A KUMMER CONSTRUCTION FOR GRAVITATIONAL INSTANTONS

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Abstract. We give a simple and uniform construction of essentially all known deformation classes of gravitational instantons with ALF, ALG or ALH asymptotics and nonzero injectivity radius. We also construct new ALH Ricci flat Kähler metrics asymptotic to the product of a real line with a flat 3-manifold.

Introduction

The aim of this paper is to give a direct and uniform construction for several Ricci-flat Kähler four-manifolds with prescribed asymptotics, 'ALF', 'ALG' or 'ALH'. This basically means that these complete Riemannian manifolds have only one end, which is diffeomorphic, up to a finite covering, to the total space of a $\mathbb{T}^{4-m}$-fibration π over $\mathbb{R}^m$ minus a ball and carries a metric that is asymptotically adapted to this fibration in the following sense. When the fibration at infinity is trivial, the metric merely goes to a flat metric on $\mathbb{R}^m \times \mathbb{T}^{4-m}$ (without holonomy). When $m = 3$, the fibration at infinity may be non-trivial and in this case, the metric goes to $\pi^* g_{\mathbb{R}^3} + \eta^2$ where η is a connection one-form on the $\mathbb{S}^1$-fibration, up to scaling (cf. [23] for details). The metric is called ALF when the 'dimension at infinity' m is 3, ALG when $m = 2$, ALH when $m = 1$. These asymptotics are generalizations of the familiar ALE (Asymptotically Locally Euclidean) case, for which the model at infinity is the Euclidean four-space (in a nutshell: $m = 4$).

Most of our examples of Ricci-flat Kähler four-manifolds are simply-connected, hence hyperkähler (i.e. with holonomy inside $SU(2)$), providing examples of gravitational instantons, namely non-compact hyperkähler four-manifolds with decaying curvature at infinity. These manifolds are of special interest in quantum gravity and string theory, hence some motivation to understand examples. Previous constructions of gravitational instantons were either explicit [9, 10], based on hyperkähler reduction [17, 8], gauge theory [2, 5, 6], twistorial methods [14, 4] or on the Monge-Ampère method of Tian and Yau [27, 28], see also [15, 16, 13, 26].

The technique we advertise here is inspired from the famous Kummer construction for $K3$ surfaces: starting from the torus $\mathbb{T}^4 = \mathbb{C}^2/\mathbb{Z}^4$, we may consider the complex orbifold $\mathbb{T}^4/\pm$, which has 16 singularities isomorphic to $\mathbb{C}^2/\pm$; blowing them up, we obtain a $K3$ surface. The physicist D. Page [25] noticed that this point of view makes it possible to grasp some idea of what the Ricci flat Kähler metric provided by Yau theorem on the $K3$ surface looks like. The recipe is the following. The desingularization of $\mathbb{C}^2/\pm$ carries an explicit Ricci-flat Kähler metric known as the Eguchi-Hanson metric [9]. So the Ricci flat metric on the

Date: 1st June 2011.

$K3$ surface should look like this Eguchi-Hanson metric near each exceptional divisor and resemble the flat metric (issued from $\mathbb{T}^4$) away from them. Such an idea has been carried out rigorously by twistorial methods in [19, 29]. N. Hitchin [14] also pointed out a twistorial argument leading to a Ricci-flat metric on the (non-compact) minimal resolution of $\{x^2 - zy^2 = z\} \subset \mathbb{C}^3$.

In this short paper, we carry out a rather elementary deformation argument providing Ricci-flat Kähler metrics on the minimal resolution of numerous Ricci-flat Kähler orbifolds. Here is the list of examples that can be constructed using this technique:

(1) ALF case: there are two classes of ALF gravitational instantons, A_k and D_k gravitational instantons, whose boundary is the quotient of the 3-sphere by a cyclic group (A_k) or a binary dihedral group (D_k). There are simple explicit formulas for the A_k metrics, in terms of the Gibbons-Hawking ansatz. We give a construction of D_k metrics: we start from the quotient X of the A_0 ALF gravitational instanton (this is the standard Taub-NUT metric on $\mathbb{R}^4$ [12, 18]), by the binary dihedral group of order $4(k-2)$, where $k > 2$. We note $\hat{X}$ the crepant resolution of the orbifold X: the singular point is replaced by a configuration of $p = k$ (-2)-rational curves.

(1') ALF D_2 case: this is the Hitchin metric, it can be constructed starting from $X = \mathbb{R}^3 \times \mathbb{S}^1/\mathbb{Z}_2$, with 2 orbifold points, so the desingularization $\hat{X}$ has $p = 2$ exceptional curves.

(2) ALG case: we start from the possible quotients of $\mathbb{R}^2 \times \mathbb{T}^2$, that is from a flat orbifold $X = (\mathbb{R}^2 \times \mathbb{T}^2)/\mathbb{Z}_k$, for $k = 2, 3, 4, 6$. For $k \neq 2$, the action of $\mathbb{Z}_k$ on $\mathbb{T}^2$ exists only for special lattices (square lattice for $k = 4$, hexagonal lattice for $k = 6$). Then $\hat{X}$ is the crepant resolution of X, with a configuration of p exceptional curves ($k-1$ at each fixed point).

(3) ALH hyperkähler case: we start from $X = \mathbb{R} \times \mathbb{T}^3/\pm$, this is an orbifold with 8 singular points, and the desingularization $\hat{X}$ contains $p = 8$ curves E_j of selfintersection -2. The boundary is $\mathbb{T}^3$.

(3') ALH Ricci flat Kähler case: we start from the orbifold $X_2 = \mathbb{R} \times F_2/\mathbb{Z}_2$, where F_2 is the flat 3-manifold $F_2 = \mathbb{T}^3/\mathbb{Z}_2$. Then X_2 is a Ricci flat Kähler (non hyperKähler) orbifold, it has $p = 4$ singular points of type $\mathbb{C}^2/\pm$, and the desingularization $\hat{X}_2$ has boundary F_2.

(3") ALH Ricci flat case: we start from $X_{2,2} = \mathbb{R} \times F_{2,2}/\mathbb{Z}_2 = X_2/\mathbb{Z}_2$, where $F_{2,2}$ is the flat 3-manifold with monodromy $\mathbb{Z}_2 \times \mathbb{Z}_2$, that is $F_{2,2} = \mathbb{T}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. This is not Kähler because the last $\mathbb{Z}_2$ action is antiholomorphic. The action extends to the desingularization $\hat{X}_2$, so one gets a desingularization $\hat{X}_{2,2} = X_2/\mathbb{Z}_2$.

Theorem 0.1. *Let (X, ω_0) denote any of the complex Kähler orbifolds in the list above (all examples but the last one), and $\hat{X}$ be its minimal resolution. We denote the exceptional -2 curves by $E_1, \dots, E_p$ and the Poincaré dual of E_j by $PD[E_j]$. Let $a_1, \dots, a_p$ denote some positive parameters. Then for every small enough positive number ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}$ in the cohomology* *class* $[\omega_0] - \epsilon \sum a_j PD[E_j]$*; it is moreover asymptotic to the initial metric* ω_0 *on* X*. Finally,* $(\hat{X}, \omega)$ *is hyperkähler if* (X, ω) *is hyperkähler.*

The last desingularization $\hat{X}_{2,2}$ *also carries Ricci flat metrics, obtained as quotients of* $\hat{X}_2$ *by an antiholomorphic involution.*

The proof of the theorem relies on a simple gluing procedure and is essentially self-contained. Even though most of the metrics we build have already been constructed in the literature by other ways (cf. paragraph below), we believe that our construction is interesting, because it is very simple and gives a very good approximation coming from the desingularization procedure. More precisely, the construction described in this paper provides a way to build members of nearly all known deformation families of ALF gravitational instantons, by starting from the explicit Taub-NUT metric and Kronheimer's ALE gravitational instantons [17]. The 'nearly' accounts for the Atiyah-Hitchin metric [2], which seems to play a special role. And more generally, apart from the Atiyah-Hitchin metric, we have the striking fact that this construction yields members of all known deformation families of gravitational instantons with positive injectivity radius. So we somehow get a global and concrete understanding of all these families.

