

# A mass for ALF manifolds.

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**Abstract:** We prove a positive mass theorem on ALF manifolds, i.e. complete noncompact manifolds that are asymptotic to a circle fibration over a Euclidean base, with fibers of asymptotically constant length.

## Introduction

In general relativity [ADM], one can define the mass of an asymptotically Euclidean spatial slice  $(M^n, g)$  in a spacetime by the formula

$$\mu_g = \frac{1}{\omega_n} \lim_{R \rightarrow \infty} \int_{\mathbb{S}^{n-1}(R)} [\partial_j g_{ij} - \partial_j g_{ii}] \nu_{\partial_i} dx. \quad (1)$$

where  $\mathbb{S}^{n-1}(R)$  is the standard sphere with radius  $R$  in  $\mathbb{R}^n$  and  $\omega_n$  is the volume of  $\mathbb{S}^{n-1}(1)$ . The terminology “asymptotically Euclidean” means  $M$  minus a compact subset is diffeomorphic to  $\mathbb{R}^n$  minus a ball and the metric  $g$  is asymptotic to the standard metric on  $\mathbb{R}^n$ . The positive mass conjecture asserts this mass, if defined, is nonnegative when the scalar curvature  $\text{Scal}_g$  is nonnegative, vanishing only when  $(M^n, g)$  is isometric to  $\mathbb{R}^n$ . It is a theorem in dimension  $3 \leq n \leq 8$  ([SY1, SY2]) and on spin manifolds of any dimension ([Wit]). J. Lohkamp [Loh] recently announced a proof of the positive mass theorem in all dimensions, without spin assumption. Moreover, R. Bartnik [Bart] proved the mass is a genuine Riemannian invariant (under sharp assumptions). The interested reader should certainly have a look at [LP, He1]. The mathematical interest of such a theorem is the rigidity result it involves: under a nonnegative curvature assumption, it asserts a model metric at infinity (the Euclidean metric, here) cannot be approached at any rate, the obstruction being precisely the mass.

Positive mass theorems were proved in other settings. Asymptotically (real) hyperbolic manifolds were studied from this point of view in [MO, AD, CH], while

asymptotically complex hyperbolic manifolds are treated in [He2, BH]. Motivated by theoretical physics considerations, Xianzhe Dai [Dai] pointed out a positive mass theorem for spin manifolds that are asymptotic to the product of a Euclidean space with a compact simply connected Calabi-Yau manifold.

In this paper, we are interested in manifolds that are asymptotic to a circle fibration over a Euclidean base, with fibers of asymptotically constant length. For instance, there are complete Ricci flat metrics on  $\mathbb{R}^2 \times \mathbb{S}^{n-2}$  that are asymptotic to the standard metric on  $\mathbb{R}^{n-1} \times \mathbb{S}^1$ ; in dimension four, an example is provided by the Euclidean Schwarzschild metric

$$g = \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\omega^2 + \left(1 - \frac{2\mu}{r}\right) dt^2$$

where  $(r, \omega, t) \in \mathbb{R}_+^* \times \mathbb{S}^{n-1} \times \mathbb{S}^1$ . Another kind of example, adapted to the Hopf fibration at infinity, is the hyperkähler Taub-NUT metric on  $\mathbb{R}^4$ :

$$g = \left(1 + \frac{2\mu}{r}\right) (dr^2 + r^2 d\omega^2) + \frac{1}{1 + \frac{2\mu}{r}} \eta^2,$$

where  $\eta$  is the standard contact form on the three-spheres (lots of details will be given about these formulas in the text). In these expressions, the parameter  $\mu$  is bound to be nonnegative (so as to get a complete metric) and it is interpreted as a mass by physicists. It is tempting to ask for a positive mass theorem involving this  $\mu$  in these examples.

Some motivation comes from the study of gravitational instantons, i.e. hyperkähler four-manifolds with decaying curvature at infinity (an example is provided by the Taub-NUT metric). These manifolds arise in Euclidean quantum gravity and string theory, but also in gauge theory, so that it is of interest to try and understand their geometry. The main result of [Mi2] asserts that gravitational instantons with cubic volume growth are always asymptotic to circle fibrations over a Euclidean base (up to a finite group action) with an approximation rate of order  $r^{-1}$ , where  $r$  is the distance to some point. More generally, the result applies to a class of Ricci flat metrics, including Schwarzschild examples for instance. With this in mind, it seems interesting to wonder if there are non trivial examples of Ricci flat metrics  $g$  that are closer to their model at infinity, namely for instance:  $g = g_{\mathbb{R}^3 \times \mathbb{S}^1} + \mathcal{O}(r^{-1-\epsilon})$ , with  $\epsilon > 0$ ? The corresponding question with a nontrivial circle fibration at infinity can also be addressed. This paper will prove in particular that the answer is no, because of a positive mass theorem.

To introduce the class of metrics we will work with, we first consider the total space  $\mathcal{X}^{m+1}$  of a principal  $\mathbb{S}^1$ -bundle  $\pi$  over  $\mathbb{R}^m \setminus \mathbb{B}^m$  (the exterior of the unit ball in  $\mathbb{R}^m$ ,  $m \geq 3$ ). Given a positive number  $L$ , we introduce the vector field  $T$  that is equal to  $\frac{L}{2\pi}$  times the infinitesimal generator of the  $\mathbb{S}^1$  action and consider a “model” metric  $h$  on  $\mathcal{X}$ , given by

$$h = \pi^* g_{\mathbb{R}^m} + \eta^2,$$

where  $\eta$  is a connection one-form, namely a  $\mathbb{S}^1$  invariant one-form on  $\mathcal{X}$  such that  $\eta(T) = 1$ . Observe the fibers have length  $L$ . For such a one-form  $\eta$ , one may write  $d\eta = \pi^* \omega$  for some two-form  $\omega$  on the base  $\mathcal{B}$  and we will assume this

“curvature” two-form  $\omega$  decays at infinity, in that there is a positive number  $\tau$  such that

$$D^i \omega = \mathcal{O}(r^{-\tau-1-i}), \quad 0 \leq i \leq 2. \quad (2)$$

We will consider Riemannian manifolds  $(M^{m+1}, g)$  such that  $M$  minus a compact set is diffeomorphic to such an  $\mathcal{X}$  and such that  $g$  is asymptotic to  $h$  in the following sense :

$$\nabla^{h,i}(g-h) = \mathcal{O}(r^{-\tau-i}), \quad 0 \leq i \leq 3. \quad (3)$$

Such metrics  $g$  are known as “ALF metrics”.

Our main result is a positive mass theorem, valid for any circle fibration at infinity under a nonnegativity assumption on the Ricci curvature. We will denote by  $B_R$  the preimage by  $\pi$  of the ball of radius  $R$  on the base.

**Theorem 1.** *Let  $(M^{m+1}, g)$ ,  $m \geq 3$ , be a complete oriented manifold with nonnegative Ricci curvature. We assume that, for some compact subset  $K$ ,  $M \setminus K$  is the total space of a circle fibration over  $\mathbb{R}^m \setminus \mathbb{B}^m$ , which can be endowed with a model metric  $h$  such that (2) and (3) hold with  $\tau > \frac{m-2}{2}$ . Then the quantity defined by*

$$\mu_g^{GB} = -\frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *h \left( \operatorname{div}_h g + d \operatorname{Tr}_h g - \frac{1}{2} d g(T, T) \right).$$

*is a nonnegative Riemannian invariant and vanishes exactly when  $(M, g)$  is the standard  $\mathbb{R}^m \times \mathbb{S}^1$ .*

We stress the fact that the assumption  $\operatorname{Ric} \geq 0$  is too strong for the asymptotically Euclidean setting (for Bishop-Gromov theorem immediately implies the manifold is isometric to  $\mathbb{R}^n$ ), while it allows many interesting examples when the volume growth is slower. The assumption  $\tau > \frac{m-2}{2}$  is classical (cf. [Bart] for instance). Considering products of asymptotically Euclidean manifolds with  $\mathbb{S}^1$ , one can rely on [DS] to see that this hypothesis is optimal (see also the remark before theorem 4.3 in [Bart]). The assumption on the third derivative of  $g$  is only used to ensure the mass is a Riemannian invariant, namely does not depend on  $h$ .

A cheap adaptation of our argument yields another positive mass theorem, valid for asymptotic trivial fibrations, with a spin assumption and a nonnegativity assumption on the scalar curvature.

**Theorem 2.** *Let  $(M^{m+1}, g)$ ,  $m \geq 3$ , be a complete spin manifold with nonnegative scalar curvature. We assume there is a compact set  $K$  and a spin preserving diffeomorphism between  $M \setminus K$  and  $\mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$  such that for some  $\tau > \frac{m-2}{2}$  :*

$$D^i(g - g_{\mathbb{R}^m \times \mathbb{S}^1}) = \mathcal{O}(r^{-\tau-i}), \quad 0 \leq i \leq 3.$$

*Then the quantity defined by*

$$\mu_g^D = -\frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *_{h_0} \left( \operatorname{div}_{h_0} g + d \operatorname{Tr}_{h_0} g \right).$$

*is a nonnegative Riemannian invariant and vanishes exactly when  $(M, g)$  is the standard  $\mathbb{R}^m \times \mathbb{S}^1$ .*

There is a definite link with Dai's work [Dai]. In his paper, he points out he cannot include Schwarzschild-like metrics in the discussion because the spin structure on the circles at infinity is not trivial. Indeed, physicists even built examples of complete spin manifolds asymptotic to  $\mathbb{R}^3 \times \mathbb{S}^1$ , with nonnegative scalar curvature and with a *negative* mass ([BrH,CJ])! Here, we can “justify” the positivity of the mass of genuine Schwarzschild metrics, relying on the fact that their Ricci tensor vanishes (see section 4).

Moreover, we can cope with nontrivial fibrations at infinity, which is interesting in view of the Taub-NUT example. In this case, the “model” at infinity cannot be realized by a complete metric, so that the mass cannot vanish. It seems that it is the first positive theorem with such a feature.

In these positive mass theorems, the base of the circle fibration at infinity is a Euclidean space. One might ask for a generalization to the case of Riemannian manifolds asymptotic to a circle fibration over a finite quotient of  $\mathbb{R}^m$  (this occurs for the so-called  $D_k$  gravitational instantons). It fails. For instance, M. Atiyah and N. Hitchin [AH,Hit] introduced a famous hyperkähler (hence Ricci-flat) metric on the moduli space of two monopoles. Its asymptotic shape is that of  $(\mathbb{R}^3 \times \mathbb{S}^1)/\mathbb{Z}_2$  endowed with a metric of Taub-NUT form, except that the mass parameter  $\mu$  is bound to be negative ! This problem with finite quotients was already noticed for Asymptotically Locally Euclidean manifolds [Dah,Le1,Nak].

The paper is organized as follows. In a first section, we introduce the class of metrics we are interested in. In a second section, we describe the analytical tools required for our arguments. We could have relied on Mazzeo-Melrose machinery; yet, we have chosen to include elementary proofs for everything we need. An advantage is we never use complete Taylor expansions at infinity; the theory is thus simpler, closer to what is usually done in the asymptotically Euclidean case and it does not require any familiarity with pseudo-differential calculus. We emphasize nonetheless that the paper [HHM] greatly inspired us, as well as the text of some lectures given by Frank Pacard [Pac]. In a third section, we prove the positive mass theorems. In the last section, we discuss examples and counter-examples.

We will often use an Einstein summation convention without mentioning it: when an index is repeated, the expression should be summed over this index (with a range depending on the context).

## 1. Metrics adapted to a circle fibration at infinity

Let us work on the total space  $\mathcal{X}^{m+1}$  of a principal circle fibration  $\pi$  over  $\mathcal{B}^m := \mathbb{R}^m \setminus \mathbb{B}^m$ ,  $m \geq 3$ . From a topological point of view, there are two families of such fibrations. In any dimension, we can consider the trivial circle fibration  $\pi_0$  over  $\mathcal{B} = \mathbb{R}^m \setminus \mathbb{B}^m$ . The total space is  $\mathcal{X} = \mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$ . In dimension four (i.e.  $m = 3$ ), the Hopf fibration  $\mathbb{S}^3 \rightarrow \mathbb{S}^2$  induces non trivial fibrations  $\pi_k$ ,  $k \in \mathbb{N}^*$ , where the total space  $\mathcal{X}$  is the quotient of  $\mathbb{R}^4 \setminus \mathbb{B}^4$  by  $\mathbb{Z}_k$ ; the action of  $\mathbb{Z}_k$  is the complex scalar action of the  $k$ th unit root group on  $\mathbb{C}^2 = \mathbb{R}^4$ .

We wish to define a class of “model” metrics  $h$  on  $\mathcal{X}$ . Their first feature is that the fibers of  $\pi$  will have *constant* length  $L$  ( $0 < L < +\infty$ ). It is therefore natural to introduce the vertical vector field  $T$  that is equal to  $\frac{L}{2\pi}$  times the infinitesimal generator of the  $\mathbb{S}^1$ -action; the flow of  $T$  goes around the fibers in time  $L$ . As a second feature,  $h$  should pullback the Euclidean metric on  $\mathbb{R}^m$

as much as possible, i.e. on the orthogonal of the fibers. The way to do this is to pick a ‘‘connection’’ one-form  $\eta$ , namely a  $\mathbb{S}^1$ -invariant one-form  $\eta$  such that  $\eta(T) = 1$ . We define the corresponding model metric on  $\mathcal{X}$  by

$$h := \pi^* g_{\mathbb{R}^m} + \eta^2 = dx^2 + \eta^2,$$

where  $x_i := \pi^* \check{x}_i$ ,  $1 \leq i \leq m$ , denote the lifts of the canonical coordinates  $\check{x}_i$  on the base  $\mathcal{B} \subset \mathbb{R}^m$ . In addition, we will require some decay for the curvature  $d\eta$  of this connection. To express it, let us define  $r := \sqrt{x_1^2 + \dots + x_m^2}$  and observe  $d\eta = \pi^* \omega$  for some two-form  $\omega$  on the base  $\mathcal{B}$  ( $d\eta$  is  $\mathbb{S}^1$ -invariant and Cartan’s magic formula  $L_T \eta = \iota_T d\eta + d(\iota_T \eta)$  implies  $\iota_T d\eta = 0$ ). We will assume

$$\omega = \mathcal{O}(r^{-\tau-1}) \quad \text{and} \quad D\omega = \mathcal{O}(r^{-\tau-2})$$

for some positive number  $\tau$  ( $D$  is the flat connection on  $\mathbb{R}^m$ ).

*Example 1 (Trivial fibration).* In the trivial fibration case, we can choose a trivialization  $(x_1, \dots, x_m, e^{it})$  and put  $\eta = \eta_0 := dt$ . Since  $d\eta_0 = 0$ , any  $\tau$  is convenient. Observe  $\eta_0$  defines *foliations* over  $r^{-1}(R)$  for any  $R$ , corresponding to the product foliation of  $\mathbb{R}^m \times \mathbb{S}^1$  by circles. The model metric is the flat  $h_0 := dx^2 + dt^2$ .

*Example 2 (Hopf fibration).* In the Hopf fibration case, we define a connection form  $\eta$  on  $\mathbb{R}^4 \setminus \{0\} = \mathbb{C}^2 \setminus \{0\}$  by the formula  $\eta(z) := |z|^{-2} g_{\mathbb{R}^4}(T, \cdot)$ , where  $T$  is the infinitesimal generator of the scalar action of  $\mathbb{S}^1 \subset \mathbb{C}^*$ . Then  $\tau = 1$  is convenient. Note  $\eta$  is nothing but the standard contact form on  $\mathbb{S}^3$  and  $h := dx^2 + \eta^2$  is the model at infinity for the Riemannian Taub-NUT metric (see example 4.3 at the end of the paper).

