i, j, k \in \{1, 2, 3\}$. Explain why $g_\lambda\left(\nabla^\lambda_{X_i} X_j, X_k\right) = -g_\lambda\left(\nabla^\lambda_{X_i} X_k, X_j\right)$ and why $g_\lambda\left(\nabla^\lambda_{X_i} X_j, X_k\right) = 0$ as soon as two of the indices i, j, k coincide. Using these remarks, compute $\nabla^\lambda X_1$, $\nabla^\lambda X_2$ and $\nabla^\lambda X_3$.

10. Deduce that any integral line of X_1, X_2 or X_3 is a geodesic. What can you say about the injectivity radius of (S^3, g_λ) as λ goes to zero?

11. Compute, at every point of S^3, the sectional curvature of the planes $\mathrm{Vect}(X_1, X_2)$ and $\mathrm{Vect}(X_2, X_3)$ with respect to g_λ. Deduce all the sectional curvatures of (S^3, g_λ).

M. Gromov introduced a very useful metric structure on the set of all Riemannian manifolds and the family of Berger spheres illustrate an interesting phenomenon within this theory. Let us give a general definition. Let (X, d_X) and (Y, d_Y) denote two compact metric spaces. Given a positive number ϵ, we say that $f : X \to Y$ is an ϵ-approximation if the following properties are satisfied:

– for every $y \in Y$, there is an $x \in X$ such that $d_Y(f(x), y) \leqslant \epsilon$;

– for every $x, x' \in X$, $|d_Y(f(x), f(x')) - d_X(x, x')| \leqslant \epsilon$.

Note we do not require f to be continuous. The Gromov-Hausdorff distance $d_{GH}(X, Y)$ between X and Y is defined as the infimum of all $\epsilon > 0$ such that there exists an ϵ-approximation $f_1 : X \to Y$ and an ϵ-approximation $f_2 : Y \to X$. This yields in particular a distance between compact connected Riemannian manifolds (for the Riemannian distance). Beware that two Riemannian manifolds can be Gromov-Hausdorff close, while being very different as manifolds, for instance, they can obviously have different dimensions.

12. Let (M, g) be a compact connected Riemannian manifold with diameter D and let P denote the one-point metric space. Prove that $d_{GH}\left((M, g), P\right)$ is at most D.

By scaling, it is therefore easy to obtain sequences of Riemannian manifolds that converge in Gromov-Hausdorff sense to a point. But the curvature blows up in this process.

13. Pick a compact Riemannian manifold (M^n, g) and an integer $m \geqslant 0$. Construct a sequence of metrics g_i on $M \times \mathbb{T}^m$ with uniformly bounded curvature and which converges to (M, g) in Gromov-Hausdorff topology.

A sequence of Riemannian manifolds $(M^n, g_i)_i$ with uniformly bounded curvature which converges to a lower dimensional Riemannian manifold is said to *collapse*. The previous question provides an example, but it might seem a bit artificial. Now let us estimate the Gromov-Hausdorff distance between a Berger sphere and $\mathbb{C}P^1$.

14. Let $x \in S^3$ and let $\gamma : [0, L] \to \mathbb{C}P^1$ be a smooth path with $\gamma(0) = \pi(x)$. Prove that there is a unique path $\widehat{\gamma} : [0, L] \to S^3$ such that $\widehat{\gamma}(0) = x$, $\pi \circ \widehat{\gamma} = \gamma$ and, for all $t \in [0, L]$, $\frac{d\widehat{\gamma}(t)}{dt} \in H_{\widehat{\gamma}(t)}$ (the distribution H is defined in 2.). What is the length of $\widehat{\gamma}$ with respect to g_λ?

15. Prove that $d_{GH}\left((S^3, g_\lambda), (\mathbb{C}P^1, g_{\mathbb{C}P^1})\right)$ goes to zero as λ goes to zero.

In other words, with 11., this means that the spheres (S^3, g_λ) collapse onto $\mathbb{C}P^1$, which is also the round sphere S^2 of radius $1/2$.

3.12. Bibliography

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