Let us detail the relations with the existing constructions. In case (1), the metrics that we obtain probably coincide with the D_k gravitational instantons of [5, 4]. The same procedure using an action of a cyclic group instead of D_k would lead to the ALF gravitational instantons of cyclic type, that is multi-Taub-NUT metrics, given by the Gibbons-Hawking ansatz [23]. In case (1'), we obtain a direct PDE construction of the Hitchin metric mentioned above [14], complementing the twistorial initial description; this is a D_2 ALF gravitational instanton of dihedral type (in the sense of [23]). Together, cases (1) and (1') give constructions of ALF gravitational instantons in all known ALF deformation classes, except D_0 (the Atiyah-Hitchin metric) and D_1 (its double cover), that cannot be obtained in this way because they have 'negative mass'.

The ALG and ALH examples (2) and (3) have been constructed recently in a general Tian-Yau framework on rational elliptic surfaces in [13] (see also [16, 26]); in [13], they correspond to isotrivial elliptic fibrations. Here we obtain exactly all the classes with positive injectivity radius, but we miss the classes where the injectivity radius goes to 0 at infinity.

Finally, the non hyperkähler examples (3') and (3") are just global quotients of a previous ALH space, but can be obtained by desingularization of a flat orbifold space, as shows theorem 0.1. They are asymptotic to $\mathbb{R}_+ \times F_2$ and $\mathbb{R}_+ \times F_{2,2}$, where F_2 and $F_{2,2}$ are the compact flat orientable 3-manifolds arising as flat $\mathbb{T}^2$ bundles over $\mathbb{S}^1$ with monodromy $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is natural to ask whether there are ALH Ricci-flat Kähler metrics with an end asymptotic to $\mathbb{R}_+ \times F$, where F is one of the other orientable flat 3-manifolds: $F = F_3, F_4, F_6$, where F_j $(j = 3, 4, 6)$ is the flat 3-manifold with monodromy $\mathbb{Z}_j$. We answer positively to this question:

Theorem 0.2. *For each orientable flat 3-manifold* F*, there exists an ALH Ricci flat manifold with one end asymptotic to* $\mathbb{R}_+ \times F$*. It is Kähler except in the case* $F = F_{2,2}$*, where it is only locally kähler.*

The new metrics do not arise as desingularizations of flat orbifolds, but as global quotients of ALH hyperkähler manifolds. The proof of the theorem consists therefore in constructing ALH hyperkähler manifolds admitting actions of finite groups: the examples come from certain elliptic fibrations, and the metric arises from the solution of a Monge-Ampère equation, invariant under a finite group action (solving the Monge-Ampère equation in this setting is now more or less standard, so we just refer to the literature: the point here is the construction of examples admitting finite group actions).

Finally, the classification of gravitational instantons is an important open question. The ALE gravitational instantons were classified by Kronheimer, and the A_k ALF instantons by the second author [23]. The other possible class of ALF instantons is the class of D_k ALF gravitational instantons, with boundary $\mathbb{S}^3/D_k$, where D_k is the binary dihedral group of order $4(k-2)$, for $k>2$ (and we have proposed a new construction of these instantons in theorem 0.1). This gives all the possible topologies for the boundaries of an ALF gravitational instantons, but the possible orientations of the boundary are less clear (the two possible orientations correspond to 'positive mass' or 'negative mass'). For $k=0$, the D_0 gravitational instanton (the Atiyah-Hitchin metric) has the same boundary as the D_4 one, endowed with the opposite orientation. The same is true for the D_1 instanton (the double cover of the D_0 one) and the D_3 instanton. Finally the flat space $\mathbb{R}^3\times\mathbb{S}^1/\pm$ admits an orientation reversing isometry, so the opposite orientation of the boundary of the D_2 instanton is realized by the same space. We prove that these are the only possible cases:

Theorem 0.3. *There is no dihedral ALF gravitational instanton with boundary equal to $\mathbb{S}^3/D_k$ with negative orientation for $k>4$.*

The theorem is a consequence of theorem 4.1, where we relate the Euler characteristic of an ALF gravitational instanton with the adiabatic η invariant of its boundary: this determines the b_2 of an ALF gravitational instanton in terms of its oriented boundary, since the η invariant is sensitive to the orientation.

The paper is organized as follows. In the first section, we have chosen to give a detailed proof of theorem 0.1 in the simplest case, that is $X=\mathbb{R}\times\mathbb{T}^3/\pm$. The necessary adaptations for the other cases are explained in the second section. In the third section, we give a construction of the other ALH Ricci-flat Kähler surfaces, proving theorem 0.2. The last section contains the proof of the formula on the η invariant, leading to theorem 0.3. Finally, an appendix briefly reviews some facts about the weighted analysis on ALF, ALG or ALH manifolds, needed for the construction.

Acknowledgments. We thank Sergei Cherkis for useful discussions, and Hans-Joachim Hein for carefully checking the paper and suggesting proposition 2.2. Finally we thank the referee for his useful suggestions, making the paper much more readable.

1. A Kummer construction

In this section, we carefully build an ALH gravitational instanton asymptotic to $\mathbb{R}\times\mathbb{T}^3$. First, we consider the quotient $X := \mathbb{R}\times\mathbb{T}^3/\pm$. This is a complex orbifold with eight singular points, corresponding to the fixed points of minus identity on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. These are all rational double points and we may blow them up to get a non-singular complex manifold $\hat{X}$. We will build an *approximately* Ricci-flat metric on $\hat{X}$ by patching together two type of metrics : the flat metric away from the exceptional divisors and the Eguchi-Hanson metric near the divisors.

1.1. **The approximately Ricci-flat Kähler metric.** The Eguchi-Hanson metric can be described as follows. Take $\mathbb{C}^2/\pm$ and blow up the origin to get the minimal resolution $\pi : T^*\mathbb{C}P^1 \longrightarrow \mathbb{C}^2/\pm$. Outside the zero section $\mathbb{C}P^1 = \pi^{-1}(0)$ in $T^*\mathbb{C}P^1$, this map π is a biholomorphism. Let $z = (z_1, z_2)$ denote the standard complex coordinates on $\mathbb{C}^2$. The formula

$$\phi_{EH}(z_1,z_2) := \frac{1}{2}\left(\sqrt{1+|z|^4} + 2\log|z| - \log\left(1+\sqrt{1+|z|^4}\right)\right),$$

defines a function on $\mathbb{C}^2/\pm$ and therefore on $T^*\mathbb{C}P^1\backslash\pi^{-1}(0)$. Then $\omega_{EH} := dd^c\phi_{EH}$ extends on the whole $T^*\mathbb{C}P^1$ as a Kähler form, which turns out to be Ricci-flat. Moreover, the Kähler form ω_{EH} is asymptotic to the flat Kähler form $\omega_{EH,0} = dd^c\phi_{EH,0}$, with $\phi_{EH,0}(z) = \frac{|z|^2}{2}$:

$$\nabla^k(\phi_{EH} - \phi_{EH,0}) = \mathcal{O}(|z|^{-2-k}) \quad \text{and} \quad \nabla^k(\omega_{EH} - \omega_{EH,0}) = \mathcal{O}(|z|^{-4-k}).$$

For future reference, let us point that the Eguchi-Hanson metric admits a parallel symplectic $(2,0)$ form, extending (the pull-back of) $dz_1 \wedge dz_2$.