Let us describe a few features of such metrics. The coframe  $(dx_1, \dots, dx_m, \eta)$  is obviously orthonormal. We let  $(X_1, \dots, X_m, T)$  be the dual frame. Uniqueness of the Levi-Civita connection ensures that, for horizontal vector fields  $X, Y$  (namely  $\eta(X) = \eta(Y) = 0$ ),  $\pi_* \nabla_X^h Y = D_{\pi_* X} \pi_* Y$ . The computation of brackets of horizontal vector fields, together with  $T$ , contains the geometric information about the fibration. First, since  $\eta([X_i, T]) = -d\eta(X_i, T) = 0$  and  $\pi_*[X_i, T] = [\pi_* X_i, \pi_* T] = 0$ ,  $T$  commutes with any vector field  $X_i$ . Besides, with  $\eta([X_i, X_j]) = -d\eta(X_i, X_j)$  and  $\pi_*[X_i, X_j] = [\pi_* X_i, \pi_* X_j] = 0$ , we obtain  $[X_i, X_j] = -d\eta(X_i, X_j)T$ . Koszul formula then yields:

$$\nabla^h \eta = \frac{d\eta}{2} \quad \text{and} \quad \nabla^h dx_i = \frac{\iota_{X_i} d\eta \otimes \eta + \eta \otimes \iota_{X_i} d\eta}{2}. \quad (4)$$

In other words, the only non trivial Christoffel coefficients are

$$h(\nabla_{X_i}^h T, X_j) = h(\nabla_T^h X_i, X_j) = -h(\nabla_{X_i}^h X_j, T) = \frac{d\eta(X_i, X_j)}{2}.$$

It is important to keep in mind that  $\nabla^h X_i$  and  $\nabla^h T$  are  $\mathcal{O}(r^{-\tau-1})$ .

Moreover, a short computation yields  $\Delta_h u = -X_i \cdot X_i \cdot u - T \cdot T \cdot u$ . Given a local section of the circle bundle, one may consider a local vertical coordinate  $t$  and work in the coordinates  $(x_1, \dots, x_m, t)$ . Observe  $\partial_t = T$  and  $\eta = dt + A_i dx_i$  for some functions  $A_i$  independent of  $t$ . Then  $X_k = \partial_k - A_k \partial_t$  and

$$\Delta_h = -\partial_{kk} - \partial_{tt} + 2A_k \partial_{kt} - A_k^2 \partial_{tt} + \partial_k A_k \partial_t. \quad (5)$$

## 2. Analysis on asymptotic circle fibrations

In this section, we work on a complete Riemannian manifold  $(M^{m+1}, g)$ ,  $m \geq 3$ , such that for some compact subset  $K$ ,  $M \setminus K$  is diffeomorphic to the total space  $\mathcal{X}$  of a principal circle fibration  $\pi$  over  $\mathcal{B} := \mathbb{R}^m \setminus \mathbb{B}^m$ , with a model metric  $h$  as in section 1, such that:

$$g = h + \mathcal{O}(r^{-\tau}) \quad \text{and} \quad \nabla^h g = \mathcal{O}(r^{-\tau-1}).$$

Recall  $\tau$  is assumed to be positive. We intend to study the equation  $\Delta_g u = f$  in weighted  $L^2$  spaces. The presentation is deeply influenced by [HHM] and [Pac]. Given a real number  $\delta$  and a subset  $\Omega$  of  $M$ , we first define the weighted Lebesgue space

$$L_\delta^2(\Omega) := \left\{ u \in L_{loc}^2 \quad / \quad \int_{\Omega \setminus K} u^2 r^{-2\delta} dvol_h < \infty \right\}$$

and endow it with the norm  $\|u\|_{L_\delta^2(\Omega)} := \left( \int_{\Omega \cap K} u^2 dvol_g + \int_{\Omega \setminus K} u^2 r^{-2\delta} dvol_h \right)^{\frac{1}{2}}$ .

Changing  $K$  only produces equivalent norms. We will often write  $L_\delta^2$  for  $L_\delta^2(M)$ . The following should be kept in mind:  $r^a \in L_\delta^2(M \setminus K) \Leftrightarrow \delta > \frac{m}{2} + a$ . Note the Riemannian measures  $dvol_g$  and  $dvol_h$  on  $M \setminus K$  can always be interchanged, thanks to our asymptotic assumption. For the same reason, the Riemannian connections  $\nabla^g$  and  $\nabla^h$  will be completely equivalent outside  $K$ .

Any function  $u \in L_{loc}^2(M \setminus K)$  can be written  $u = \Pi_0 u + \Pi_\perp u$  where  $\Pi_0 u$  is obtained by computing the mean value of  $u$  along the fibers:  $(\Pi_0 u)(x) = \frac{1}{L} \int_{\pi^{-1}(x)} u \eta$ . It corresponds to a Fourier series decomposition along the fibers:  $\Pi_0 u$  is the part in the kernel of  $-T^2 = -\partial_{tt}$  while  $\Pi_\perp u$  lies in the positive eigenspaces of  $-\partial_{tt}$ . The point is  $\Delta_h$  commutes with the projectors  $\Pi_0$  and  $\Pi_\perp$ : for any function  $u$ , one can use (5) to write locally  $(\Pi_0 \Delta_h u)(x)$  as

$$\frac{1}{L} \int (-\partial_{kk} - \partial_{tt} + 2A_k(x)\partial_{kt} - A_k(x)^2\partial_{tt} + \partial_k A_k(x)\partial_t) u(x, t) dt.$$

This simplifies into

$$(\Pi_0 \Delta_h u)(x) = -\frac{1}{L} \int \partial_{xx} u(x, t) dt = -\partial_{xx} \frac{1}{L} \int u(x, t) dt = \Delta_h \left( \frac{1}{L} \int u(x, t) dt \right),$$

which ensures  $\Pi_0 \Delta_h = \Delta_h \Pi_0$  and also  $\Pi_\perp \Delta_h = \Delta_h \Pi_\perp$ .

Since this decomposition will prove useful, we introduce the following Hilbert spaces, depending on two real parameters  $\delta$  and  $\epsilon$ :

$$L_{\delta, \epsilon}^2(\Omega) := \left\{ u \in L_{loc}^2(\Omega) \quad / \quad \|\Pi_0 u\|_{L_\delta^2(\Omega \setminus K)} < \infty \text{ and } \|\Pi_\perp u\|_{L_\epsilon^2(\Omega \setminus K)} < \infty \right\}.$$

They are endowed with the following Hilbert norm:

$$\|u\|_{L_{\delta, \epsilon}^2(\Omega)} := \left( \|u\|_{L^2(K \cap \Omega)}^2 + \|\Pi_0 u\|_{L_\delta^2(\Omega \setminus K)}^2 + \|\Pi_\perp u\|_{L_\epsilon^2(\Omega \setminus K)}^2 \right)^{\frac{1}{2}}.$$

We also introduce the Sobolev space

$$H_\delta^2 := \left\{ u \in H_{loc}^2 / \left\| \nabla^g d\Pi_0 u \right\|_{L_{\delta-2}^2(K^c)} + \left\| d\Pi_0 u \right\|_{L_{\delta-1}^2(K^c)} + \left\| \Pi_0 u \right\|_{L_\delta^2(K^c)} < \infty \right. \\ \left. \text{and } \left\| \nabla^g d\Pi_\perp u \right\|_{L_{\delta-2}^2(K^c)} + \left\| d\Pi_\perp u \right\|_{L_{\delta-2}^2(K^c)} + \left\| \Pi_\perp u \right\|_{L_{\delta-2}^2(K^c)} < \infty \right\}$$

endowed with the obvious Hilbert norm.

In what follows, we will always write  $A_R$  for the ‘‘annulus’’ defined by  $R \leq r \leq 2R$  and  $A_R^\kappa$  for  $2^{-\kappa}R \leq r \leq 2^{\kappa+1}R$  ( $\kappa \geq 0$ ). Similarly, the ‘‘balls’’  $K \cup \{r \leq R\}$  will be denoted by  $B_R$ . The letter  $c$  will always denote a positive constant whose value changes from line to line. Its precise dependence on parameters will often be clear in the context; otherwise, we will write  $c(\dots)$  to clarify it.

## 2.1. A priori estimates.

*2.1.1. A priori estimates on the kernel of  $\Pi_\perp$ .* We aim at some a priori estimates for  $\Delta_h$  on  $\mathcal{X}$ . Let us begin with the kernel of  $\Pi_\perp$ : basically, we work with functions defined on exterior domains in  $\mathbb{R}^m$  and the Laplace operator is the standard one! What follows is therefore standard (it is in [Pac] and can be compared with [LP, Bart, HHM]). We will nonetheless provide details of the proofs, in order to use them later.

**Lemma 1.** *Given  $\delta$  in  $\mathbb{R}$ , there is a positive number  $c = c(m, \delta)$  such that for any  $R_0 \geq 1$  and any  $u$  in  $L_\delta^2 \cap \text{Ker } \Pi_\perp$ ,*

$$\left\| \nabla^h du \right\|_{L_{\delta-2}^2(B_{2R_0}^c)} + \left\| du \right\|_{L_{\delta-1}^2(B_{2R_0}^c)} \leq c \left( \left\| \Delta_h u \right\|_{L_{\delta-2}^2(B_{R_0}^c)} + \left\| u \right\|_{L_\delta^2(B_{R_0}^c)} \right).$$

*Proof.* Scaling the usual Garding inequality for the Laplacian on  $\mathbb{R}^m$  yields

$$\left\| Ddu \right\|_{L_{\delta-2}^2(A_R)} + \left\| du \right\|_{L_{\delta-1}^2(A_R)} \leq c \left( \left\| \Delta_h u \right\|_{L_{\delta-2}^2(A_R^1)} + \left\| u \right\|_{L_\delta^2(A_R^1)} \right).$$

The formulas for the connection of  $h$  imply  $|\nabla^h du| \leq c|Ddu| + cr^{-\tau-1}|du|$  so we can replace  $D$  by  $\nabla^h$  in the estimate above. Using it for  $R = 2^{i+1}R_0$ ,  $i \in \mathbb{N}$ , and summing over  $i$ , we get the desired inequality.  $\square$

To carry on, we need to use a  $L^2$  spectral decomposition for the Laplace operator  $\Delta_S$  on the unit sphere  $\mathbb{S}^{m-1}$  in  $\mathbb{R}^m$ . Its eigenvalues are  $\lambda_j := j(m-2+j)$ , with  $j \in \mathbb{N}$ , and we denote by  $E_j$  the corresponding eigenspaces. We also set

$$\delta_j := \frac{m}{2} + j \quad \text{and} \quad \nu_j^\pm := \frac{2-m}{2} \pm (\delta_j - 1)$$

for  $j \in \mathbb{N}$ . This simply means  $\nu_j^+ = j$  and  $\nu_j^- = 2 - m - j$ . These numbers  $\nu_j^\pm$  are usually called ‘‘indicial roots’’. Their basic property is the following: given an element  $\phi_j$  of  $E_j$ , we have  $\Delta(r^{\nu_j} \phi_j(\omega)) = 0$  outside 0. It can easily be seen from the formula

$$\Delta = -\partial_{rr} - \frac{m-1}{r} \partial_r + \frac{1}{r^2} \Delta_S - \partial_{tt}. \quad (6)$$

**Definition 1.** *We will say  $\delta$  is critical if  $\delta = \delta_j$  or  $\delta = 2 - \delta_j$  for some  $j$  in  $\mathbb{N}$ .*



**Fig. 1.** The critical exponents.

**Lemma 2.** *If  $\delta$  is not critical, there is a positive number  $c = c(m, \delta)$  such that for any  $R_0 \geq 1$  and any  $u$  in  $L^2_\delta \cap \text{Ker } \Pi_\perp$ ,*

$$\|u\|_{L^2_\delta(B_{2R_0}^c)} \leq c \left( \|\Delta_h u\|_{L^2_{\delta-2}(B_{R_0}^c)} + \|u\|_{L^2_\delta(A_{R_0})} \right).$$

*Proof.* If we first perform a spectral decomposition with respect to  $\Delta_S$ , our problem reduces to find an estimate on solutions  $u_j \in L^2(\mathbb{R}_+, r^{m-1-2\delta} dr)$  to the radial equation

$$-\partial_{rr} u_j - \frac{m-1}{r} \partial_r u_j + \frac{\lambda_j}{r^2} u_j = f_j.$$

Setting  $r := e^s$ ,  $v_j(s) := u(r)$  and  $g_j(s) = r^2 f_j(r)$ , we turn the equation into

$$\frac{d}{ds} \left( v_j'(s) e^{(m-2)s} \right) = (\lambda_j v_j(s) - g_j(s)) e^{(m-2)s} \quad (7)$$

And we work in  $L^2(\mathbb{R}_+, d\mu_\delta)$ , with  $d\mu_\delta := e^{(m-2\delta)s} ds$ . Given a radial truncature function  $\chi$  vanishing on  $B_{R_0}$  and equal to 1 outside  $B_{2R_0}$ , integration by parts provides

$$\begin{aligned} & \int ((\chi v_j)')^2 d\mu_\delta \\ &= \int \chi'^2 v_j^2 d\mu_\delta + \int (\chi^2 v_j)' (v_j' e^{(m-2)s}) e^{(2-2\delta)s} ds \\ &= \int \chi'^2 v_j^2 d\mu_\delta - \int (\chi^2 v_j) (\lambda_j v_j - g_j) d\mu_\delta - (1-\delta) \int \chi^2 (v_j^2)' e^{(m-2\delta)s} ds \\ &= \int [\chi'^2 + (1-\delta)(\chi^2)'] v_j^2 d\mu_\delta \\ & \quad + [(1-\delta)(m-2\delta) - \lambda_j] \int (\chi v_j)^2 d\mu_\delta + \int \chi^2 v_j g_j d\mu_\delta. \end{aligned}$$

One might then use the Hardy inequality  $\frac{(m-2\delta)^2}{4} \int (\chi v_j)^2 d\mu_\delta \leq \int ((\chi v_j)')^2 d\mu_\delta$  to deduce the estimate

$$(\delta_j - \delta)(\delta_j + \delta - 2) \int (\chi v_j)^2 d\mu_\delta \leq \int [\chi'^2 + (1-\delta)(\chi^2)'] v_j^2 d\mu_\delta + \int \chi^2 v_j g_j d\mu_\delta.$$

For all indices  $j$  such that  $(\delta_j - \delta)(\delta_j + \delta - 2) > 0$ , Young inequality yields

$$\int_{2R_0}^\infty (\chi v_j)^2 d\mu_\delta \leq c \int_{R_0}^{2R_0} v_j^2 d\mu_\delta + c \int \chi^2 g_j^2 d\mu_\delta,$$

which is what we need. For the finitely many indices  $j$  such  $2 - \delta < \delta_j < \delta$ , we consider the function  $w_j := \frac{1}{2(1-\delta_j)} (w_{j,+} - w_{j,-})$ , with

$$w_{j,\pm}(s) := \int_{R_0}^s e^{\nu_j^\pm(s-\sigma)} g_j(\sigma) d\sigma. \quad (8)$$

Since  $w'_{j,\pm} = \nu_j^\pm w_{j,\pm} + g_j$ , integration by parts leads to

$$\begin{aligned} & (m-2\delta) \int_{R_0}^{R_\infty} w_{j,\pm}^2(s) e^{(m-2\delta)s} \\ &= -2\nu_j^\pm \int_{R_0}^{R_\infty} w_{j,\pm}^2 d\mu_\delta - 2 \int_{R_0}^{R_\infty} w_{j,\pm} g_j d\mu_\delta + \underbrace{w_{j,\pm}^2(R_\infty) e^{(m-2\delta)R_\infty}}_{\geq 0}. \end{aligned}$$

Since  $m - 2\delta + 2\nu_j^\pm > 0$ , it implies (Cauchy-Schwarz inequality):

$$\int_{R_0}^{\infty} w_{j,\pm}^2 d\mu_\delta \leq c \int_{R_0}^{\infty} g_j^2 d\mu_\delta.$$

Now,  $w_j$  is a solution of (7) as well as  $v_j$ , so from ODE theory,  $w_j - v_j$  is a linear combination of  $e^{\nu_j^\pm s}$ . Its  $L^2(d\mu_\delta)$ -norm over  $[R_0, +\infty[$  can therefore be estimated by its  $L^2(d\mu_\delta)$ -norm over  $[R_0, 2R_0[$ . It follows that  $v_j$  can be estimated by

$$\int_{R_0}^{\infty} v_j^2 d\mu_\delta \leq c \int_{R_0}^{\infty} g_j^2 d\mu_\delta + c \int_{R_0}^{2R_0} v_j^2 d\mu_\delta.$$

The remaining case,  $\delta < \delta_j < 2 - \delta$  can be dealt with in a similar way. One defines an explicit solution  $w_j$  as above, replacing only  $R_0$  by  $+\infty$  in formula (8). The integration by parts argument still works (because  $w_{j,\pm} = o(e^{(\delta-\frac{\sigma}{2})s})$ ) and the fact that  $v_j$  and  $w_j$  belong to  $L^2(\mathbb{R}_+, d\mu_\delta)$  forces them to coincide.  $\square$

*2.1.2. A priori estimates on the kernel of  $\Pi_0$ .* The following lemma shows that estimates are basically *better* on the kernel of  $\Pi_0$ .