Now pick a smooth non-increasing function χ on $\mathbb{R}_+$ that is identically 1 on $[0,1]$ and vanishes on $[2,+\infty)$. Given a small positive number ϵ, we introduce the cut-off function $\chi_\epsilon(z) := \chi(\sqrt{\epsilon}\,|z|)$, on $T^*\mathbb{C}P^1$, and we define

$$\phi_{EH,\epsilon} := \chi_\epsilon\phi_{EH} + (1-\chi_\epsilon)\phi_{EH,0}. \tag{1}$$

Then $\omega_{EH,\epsilon} := dd^c\phi_{EH,\epsilon}$ is a $(1,1)$-form on $T^*\mathbb{C}P^1$ which coincides with the Eguchi-Hanson Kähler form for $|z| \le \frac{1}{\sqrt{\epsilon}}$ and coincides with the flat Kähler form on $\mathbb{C}^2/\pm$ for $|z| \ge \frac{2}{\sqrt{\epsilon}}$. In between, we have the following controls:

$$\left|\nabla^k(\phi_{EH,\epsilon} - \phi_{EH,0})\right| \le c(k)\sqrt{\epsilon}^{2+k} \quad \text{and} \quad \left|\nabla^k(\omega_{EH,\epsilon} - \omega_{EH,0})\right| \le c(k)\sqrt{\epsilon}^{4+k}.$$

In particular, for small ϵ, $\omega_{EH,\epsilon}$ is a Kähler form on the whole $T^*\mathbb{C}P^1$. (The letter c will always denote a positive constant whose value changes from line to line and we sometimes write $c(\dots)$ to insist on the dependence upon some parameters.)

We can now describe precisely our approximately Ricci flat Kähler metric ω_ϵ on $\hat{X}$. Let ρ denote the distance to the singular points in the flat orbifold $\mathbb{R}\times\mathbb{T}^3/\pm$. The function ρ can also be seen as the ω_0-distance to the exceptional divisors in $\hat{X}$. We will denote the connected components of $N := \left\{\rho \le \frac{1}{8}\right\}$ by N_j, $1 \le j \le 8$, and the remaining part of $\hat{X}$ by W. Each N_j contains an exceptional divisor E_j and $N_j\backslash E_j$ is naturally a punctured ball of radius $\frac{1}{8}$ and centered in 0 in

$\mathbb{C}^2/\pm \cong T^*\mathbb{C}P^1\backslash\mathbb{C}P^1$. Thanks to a $\frac{1}{\epsilon}$-dilation, we may therefore identify N_j with the set $N(\epsilon) := \left\{|z| \leq \frac{1}{8\epsilon}\right\}$ in $T^*\mathbb{C}P^1$ and then define ω_ϵ on N_j by $\omega_\epsilon := \epsilon^2\omega_{EH,\epsilon}$. On W, we then let ω_ϵ coincide with the flat ω_0. Owing to the shape of $\omega_{EH,\epsilon}$, this defines a smooth Kähler metric on the whole $\hat{X}$. By construction, ω_ϵ is the flat ω_0 for $\rho \geq 2\sqrt{\epsilon}$, a (scaled) Eguchi-Hanson metric for $\rho \leq \sqrt{\epsilon}$ and obeys

$$\left|\nabla^k(\omega_\epsilon - \omega_0)\right| \leq c(k)\epsilon^{2-\frac{k}{2}} \tag{2}$$

between these two areas (beware the scaling induces a nasty ϵ^{-k-2} factor). The Ricci form Ric_ϵ vanishes outside the domain $\{\sqrt{\epsilon} \leq \rho \leq 2\sqrt{\epsilon}\}$ and obeys $|\mathrm{Ric}_\epsilon| \leq c\epsilon$: it is small. We now wish to deform this approximately Ricci-flat Kähler metric into a genuine Ricci-flat Kähler metric.

1.2. **The nonlinear equation.** In view of building a Ricci-flat Kähler form $\omega = \omega_\epsilon + dd^c\phi$ in the Kähler class of ω_ϵ, we wish to solve the complex Monge-Ampère equation

$$(\omega_\epsilon + dd^c\psi)^2 = e^{f_\epsilon}\omega_\epsilon \wedge \omega_\epsilon, \tag{3}$$

where f_ϵ is essentially a potential for the Ricci form Ric_ϵ of ω_ϵ : $\mathrm{Ric}_\epsilon = \frac{1}{2}dd^c f_\epsilon$. This is the classical approach of Aubin-Calabi-Yau (cf. [15] for instance). Our framework gives an explicit function f_ϵ, which we can describe as follows. Let ζ_1, ζ_2 denote the complex coordinates on $\mathbb{C}^2$ and consider the $(2,0)$ form $d\zeta_1 \wedge d\zeta_2$ on $\mathbb{R} \times \mathbb{T}^3 = \mathbb{C}^2/\mathbb{Z}^3$. It is still defined on $X = \mathbb{R} \times \mathbb{T}^3/\pm$ and then lifts into a holomorphic $(2,0)$ form Ω on $\hat{X}$. Since $\hat{X}$ is a crepant resolution of X, this $(2,0)$ form Ω does not vanish along the exceptional divisors, providing a genuine symplectic $(2,0)$ form. We can then choose the following function f_ϵ:

$$f_\epsilon := \log\left(\frac{\Omega \wedge \bar{\Omega}}{\omega_\epsilon \wedge \omega_\epsilon}\right). \tag{4}$$

In other words, the right-hand side of (3) is simply $\Omega \wedge \bar{\Omega}$. Observe f_ϵ is compactly supported inside $\{\sqrt{\epsilon} \leq \rho \leq 2\sqrt{\epsilon}\}$ and obeys:

$$\left|\nabla^k f_\epsilon\right| \leq c(k)\epsilon^{2-\frac{k}{2}}. \tag{5}$$

1.3. **The linear estimate.** The linearization of the Monge-Ampère operator is essentially the Laplace operator Δ_ϵ. We need to show that it is an isomorphism between convenient Banach spaces and that its inverse is uniformly bounded for small ϵ. Let us introduce the relevant functional spaces. We denote the $\mathbb{R}$-variable in $\mathbb{R} \times \mathbb{T}^3$ by t and let $r := |t|$. We may choose a smooth positive function r_ϵ with the following properties:

- it is equal to r wherever r is larger than 1;
- it coincides with the distance ρ to the exceptional divisors wherever $2\epsilon \leq \rho \leq \frac{1}{8}$;
- it is identically ϵ wherever $\rho \leq \epsilon$;
- it is a non-decreasing function of ρ in the domain where $\epsilon \leq \rho \leq 2\epsilon$;
- it remains bounded between $\frac{1}{8}$ and 1 on the part of $\hat{X}$ where $\rho > \frac{1}{8}$ and $r < 1$.

Let k be a nonnegative integer and α be a number in $(0,1)$. Given positive real numbers a and b, we let $w_{\epsilon,i}$ denote the continuous function which coincides with r_ϵ^{a+i} on W and r_ϵ^{b+i} on N. Then we define the Banach space $C^{k,\alpha}_{\epsilon,a,b}$ as the set of $C^{k,\alpha}$ functions u for which the following quantity is finite:

$$\|u\|_{C^{k,\alpha}_{\epsilon,a,b}} := \sum_{i=0}^{k} \sup \left| w_{\epsilon,i} \nabla_\epsilon^i u \right|_\epsilon + \sup_{d_\epsilon(x,y)<\mathrm{inj}_\epsilon} \left| \min(w_{\epsilon,k+\alpha}(x), w_{\epsilon,k+\alpha}(y)) \frac{\nabla_\epsilon^k u(x) - \nabla_\epsilon^k u(y)}{d_\epsilon(x,y)^\alpha} \right|_\epsilon .$$

This formula provides the norm. The subscripts ϵ mean that everything is computed with respect to the metric g_ϵ; in particular, inj_ϵ denotes the *positive* injectivity radius of g_ϵ. In this definition, since $d_\epsilon(x,y) < \mathrm{inj}_\epsilon$, we can compare the values of $\nabla^k u_\epsilon$ at x and y through the parallel transport along the *unique* minimizing geodesic from x to y.

On the exterior domain W, a $C^{k,\alpha}_{\epsilon,a,b}$ control on u means it decays like r^{-a} (with a corresponding natural estimate on the derivatives). As expected, the parameter ϵ only matters near the exceptional divisors. Recall each N_j is identified with the domain $N(\epsilon) := \left\{ |z| \le \frac{1}{8\epsilon} \right\}$ in $T^*\mathbb{C}P^1$. Accordingly, we may carry any function u on N_j to a function u_ϵ on $N(\epsilon)$. A $C^0_{\epsilon,a,b}$ control on u basically means $|u_\epsilon| \le c\epsilon^{-b} |z|^{-b}$ for $2 \le |z| \le \frac{1}{8\epsilon}$ and $|u_\epsilon| \le c\epsilon^{-b}$ where $|z| \le 2$. Note that standard (scaled) Schauder estimates imply the following control:

$$\|u\|_{C^{k+2,\alpha}_{\epsilon,a,b}} \le c(k,\epsilon) \left(\|u\|_{C^0_{\epsilon,a,b}} + \|\Delta_\epsilon u\|_{C^{k,\alpha}_{\epsilon,a+2,b+2}} \right). \tag{6}$$

In this asymptotically cylindrical setting, we may have used exponential weights instead of polynomial weights. We have made this choice because 1) these weights suffice for our purpose here and 2) they will be adapted to the other settings (ALG, ALF, cf. appendix).

The relevant Banach spaces for us will turn out to be

$$E_\epsilon^{a,b} := \mathbb{R}\tilde{r} \oplus C^{2,\alpha}_{\epsilon,a,b} \quad \text{and} \quad F_\epsilon^{a,b} := C^{0,\alpha}_{\epsilon,a+2,b+2}.$$

The notation $\tilde{r}$ stands for the function $\phi \circ r$, where ϕ is any smooth non-decreasing function on $\mathbb{R}_+$ which is identically 1 on $[0,1]$ and coincides with the identity on $[2,+\infty)$. We fix the norm on $\mathbb{R}\tilde{r}$ by setting $\|\tilde{r}\| := 1$ and we endow $E_\epsilon^{a,b}$ with the sum of the norms of the two factors. Note we have dropped the dependence on α, which will not play a major role. The Laplacian Δ_ϵ defines a bounded operator between $E_\epsilon^{a,b}$ and $F_\epsilon^{a,b}$ and we need to invert it.

Lemma 1.1. *The map* $\Delta_\epsilon : E_\epsilon^{a,b} \longrightarrow F_\epsilon^{a,b}$ *is an isomorphism.*

Proof. Any function u in $E_\epsilon^{a,b}$ has the following asymptotics: $u = \lambda r + \mathcal{O}(r^{-a})$. When u is harmonic, integration by part yields for large R:

$$0 = \int_{B_R} \Delta_\epsilon u = -\int_{\partial B_R} \partial_r u = -\lambda + \mathcal{O}(R^{-a-1}),$$

with $B_R = \{r \leq R\}$. So $\lambda = 0$ and u goes to zero at infinity, hence vanishes. This proves injectivity. Let us turn to surjectivity. Given $f \in F_\epsilon^{a,b}$, weighted analysis (cf. appendix) provides a solution u to $\Delta_\epsilon u = f$, with

$$u = \lambda \tilde{r} + \eta + v$$

for some constants λ and η, and some $C^{2,\alpha}_{\mathrm{loc}}$ function v bounded in $C^{2,\alpha}_{\epsilon,a,b}$ (this estimate follows from (A.3) and (6)). Of course, the constant η can be dropped (for there is only one end) and we have proved the surjectivity. Note the parameter b is not relevant here. It enters the picture only when one seeks uniform bounds, which is the next topic we tackle. □

Lemma 1.2. *Let a and b be positive numbers, with $b < 2$. Then there is a constant c such that for every small ϵ and every $u \in E_\epsilon^{a,b}$,*

$$\|u\|_{E_\epsilon^{a,b}} \leq c \, \|\Delta_\epsilon u\|_{F_\epsilon^{a,b}} .$$

Proof. Assume this statement is false. Then there is a sequence of positive numbers ϵ_i and a sequence of functions u_i such that: $(\epsilon_i)_i$ goes to zero, $\|u_i\|_{E_{\epsilon_i}^{a,b}} = 1$ and $\|\Delta_{\epsilon_i} u_i\|_{F_{\epsilon_i}^{a,b}}$ goes to zero. Then $u_i = \lambda_i \tilde{r} + v_i$, with (λ_i) bounded in $\mathbb{R}$ and (v_i) uniformly bounded in $C^{2,\alpha}_{\epsilon_i,a,b}$. In particular, we may assume that (λ_i) goes to some λ_∞ and, with Arzela-Ascoli theorem, that (v_i) converges to v_∞ on compact subsets of $\hat{X}$ minus the exceptional divisors. Moreover, $u_\infty := \lambda_\infty \tilde{r} + v_\infty$ is a harmonic function on the regular part of $\mathbb{R} \times \mathbb{T}^3/\pm$ (with its flat metric) and is bounded by a constant times ρ^{-b} near each singular point, which is less singular than the Green function on $\mathbb{R}^4$, for $b < 2$. It follows that u_∞ can be lifted into a smooth harmonic function $\hat{u}_\infty$ on the whole $\mathbb{R} \times \mathbb{T}^3$. Moreover, $r^a v_\infty$ is bounded so $\hat{u}_\infty = \lambda_\infty r + \mathcal{O}(r^{-a})$. Integration by part yields for large R:

$$0 = \int_{B_R} \Delta \hat{u}_\infty = -\int_{\partial B_R} \partial_r \hat{u}_\infty = -2\lambda_\infty + \mathcal{O}(R^{-a}).$$

So $\lambda_\infty = 0$ and $\hat{u}_\infty$ goes to zero at infinity, hence vanishes: $v_\infty = 0$. It follows that $\|v_i\|_{C^0(K)}$ goes to zero for every compact set K outside the exceptional divisors. Using Lemma A.3, we then see that, for any smooth and compactly supported function τ which is identically 1 near the exceptional divisors, the functions $w_i := (1-\tau) v_i$ satisfy

$$\|r^a w_i\|_{L^\infty} = \|r^a G_{R_0} \Delta_0 w_i\|_{L^\infty} \leq c \, \|r^a \Delta_0 w_i\|_{L^\infty} \overset{i \to \infty}{\longrightarrow} 0.$$

With the scaled Schauder estimate

$$\|v_i\|_{C_a^{2,\alpha}(B_{2R_0}^c)} \leq \; c \left(\|r^a v_i\|_{L^\infty(B_{R_0}^c)} + \|\Delta_0 v_i\|_{C_a^{0,\alpha}(B_{R_0}^c)} \right)$$

(where we dropped the irrelevant ϵ and b) and the convergence over compact subsets, we therefore obtain that $\|v_i\|_{C_a^{2,\alpha}(W)}$ goes to zero; and more generally, this remains true when W is replaced by the complement of any compact neighbourhood of the exceptional divisors.