**Lemma 3.** *Given  $\delta \in \mathbb{R}$  and a large number  $R_0$ , there is a constant  $c$  such that for any function  $u$  in  $L_\delta^2 \cap \text{Ker } \Pi_0$ ,*

$$\|\nabla^h du\|_{L_\delta^2(B_{2R_0}^c)} + \|du\|_{L_\delta^2(B_{2R_0}^c)} + \|u\|_{L_\delta^2(B_{2R_0}^c)} \leq c \left( \|\Delta_h u\|_{L_\delta^2(B_{R_0}^c)} + \|u\|_{L_\delta^2(A_{R_0})} \right).$$

*Proof.* Given parameters  $R \gg 1$  and  $0 \leq \kappa < \kappa' \leq 1$ , we can always choose a smooth nonnegative function  $\chi$  that is equal to 1 on  $A_R^\kappa$ , vanishes outside  $A_R^{\kappa'}$ , is  $\mathbb{S}^1$ -invariant and has gradient bounded by  $c(\kappa, \kappa')R^{-1}$ . The integration by part formula

$$\int |d(\chi u)|^2 d\text{vol}_h = \int |d\chi|^2 u^2 d\text{vol}_h + \int \chi^2 u \Delta_h u d\text{vol}_h.$$

can be used together with Young inequality to obtain

$$\|du\|_{L^2(A_R^\kappa)}^2 \leq c(\epsilon) \|\Delta_h u\|_{L^2(A_R^{\kappa'})}^2 + \epsilon \|u\|_{L^2(A_R^{\kappa'})}^2, \quad (9)$$

where  $\epsilon$  can be chosen arbitrarily small provided  $R$  is sufficiently large. Since  $\Pi_0 u = 0$ , we have

$$\int_{\text{fiber}} (-T^2 u) u \eta = \int_{\text{fiber}} du (T)^2 \eta \geq c \int_{\text{fiber}} u^2 \eta.$$

This implies  $\|u\|_{L^2(A_R^\kappa)} \leq c \|du\|_{L^2(A_R^\kappa)}$  and therefore, with (9):

$$\|u\|_{L^2(A_R^\kappa)}^2 + \|du\|_{L^2(A_R^\kappa)}^2 \leq c(\epsilon) \|\Delta_h u\|_{L^2(A_R^{\kappa'})}^2 + \epsilon \|u\|_{L^2(A_R^{\kappa'})}^2. \quad (10)$$

To get an order two estimate, we first write

$$\int \chi^2 |\nabla^h du|^2 = \int (\nabla^h(\chi^2 du), \nabla^h du) - 2 \int \chi(d\chi \otimes du, \nabla^h du).$$

The Bochner Laplacian  $\nabla^{h,*}\nabla^h$  and the Hodge Laplacian  $\Delta_h = dd_h^* + d_h^*d$  only differ by the Ricci endomorphism, so an integration by parts provides:

$$\int \chi^2 |\nabla^h du|^2 = \int \chi^2 (\Delta_h du, du) - \int \chi^2 \text{Ric}_h(du, du) - 2 \int \chi(d\chi \otimes du, \nabla^h du).$$

Since the Hodge Laplacian commutes with  $d$ , another integration by part yields

$$\int \chi^2 (\Delta_h du, du) = \int (d\Delta_h u, \chi^2 du) = \int \chi^2 |\Delta_h u|^2 - 2 \int \chi(d\chi, du) \Delta_h u.$$

Putting these formulas altogether and using Young inequality, we obtain:

$$\int \chi^2 |\nabla^h du|^2 \leq 4 \int \chi^2 |\Delta_h u|^2 + 6 \int |d\chi|^2 |du|^2 - 2 \int \chi^2 \text{Ric}_h(du, du).$$

The upshot of this formula is the following estimate (observe  $\text{Ric}_h$  is bounded):

$$\|\nabla^h du\|_{L^2(A_R^\kappa)}^2 \leq c \|\Delta_h u\|_{L^2(A_R^{\kappa'})}^2 + c \|du\|_{L^2(A_R^{\kappa'})}^2.$$

With (10), we deduce:

$$\|u\|_{L^2(A_R^\kappa)}^2 + \|du\|_{L^2(A_R^\kappa)}^2 + \|\nabla^h du\|_{L^2(A_R^\kappa)}^2 \leq c(\epsilon) \|\Delta_h u\|_{L^2(A_R^{\kappa'})}^2 + \epsilon \|u\|_{L^2(A_R^{\kappa'})}^2.$$

We then set  $\epsilon = 0.5$ ,  $\kappa = 0$ ,  $\kappa' = 1$ , we multiply these inequalities by  $R^{-2\delta}$  and sum them for  $R = 2^k R_0$ ,  $k \in \mathbb{N}^*$  ( $R_0$  is chosen large) to find:

$$\|u\|_{L_\delta^2(B_{2R_0}^c)}^2 + \|du\|_{L_\delta^2(B_{2R_0}^c)}^2 + \|\nabla^h du\|_{L_\delta^2(B_{2R_0}^c)}^2 \leq c \|f\|_{L_\delta^2(B_{R_0}^c)}^2 + c \|u\|_{L_\delta^2(A_{R_0})}^2.$$

□

*Remark 1.* When the fibration is trivial, one can also study the integral kernel of the resolvent, as in [Dav]. Using a Fourier decomposition, we are left to prove that for positive  $k$ , the integral kernel

$$P(x, y) := \left( \frac{1 + d(o, y)}{1 + d(o, x)} \right)^\delta (\Delta_{\mathbb{R}^m} + k^2)^{-1}(x, y)$$

defines a bounded operator on  $L^2(\mathbb{R}^m)$ . Indeed, it is true for instance on a complete manifold with  $\text{Ric} \geq 0$  and  $\forall t \geq 1, At^\nu \leq \text{vol} B(x, t) \leq Bt^\nu$  for some  $\nu > 2$ . Noticing

$$(\Delta_{\mathbb{R}^m} + k^2)^{-1} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t\Delta} e^{-k^2 t}}{\sqrt{t}} dt \quad \text{and} \quad \frac{1 + d(o, y)}{1 + d(o, x)} \leq 1 + d(x, y),$$

we can use Li-Yau Gaussian estimate on the heat kernel [LY] to get

$$P(x, y) \leq c(1 + d(x, y)) \int_0^\infty \frac{e^{-\frac{d(x, y)^2}{ct}} e^{-k^2 t}}{\text{vol} B(x, \sqrt{t}) \sqrt{t}} dt.$$

It implies  $P(x, y) \leq ce^{-\frac{d(x, y)}{c}}$  and therefore  $\int P(x, y) dx \leq c$  and  $\int P(x, y) dy \leq c$ , which is enough to ensure the boundedness of  $P$  on  $L^p$ ,  $1 < p < \infty$ .

### 2.1.3. The main estimate.

**Proposition 1.** *If  $\delta$  is not critical, there is a constant  $c$  and a compact set  $B$  such that for any  $u$  in  $L^2_{\delta, \delta-2}$ ,  $\|u\|_{H^2_\delta} \leq c \left( \|\Delta_g u\|_{L^2_{\delta-2}} + \|u\|_{L^2(B)} \right)$ .*

*Proof.* Since  $\Delta_h$  commutes with  $\Pi_0$  and  $\Pi_\perp$ , we can write the equation  $\Delta_g u = f$  outside a large ball  $B_{R_0}$  as follows:

$$\begin{cases} \Delta_h u_0 = f_0 + \Pi_0(\Delta_h - \Delta_g)u \\ \Delta_h u_\perp = f_\perp + \Pi_\perp(\Delta_h - \Delta_g)u \end{cases}$$

We have denoted  $\Pi_0 u$  by  $u_0$ ,  $\Pi_\perp u$  by  $u_\perp$ , etc, to make the equations easier to read. We apply lemmata 1 and 2 to the first equation and lemma 3 to the second equation, which results in

$$\begin{aligned} & \|\nabla^h du_0\|_{L^2_{\delta-2}(B^c_{2R_0})} + \|du_0\|_{L^2_{\delta-1}(B^c_{2R_0})} + \|u_0\|_{L^2_\delta(B^c_{2R_0})} \\ & + \|\nabla^h du_\perp\|_{L^2_{\delta-2}(B^c_{2R_0})} + \|du_\perp\|_{L^2_{\delta-2}(B^c_{2R_0})} + \|u_\perp\|_{L^2_{\delta-2}(B^c_{2R_0})} \\ & \leq c \|f\|_{L^2_{\delta-2}(B^c_{R_0})} + c \|u\|_{L^2_\delta(B_{2R_0})} + c \|(\Delta_h - \Delta_g)u\|_{L^2_{\delta-2}(B^c_{R_0})}. \end{aligned}$$

Since

$$|(\Delta_h - \Delta_g)u| \leq c |g - h| |\nabla^h du| + c |\nabla^g - \nabla^h| |du|, \quad (11)$$

there is a function  $\epsilon$  going to zero at infinity such that

$$\|(\Delta_h - \Delta_g)u\|_{L^2_{\delta-2}(B^c_{R_0})} \leq \|\epsilon(r) \nabla^h du\|_{L^2_{\delta-2}(B^c_{R_0})} + \|r^{-1} \epsilon(r) du\|_{L^2_{\delta-2}(B^c_{R_0})}.$$

Therefore,  $\|(\Delta_h - \Delta_g)u\|_{L^2_{\delta-2}(B_{R_0}^c)}$  can be bounded by

$$\begin{aligned} & \epsilon(R_0) \left( \|\nabla^h du_0\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|\nabla^h du_\perp\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|du_0\|_{L^2_{\delta-1}(B_{2R_0}^c)} \right) \\ & + \epsilon(R_0) \left( \|du_\perp\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|\nabla^h du\|_{L^2_{\delta-2}(A_{R_0})} + \|du\|_{L^2_{\delta-2}(A_{R_0})} \right). \end{aligned}$$

Using this and choosing  $R_0$  large enough, we find

$$\begin{aligned} & \|\nabla^h du_0\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|du_0\|_{L^2_{\delta-1}(B_{2R_0}^c)} + \|u_0\|_{L^2_\delta(B_{2R_0}^c)} \\ & + \|\nabla^h du_\perp\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|du_\perp\|_{L^2_{\delta-2}(B_{2R_0}^c)} + \|u_\perp\|_{L^2_{\delta-2}(B_{2R_0}^c)} \\ & \leq c \|f\|_{L^2_{\delta-2}(B_{R_0}^c)} + c \|u\|_{L^2(B_{2R_0})} + c \|\nabla^h du\|_{L^2(B_{2R_0})} + c \|du\|_{L^2(B_{2R_0})}. \end{aligned}$$

Owing to the asymptotic of the metric,  $\nabla^h$  can be changed into  $\nabla^g$  in this estimate. Since the standard Garding inequality provides

$$\|\nabla^g du\|_{L^2(B_{2R_0})} + \|du\|_{L^2(B_{2R_0})} \leq c \|f\|_{L^2(B_{4R_0})} + c \|u\|_{L^2(B_{4R_0})},$$

we can conclude:  $\|u\|_{H^2_\delta} \leq c \|f\|_{L^2_{\delta-2,\delta-2}} + c \|u\|_{L^2(B_{4R_0})}$ .  $\square$

*2.2. Mapping properties.* We are interested in the unbounded operator

$$\begin{aligned} P_\delta : \mathcal{D}(P_\delta) &\longrightarrow L^2_{\delta-2,\delta-2} \\ u &\longmapsto \Delta_g u \end{aligned}$$

whose domain  $\mathcal{D}(P_\delta)$  is the dense subset of  $L^2_{\delta,\delta-2}$  whose elements  $u$  have their Laplacian in  $L^2_{\delta-2,\delta-2}$ .

*2.2.1. Fredholmness.* Proposition 1 has a direct consequence. In view of Rellich's theorem and Peetre's lemma (cf. [LiM], p. 171), it implies  $\text{Ker } P_\delta$  is finite dimensional (for any  $\delta$ ) and  $\text{Ran } P_\delta$  is closed (for any noncritical  $\delta$ ). The usual  $L^2$  pairing identifies the topological dual space of  $L^2_{\delta,\delta-2}$  (resp.  $L^2_{\delta-2,\delta-2}$ ) with  $L^2_{-\delta,2-\delta}$  (resp.  $L^2_{2-\delta,2-\delta}$ ). For this identification, the adjoint  $P_\delta^*$  of  $P_\delta$  is

$$\begin{aligned} P_\delta^* : \mathcal{D}(P_\delta^*) &\longrightarrow L^2_{-\delta,2-\delta} \\ u &\longmapsto \Delta_g u \end{aligned}$$

where the domain  $\mathcal{D}(P_\delta^*)$  is the dense subset of  $L^2_{2-\delta,2-\delta}$  whose elements  $u$  have their Laplacian in  $L^2_{-\delta,2-\delta}$ . Observing  $\text{Ker } P_\delta^* \subset \text{Ker } P_\eta$  for some large non critical  $\eta$ , we see that  $\text{Ker } P_\delta^*$  is always finite dimensional. We have proved the

**Proposition 2.** *If  $\delta$  is not critical, then  $P_\delta$  is Fredholm.*

*2.2.2. Solving an exterior problem.* We first solve the equation  $\Delta_h u = f$  on an exterior domain.

**Lemma 4.** *Given a noncritical  $\delta$  and a large number  $R_0$ , there is a bounded operator  $G_h : L^2_{\delta-2}(B^c_{R_0}) \longrightarrow H^2_{\delta}(B^c_{R_0})$  such that  $\Delta_h \circ G_h = \text{id}$ .*

*Proof.* We will define  $G_h$  in three steps, relying on an orthogonal decomposition of  $L^2_{\delta-2}(B^c_{R_0})$ .

For the first step, we pick a function  $f$  in  $L^2_{\delta-2} \cap \text{Ker } \Pi_{\perp}$  and we assume it has no component along the eigenspaces  $E_j$  of  $\Delta_S$  such that  $(\delta_j - \delta)(\delta_j + \delta - 2) > 0$ . Given  $R \gg R_0$ , we can use standard elliptic theory to solve the equation  $\Delta_h u_R = f$  in  $L^2(B_R \setminus B_{R_0})$ , with Dirichlet boundary condition. Adapting the proof of lemma 2 (on  $B_R \setminus B_{R_0}$ , with  $\chi = 1$ ), we obtain

$$\|u_R\|_{L^2_{\delta}(B_R \setminus B_{R_0})} \leq c \|f\|_{L^2_{\delta-2}(B_R \setminus B_{R_0})}, \quad (12)$$

with  $c$  independent of  $R$ . Elliptic regularity then bounds the  $H^2$  norm of  $u_R$  over compact subsets in terms of  $\|f\|_{L^2_{\delta-2}}$ , so that we can use Rellich theorem and a diagonal argument to extract a sequence  $u_R$  converging to a function  $u$  in  $H^1_{loc}$ , with  $\Delta_h u = f$  and  $u = 0$  on  $\partial B_{R_0}$ . Taking a limit in (12), one gets  $\|u\|_{L^2_{\delta-2}(B^c_{R_0})} \leq c \|f\|_{L^2_{\delta-2}(B^c_{R_0})}$ . From (1) (plus standard elliptic arguments near  $\partial B_{R_0}$ ), we deduce an estimate on the derivatives:

$$\|u\|_{H^2_{\delta}(B^c_{R_0})} \leq c \|f\|_{L^2_{\delta-2}(B^c_{R_0})}. \quad (13)$$

We need to show that such a  $u$  is uniquely defined, i.e. independent of the choice of extracted sequence. The difference  $v$  between two such functions  $u$  obeys  $\Delta_{\mathbb{R}^m} v = 0$  so, from ODE theory, its modes read  $v_j(r, \omega) = r^{\nu_j^+} \phi_j^+(\omega) + r^{\nu_j^-} \phi_j^-(\omega)$  with  $\phi_j^{\pm}$  in  $E_j$ . Since  $v_j$  is in  $L^2_{\delta}$  and vanishes on the boundary, we have  $\phi_j^{\pm} = 0$ , so  $v = 0$ . We can therefore set  $G_h f := u$ .

As a second step, we observe the same approach can be used for a function  $f$  in  $L^2_{\delta-2} \cap \text{Ker } \Pi_0$ . We can still obtain the functions  $u_R$  by solving the same problem and the proof of lemma 3 can be adapted to provide the estimate (12), provided  $R_0$  is large enough. This makes it possible to extract a converging subsequence as above, yielding a function  $u$  such that  $\Delta_h u = f$ ,  $u = 0$  on  $\partial B_{R_0}$  and satisfying (13). As for unicity, we consider the difference  $v$  between two such functions  $u$ :  $v$  is in  $L^2_{\delta, \delta-2} \cap \text{Ker } \Pi_0$ , vanishes on  $\partial B_{R_0}$  and obeys  $\Delta_h v = 0$ . Given a large number  $R$ , we can choose a cutoff function  $\chi$  that is equal to 1 on  $A_R$ , vanishes on  $A^1_R$  and has gradient bounded by  $10/R$ . Then

$$\int |d(\chi v)|^2 = \int |d\chi|^2 v^2 + \int \chi^2 v \Delta_h v$$

implies  $\|dv\|_{L^2_{\eta}(A_R)} \leq c \|v\|_{L^2_{\eta-1}(A^1_R)}$  for any exponent  $\eta$ . Since  $\Pi_0 v = 0$ , we get

$$\|v\|_{L^2_{\eta}(A_R)} \leq c \|dv\|_{L^2_{\eta}(A_R)} \leq c \|v\|_{L^2_{\eta-1}(A^1_R)} \quad \text{for any exponent } \eta.$$

This improves  $L^2_{\eta}$  estimates into  $L^2_{\eta-1}$  estimates. So  $v \in L^2_{\delta}$  implies  $v \in L^2_{\eta}$  for any  $\eta$ . In particular,  $v$  is in  $L^2$  and thus vanishes. So  $u$  is well defined and we can set  $G_h f := u$ .

As a third step, we consider those  $f$  in  $L_{\delta-2}^2 \cap \text{Ker } \Pi_{\perp}$  whose only component in the spectral decomposition of  $\Delta_S$  is in  $E_j$ , with  $(\delta_j - \delta)(\delta_j + \delta - 2) < 0$ . In case  $2 - \delta < \delta_j < \delta$ , we set  $G_h f := u$ , with

$$u(r, \omega) := \frac{1}{2(1 - \delta_j)} \left( r^{\nu_j^+} \int_{R_0}^r t^{1-\nu_j^+} f(t, \omega) dt - r^{\nu_j^-} \int_{R_0}^r t^{1-\nu_j^-} f(t, \omega) dt \right), \quad (14)$$

The proof of lemma 2 yields

$$\|u\|_{L_{\delta}^2(B_{R_0}^c)} \leq c \|f\|_{L_{\delta-2}^2(B_{R_0}^c)}. \quad (15)$$

With lemma 1, we obtain:

$$\|u\|_{H_{\delta}^2(B_{2R_0}^c)} \leq c \|f\|_{L_{\delta-2}^2(B_{R_0}^c)}. \quad (16)$$

Since  $u$  vanishes on  $\partial B_{R_0}$ , standard elliptic estimates improve (16) into

$$\|u\|_{H_{\delta}^2(B_{R_0}^c)} \leq c \|f\|_{L_{\delta-2}^2(B_{R_0}^c)}.$$

The case  $\delta < \delta_j < 2 - \delta$  is dealt with similarly, replacing  $R_0$  by  $+\infty$  in (14). Note  $u$  does not vanish on  $\partial B_{R_0}$  in this setting. But it can be bounded by  $c \|f\|_{L_{\delta-2}^2(B_{R_0}^c)}$  on  $\partial B_{R_0}$ ; since  $\Delta_S u_j = \lambda_j u_j$ ,  $\|u\|_{H^2(\partial B_{R_0})}$  is then bounded by  $c \|f\|_{L_{\delta-2}^2(B_{R_0}^c)}$ , so the argument above still works.  $\square$

A perturbation argument extends this result to a more general setting.

**Proposition 3.** *Given a noncritical  $\delta$  and a large number  $R_0$ , we can define a bounded operator  $G_g : L_{\delta-2}^2(B_{R_0}^c) \rightarrow H_{\delta}^2(B_{R_0}^c)$  such that  $\Delta_g \circ G_g = \text{id}$ .*

*Proof.* Thanks to lemma 4, we can write  $\Delta_g = \Delta_h [\text{id} + G_h (\Delta_g - \Delta_h)]$ . With (11), we can estimate  $\|(\Delta_h - \Delta_g)u\|_{L_{\delta, \delta-2}^2(B_{R_0}^c)}$  by  $\epsilon(R_0) \|u\|_{H_{\delta}^2(B_{R_0}^c)}$ . Since  $G_h$  is bounded from  $L_{\delta, \delta-2}^2(B_{R_0}^c)$  to  $H_{\delta}^2(B_{R_0}^c)$ , we deduce that  $G_h (\Delta_g - \Delta_h)$  defines a bounded operator on  $H_{\delta}^2(B_{R_0}^c)$ , with norm strictly inferior to 1 for  $R_0$  large enough. So  $\text{id} + (\Delta_g - \Delta_h) G_h^{-1}$  is an automorphism of  $H_{\delta}^2(B_{R_0}^c)$  and  $G_g := [\text{id} + G_h (\Delta_g - \Delta_h)]^{-1} G_h$  is a bounded operator from  $L_{\delta, \delta-2}^2(B_{R_0}^c)$  to  $H_{\delta}^2(B_{R_0}^c)$ , with  $\Delta_g G_g = \Delta_h G_h = \text{id}$ .  $\square$

This lemma can be used to build functions which are harmonic outside a compact set and have some prescribed asymptotics.

**Corollary 1.** *Given  $j \in \mathbb{N}$  and  $\phi \in E_j$ , there are functions  $\aleph_{j, \phi}^{\pm}$  that are harmonic outside a compact set and can be written  $\aleph_{j, \phi}^{\pm} = r^{\nu_j^{\pm}} \phi + v_{\pm}$  with  $v_{\pm}$  in  $H_{\eta}^2$  for any  $\eta > \delta_j - \tau$  and  $v_{-}$  in  $H_{\eta}^2$  for any  $\eta > 2 - \delta_j - \tau$ .*

*Proof.* For noncritical  $\delta > \delta_j$ , since  $r^{\nu_j^+} \phi$  is in  $L_{\delta}^2$  and  $\Delta_h(r^{\nu_j^+} \phi) = 0$ , lemma 1 implies  $r^{\nu_j^+} \phi$  is in  $H_{\delta}^2$ . With (11), we deduce  $\Delta_g(r^{\nu_j^+} \phi) \in r^{-\tau} L_{\delta-2}^2 = L_{\delta-\tau-2}^2$ . Now we can use lemma 3 to solve  $\Delta_g u = -\Delta_g(r^{\nu_j^+} \phi)$  outside a compact set and put  $\aleph_{j, \phi}^+ := \chi(r^{\nu_j^+} \phi + u)$  for some smooth nonnegative function  $\chi$  which vanishes on a large compact set and is equal to 1 outside a larger compact set. The construction of  $\aleph_{j, \phi}^-$  follows the same lines.  $\square$

*2.2.3. Decay jumps.* The following lemma is the key to understand the growth of solutions to our equations.

**Lemma 5.** *Suppose  $\Delta_h u = f$  with  $u$  in  $L^2_\delta(B_{R_0}^c)$  and  $f$  in  $L^2_{\delta'-2}(B_{R_0}^c)$  for noncritical exponents  $\delta > \delta'$  and a large number  $R_0$ . Then there is an element  $v$  of  $L^2_{\delta',\delta'-2}(B_{R_0}^c)$  such that  $u - v$  is a linear combination of the following functions:*

- $r^{\nu_j^+} \phi_j$  with  $\phi_j$  in  $E_j$  and  $\delta' < \delta_j < \delta$ ;
- $r^{\nu_j^-} \phi_j$  with  $\phi_j$  in  $E_j$  and  $\delta' < 2 - \delta_j < \delta$ .

*Proof.* We will build  $v$  step by step, starting from the solution  $\tilde{v}$  of  $\Delta_h \tilde{v} = f$  provided by lemma 4:  $\tilde{v}$  is in  $L^2_{\delta'-2,\delta'}(B_{R_0}^c)$ . Consider  $w := \Pi_0(u - \tilde{v})$  and look at its modes  $w_j$ . The equation  $\Delta_h w_j = 0$  implies  $w_j = r^{\nu_j^+} \phi_j^+ + r^{\nu_j^-} \phi_j^-$  with  $\phi_j^\pm$  in  $E_j$ . Observing  $r^{\nu_j^+} \in L^2_\eta \Leftrightarrow \eta > \delta_j$  and  $r^{\nu_j^-} \in L^2_\eta \Leftrightarrow \eta > 2 - \delta_j$ , one can see that each term is either in  $L^2_{\delta',\delta'-2}$ , so that we can add it to  $\tilde{v}$  and forget it, or satisfies the conditions in the statement. What about  $z := \Pi_\perp(u - \tilde{v})$ ? This  $z$  satisfies  $\Delta_h z = 0$ ,  $\Pi_0 z = 0$  and  $z \in L^2_\delta$ . As in the proof of lemma 4, for any  $\eta$ ,  $L^2_\eta$  estimates on  $z$  improve into  $L^2_{\eta-1}$  estimates, so  $z \in L^2_\delta$  implies  $z \in L^2_\eta$  for any  $\eta$  and the proof is complete.  $\square$

Let us generalize this to  $\Delta_g$ .

**Proposition 4.** *Suppose  $\Delta_g u = f$  with  $u$  in  $L^2_\delta(K^c)$  and  $f$  in  $L^2_{\delta'-2}(K^c)$  for noncritical exponents  $\delta > \delta'$ . Then, up to enlarging  $K$ , there is an element  $v$  of  $L^2_{\delta',\delta'-2}(K^c)$  such that  $u - v$  is a linear combination of the following functions:*

- $\aleph_{j,\phi_j}^+$  with  $\phi_j$  in  $E_j$ , if  $\delta' < \delta_j < \delta$ ;
- $\aleph_{j,\phi_j}^-$  with  $\phi_j$  in  $E_j$ , if  $\delta' < 2 - \delta_j < \delta$ .

*Proof.* Thanks to lemma 1, the equation  $\Delta_g u = f$ , with  $u$  in  $L^2_{\delta,\delta-2}$  and  $f$  in  $L^2_{\delta'-2}$ , ensures  $u \in H^2_\delta$ . With (11), we obtain

$$\Delta_h u \in L^2_{\delta'-2} + L^2_{\delta-\tau-2}.$$

So if we pick any noncritical  $\eta \geq \max(\delta', \delta - \tau)$ , we have  $\Delta_h u \in L^2_{\eta-2}$ . Lemma 5 then says  $u$  admits a decomposition

$$u = u_1 + \sum_{j_+} r^{\nu_{j_+}^+} \phi_{j_+}^+ + \sum_{j_-} r^{\nu_{j_-}^-} \phi_{j_-}^-$$

where  $u_1$  belongs to  $L^2_{\eta,\eta-2}$ ,  $\eta < \delta_{j_+} < \delta$  and  $2 - \delta < 2 - \delta_{j_-} < 2 - \eta$ . With lemma 1, we can therefore write

$$u = u_2 + \sum_{j_+} \aleph_{j_+,\phi_{j_+}^+}^+ + \sum_{j_-} \aleph_{j_-,\phi_{j_-}^-}^-$$

with  $u_2$  in  $L^2_{\eta,\eta-2}$ ,  $\eta < \delta_{j_+} < \delta$  and  $2 - \delta < 2 - \delta_{j_-} < 2 - \eta$ .

If  $\delta - \tau \leq \delta'$ , we are done.

If not, observe  $\Delta_g u_2 = f$  outside a compact set and  $u_2$  belongs to  $L^2_{\eta,\eta-2}$ . So we can repeat the argument with  $u_2$  in the role of  $u$  and  $\eta$  in the role of  $\delta$ . In a finite number of steps, we are in the first case.  $\square$

**Corollary 2.**  $P_\delta$  is surjective for any noncritical value  $\delta > 2 - \frac{m}{2}$ .

*Proof.* For any noncritical  $\delta \geq 1$ ,  $\text{Ker } P_\delta^*$  consists of harmonic functions  $u \in L_1^2$ . For every large number  $R$ , there is a smooth function  $0 \leq \chi_R \leq 1$  such that  $\chi_R = 1$  on  $B_R$ ,  $\chi_R = 0$  outside  $B_{2R}$  and  $|d\chi_R| \leq \frac{10}{R}$ . Then

$$\int_M |d(\chi_R u)|^2 = \int_M |d\chi_R|^2 u^2 + \int_M \chi_R^2 u \Delta_g u \quad \text{implies} \quad \int_{B_R} |du|^2 \leq c \int_{A_R} u^2 r^{-2}$$

and, letting  $R$  go to infinity, one finds  $du = 0$ , hence  $u = 0 : \text{Ker } P_\delta^* = \{0\}$ . So  $P_\delta$  is surjective as soon as  $\delta \geq 1$  is noncritical (recall it is Fredholm). Now, choose some  $\delta'$  in  $]2 - \frac{m}{2}, 1[$  and pick some function  $f$  in  $L_{\delta', -2}^2$ . In particular,  $f$  is in  $L_{1-2}^2$ , so there is a solution  $u \in L_{1, 1-2}^2$  of  $\Delta_g u = f$ . Proposition 4, with  $\delta = 1$  and  $\delta' = \delta'$ , implies  $u \in L_{\delta', \delta' - 2}^2$ , for there is no critical exponent in  $]\delta', 1[$ !  $P_\delta$  is therefore surjective.  $\square$

*2.3. An extension to the Dirac operator.* If we endow  $\mathbb{R}^m \times \mathbb{S}^1$  with its trivial spin structure, we can define a Dirac operator  $\mathcal{D}_{h_0}$  whose square  $\mathcal{D}_{h_0}^2$  acts diagonally as the Laplace operator  $\Delta_{h_0}$  in a constant trivialization. As a consequence, the a priori estimates that we have proved for  $\Delta_{h_0}$  are also available for  $\mathcal{D}_{h_0}^2$ . We wish to work on a complete spin Riemannian manifold  $(M^{m+1}, g)$ ,  $m \geq 3$ , such that for some compact subset  $K$ ,  $M \setminus K$  is spin-diffeomorphic to  $\mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$ , with:  $g = h_0 + \mathcal{O}(r^{-\tau})$ ,  $\nabla^{h_0} g = \mathcal{O}(r^{-\tau-1})$  and  $\nabla^{h_0, 2} g = \mathcal{O}(r^{-\tau-2})$ ,  $\tau > 0$ . A perturbation argument easily yields the

**Proposition 5.** *If  $\delta$  is not critical, there is a constant  $c$  and a compact set  $B$  such that for any  $\psi$  in  $L_{\delta, \delta-2}^2$ ,*

$$\|\psi\|_{H_\delta^2} \leq c \left( \|\mathcal{D}_g^2 \psi\|_{L_{\delta-2}^2} + \|\psi\|_{L^2(B)} \right).$$

The functional spaces are defined in the obvious way, using a constant trivialization. The proof is basically the same as that of proposition 1. Note however the estimate on the second derivative of  $g$ : we need this to control the difference between the model Dirac Laplacian  $\mathcal{D}_{h_0}^2$  and the operator  $\mathcal{D}_g^2$ , and more precisely the 0th order term (which of course vanishes for the Laplace operator on functions). Indeed, we bound  $|\mathcal{D}_g^2 \psi - \mathcal{D}_{h_0}^2 \psi|$  by a constant times

$$|g - h_0| |\nabla^{h_0, 2} \psi| + |\nabla^{h_0} g| |\nabla^{h_0} \psi| + \left( |\nabla^g - \nabla^{h_0}|^2 + |\nabla^{h_0, 2} g| \right) |\psi|.$$

The Fredholmness of the corresponding operator  $P_\delta$ ,  $\delta$  noncritical, follows immediately. The decay jump phenomenons carry over to this setting in exactly the same way. The proof of corollary 2 can be adapted to get the

**Corollary 3.** *If  $\text{Scal}_g \geq 0$ ,  $P_\delta$  is surjective for any noncritical value  $\delta > 2 - \frac{m}{2}$ .*

*Proof.* Lichnerowicz formula and the assumption  $\text{Scal}_g \geq 0$  ensure the  $L_1^2$  kernel of  $\mathcal{D}$  is trivial: just use the integration by part formula

$$\int_M |\nabla(\chi_R \psi)|^2 = \int_M |d\chi_R|^2 |\psi|^2 + \int_M \chi_R^2 (\psi, \mathcal{D}^2 \psi) - \frac{1}{4} \int_M \chi_R^2 \text{Scal} |\psi|^2.$$

The end of the proof is the same as for the Laplace operator.  $\square$

### 3. Towards a mass

In this section, we introduce a notion of mass for ALF metrics. The first paragraph contains a few (classic) algebraic computations which are useful in the sequel. The second and third paragraph develop two points of view corresponding to the two standard examples of Dirac type operators.