Next, we focus on what happens around the jth exceptional divisor. We consider the function $V_i := \epsilon_i^b (v_i)_{\epsilon_i}$, defined on $N(\epsilon_i)$. The bound on $\|v_i\|_{C^{2,\alpha}_{\epsilon_i,b}(N_j)}$ makes it possible to extract a subsequence V_i which converges to a function V_∞ on every compact subset of $T^*\mathbb{C}P^1$. Moreover, this function V_∞ is harmonic and uniformly bounded by a constant times $(1+|z|)^{-b}$. Since b is positive, we deduce that $V_\infty = 0$, as above. Now assume that $\|V_i\|_{C^0_b}$ remains bounded from below (up to subsequence). Then we can find a sequence of points p_i such that $R_i := |z(p_i)|$ goes to infinity and $R_i^b\,|V_i(p_i)|$ is bounded from below. Let us rescale the metric into $\xi_i := R_i^{-2}\omega_{\epsilon_i}$ and consider the functions $W_i := R_i^b V_i$. Since ω_{ϵ_i} is closer and closer to the Eguchi-Hanson metric and blowing down the Eguchi-Hanson metric results in the orbifold $\mathbb{R}^4/\pm$, we see that ξ_i converges to the flat metric on the orbifold $\mathbb{R}^4/\pm$. Each W_i is defined on a ball of radius $\frac{1}{8\epsilon_i R_i}$ with respect to ξ_i and we may assume that $\epsilon_i R_i$ goes to zero, in view of the convergence of v_i to zero away from the exceptional divisors (proved in the first step). Since (W_i) is uniformly bounded in $C^{2,\alpha}_b$ (defined with respect to ξ_i), we can again make it converge to a harmonic function W_∞ on $\mathbb{R}^4/\pm$ minus the origin. Let us denote by $|x|$ the distance to the origin in $\mathbb{R}^4/\pm$. By construction, $W_\infty(x_\infty) > 0$ for some point x_∞ with $|x_\infty| = 1$. But one can check that $|x|^b\, W_\infty(x)$ is uniformly bounded, so that the harmonic function W_∞ is bound to vanish, as above. This is a contradiction, so $\|V_i\|_{C^0_b}$ goes to zero. Scaled Schauder estimates imply that $\|V_i\|_{C^{2,\alpha}_b}$ and therefore $\|v_i\|_{C^{2,\alpha}_{\epsilon_i,b}(N_j)}$ go to zero. Playing the same game around each component of the exceptional divisor, we obtain a contradiction. □

1.4. **The deformation.** We will use the following version of the implicit function theorem, whose proof is an immediate application of Banach's fixed point theorem.

Lemma 1.3. *Let $\Phi : E \longrightarrow F$ be a smooth map between Banach spaces and define $Q := \Phi - \Phi(0) - d_0\Phi$. We assume there are positive constants q, r_0 and c such that*

(1) $\|Q(x) - Q(y)\| \le q\,\|x-y\|\,(\|x\| + \|y\|)$ *for every x and y in $B_E(0, r_0)$;*
(2) $d_0\Phi$ *is an isomorphism with inverse bounded by c.*

Pick $r < \min(r_0, \frac{1}{2qc})$ and assume $\|\Phi(0)\| \le \frac{r}{2c}$. Then the equation $\Phi(x) = 0$ admits a unique solution x in $B_E(0, r)$.

We apply this to the operator

$$\Phi_\epsilon \;:\; \psi \mapsto \frac{(\omega_\epsilon + dd^c\psi)^2}{\omega_\epsilon \wedge \omega_\epsilon} - e^{f_\epsilon},$$

between the Banach spaces $E^{a,b}_\epsilon$ and $F^{a,b}_\epsilon$ for some positive number a and a positive number b, with $b < 2$. The linearization of Φ_ϵ is $-\Delta_\epsilon$ so condition (2) stems from Lemmata 1.1 and 1.2, with a constant c independent of ϵ.

We need some smallness on $\Phi_\epsilon(0) = 1 - e^{f_\epsilon}$. In view of the shape of f_ϵ, including (5), we have

$$\|f_\epsilon\|_{C^{0,\alpha}_{\epsilon,a+2,b+2}} \le c\epsilon^{3+\frac{b}{2}}, \tag{7}$$

which leads to

$$\|\Phi_\epsilon(0)\|_{F^{a,b}_\epsilon} \le c\epsilon^{3+\frac{b}{2}}. \tag{8}$$

In view of condition (1), observe the non-linear term is the quadratic map given by:

$$Q_\epsilon(\psi) = \frac{dd^c\psi \wedge dd^c\psi}{\omega_\epsilon \wedge \omega_\epsilon}$$

For every ψ_1 and ψ_2 in $C^{2,\alpha}_{\epsilon,a,b}$, we have the following estimate:

$$\|Q_\epsilon(\psi_1) - Q_\epsilon(\psi_2)\|_{C^{0,\alpha}_{\epsilon,a+2,b+2}} \leq c\,\epsilon^{-b-2}\,\|\psi_1 - \psi_2\|_{C^{2,\alpha}_{\epsilon,a,b}} \left(\|\psi_1\|_{C^{2,\alpha}_{\epsilon,a,b}} + \|\psi_2\|_{C^{2,\alpha}_{\epsilon,a,b}}\right),$$

where c does not depend on ϵ. This uniform $C^{0,\alpha}_{\epsilon,a+2,b+2}$ control is pretty clear on W, without any ϵ in the constant, indeed. Near the divisors, we need to compensate the (small) weight, hence this unpleasant ϵ^{-b-2}. It is straightforward to extend to the case where ψ_1 and ψ_2 in $E^{a,b}_\epsilon$:

$$\|Q_\epsilon(\psi_1) - Q_\epsilon(\psi_2)\|_{F^{a,b}_\epsilon} \leq q_\epsilon\,\|\psi_1 - \psi_2\|_{E^{a,b}_\epsilon} \left(\|\psi_1\|_{E^{a,b}_\epsilon} + \|\psi_2\|_{E^{a,b}_\epsilon}\right),$$

with

$$q_\epsilon = c\,\epsilon^{-b-2}. \tag{9}$$

In order to use Lemma 1.3, we compare $\|\Phi_\epsilon(0)\|_{F^{a,b}_\epsilon}$ to $\frac{1}{q_\epsilon}$: in view of (8) and (9), we have

$$\|\Phi_\epsilon(0)\|_{F^{a,b}_\epsilon} \leq c\epsilon^{1-\frac{b}{2}} q_\epsilon^{-1}. \tag{10}$$

Then, since $b < 2$, we see that for small ϵ, $\|\Phi_\epsilon(0)\|_{F^{a,b}_\epsilon}$ is much smaller than $\frac{1}{q_\epsilon}$ and we can use Lemma 1.3 to prove the following theorem. Recall that we denote the exceptional divisors by $E_1, \dots, E_p$ and the Poincaré dual of E_j by $PD[E_j]$. Observe we can pick different deformation parameters $\epsilon_j = a_j\epsilon$ around the divisors E_j, in order to get a larger range of examples.

Theorem 1.4. *Let $a_1, \dots, a_8$ denote some positive parameters. Then for every small enough positive number ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}$ in the cohomology class $[\omega_0] - \epsilon \sum a_j PD[E_j]$. These provide ALH gravitational instantons asymptotic to $\mathbb{R}_+ \times \mathbb{T}^3$: $\omega = \omega_0 + \mathcal{O}(r^{-\infty})$.*

The notation $\mathcal{O}(r^{-\infty})$ denotes a function decaying faster than any (negative) power of r (cf. appendix, after Lemma A.2). By working with exponential weights, we may prove that the decay rate to the flat metric is indeed exponential.

Proof. We can apply Lemma 1.3 to solve the Monge-Ampère equation (3) for small enough ϵ, which provides a $(1,1)$-form $\omega = \omega_\epsilon + dd^c\psi$, with $\psi \in E^{a,b}_\epsilon$. Since ω is asymptotic to ω_ϵ, it is positive outside a compact set. From (3), we know that ω is everywhere non-degenerate, so it must remain positive on $\hat{X}$: it is a Kähler form. Moreover, its Ricci form is given by $\rho_\epsilon - \frac{1}{2}dd^c f_\epsilon$, which vanishes in view of our choice of f_ϵ, so ω is Ricci-flat. Finally, a Ricci-flat Kähler structure on a simply-connected four-dimensional manifold is the same as a hyperkähler structure, so we only need to check that $\hat{X}$ is simply connected. First, observe that $X = \mathbb{R} \times \mathbb{T}^3/\pm$ retracts onto $\mathbb{T}^3/\pm$, which is covered by two open sets U_1 and U_2 with connected intersection, such that U_1 and U_2 are homeomorphic to $[0, \frac{1}{2}) \times \mathbb{T}^2/\sim$ where, for any $x \in \mathbb{T}^2$, $(0,x) \sim (0,-x)$. Since $[0, \frac{1}{2}) \times \mathbb{T}^2/\sim$ retracts onto $\mathbb{T}^2/\pm$, which is homeomorphic to the 2-sphere, we eventually see that X, and therefore $\hat{X}$, is simply-connected. □

2. Other similar constructions

2.1. New ALH Ricci-flat manifolds. In the previous example, it is natural to try and replace $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ by another compact flat orientable three-manifold. Let F_2 and $F_{2,2}$ denote the smooth flat three-manifolds obtained as $F_2 := \mathbb{T}^3/\sigma$ and $F_{2,2} := \mathbb{T}^3/\langle\sigma,\tau\rangle$, where σ and τ are the two commuting involutions

$$\sigma(x,y,z) = (x+\frac{1}{2}, -y, -z+\frac{1}{2}),$$
$$\tau(x,y,z) = (-x, -y+\frac{1}{2}, z+\frac{1}{2}).$$

Then $\mathbb{R}\times F_2$ is naturally a complex flat Kähler manifold—a quotient of $\mathbb{C}^2$, indeed. More specifically, if t is the coordinate along $\mathbb{R}$, we take $t+ix$ and $y+iz$ as complex coordinates.