*3.1. Algebraic preliminaries.* Let  $M^n$  be a Riemannian manifold and let  $E$  be a bundle of left modules over the Clifford algebra bundle  $Cl(TM)$ . For definitions and basic facts about spin geometry, we refer to [LM]. We assume  $E$  is endowed with a compatible Euclidean metric  $(\cdot, \cdot)$  and metric connection  $\nabla$ , whose curvature tensor is  $R$ . This data determines a Dirac type operator  $\mathcal{D}$  on  $E$  and a section  $\mathcal{R}$  of  $\text{End } E$ . In a local orthonormal basis  $(e_a)_a$ , these are

$$\mathcal{D} = \sum_{a=1}^n e_a \cdot \nabla_{e_a} \quad \text{and} \quad \mathcal{R} = \frac{1}{4} \sum_{a,b=1}^n [e_a \cdot, e_b \cdot] R_{e_a, e_b}.$$

*Remark 2.* Commutators are easier to handle than Clifford products, for the latter are not antisymmetric. The identity

$$[e_a \cdot, e_b \cdot] = 2(\delta_{ab} + e_a \cdot e_b \cdot) \quad (17)$$

makes the translation. Moreover, brackets are skew-symmetric with respect to the Euclidean metric:

$$([e_a \cdot, e_b \cdot] \psi, \psi) = 0. \quad (18)$$

Given sections  $\alpha$  and  $\beta$  of  $E$ , we define a one-form  $\zeta_{\alpha, \beta}$  on  $M$  by the following formula (as in [AD, He2, Dai]):

$$\zeta_{\alpha, \beta}(X) := (\nabla_X \alpha + X \cdot \mathcal{D} \alpha, \beta).$$

The point is:  $d^* \zeta_{\alpha, \beta} = (\mathcal{D} \alpha, \mathcal{D} \beta) - (\nabla \alpha, \nabla \beta) - (\mathcal{R} \alpha, \beta)$ . We can integrate this Lichnerowicz-type formula over a domain  $\Omega$  to get

$$\int_{\Omega} [(\nabla \alpha, \nabla \beta) + (\mathcal{R} \alpha, \beta) - (\mathcal{D} \alpha, \mathcal{D} \beta)] dvol = - \int_{\Omega} d^* \zeta_{\alpha, \beta} dvol = \int_{\Omega} d * \zeta_{\alpha, \beta}$$

Stokes formula then provides

$$\int_{\Omega} \left[ (\nabla \alpha, \nabla \beta) + (\mathcal{R} \alpha, \beta) - \int_{\Omega} (\mathcal{D} \alpha, \mathcal{D} \beta) \right] dvol = \int_{\partial \Omega} * \zeta_{\alpha, \beta}. \quad (19)$$

If  $\omega_{\alpha, \beta}$  is the two-form defined by  $\omega_{\alpha, \beta}(X, Y) = ([X \cdot, Y \cdot] \alpha, \beta)$ , one can compute  $d^* \omega_{\alpha, \beta} = 4\zeta_{\alpha, \beta} - 4\zeta_{\beta, \alpha}$ . Stokes theorem thus implies:

$$\int_{\partial \Omega} * \zeta_{\alpha, \beta} = \int_{\partial \Omega} * \zeta_{\beta, \alpha}. \quad (20)$$

*3.2. The Gauss-Bonnet case.* Let us consider a complete oriented Riemannian manifold  $(M^{m+1}, g)$ ,  $m \geq 3$ , such that for some compact subset  $K$ ,  $M \setminus K$  is diffeomorphic to the total space of a circle fibration  $\pi$  over  $\mathbb{R}^m \setminus \mathbb{B}^m$ , which we endow with a model metric  $h = dx^2 + \eta^2$  as in section 1. Recall  $d\eta = \pi^*\omega$ , with

$$\omega = \mathcal{O}(r^{-\tau-1}) \quad \text{and} \quad D\omega = \mathcal{O}(r^{-\tau-2}). \quad (21)$$

We assume  $g$  is asymptotic to  $h$  in the following sense:

$$g = h + \mathcal{O}(r^{-\tau}), \quad \nabla^h g = \mathcal{O}(r^{-\tau-1}) \quad \text{and} \quad \nabla^{h,2} g = \mathcal{O}(r^{-\tau-2}). \quad (22)$$

For what follows, it is important to require :  $\tau > \frac{m-2}{2}$ . To simplify the statement of lemma 6, we will also suppose  $\tau \leq m-2$  (if not, take  $\tau := m-2$ ).

In this paragraph, we work on the exterior bundle  $\Lambda M (= E)$ , endowed with the Levi-Civita connection  $\nabla^g$  and we use the Gauss-Bonnet operator  $d + d^*$  as Dirac-type operator  $\mathcal{D}$ . The Clifford product  $v \cdot := \epsilon_v - \iota_v$  is obtained from the exterior product  $\epsilon_v := (v, \cdot) \wedge$  and the interior product  $\iota_v$ . In this setting,  $\mathcal{R}$  preserves the form degree. In degree 1, it reduces to the natural action of the Ricci tensor.

Let us introduce the vector space  $\mathcal{Z}$  spanned by  $X_1, \dots, X_m$ . As a first step toward the definition of a mass, we wish to build  $g$ -harmonic one-forms that are asymptotic to any element of the dual vector space  $\mathcal{Z}^*$ . We fix a small positive number  $\epsilon$  (say  $0 < \epsilon < 1/2$  and  $\epsilon < \tau - \frac{m-2}{2}$ ).

**Lemma 6.** *For any element  $\tilde{\alpha}$  in  $\mathcal{Z}^*$ , there is a one-form  $\alpha$  on  $M$  such that  $(d + d^*)\alpha = 0$  and  $\alpha = \tilde{\alpha} + \beta$ , with  $\beta = \mathcal{O}(r^{\epsilon-\tau})$  and  $\nabla^g \beta \in L^2_{\epsilon-\tau-1+\frac{m}{2}}$ .*

*Remark 3.* It is important to stick to asymptotically horizontal forms: this lemma cannot be generalized by choosing  $\tilde{\alpha} = \eta$ . A counter-example is provided by the Taub-NUT metric :  $g = Vdx^2 + \frac{1}{V}\eta^2$  with  $V = 1 + \frac{2m}{r}$  (cf. the end of the paper for more details). In this Riemannian manifold, if there were a one-form  $\alpha$  with  $(d + d^*)\alpha = 0$  and  $\alpha$  asymptotic to  $\eta$  (in the sense above), it would have to be  $\eta/V$  (for it is harmonic and asymptotic to this one-form ; a similar argument can be found in the proof of corollary 3). But this one-form is not closed, hence is not suitable. This fact explains the difference between the formulas for  $\mu_g^{GB}$  and  $\mu_g^D$ . In the Gauss-Bonnet case, we will define the mass by taking a trace over the horizontal directions (because of this lemma), whereas we would have needed a full trace to recover the same mass as in the Dirac setting.

We will need a Sobolev inequality. Indeed, the assumptions on  $h$  and  $g$  imply  $\text{Ric}_g \geq cr^{-2}$ , so we can use [SC] (or [Mi1]) to ensure the following family of scaled Sobolev inequalities: with  $n = m+1$ , any smooth function  $u$  with compact support in  $A_R$  satisfies

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq c \frac{R}{(\text{vol } B_R)^{\frac{1}{n}}} \|du\|_{L^2}.$$

This allows Moser iterations, turning integral estimates into pointwise estimates ([GT]; more precisely, lemma 3.9 in [TV] and lemma A.3 in [Mi2]):

$$\|u\|_{L^\infty} \leq \frac{c}{(\text{vol } B_R)^{\frac{1}{2}}} \|u\|_{L^2} + cR^2 \|\Delta_g u\|_{L^\infty}. \quad (23)$$

Since  $\text{Ric}_g \geq cr^{-2}$ , this estimate also holds for one-forms ( $\Delta_g$  is then replaced by the Hodge Laplacian  $(d + d^*)^2$ ) and for spinors (with the Dirac Laplacian  $\mathbb{D}^2$ ).

*Proof.* It is enough to prove the claim for  $\tilde{\alpha} := dx_k$ . Let us choose a truncature function  $\chi$  that vanishes on a large compact set and is equal to 1 outside a larger compact set. Since  $\Delta_g x_k$  is in  $L^2_{\delta'-2}$  for any  $\delta' > 1 + \frac{m}{2} - \tau$ , we can apply weighted analysis (corollary 2 ; observe  $\delta' > 2 - \frac{m}{2}$  because of our assumption  $\tau \leq m - 2$ ) to construct a function  $u_k$  in  $H^2_{\delta'}$  such that  $\Delta_g u_k = -\Delta_g(\chi x_k)$ , for any noncritical  $\delta' > 1 + \frac{m}{2} - \tau$ . Setting  $\beta_k := du_k$ , we therefore obtain  $(d + d^*)\beta_k = -(d + d^*)d(\chi x_k)$ , with  $\beta_k \in L^2_{\delta}$  and  $\nabla\beta_k \in L^2_{\delta-1}$ , for any  $\delta > \frac{m}{2} - \tau$  such that  $\delta' = \delta + 1$  is not critical. We may choose  $\delta = \frac{m}{2} - \tau + \epsilon$ . Then the one-form  $\alpha_k := d(\chi x_k) + \beta_k$  obeys  $(d + d^*)\alpha_k = 0$ . The pointwise estimate on  $\beta_k$  follows from (23), using  $\beta_k \in L^2_{\delta}$  and  $\Delta_g \beta_k = \mathcal{O}(r^{-\tau-2})$  (this coincides with  $d\Delta_g x_k$  outside a compact set).  $\square$

Now, the basic formula we need is given by (19) and (20): for any one-form  $\alpha$  as in lemma 6 and any large number  $R$ , we have

$$\int_{B_R} [|\nabla^g \alpha|^2 + \text{Ric}_g(\alpha, \alpha)] dvol = \int_{\partial B_R} *\zeta_{\tilde{\alpha}, \tilde{\alpha}} + 2 \int_{\partial B_R} *\zeta_{\tilde{\alpha}, \beta} + \int_{\partial B_R} *\zeta_{\beta, \beta}. \quad (24)$$

Our next aim consists in understanding the asymptotic behaviour of the right hand-side as much as possible. The mass is to be the limit of this quantity when the domains  $B_R$  are larger and larger. Let us tackle the second and third terms on the right-hand side. The assumption  $\tau > \frac{m-2}{2}$  kills them at infinity.

**Lemma 7.** *There is a sequence  $(R_i)_i$  going to infinity such that*

$$\lim_{i \rightarrow \infty} \int_{\partial B_{R_i}} *\zeta_{\tilde{\alpha}, \beta} = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{\partial B_{R_i}} *\zeta_{\beta, \beta} = 0$$

*Proof.* Since  $\beta$  is in  $L^2_{\frac{m}{2} + \epsilon - \tau}$  and  $\nabla\beta$  is in  $L^2_{\frac{m}{2} + \epsilon - \tau - 1}$ , one can find a sequence  $R_i$  going to infinity such that  $\int_{\partial \Omega_{R_i}} (|\beta|^2 + r^2 |\nabla^g \beta|^2) dvol = o(R_i^{m-2\tau+2\epsilon-1})$ .

The lemma then follows from Cauchy-Schwarz inequality and  $\nabla^g \tilde{\alpha} = \mathcal{O}(r^{-\tau-1})$ , with  $\tau > \frac{m-2}{2}$ .  $\square$

We need to compute the first term in the right-hand side of (24). Observe  $h$  identifies  $\mathcal{Z}$  and  $\mathcal{Z}^*$ , so any  $Z$  in  $\mathcal{Z}$  corresponds to a well defined  $\tilde{\alpha} = \tilde{\alpha}_Z$  in  $\mathcal{Z}^*$ .

**Lemma 8.**  $\zeta_{\tilde{\alpha}, \tilde{\alpha}} = -(\text{div}_h g)(Z)\tilde{\alpha} - \frac{1}{2} [d(\text{Tr}_h g)(Z)\tilde{\alpha} + d(g(Z, Z))] + \mathcal{O}(r^{-2\tau-1})$ .