We then consider the complex flat Kähler orbifold $X_2 := \mathbb{R}\times F_2/\pm$. The reader may check that this involution is well defined and has four fixed points, yielding rational double points. We blow them up to obtain the complex manifold $\hat{X}_2$.

Proposition 2.1. *Let $a_1,\dots,a_4$ denote some positive parameters. Then for every small enough positive number ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}_2$ in the cohomology class $[\omega_0] - \epsilon\sum a_j PD[E_j]$. These provide ALH manifolds asymptotic to the flat metric on $\mathbb{R}_+\times F_2$: $\omega = \omega_0 + \mathcal{O}(r^{-\infty})$.*

In this statement, ω_0 denotes again the pull-back of the flat Kähler form on X_2 and $r = |t|$. Actually the proposition is a direct consequence of theorem 1.4, since one can perform first the quotient by $\pm$ and then by σ. Indeed the involution σ acts freely on $\mathbb{R}\times\mathbb{T}^3/\pm$ and on its desingularization, say $\hat{X}_1$; if the Kähler class in theorem 1.4 is invariant, then it is obvious that the whole construction can be made σ invariant, so the resulting metric descends on $\hat{X}_2$.

It follows that the fundamental group of $\hat{X}_2$ is $\mathbb{Z}_2$. The metrics are not hyperkähler because the holomorphic symplectic form Ω on $\hat{X}_1$ satisfies $\sigma^*\Omega = -\Omega$, so there is only a multivalued symplectic form on $\hat{X}_2$. (This is also apparent on the flat model $\mathbb{R}\times F_2$, whose holonomy is not in $SU(2)$.)

Finally the involution τ is real with respect to the above choice of complex structure, and acts freely on X_2 and $\hat{X}_2$, on which it exchanges the four curves E_j, say for example $\tau E_1 = -E_2$ and $\tau E_3 = -E_4$. This leads to:

Proposition 2.2. *With the same notations as above, if $a_1 = a_2$ and $a_3 = a_4$ then the metric of proposition 2.1 is τ invariant so descends to a Ricci flat, locally Kähler metric on $\hat{X}_{2,2} := \hat{X}_2/\tau$. This is an ALH metric with an end asymptotic to $\mathbb{R}_+\times F_{2,2}$.*

Again the whole construction can be made σ and τ invariant (in particular, looking for a τ invariant potential), so the proposition is immediate.

2.2. **ALG gravitational instantons.** In order to build ALG examples with the same technique, we may start from $\mathbb{R}^2 \times \mathbb{T}^2$ and consider 'crystallographic' quotients. The basic example is $X_2 = \mathbb{R}^2 \times \mathbb{T}^2/\pm$, which is a complex flat Kähler orbifold with four rational double points. When $\mathbb{T}^2$ is obtained from a square lattice in $\mathbb{R}^2$, we may also consider $X_2 = \mathbb{R}^2 \times \mathbb{T}^2/\mathbb{Z}_4$, where the action of $\mathbb{Z}_4$ is induced by the rotation of angle $\frac{\pi}{2}$ on both factors. In this case, there are two $\mathbb{C}^2/\mathbb{Z}_4$ singularities and one $\mathbb{C}^2/\mathbb{Z}_2$ singularity. Similarly, starting from a hexagonal lattice and using rotations of angle $\frac{\pi}{3}$ and $\frac{\pi}{6}$, we may work with $X_k = \mathbb{R}^2 \times \mathbb{T}^2/\mathbb{Z}_k$ for $k = 3$ or 6. The orbifold X_3 has three $\mathbb{C}^2/\mathbb{Z}_3$ singularities, while X_6 has one $\mathbb{C}^2/\mathbb{Z}_6$ singularity, one $\mathbb{C}^2/\mathbb{Z}_2$ singularity and one $\mathbb{C}^2/\mathbb{Z}_3$ singularity. In any case, we may blow up the singularities to get the smooth complex manifold $\hat{X}_k$, $k = 2, 3, 4, 6$. Every $\mathbb{C}^2/\mathbb{Z}_k$ singularity can be endowed with an asymptotically locally Euclidean (ALE) Ricci-flat Kähler metric: the Gibbons-Hawking or multi-Eguchi-Hanson metrics [10, 15]. We may use them in the gluing procedure. We do not have an explicit Kähler potential ϕ but for instance Theorem 8.2.3 in [15] gives a potential $\phi = \frac{|z|^2}{2} + \mathcal{O}(|z|^{-2})$, which is what we need.

Theorem 2.3. *Pick $k = 2, 3, 4, 6$ and let $a_1, \dots, a_p$ denote some positive parameters (p is the number of singularities). Then for every small enough positive number ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}$ in the cohomology class $[\omega_0] - \epsilon \sum a_j PD[E_j]$. These provide ALG gravitational instantons asymptotic to $\mathbb{R}^2 \times \mathbb{T}^2/\mathbb{Z}_k$: $\omega = \omega_0 + \mathcal{O}(r^{-k-2+\delta})$, for every positive δ.*

Proof. The proof follows the same lines as that of theorem 1.4, so we just point out the necessary adaptations. To begin with, we may check that $\hat{X}_k$ is simply connected: this follows from the fact that $\mathbb{T}^2/\mathbb{Z}_k$ is homeomorphic to the two-sphere (it is for instance a consequence of the Gauss-Bonnet formula for closed surfaces with conical singularities). In view of weighted analysis, we can work with $0 < a < k$ (because there is no harmonic function on $\mathbb{R}^2/\mathbb{Z}_k$ that decays like r^{-a}) and in the definition of $E^{a,b}_\epsilon$, the $\mathbb{R}\tilde{r}$ summand has to be replaced by $\mathbb{R}\widetilde{\log r}$ (a smooth function equal to $\log r$ outside a compact set). □

2.3. **Hitchin's ALF gravitational instanton.** In [14], N. Hitchin built a hyperkähler structure on the desingularization $\hat{X}$ of $\mathbb{R}^3 \times \mathbb{S}^1/\pm$ through twistor theory. Beware $\mathbb{S}^1$ is again seen as $\mathbb{R}/\mathbb{Z}$ (so the involution is not an antipodal map). Our direct analytical approach gives another construction of this hyperkähler manifold.

Theorem 2.4. *Let a_1, a_2 denote some positive parameters. Then for every small enough positive number ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}$ in the cohomology class $[\omega_0] - \epsilon \sum a_j PD[E_j]$. These provide ALF gravitational instantons asymptotic to $\mathbb{R}^3 \times \mathbb{S}^1/\pm$: $\omega = \omega_0 + \mathcal{O}(r^{-3+\delta})$, for every positive δ.*

Proof. Again, it is a simple adaptation of the proof of theorem 1.4. We work with $0 < a < 1$, so that weighted analysis ensures the Laplacian is an isomorphism. In the definition of $E^{a,b}_\epsilon$, the $\mathbb{R}\tilde{r}$ must therefore be dropped. The analysis can then be done similarly and we just need to check that $\hat{X}$ is simply connected: this is immediate, for $\mathbb{R}^3 \times \mathbb{S}^1/\pm$ turns out to be contractible. □

2.4. **ALF gravitational instantons of dihedral type.** In this section, we start from the Taub-NUT metric g_{TN} on $\mathbb{R}^4$. It is given by the following explicit formulas. We refer to [18] for details. To begin with, we identify $\mathbb{R}^4$ with $\mathbb{C}^2$, with complex coordinates w_1, w_2. The Hopf fibration $\pi = (x_1, x_2, x_3) : \mathbb{C}^2 \longrightarrow \mathbb{R}^3$ is given by

$$x_1 = 2\,\mathrm{Re}(w_1\bar{w_2}), \quad x_2 = 2\,\mathrm{Im}(w_1\bar{w_2}), \quad x_3 = |w_1|^2 - |w_2|^2 .$$

Let us fix a positive number m and define

$$V = 1 + \frac{2m}{|x|}, \quad \theta = 4m\frac{\mathrm{Im}(\bar{w}dw)}{|w|^2}. \tag{11}$$

Then $\frac{\theta}{4m}$ is a connection one-form on the Hopf fibration, with curvature $\frac{*_{\mathbb{R}^3}dV}{4m}$, and the Taub-NUT metric is given by

$$g_{TN} = V(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{V}\theta^2. \tag{12}$$

It turns out to be a complete Kähler metric, with respect to the complex structure I mapping dx_1 to dx_2 and dx_3 to $\frac{\theta}{V}$. The corresponding Kähler form is

$$\omega_{TN} = V dx_1 \wedge dx_2 + dx_3 \wedge \theta.$$

Moreover, it is endowed with a parallel symplectic $(2,0)$ form:

$$\Omega = (V dx_2 \wedge dx_3 + dx_1 \wedge \theta) + i(V dx_3 \wedge dx_1 + dx_2 \wedge \theta).$$

This holomorphic symplectic structure is in fact isomorphic to the standard one on $\mathbb{C}^2$ [18]. The Taub-NUT metric is therefore hyperkähler.

This hyperkähler structure is preserved by an action of the binary dihedral group D_k (of order $4(k-2)$) for every $k > 2$. Explicitly, we see D_k as the group generated by the following diffeomorphisms τ and ζ_k of $\mathbb{C}^2$:

$$\tau(w_1, w_2) = (\bar{w_2}, -\bar{w_1}), \quad \zeta_k(w_1, w_2) = (e^{\frac{i\pi}{k-2}}w_1, e^{\frac{i\pi}{k-2}}w_2).$$

And the reader may check that this action preserves the whole hyperkähler structure.

We then let X be the orbifold obtained as the quotient of the Taub-Nut manifold by this action of D_k. It has one complex singularity, isomorphic to the standard $\mathbb{C}^2/D_k$ (with D_k in $SU(2)$). Let us denote the minimal resolution of X by $\hat{X}$. Again, we need approximately Ricci-flat metrics on $\hat{X}$. Near the exceptional divisor, it is natural to glue one of the D_k ALE gravitational instantons introduced by P. B. Kronheimer in [17]. This yields a potential $\phi_{ALE} = \frac{|z|^2}{2} + \mathcal{O}(|z|^{-2})$. If we implemented the same gluing as above, we would be in trouble, basically because g_{TN} is not flat. Technically, we would end up with $f_\epsilon = \mathcal{O}(\epsilon)$ (instead of ϵ^2 in (5)); therefore, we would lose an ϵ in (7) and (8) and the exponent in (10) would be $-\frac{b}{2}$ instead of $1-\frac{b}{2}$; since b has to be positive, this exponent would be negative and we would never find a ball where to perform the fixed point argument. So we need to refine the gluing.

Near the origin, we can find complex coordinates $s = (s_1, s_2)$ in which ω_{TN} is the standard flat Kähler form $\omega_0 = dd^c\frac{|s|^2}{2}$ up to $\mathcal{O}(|s|^2)$ and, more precisely, we can find a potential ϕ_{TN} for ω_{TN} with the expansion

$$\phi_{TN} = \frac{|s|^2}{2} + \theta_4(s) + \mathcal{O}(|s|^5), \tag{13}$$

where $\theta_4(s)$ is a D_k-invariant quartic expression in s (and $\bar{s}$). Moreover, since ω_{TN} is Ricci-flat, we have the Monge-Ampère equation

$$dd^c \log\left(\frac{\omega_{TN}^2}{\omega_0^2}\right) = 0,$$

which can be expanded into $dd^c\Delta_{\omega_0}\theta_4 = \mathcal{O}(|s|)$. Since $\Delta_{\omega_0}\theta_4$ is a quadratic form, it is bound to vanish, so θ_4 is harmonic. We then identify a neighbourhood of 0 in $\hat{X}$ with a large domain in the D_k ALE gravitational instanton, in the same manner as previously, with an ϵ-dilation $s \mapsto \epsilon z$. Then $\theta_4(s) = \epsilon^4\theta_4(z)$. Since θ_4 is harmonic with respect to the flat metric, we see that $\Delta_{ALE}\theta_4 = \mathcal{O}(|z|^{-2})$. From weighted analysis, we may then find a Δ_{ALE}-harmonic function h_4 with $h_4 = \theta_4(z) + \mathcal{O}(|z|^{-2})$. Instead of gluing the Taub-NUT metric with the (scaled) ALE metric, we will patch the Taub-NUT potential ϕ_{TN} together with $\epsilon^2(\phi_{ALE} + \epsilon^2 h_4)$, namely the approximately Ricci-flat metric ω_ϵ we use in this context is given by the potential

$$\phi_\epsilon := \chi_\epsilon\left[\epsilon^2(\phi_{ALE} + \epsilon^2 h_4)\right] + (1 - \chi_\epsilon)\phi_{TN},$$

with a cutoff function χ_ϵ like in (1). The (1,1)-form $\omega_{loc,\epsilon} = dd^c(\phi_{ALE} + \epsilon^2 h_4)$ therefore plays the role of the Eguchi-Hanson metric ω_{EH} in this context. Beware it depends on ϵ and defines a Kähler metric (a priori) only on some ball $\left\{|z| \leq \frac{c}{\epsilon}\right\}$, owing to the estimate

$$|\omega_{loc,\epsilon} - \omega_{ALE}| = \epsilon^2\left|dd^c h^4\right| \leq c\epsilon^2 |z|^2. \tag{14}$$

We need a control on the function f_ϵ given by (4). Since Taub-NUT is Ricci flat, f_ϵ vanishes for $\rho \geq 2\sqrt{\epsilon}$. On $\{\rho \leq \sqrt{\epsilon}\}$, we may use the Ricci-flat ω_{ALE} and observe

$$e^{-f_\epsilon} = \frac{\omega_{loc,\epsilon}^2}{\omega_{ALE}^2} = 1 + 2\epsilon^2\Delta_{ALE}h_4 + \epsilon^4\frac{(dd^c h_4)^2}{\omega_{ALE}^2} = 1 + 0 + \epsilon^4\mathcal{O}(|z|^4) = \mathcal{O}(\rho^4),$$

which results in $\left|\nabla^k f_\epsilon\right| \leq c(k)\epsilon^{2-\frac{k}{2}}$. Finally, on the transition area $\{\sqrt{\epsilon} \leq \rho \leq 2\sqrt{\epsilon}\}$, we use the expansions (13) and

$$\phi_{ALE} = \frac{|z|^2}{2} + \mathcal{O}(|z|^{-2}) = \epsilon^{-2}\frac{|s|^2}{2} + \epsilon^2\mathcal{O}(|s|^{-2})$$

to obtain

$$\omega_\epsilon - \omega_{TN} = dd^c\left(\epsilon^4\chi_\epsilon\mathcal{O}(|s|^{-2}) + \chi_\epsilon\mathcal{O}(|s|^5)\right).$$

Note that without the trick consisting in plugging this function h_4 into the potential, the last exponent would have been 4 instead of 5, resulting in the bad estimate $f_\epsilon = \mathcal{O}(\epsilon)$. Instead, here, we find

$$\left|\nabla^k(\omega_\epsilon - \omega_{TN})\right| \leq c(k)\epsilon^{\frac{3-k}{2}}$$

and eventually

$$\left|\nabla^k f_\epsilon\right| \leq c(k)\epsilon^{\frac{3-k}{2}}.$$

If we follow the proof detailed above, this leads to an exponent $\frac{1-b}{2}$ in (10), which is good enough since we can choose any b in $(0,1)$. The other arguments can be adapted. In particular, the proof of Lemma 1.2 still works, because $\omega_{loc,\epsilon}$ gets closer and closer to the ALE metric on larger and larger domains, cf. (14).