*Proof.* Let us  $g$ -orthonormalize the frame field  $(X_1, \dots, X_m, T)$  into  $(e_a)_a$  and set  $\omega_{cd} := g(\nabla^g e_c, e_d) - g(\nabla^h e_c, e_d)$ . Since the connection on the *tangent* bundle  $TM$  reads  $\nabla^g = \nabla^h + \omega_{cd} e^c \otimes e_d$ , the connection on the *cotangent* bundle satisfies:  $\nabla^g \tilde{\alpha} = \nabla^h \tilde{\alpha} - \omega_{dc} \tilde{\alpha}(e_c) e^d$ . Therefore, we can write  $\zeta_{\tilde{\alpha}, \tilde{\alpha}} = \xi_1 + \xi_2$ , with

$$\xi_1 = \frac{1}{2} g([e_a \cdot, e_b \cdot] \nabla_{e_b}^h \tilde{\alpha}, \tilde{\alpha}) e^a \quad \text{and} \quad \xi_2 = -\frac{1}{2} \omega_{dc}(e_b) \tilde{\alpha}(e_c) g([e_a \cdot, e_b \cdot] e^d, \tilde{\alpha}) e^a.$$

We are lead to estimate terms like

$$g([e_a \cdot, e_b \cdot]e^d, \tilde{\alpha}) = g([\epsilon_{e_a} - \iota_{e_a}, \epsilon_{e_b} - \iota_{e_b}]e^d, \tilde{\alpha}) = -g([\iota_{e_a}, \epsilon_{e_b}]e^d + [\epsilon_{e_a}, \iota_{e_b}]e^d, \tilde{\alpha})$$

(recall  $\tilde{\alpha}$  is a one-form). A little algebra provides  $[\epsilon_{e_a}, \iota_{e_b}]e^d = 2\delta_{bd}e^a - \delta_{ab}e^d$  and therefore:

$$\frac{1}{2}g([e_a \cdot, e_b \cdot]e^d, \tilde{\alpha}) = \delta_{ad}\tilde{\alpha}(e_b) - \delta_{bd}\tilde{\alpha}(e_a). \quad (25)$$

Formula (25) implies  $\xi_1$  reads  $g(\nabla_{e_b}^h \tilde{\alpha}, e^d) [\delta_{ad}\tilde{\alpha}(e_b) - \delta_{bd}\tilde{\alpha}(e_a)] e^a$ , namely  $\xi_1 = \nabla_Z^h \tilde{\alpha} - g(\nabla_{e_b}^h \tilde{\alpha}, e^b) \tilde{\alpha}$ . With (4), we deduce  $\xi_1 = \nabla_Z^h \tilde{\alpha} + (d_h^* \tilde{\alpha}) \tilde{\alpha} + \mathcal{O}(r^{-2\tau-1}) = \mathcal{O}(r^{-2\tau-1})$  is a negligible term. As for  $\xi_2$ , we use (25) again to obtain  $\xi_2 = \omega_{dc}(e_b) \tilde{\alpha}(e_c) [\delta_{bd}\tilde{\alpha}(e_a) - \delta_{ad}\tilde{\alpha}(e_b)] e^a$ . Since Koszul formula provides

$$\begin{aligned} 2\omega_{xy}(e_w) &= -g(e_w, [e_x, e_y]) - g(e_x, [e_w, e_y]) + g(e_y, [e_w, e_x]) - 2g(\nabla_{e_w}^h e_x, e_y) \\ &= -g(e_w, \nabla_{e_x}^h e_y) - g(e_x, \nabla_{e_w}^h e_y) + g(e_y, \nabla_{e_w}^h e_x) + g(e_w, \nabla_{e_y}^h e_x) \\ &\quad + g(e_x, \nabla_{e_y}^h e_w) - g(e_y, \nabla_{e_x}^h e_w) - 2g(\nabla_{e_w}^h e_x, e_y) \\ &= (\nabla_{e_x}^h g)(e_y, e_w) - (\nabla_{e_y}^h g)(e_x, e_w) + (\nabla_{e_w}^h g)(e_x, e_y), \end{aligned}$$

we eventually find:

$$\begin{aligned} 2\xi_2 &= (\nabla_{e_b}^h g)(Z, e_b) \tilde{\alpha} - (\nabla_Z^h g)(e_b, e_b) \tilde{\alpha} + (\nabla_{e_b}^h g)(e_b, Z) \tilde{\alpha} \\ &\quad - (\nabla_{e_a}^h g)(Z, Z) e^a + (\nabla_Z^h g)(e_a, Z) e^a - (\nabla_Z^h g)(e_a, Z) e^a. \end{aligned}$$

We then use the asymptotic of  $g$  and (4) simplify this expression into the promised  $-2(\operatorname{div}_h g)(Z) \tilde{\alpha} - d(\operatorname{Tr}_h g)(Z) \tilde{\alpha} - d(g(Z, Z))$ , up to  $\mathcal{O}(r^{-2\tau-1})$ .  $\square$

The computations are done, it is time to draw a theorem, which requires a definition.

**Definition 2.** *On  $M \setminus K$ , we define a one-form  $q_{g,h}$  with values in the quadratic forms on  $\mathcal{Z}$  by the formula*

$$q_{g,h}(Z) = -(\operatorname{div}_h g)(Z) \tilde{\alpha}_Z - \frac{1}{2} [d(\operatorname{Tr}_h g)(Z) \tilde{\alpha}_Z + d(g(Z, Z))].$$

The “mass” quadratic form  $\mathcal{Q}_{g,h}$  is the quadratic form defined on  $\mathcal{Z}$  by:

$$\mathcal{Q}_{g,h}(Z) := \frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *_h q_{g,h}(Z),$$

where  $\omega_m$  is the volume of  $\mathbb{S}^{m-1}$  and  $L$  is the asymptotic length of fibers.

Why this normalization constant? The factor  $1/L$  is there to make the mass independent of the length of the asymptotic circles. The normalization by the volume of a sphere is more anecdotic. This corresponding positive mass theorem is the following.

**Theorem 3.** *Let  $(M^{m+1}, g)$ ,  $m \geq 3$ , be a complete oriented manifold with non-negative Ricci curvature. We assume that, for some compact subset  $K$ ,  $M \setminus K$  is the total space of a circle fibration over  $\mathbb{R}^m \setminus \mathbb{B}^m$ , which can be endowed with a model metric  $h$  such that (21) and (22) hold with  $\tau > \frac{m-2}{2}$ . Then  $\mathcal{Q}_{g,h}$  is a nonnegative quadratic form. It vanishes exactly when  $(M, g)$  is the standard  $\mathbb{R}^m \times \mathbb{S}^1$ .*

*Proof.* Formula (24), together with  $\text{Ric} \geq 0$ , lemma 7 and lemma 8, provides

$$\int_M \left[ |\nabla^g \alpha_Z|^2 + \text{Ric}_g(\alpha_Z, \alpha_Z) \right] d\text{vol} \leq \omega_m L \mathcal{Q}_{g,h}(Z). \quad (26)$$

Since  $\text{Ric} \geq 0$ , we deduce  $\mathcal{Q}_{g,h}(Z) \geq 0$ .

Now we assume  $\mathcal{Q}_{g,h} = 0$ . In view of (26), the  $g$ -harmonic one-forms  $\alpha_Z$  are then  $g$ -parallel. We have therefore built  $m$   $g$ -parallel one-forms  $\alpha_1, \dots, \alpha_m$  that are asymptotic to  $dx_1, \dots, dx_m$ . We also put  $\alpha_{m+1} := *_g(\alpha_1 \wedge \dots \wedge \alpha_m)$ , so as to obtain  $m+1$   $g$ -parallel one-forms that are linearly independent outside a compact set, hence linearly independent on the whole  $M$ . This yields a global parallel coframe field on  $M$ . In particular,  $(M, g)$  is flat, so it is a flat vector bundle over a compact flat manifold and  $g$  is naturally induced by the flat connection ([CG] p. 281). Since the volume growth of  $(M, g)$  is comparable to that of  $\mathbb{R}^m$ , the fibers of this bundle are  $m$ -dimensional. This means  $(M, g)$  is a flat  $\mathbb{R}^m$ -bundle over  $\mathbb{S}^1$ , namely it is obtained as a quotient of  $\mathbb{R}^{m+1}$  by the group generated by a rigid motion  $\rho: \rho(x) = Ax + v$ , with  $A \in O(m+1)$ ,  $v \neq 0$  and  $Av = v$ . For such a manifold, the holonomy group is generated by  $A$ . But we have built a *global* parallel coframe on  $(M, g)$ , so the holonomy group is trivial:  $A$  is the identity. It follows that  $(M, g)$  is obtained as a quotient of  $\mathbb{R}^{m+1}$  by a translation : it is isometric to the standard  $\mathbb{R}^m \times \mathbb{S}^1$ .  $\square$

Note  $\mathcal{Q}_{g,h}$  vanishes if and only if its trace vanishes. So  $\mu_{g,h}^{GB} := \text{Tr} \mathcal{Q}_{g,h}$  plays the role of a numerical mass. We have:

$$\mu_{g,h}^{GB} = -\frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *_h \left( \text{div}_h g + d \text{Tr}_h g - \frac{1}{2} d(g(T, T)) \right). \quad (27)$$

In this formula, the integrand can be replaced by the Hodge star of

$$\left[ (-\nabla_{X_j}^h g)(X_i, X_j) + X_i \cdot g(X_j, X_j) - (\nabla_T^h g)(X_i, T) + \frac{d(g(T, T))(X_i)}{2} \right] dx_i.$$

This can be simplified for we can use (4) and the closeness of  $g$  to  $h$  to get  $(\nabla_T^h g)(X_i, T) *_h dx_i = T \cdot g(X_i, T) *_h dx_i + \mathcal{O}(r^{-2\tau-1})$ , which can be rephrased as  $d(g(X_i, T) *_h (dx_i \wedge \eta)) + X_i \cdot g(X_i, T) *_h \eta + \mathcal{O}(r^{-2\tau-1})$ . So this part integrates to zero at infinity! In the same spirit,  $(\nabla_{X_j}^h g)(X_i, X_j)$  equals  $X_j \cdot g(X_i, X_j)$  up to  $\mathcal{O}(r^{-2\tau-1})$ , so the mass  $\mu_{g,h}^{GB}$  reduces to

$$\mu_{g,h}^{GB} = \frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *_h \left( [X_j \cdot g(X_i, X_j) - X_i \cdot g(X_j, X_j)] dx_i - \frac{d(g(T, T))}{2} \right).$$

The term between brackets is similar to the usual expression of the mass in the asymptotically Euclidean setting (see (1) in the introduction). It is the contribution from the base. More precisely, one can average the metric along the fibers and compute the mass of the asymptotically Euclidean metric induced on the base: it is this term. The other term is related to the variation of the length of fibers.

We now turn to the geometric invariance of the mass: does the mass really depends on  $h$  or is it a Riemannian invariant of  $g$ ? To ensure the answer is yes, we replace the assumptions (21) and (22) by

$$D^i \omega = \mathcal{O}(r^{-\tau-1-i}), \quad 0 \leq i \leq 2. \quad (28)$$

and

$$\nabla^{h,i}(g-h) = \mathcal{O}(r^{-\tau-i}), \quad 0 \leq i \leq 3. \quad (29)$$

If there is a model metric  $h$  such that these estimates hold, the corresponding mass  $\mu_{g,h}^{GB}$  does not depend on  $h$  but only on  $g$  : it is a Riemannian invariant. Before embarking into the proof, let us explain what the assumptions (28) and (29) are for. If we go back to the crucial lemma 6, we observe the derivative of  $\beta$  is only controlled in an  $L^2$  norm, which explains why we have to choose a convenient sequence of radii  $R_i$  in lemma 7 and why we define the mass with a limsup instead of a simple limit. Our new assumptions provide a  $C^1$  estimate on  $\beta$ , which allows us to get rid of the limsup.

**Lemma 9.** *If (28) and (29) hold, then the mass can be written as*

$$\begin{aligned} \mu_{g,h}^{GB} &= \frac{1}{\omega_m L} \lim_{R \rightarrow \infty} \int_{\partial B_R} *h q_{g,h}(X_k) \\ &= \frac{1}{\omega_m L} \sum_{k=1}^m \int_M \left[ |\nabla \alpha_k|^2 + (\text{Ric}(\alpha_k), \alpha_k) \right] dvol. \end{aligned}$$

*Proof.* As explained above, it amounts to obtain a  $C^1$  bound on the one-form  $\beta_k$  obtained in lemma 6 if  $\tilde{\alpha} = dx_k$ . Let  $\bar{\Delta}$  denote the rough Laplacian  $\nabla^* \nabla$ . A slight computation yields  $\bar{\Delta} \nabla \beta_k = \nabla \bar{\Delta} \beta_k + \text{Rm} \otimes \nabla \beta_k + \nabla \text{Rm} \otimes \beta_k$ , where  $\otimes$  denotes any bilinear pairing obtained by contracting with respect to the metric. The Bochner formula  $\bar{\Delta} \beta_k = (dd^* + d^*d)\beta_k - \text{Ric}(\beta_k)$  implies that, outside a compact set, we have  $\bar{\Delta} \beta_k = -d\Delta_g x_k - \text{Ric}(\beta_k)$ , hence

$$(\bar{\Delta} - \text{Rm} \otimes) \nabla \beta_k = -\nabla d\Delta_g x_k + \nabla \text{Rm} \otimes \beta_k = \mathcal{O}(r^{-\tau-3}) \quad (30)$$

thanks to (28) and (29). Since  $\text{Rm} = \mathcal{O}(r^{-\tau-2})$ , the term  $\text{Rm} \otimes$  can be treated as a perturbation and we can use a Moser iteration (cf. lemma A.3 in [Mi2]) in order to find

$$\|\nabla \beta_k\|_{L^\infty(A_R)} \leq \frac{c}{(\text{vol } B_R)^{\frac{1}{2}}} \|\nabla \beta_k\|_{L^2(A'_R)} + cR^2 \|(\bar{\Delta} - \text{Rm} \otimes) \nabla \beta_k\|_{L^\infty(A_R)},$$

with  $A_R = B_{2R} \setminus B_R$  and  $A'_R = B_{2.5R} \setminus B_{0.5R}$ . With (30) and lemma 6 (to get an  $L^2$  bound on  $\nabla \beta_k$ ), we end up with  $\|\nabla \beta_k\| = \mathcal{O}(r^{\epsilon-\tau-1})$ . With this and lemma 6, we can deduce from formula (24) and lemma 8 that

$$\begin{aligned} \int_{B_R} \left[ |\nabla \alpha_k|^2 + (\text{Ric}(\alpha_k), \alpha_k) \right] dvol &= \int_{\partial B_R} *h \zeta_{dx_k, dx_k} + o(1) \\ &= \int_{\partial B_R} *h q_{g,h}(X_k) + o(1), \end{aligned}$$

with  $\alpha_k = dx_k + \beta_k$ . Since  $\text{Ric} \geq 0$ , the left-hand side admits a limit in  $[0, +\infty]$ , so  $\int_{\partial B_R} *h q_{g,h}(X_k)$  goes to an element of  $[0, +\infty]$  as  $R$  goes to infinity. Summing over  $k$ , we see that the limsup in the definition of  $\mu_{g,h}^{GB}$  can be turned into a genuine limit, in  $[0, +\infty]$ , hence the lemma.

**Proposition 6.** *If  $h$  and  $h'$  are two model metrics satisfying (28) and (29), then  $\mu_{g,h}^{GB} = \mu_{g,h'}^{GB}$ .*

*Proof.* The metrics  $h = dx^2 + \eta^2$  and  $h' = dx'^2 + \eta'^2$  come with two circle fibrations  $\pi$  and  $\pi'$ , with connections as above. We can assume (28) and (29) hold for  $h$  and  $h'$  with the same  $\tau$ . The proof of the unicity of the mass is in three steps (compare [Bart, CH, Chr]). We first prove that, given an adapted (i.e. as above)  $h$ -orthonormal frame field  $(X_1, \dots, X_m, T)$ , we can find an adapted  $h'$ -orthonormal frame field  $(X'_1, \dots, X'_m, T')$  that is  $r^{-\tau}$ -close to  $(X_1, \dots, X_m, T)$ . Secondly, we show that the computation of the mass term does not depend on the choice of “spheres at infinity” ( $\partial B_R = r^{-1}(R)$  or  $\partial B'_R = r'^{-1}(R)$ ). And finally, we prove that the difference between the integrands corresponding to  $h$  and  $h'$  is the sum of an exact form and of a negligible term.

For the first step, we need to prove that the infinitesimal generators  $T$  and  $T'$  of both circle actions are  $r^{-\tau}$ -close. Note that, since  $h$  and  $h'$  are asymptotic to  $g$  (and because the fibers of  $\pi$  and  $\pi'$  have bounded length), the quotient  $r/r'$  is bounded from above and below outside a compact set and, indeed, both  $r$  and  $r'$  are comparable to the  $g$ -distance to a fixed point. Observe also that by assumption :

$$|T'|_h = 1 + \mathcal{O}(r^{-\tau}) = |T|_h + \mathcal{O}(r^{-\tau}). \quad (31)$$

For every point  $x$  with  $r(x)$  sufficiently large, we introduce the smooth loop  $\beta$  defined by  $\beta(0) = x$  and  $\dot{\beta}(t) = T'_{\beta(t)}$ ; we can push it into a smooth loop  $\gamma := \pi \circ \beta$  on  $\mathbb{R}^m$ . Observe  $L = L'$  is well defined since it is the injectivity radius at infinity. The idea of the proof is the following. Suppose  $T'_x$  were not close to  $T_x$  (nor to  $-T_x$ ). Then its  $h$ -horizontal component would be substantial, so that the curve  $\gamma$  would have a substantial initial speed vector  $(\pi_* T')_x$ . We will prove that  $\gamma$  has a very small acceleration, so that it remains close to a straight line, which is not compatible with the fact that  $\gamma$  has to come back to its origin in a time  $L$ . So  $T'_x$  has to be close to  $T_x$ . To make this argument effective, we decompose  $T'$  as the sum of its  $h$ -horizontal part  $H$  and  $h$ -vertical part  $W$ ; observe (31) bounds the norm of  $H$  and  $W$  by a constant. Since  $\pi$  is a Riemannian submersion between  $h$  and the flat metric on  $\mathbb{R}^m$ , we have

$$D_{\pi_* T'} \pi_* T' = \pi_*(\nabla_H^h H) = \pi_*(\nabla_H^h T') - \pi_*(\nabla_H^h W). \quad (32)$$

The first term,  $\pi_*(\nabla_H^h T')$ , is bounded by a constant times  $|\nabla^h T'|_h$ , which is  $\mathcal{O}(r^{-\tau-1})$  in view of (28), (29) and (4). To bound the second term, we use (4) to compute

$$h(\nabla_H^h W, X_i) = -h(W, \nabla_H^h X_i) = \frac{1}{2} \eta(W) d\eta(H, X_i).$$

This is  $\mathcal{O}(r^{-\tau-1})$ , by (28). With (32), this ensures  $D_{\pi_* T'} \pi_* T' = \mathcal{O}(r^{-\tau-1})$ . Given a point  $x$ , the acceleration  $\ddot{\gamma}$  of the corresponding loop  $\gamma$  is  $D_{\dot{\gamma}} \dot{\gamma} = D_{\pi_* T'} \pi_* T'$  (by definition of  $\gamma$ ), so it obeys  $|\ddot{\gamma}| \leq cr(x)^{-\tau-1}$ . Since  $\gamma(L) = \gamma(0)$ , Taylor formula provides  $L\dot{\gamma}(0) = \int_0^L t\ddot{\gamma}(t)dt$ . With the bound on  $\ddot{\gamma}$ , we deduce  $|\dot{\gamma}(0)| \leq cr(x)^{-\tau-1}$ . Since, by definition of  $\gamma$ ,  $\dot{\gamma}(0) = (\pi_* T')_x$ , we can conclude that the  $h$ -horizontal part  $H_x$  of  $T'_x$  is bounded by a constant times  $r(x)^{-\tau-1}$ . Therefore, up to an error term of order  $r^{-\tau-1}$ , the vector fields  $T$  and  $T'$  are colinear. Since they have the same  $h$ -norm up to  $\mathcal{O}(r^{-\tau})$  by (31), we may assume  $T = T' + \mathcal{O}(r^{-\tau})$  (changing  $T'$  into  $-T'$  if necessary). Now, given an adapted  $h$ -orthonormal frame field  $(X_1, \dots, X_m, T)$ , we average the vectors  $\pi'_* X_i$  along the fibers of  $\pi'$ , so as

to get a frame field  $(\check{X}_1, \dots, \check{X}_m)$  on the base  $\mathbb{R}^m$  of  $\pi'$ . Now we can proceed on the base as in the asymptotically Euclidean setting (cf. [Chr]). Since each  $\check{X}_i$  has derivative bounded by  $r^{-1-\tau}$ ,  $\check{X}_i = v_i + \mathcal{O}(r^{-\tau})$  for some (constant) vector  $v_i$  on  $\mathbb{R}^m$  (Cauchy criterion); moreover,  $(v_1, \dots, v_m)$  is an orthonormal frame in  $\mathbb{R}^m$ . We define  $X'_i$  as the  $h'$ -horizontal lift of  $v_i$ . Then  $(X'_1, \dots, X'_m, T')$  is an  $h'$ -adapted frame field that is  $r^{-\tau}$ -close to  $(X_1, \dots, X_m, T)$ .

To complete the second step of the proof, we rely on lemma 9 :

$$\begin{aligned} \mu_{g,h'}^{GB} &= \frac{1}{\omega_m L} \lim_{R \rightarrow \infty} \int_{\partial B'_R} *_{h'} q_{g,h'}(X'_k) \\ &= \frac{1}{\omega_m L} \sum_{k=1}^m \int_M \left[ |\nabla \alpha'_k|^2 + (\text{Ric}(\alpha'_k), \alpha'_k) \right] dvol. \end{aligned}$$

Since the ratio of the distance functions  $r/r'$  is bounded from above and below (as detailed in [Chr]), the end of the proof of lemma 9 can be adapted to obtain

$$\begin{aligned} \int_{B_R} \left[ |\nabla \alpha'_k|^2 + (\text{Ric}(\alpha'_k), \alpha'_k) \right] dvol &= \int_{\partial B_R} *_{h'} \zeta_{dx'_k, dx'_k} + o(1) \\ &= \int_{\partial B_R} *_{h'} q_{g,h'}(X'_k) + o(1). \end{aligned}$$

Letting  $R$  go to infinity, we find

$$\int_M \left[ |\nabla \alpha'_k|^2 + (\text{Ric}(\alpha'_k), \alpha'_k) \right] dvol = \lim_{R \rightarrow \infty} \int_{\partial B_R} *_{h'} q_{g,h'}(X'_k),$$

hence  $\mu_{g,h'}^{GB} = \frac{1}{\omega_m L} \lim_{R \rightarrow \infty} \int_{\partial B_R} *_{h'} q_{g,h'}(X'_k)$ , which can be computed by

$$\frac{1}{\omega_m L} \lim_{R \rightarrow \infty} \int_{\partial B_R} *_{h'} \left( [X'_j \cdot g(X'_i, X'_j) - X'_i \cdot g(X'_j, X'_j)] dx'_i - \frac{d(g(T', T'))}{2} \right),$$

where  $\partial B_R = r^{-1}(R)$  is defined with respect to  $h$ .

Let us turn to the third step, which consists in proving the equality

$$\begin{aligned} &\lim_{R \rightarrow \infty} \int_{\partial B_R} *_{h'} \left( [X'_j \cdot g(X'_i, X'_j) - X'_i \cdot g(X'_j, X'_j)] dx'_i - \frac{d(g(T', T'))}{2} \right) \\ &= \lim_{R \rightarrow \infty} \int_{\partial B_R} *_{h'} \left( [X_j \cdot g(X_i, X_j) - X_i \cdot g(X_j, X_j)] dx_i - \frac{d(g(T, T))}{2} \right). \end{aligned}$$

First observe the difference  $d(g(T', T')) - d(g(T, T))$  reads

$$\underbrace{(\nabla^{h'} g)(T', T') - (\nabla^{h'} g)(T, T)}_{\mathcal{O}(r^{-2\tau-1})} + \underbrace{2g(\nabla^{h'} T', T')}_{\mathcal{O}(r^{-2\tau-1})} - \frac{2g(\nabla^{h'} T, T)}{2\eta'(\nabla^{h'} T) + \mathcal{O}(r^{-2\tau-1})},$$

which implies  $*_h d(g(T', T')) - *_h d(g(T, T)) = 2 *_h \eta'(\nabla^{h'} T) + \mathcal{O}(r^{-2\tau-1})$ . Now, a slight computation (using  $[X_i, T] = 0$ ) leads to

$$\begin{aligned} *_h \eta'(\nabla^{h'} T) &= \eta'(\nabla_{X_i}^{h'} T) *_h dx_i + \eta'(\nabla_T^{h'} T) *_h \eta \\ &= \eta'(\nabla_T^{h'} X_i) *_h dx_i + \eta'(\nabla_T^{h'} T) *_h \eta \\ &= T \cdot \eta'(X_i) *_h dx_i + \eta'(\nabla_T^{h'} T) *_h \eta + \mathcal{O}(r^{-2\tau-1}) \\ &= d(*_h(\eta' \wedge \eta)) + X_i \cdot \eta'(X_i) *_h \eta + \eta'(\nabla_T^{h'} T) *_h \eta + \mathcal{O}(r^{-2\tau-1}). \end{aligned}$$

Since the terms involving  $*\eta$  integrate to zero on  $\partial B_R$ , Stokes formula yields

$$\lim_{R \rightarrow \infty} \int_{\partial B_R} [*_h d(g(T', T')) - *_h d(g(T, T))] = 0.$$

To tackle the remaining term, we expand it as follows:

$$\begin{aligned} & [X'_j \cdot g(X'_i, X'_j) - X'_i \cdot g(X'_j, X'_j)] - [X_j \cdot g(X_i, X_j) - X_i \cdot g(X_j, X_j)] \\ &= \left[ (\nabla_{X'_j}^{h'} g)(X'_i, X'_j) + g(\nabla_{X'_j}^{h'} X'_i, X'_j) + g(X'_i, \nabla_{X'_j}^{h'} X'_j) - (\nabla_{X'_i}^{h'} g)(X'_j, X'_j) \right. \\ &\quad \left. - 2g(\nabla_{X'_i}^{h'} X'_j, X'_j) \right] - \left[ (\nabla_{X_j}^{h'} g)(X_i, X_j) + g(\nabla_{X_j}^{h'} X_i, X_j) \right. \\ &\quad \left. + g(X_i, \nabla_{X_j}^{h'} X_j) - (\nabla_{X_i}^{h'} g)(X_j, X_j) - 2g(\nabla_{X_i}^{h'} X_j, X_j) \right]. \end{aligned}$$

Using the closeness of the frames and  $\nabla^{h'} g = \mathcal{O}(r^{-\tau-1})$ , one can see that the contribution of the terms involving  $\nabla^{h'} g$  is of order  $r^{-2\tau-1}$ , hence negligible. Besides, using the closeness of  $g$  to  $h'$  and (4), we have  $g(\nabla_{X'_i}^{h'} X'_j, X'_k) = \mathcal{O}(r^{-2\tau-1})$ , which ensures many terms above are lower order terms. Another simplification comes from the fact that the commutator  $[X_i, X_j]$  is  $\pi$ -vertical and of order  $r^{-\tau-1}$ : it implies  $\nabla_{X'_i}^{h'} X_j$  and  $\nabla_{X'_j}^{h'} X_i$  only differ by a lower order term. All in all, we find

$$\begin{aligned} & [X'_j \cdot g(X'_i, X'_j) - X'_i \cdot g(X'_j, X'_j)] - [X_j \cdot g(X_i, X_j) - X_i \cdot g(X_j, X_j)] \\ &= g(\nabla_{X'_i}^{h'} X_j, X_j) - g(X_i, \nabla_{X'_j}^{h'} X_j) + \mathcal{O}(r^{-2\tau-1}). \end{aligned}$$

We now introduce the  $(m-1)$ -form  $h'(X_j, X'_i) *_h(dx_i \wedge dx_j)$ . Its exterior derivative  $d(h'(X_j, X'_i) *_h(dx_i \wedge dx_j))$  is  $g(\nabla_{X'_j}^{h'} X_j, X_i) *_h dx_i - g(\nabla_{X'_i}^{h'} X_j, X_i) *_h dx_j$  plus lower order terms. After changing  $\nabla_{X'_i}^{h'} X_j$  into  $\nabla_{X'_j}^{h'} X_i$  (which costs only a lower order term) and switching summation indices  $i$  and  $j$  in the second term, we turn this into  $g(\nabla_{X'_j}^{h'} X_j, X_i) *_h dx_i - g(\nabla_{X'_i}^{h'} X_j, X_j) *_h dx_i + \mathcal{O}(r^{-2\tau-1})$ . This computation can be combined with Stokes formula to ensure

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\partial B_R} *_h [X'_j \cdot g(X'_i, X'_j) - X'_i \cdot g(X'_j, X'_j)] dx'_i \\ &= \lim_{R \rightarrow \infty} \int_{\partial B_R} *_h [X_j \cdot g(X_i, X_j) - X_i \cdot g(X_j, X_j)] dx_i. \end{aligned}$$

We have proved:  $\mu_{g, h'}^{GB} = \mu_{g, h}^{GB}$ .  $\square$

*3.3. The spin case, with a trivial fibration.* In this paragraph, we would like to explain a spin analogue to the construction above. The setting is a complete spin manifold  $(M^{m+1}, g)$ ,  $m \geq 3$  with nonnegative scalar curvature, such that for some compact subset  $K$ ,  $M \setminus K$  is diffeomorphic to  $\mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$ , which we endow with the standard flat metric  $h_0 = dx^2 + dt^2$ . We assume the metric  $g$  is asymptotic to  $h_0$  in the following sense: for some  $\tau > \frac{m-2}{2}$ ,

$$g = h_0 + \mathcal{O}(r^{-\tau}), \quad Dg = \mathcal{O}(r^{-\tau-1}) \quad \text{and} \quad D^2g = \mathcal{O}(r^{-\tau-2}).$$

As previously, we can choose  $\tau \leq m - 2$ . We furthermore assume the spin structure of  $M$  coincides outside  $K$  with the *trivial* spin structure on  $\mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$ .

We work on the spinor bundle  $\Sigma M := \Sigma^g M (= E)$  corresponding to the metric  $g$  and we endow it with the pullback of the Levi-Civita connection. Then  $\mathcal{D}$  is the standard Dirac operator  $\mathcal{D}$  and  $\mathcal{R}$  is the multiplication by  $\frac{1}{4} \text{Scal}$  ([LM]). Outside  $K$ ,  $\Sigma M$  is *not* the bundle corresponding to  $h_0$ , but we can identify them in a natural way. To make it precise, we denote by  $P$  the unique  $g$ -symmetric section of  $\text{End} TM$  such that  $h_0 = g(P, P)$ . One can also see  $P$  as a natural bijection between the principal bundles of orthonormal frames so that it lifts as an identification of the spin bundles and therefore identify  $\Sigma^{h_0}(M \setminus K)$  with  $\Sigma^g(M \setminus K) = \Sigma(M \setminus K)$ . The Levi-Civita connection  $D$  of  $h_0$  induces a flat metric connection  $\nabla^{euc}$  on  $(TM, g)$ , given by the formula  $\nabla_X^{euc} Y = PD_X(P^{-1}Y)$ . Since it is a metric connection, it induces a metric connection  $\nabla^{euc}$  on the spinor bundle  $\Sigma^g M = \Sigma M$ . To sum up, we have three connections on  $TM$ :  $\nabla^g$  is metric for  $g$  and torsionless;  $\nabla^{euc}$  is metric for  $g$ , is flat but has torsion;  $D$  is metric for  $h_0$ , flat and torsionless. Only two of them lift to  $\Sigma M$ :  $\nabla^g$  and  $\nabla^{euc}$ .

Now, the spinor bundle  $\Sigma(M \setminus K)$  is trivial and  $\nabla^{euc}$ -flat, so we can find a unit  $\nabla^{euc}$ -parallel spinor field  $\alpha_0$ . If  $\epsilon$  is a small positive number, we can adapt lemma 6.

**Lemma 10.** *There is a spinor field  $\alpha := \tilde{\alpha} + \beta$  such that  $\mathcal{D}\alpha = 0$ , with  $\tilde{\alpha} = \alpha_0$  outside a compact set,  $\beta = \mathcal{O}(r^{\epsilon-\tau})$  and  $\nabla\beta \in L^2_{\frac{m}{2}+\epsilon-\tau-1}$ .*

*Proof.* If  $\chi$  is a convenient truncature function  $\chi$ , we can see  $\tilde{\alpha}_0 := \chi\alpha_0$  as a section of  $\Sigma M$ . Set  $\gamma := -\mathcal{D}(\chi\alpha_0)$ . Since  $\gamma = \mathcal{O}(r^{-\tau-1})$ , it belongs to  $L^2_{\epsilon-\tau-1+\frac{m}{2}}$ . From analysis in weighted spaces (corollary 3), we obtain a solution  $\sigma \in H^2_{\epsilon-\tau+1+\frac{m}{2}}$  of the equation  $\mathcal{D}^2\sigma = \gamma$ . Put  $\beta = \mathcal{D}\sigma$ . The estimates on  $\beta$  follow as in lemma 6.  $\square$

As in the Gauss-Bonnet case, we can use formula 19 to find

$$\int_M \left[ |\nabla^g \alpha|^2 + \text{Scal}_g |\alpha|_g^2 \right] dvol \leq \limsup_{R \rightarrow \infty} \int_{\partial B_R} * \zeta_{\alpha_0, \alpha_0}. \quad (33)$$

**Lemma 11.**  $\zeta_{\alpha_0, \alpha_0} = -\frac{1}{4} (d \text{Tr}_{h_0} g + \text{div}_{h_0} g) + \mathcal{O}(r^{-2\tau-1})$ .

*Proof.* The proof is by now standard. We consider the frame field  $(\partial_a)_a := (\partial_1, \dots, \partial_m, \partial_t)$ . It is orthonormal for the model metric  $h_0$ . Putting  $e_a := P\partial_a$ , we obtain an orthonormal frame field for  $g$ . We need to understand  $\zeta_{\alpha_0, \alpha_0} =$

$\frac{1}{2}g([e_a \cdot, e_b \cdot] \nabla_{e_b}^g \alpha_0, \alpha_0) e^a$ . Since  $\alpha_0$  is  $\nabla^{euc}$ -parallel, with  $\omega_{cd} := g(\nabla^g e_c, e_d)$  and  $\nabla^g - \nabla^{euc} = \frac{1}{8}\omega_{cd}[e_c \cdot, e_d \cdot]$  ([LM]), we obtain

$$\zeta_{\alpha_0, \alpha_0} = \frac{1}{16}\omega_{cd}(e_b)g([e_a \cdot, e_b \cdot][e_c \cdot, e_d \cdot]\alpha_0, \alpha_0)e^a. \quad (34)$$

In order to compute the connection one-form, we resort to Koszul formula:

$$2\omega_{cd}(e_b) = -g(e_b, [e_c, e_d]) - g(e_c, [e_b, e_d]) + g(e_d, [e_b, e_c]).$$

Expanding the brackets thanks to the torsionless connection  $D$ , one finds:

$$2\omega_{cd}(e_b) = -(D_{e_d}g)(e_b, e_c) + (D_{e_c}g)(e_b, e_d) + g(D_{e_b}e_c, e_d) - g(D_{e_b}e_d, e_c). \quad (35)$$

Let us denote by  $H$  the  $g$ -symmetric endomorphism such that  $g - h_0 = g(H \cdot, \cdot)$ . From  $P^2 = I - H$  and  $e_c = P\partial_c$ , we get  $D_{e_b}e_c = -\frac{D_{e_b}H}{2}\partial_c + \mathcal{O}(r^{-2\tau-1})$ . And since  $H$  is  $g$ -symmetric, we have  $g((D_{e_b}H)\partial_c, \partial_d) = g((D_{e_b}H)\partial_d, \partial_c) + \mathcal{O}(r^{-2\tau-1})$ . So we deduce  $g(D_{e_b}e_c, e_d) - g(D_{e_b}e_d, e_c) = \mathcal{O}(r^{-2\tau-1})$ . Plugging this into (35) and then (34), we see that  $\zeta_{\alpha_0, \alpha_0}$  can be approximated by

$$\begin{aligned} & \frac{1}{32}(\partial_c g_{bd} - \partial_d g_{bc})g([e_a \cdot, e_b \cdot][e_c \cdot, e_d \cdot]\alpha_0, \alpha_0)e^a + \mathcal{O}(r^{-2\tau-1}) \\ &= \frac{1}{16}\partial_c g_{bd}g([e_a \cdot, e_b \cdot][e_c \cdot, e_d \cdot]\alpha_0, \alpha_0)e^a + \mathcal{O}(r^{-2\tau-1}) \\ &= \frac{1}{4}\partial_c g_{bd}g((\delta_{ab}\delta_{cd} + \delta_{ab}e_c e_d + \delta_{cd}e_a e_b + e_a e_b e_c e_d) \cdot \alpha_0, \alpha_0)e^a + \mathcal{O}(r^{-2\tau-1}). \end{aligned}$$

In view of (17) and (18), every term like  $g(e_a \cdot e_b \cdot \alpha_0, \alpha_0)$  can be replaced by  $-\delta_{ab}|\alpha_0|^2$ . Moreover, using (17), we can write

$$\partial_c g_{bd}g(e_a e_b e_c e_d \cdot \alpha_0, \alpha_0) = -\frac{\partial_c g_{bd}}{2}g(e_a [e_b, e_d] e_c \cdot \alpha_0, \alpha_0) + \partial_c g_{bd}g(\delta_{bd} e_a e_c \cdot \alpha_0, \alpha_0).$$

Since  $\partial_c g_{bd}$  is symmetric with respect to  $b$  and  $d$  whereas  $[e_b, e_d]$  is antisymmetric, the first term vanishes. These observations lead to :

$$\zeta_{\alpha_0, \alpha_0} = \frac{1}{4}\partial_c g_{bd}(\delta_{ab}\delta_{cd} - \delta_{ab}\delta_{cd} - \delta_{cd}\delta_{ab} - \delta_{bd}\delta_{ac})|\alpha_0|^2 e^a + \mathcal{O}(r^{-2\tau-1}),$$

which reduces to  $\zeta_{\alpha_0, \alpha_0} = \frac{1}{4}(-\partial_c g_{ac} - \partial_a g_{bb})e^a + \mathcal{O}(r^{-2\tau-1})$ .  $\square$

The corresponding positive mass theorem involves

$$\mu_{g,h}^D = -\frac{1}{\omega_m L} \limsup_{R \rightarrow \infty} \int_{\partial B_R} *_h \left( \operatorname{div}_{h_0} g + d \operatorname{Tr}_{h_0} g \right). \quad (36)$$

**Theorem 4.** *Let  $(M^{m+1}, g)$ ,  $m \geq 3$ , be a complete spin manifold with nonnegative scalar curvature. We assume there is a compact set  $K$  and a spin preserving diffeomorphism between  $M \setminus K$  and  $\mathbb{R}^m \setminus \mathbb{B}^m \times \mathbb{S}^1$  such that*

$$g = g_{\mathbb{R}^m \times \mathbb{S}^1} + \mathcal{O}(r^{-\tau}), \quad Dg = \mathcal{O}(r^{-\tau-1}) \quad \text{and} \quad D^2g = \mathcal{O}(r^{-\tau-2})$$

with  $\tau > \frac{m-2}{2}$ . Then  $\mu_{g,h}^D$  is nonnegative and vanishes exactly when  $(M, g)$  is isometric to the standard  $\mathbb{R}^m \times \mathbb{S}^1$ .

*Proof.* Since  $\text{Scal}_g \geq 0$ , formula 33 leads to  $\int_M |\nabla^g \alpha|^2 d\text{vol} \leq \frac{\omega_m L}{4} \mu_{g,h}^D$ , hence the nonnegativity of  $\mu_{g,h}^D$ . When  $\mu_{g,h}^D$  vanishes, every constant spinor  $\alpha_0$  in the model gives rise to a harmonic and parallel spinor field  $\alpha$  that is asymptotic to  $\alpha_0$ . This makes it possible to produce a parallel trivialization of the spinor bundle. It follows that  $(M, g)$  has trivial holonomy ([MS]) and we can conclude as in theorem 3.  $\square$

Finally, the proof of proposition 6 can be adapted to prove that  $\mu_{g,h}^D$  does not depend on  $h$  but only on  $g$ , under the assumptions of theorem 2.

#### 4. Examples

*4.1. Schwarzschild metrics.* These are complete Ricci flat metrics on  $\mathbb{R}^2 \times \mathbb{S}^{n-2}$ ,  $n \geq 4$ , given by the formula:

$$g_\gamma = d\rho^2 + F_\gamma(\rho)^2 d\theta^2 + G_\gamma(\rho)^2 d\omega^2.$$

$\rho, \theta$  are polar coordinates on the  $\mathbb{R}^2$  factor,  $d\omega^2$  is the standard metric on  $\mathbb{S}^{n-2}$ ,  $F_\gamma$  and  $G_\gamma$  are smooth functions defined by

$$G'_\gamma(\rho) = \sqrt{1 - \left(\frac{\gamma}{G_\gamma}\right)^{n-3}}, \quad G_\gamma(0) = \gamma \quad \text{and} \quad F_\gamma = \frac{2\gamma}{n-3} \sqrt{1 - \left(\frac{\gamma}{G_\gamma}\right)^{n-3}},$$

for some positive parameter  $\gamma$ .  $G_\gamma$  increases from  $\gamma$  to  $\infty$  and  $G_\gamma \sim \rho$  at infinity ;  $F_\gamma$  increases from 0 to  $\frac{2\gamma}{n-3}$  and  $F_\gamma \sim \rho$  near 0. Setting  $r := G_\gamma(\rho)$  and  $t = \frac{2\gamma}{n-3}\theta$ , we can write

$$g_\gamma = \frac{dr^2}{1 - \left(\frac{\gamma}{r}\right)^{n-3}} + r^2 d\omega^2 + \left[1 - \left(\frac{\gamma}{r}\right)^{n-3}\right] dt^2$$

In this way, it is apparent that  $g_\gamma$  is asymptotic to the flat metric on  $\mathbb{R}^{n-1} \times \mathbb{S}^1$ , with circle length equal to  $L := \frac{4\pi\gamma}{n-3}$  at infinity. Observe  $M = \mathbb{R}^2 \times \mathbb{S}^{n-2}$  is spin (with a unique spin structure, since it is simply connected). To compute the masses, we introduce ‘‘isotropic’’ coordinates: putting  $r = u \left[1 + \frac{1}{4} \left(\frac{\gamma}{u}\right)^{n-3}\right]^{\frac{2}{n-3}}$ , we get

$$g_\gamma = \left[1 + \frac{1}{4} \left(\frac{\gamma}{u}\right)^{n-3}\right]^{\frac{4}{n-3}} (du^2 + u^2 d\omega^2) + \left[\frac{1 - \frac{1}{4} \left(\frac{\gamma}{u}\right)^{n-3}}{1 + \frac{1}{4} \left(\frac{\gamma}{u}\right)^{n-3}}\right]^2 dt^2$$

for  $u > 0$ . We can then choose Cartesian coordinates  $x_1, \dots, x_{n-1}$  corresponding to the polar coordinates  $(u, \omega)$  and keep in mind the first order terms:

$$g_\gamma \approx \left[1 + \frac{\gamma^{n-3}}{n-3} u^{3-n}\right] dx^2 + [1 - \gamma^{n-3} u^{3-n}] dt^2.$$

With  $h_0 = dx^2 + dt^2$ , this readily provides  $\mu_{g_\gamma}^D = \gamma^{n-3}$  and  $\mu_{g_\gamma}^{GB} = \frac{n-1}{2} \gamma^{n-3}$ . For instance, in dimension  $n = 4$ , the masses reduce to the parameter  $\gamma$  (up to a universal constant), which is basically what we hoped (cf. introduction).

The mass is therefore positive, but we cannot deduce it from the ‘‘spin’’ version of the positive mass theorem: the spin structure at infinity is not the

trivial spin structure! Indeed, the spin structure on the asymptotic  $\mathbb{S}^1$  comes from the spin structure of the unit circle in the  $\mathbb{R}^2$  factor of  $\mathbb{R}^2 \times \mathbb{S}^{n-2}$ . This spin structure is therefore induced by the (unique and trivial) spin structure on the unit disk: it is the non trivial spin structure on the circle (see [Bar]). In the next paragraph, we will give examples showing that this non trivial spin structure at infinity really allows negative mass in nonnegative scalar curvature. But here, since the Ricci curvature of Schwarzschild metrics is nonnegative, we can use the Gauss-Bonnet point of view and explain the positivity of the mass: it is a consequence of  $\text{Ric} \geq 0$  and not of  $\text{Scal} \geq 0$ .

*4.2. Reissner-Nordström metrics.* In four dimensions, the Schwarzschild metric  $g_\gamma$  belongs to a broader family of complete scalar flat metrics on  $\mathbb{R}^2 \times \mathbb{S}^2$  that are asymptotic to  $\mathbb{R}^3 \times \mathbb{S}^1$ , with the non trivial spin structure at infinity. These Reissner-Nordström metrics [Dai, BrH, CJ, PK] are given by the same ansatz  $g = d\rho^2 + F(\rho)^2 d\theta^2 + G(\rho)^2 d\omega^2$ , with

$$G' = \sqrt{1 - \frac{2m}{G} - \frac{q^2}{G^2}} \quad \text{and} \quad F = \frac{G_0^2}{G_0 - m} \sqrt{1 - \frac{2m}{G} - \frac{q^2}{G^2}}$$

and  $G(0) := G_0 = m + \sqrt{m^2 + q^2}$ . The behaviour of  $G$  and  $F$  is similar to the Schwarzschild analogue. The new feature is  $m$  can be chosen negative: the metric is then still complete, provided  $q$  is nonzero. Setting  $r := G(\rho)$  and  $t = \frac{G_0^2}{G_0 - m} \theta$ , we can obtain a more familiar formula:

$$g = \frac{dr^2}{1 - \frac{2m}{r} - \frac{q^2}{r^2}} + \left[1 - \frac{2m}{r} - \frac{q^2}{r^2}\right] dt^2 + r^2 d\omega^2.$$

The formulas for doubly-warped products [Pet] (or geometric arguments as in [Bes], 3.F) make it possible to compute the curvature. The eigenvalues of the Ricci tensor are  $\frac{q^2}{G^4}$ , along  $\partial_\rho$  and  $\partial_t$ , and  $-\frac{q^2}{G^4}$ , along the  $\mathbb{S}^2$  factor. It therefore has no sign but the scalar curvature vanishes. To compute the mass  $\mu_g^D$ , we can again use isotropic coordinates. The new radial coordinate  $u$  is given by  $r = u \left[1 + \frac{m}{u} + \frac{m^2 + q^2}{4u^2}\right]$  and we find

$$g = \left[1 + \frac{m}{u} + \frac{m^2 + q^2}{4u^2}\right]^2 (du^2 + u^2 d\omega^2) + \left[\frac{1 - \frac{m^2 + q^2}{4u^2}}{1 + \frac{m}{u} + \frac{m^2 + q^2}{4u^2}}\right]^2 dt^2.$$

Comparing to the Schwarzschild formula, one can see that the asymptotic is the same up to  $\mathcal{O}(u^{-2})$ , so the mass  $\mu_g^D$  is  $2m$ . Since we can choose  $m$  negative, this yields a whole family of metrics with negative mass!

*4.3. Multi-Taub-NUT metrics.* The Taub-NUT metric is the basic non trivial example of ALF gravitational instanton. In particular, it is hyperkähler hence Ricci flat. For details, the reader is referred to [Le2, HHM]. Basically, it is a complete metric on  $\mathbb{R}^4$  which is adapted at infinity to the Hopf fibration  $\mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ . It can be written outside one point as  $g = V dx^2 + \frac{1}{V} \eta^2$  where  $V = 1 + \frac{2m}{r}$  for some positive parameter  $m$  and  $\eta$  is a connection form

with curvature  $*_{\mathbb{R}^3}dV$ . Up to a constant,  $\eta$  is the standard contact form on the three-spheres (cf. section 1). With  $h = dx^2 + \eta^2$ , we can compute  $\mu_g^{GB} = 3m$ : as expected, the mass is essentially the parameter  $m$ .

The same computations apply to the multi-Taub-NUT metrics (cf. [Le2, HHM]). These metrics are again hyperkähler and their asymptotic is adapted to a principal circle bundle over  $\mathbb{S}^2$  with Chern number  $-k$ . They can be written outside  $k$  points as  $g = Vdx^2 + \frac{1}{V}\eta^2$  with  $V = 1 + \sum_{i=1}^k \frac{2m}{|x-x_i|}$  and  $\eta$  is a connection form on a  $\mathbb{S}^1$  bundle over  $\mathbb{R}^3 \setminus \{x_1, \dots, x_k\}$  whose curvature is  $*_{\mathbb{R}^3}dV$ . The mass  $\mu^{GB}$  of these metrics is  $3km$ .

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