Theorem 2.5. *For every small ϵ, there is a Ricci-flat Kähler form ω on $\hat{X}$ in the cohomology class $[\omega_{TN}] - \epsilon^2 PD[E]$, where $PD[E]$ denotes the Poincaré dual of the exceptional divisor. These provide ALF gravitational instantons : $\omega = \omega_{TN} + \mathcal{O}(r^{-3+\delta})$, for every positive δ.*

These ALF gravitational instantons are of dihedral type in the sense of [23]. For $k = 3$ (resp. $k = 4$), they have the same asymptotics as the Atiyah-Hitchin metric, that is the D_0 ALF gravitational instanton (resp. its double cover, the D_1 ALF gravitational instanton), with the difference that they have positive mass: their metric is asymptotic to g_{TN} with a positive parameter m, in contrast with the Atiyah-Hitchin metric where the model at infinity is Taub-NUT with a negative parameter m. As we shall see in the next section, these are the only two cases where this can happen. Also note that the examples we build presumably coincide with the D_k ALF metrics of Cherkis-Dancer-Hitchin-Kapustin [4, 5, 8].

Remark. The class of ALF gravitational instantons of cyclic type (whose boundary is fibered over $\mathbb{S}^2$) is completely classified [23]: it is the class of multi-Taub-NUT metrics, with boundaries at infinity $\mathbb{S}^3$ quotiented by the cyclic group A_k ($k \geq 0$, the $k = 0$ case is the Taub-Nut metric on $\mathbb{R}^4$ described above). One should add one special case, the flat space $\mathbb{R}^3 \times \mathbb{S}^1$ which can be numbered A_{-1} (this fits well with several formulas in § 4).

As mentioned to us by S. Cherkis, one can also construct D_k ALF gravitational instantons starting from an A_{2k-5} ALF gravitational instanton (a multi-Taub-NUT metric associated to a symmetric configuration of $2k-4$ points), and taking the quotient by an involution with two fixed points. The same technique applies and provides a hyperkähler metric on the desingularization. The special case $k = 2$ leads to the construction of a D_2 ALF gravitational instanton (conventionally the Hitchin metric) from a A_{-1} one, that is from $\mathbb{R}^3 \times \mathbb{S}^1$: this is the construction in section 2.3.

3. Other ALH Ricci-flat Kähler examples

There are six oriented compact flat 3-manifolds [30]: the torus $\mathbb{T}^3$, four quotients $F_j = \mathbb{T}^3/\mathbb{Z}_j$ for $j = 2, 3, 4, 6$ and a quotient $\mathbb{T}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. In § 2.1, we constructed by quotient a Kähler Ricci-flat metric with one ALH end asymptotic to $\mathbb{R} \times F_2$. In this section we will exhibit similar examples with one end asymptotic to $\mathbb{R} \times F_j$ for $j = 3, 4, 6$. This amounts to construct suitable rational elliptic surfaces with finite group action.

Choose $\zeta_j = \exp(2\pi i/j)$ and a flat 2-torus $\mathbb{T}^2$ with an action of $\mathbb{Z}_j$. Then the flat manifold F_j is obtained as the quotient of $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{T}^2$ by the diagonal action of $\mathbb{Z}_j$ obtained by multiplication by ζ_j on both factors. The flat metric

$$dt^2 + dx^2 + dy^2 + dz^2 \tag{15}$$

on $\mathbb{R} \times \mathbb{T}^3$ descends to a flat Kähler metric on $\mathbb{R} \times F_j$, but the holomorphic-symplectic form $\Omega = (dt + idx) \wedge (dy + idz)$, which has a simple pole at infinity in the compactification $\mathbb{P}^1 \times \mathbb{T}^2$, becomes j-multivalued in the quotient: the metric is not hyperkähler since the monodromy at infinity is not a subgroup of $SU(2)$. (For the last flat 3-manifold the monodromy is not a subgroup of $U(2)$ so one can not hope to construct Kähler examples, but one can still hope to construct actions leading to ALH Ricci flat examples.)

We start from a rational elliptic surface X_j with:

- if $j = 3$, three singular fibres of type IV;
- if $j = 4$, four singular fibres of type III;
- if $j = 6$ six singular fibres of type II.

A glance at the table in [24, p. 206] shows that such surfaces exist. One can construct them in a concrete way using the Weierstrass model: if $L = \mathcal{O}_{\mathbb{P}^1}(1)$, and g_2 and g_3 are holomorphic sections of L^4 and L^6, then the surface

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 \quad \text{in } \mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_{\mathbb{P}^1}) \tag{16}$$

is a rational elliptic surface. In case $g_3 = 0$ and g_2 has four simple zeros, one gets X_4; if $g_2 = 0$ and g_3 has six simple zeros, one gets X_6; if $g_2 = 0$ and g_3 has three double zeros one gets X_3. Moreover we can choose g_2 and g_3 so that X_j has an action of $\mathbb{Z}_j$ over $\mathbb{P}^1$ which permutes the singular fibres. For example we take the standard action of $\mathbb{Z}_j$ on $\mathbb{P}^1$ by $z \mapsto \zeta_j z$ and we use $g_2(u) = u^4 - 1$ for $j = 4$, $g_3(u) = u^6 - 1$ for $j = 6$ and $g_3(u) = (u^3 - 1)^2$ for $j = 3$.

Given any fibre, there is a holomorphic symplectic form on X_j with a simple pole along this fibre, giving a section of $K(F)$. We choose $\Omega \in H^0(X_j, K(F))$ the symplectic form with a simple pole over the fibre at infinity, so that near infinity one has $\Omega \sim \frac{dz}{z} \wedge dv$, where v is a coordinate on the fibre at infinity.

The action of $\mathbb{Z}_j$ on $\mathbb{P}^1$ has fixed points 0 and ∞. The action can be chosen so that it is free on the fibre over the origin (translation), but has fixed points on the fibre at infinity, giving Kleinian singularities of type $\mathbb{C}^2/\mathbb{Z}_j$ on the quotient $X_j/\mathbb{Z}_j$. The minimal desingularization $\hat{X}_j$ is again an elliptic surface over $\mathbb{P}^1$, with a multiple fibre of order j over the origin, a singular fibre of type IV^* ($j = 3$), III^* ($j = 4$) or II^* ($j = 6$) over the point at infinity, and similarly a singular fibre of type IV, III or II over $u = 1$. Moreover, the section Ω^j descends as a section $\hat{\Omega} \in H^0(\hat{X}_j, K^j(F))$ which does not vanish on $\hat{X}_j$ and has a simple pole over ∞ (in other words, $\hat{\Omega}^{\frac{1}{j}}$ is a multivalued holomorphic symplectic form outside the fibre over ∞).

Given a Kähler form ω which is asymptotic to (15), an ALH Kähler Ricci flat metric on $\hat{X}_j$ is given by a solution of the Monge-Ampère equation

$$(\omega + i\partial\bar{\partial} f)^2 = \Omega^{\frac{1}{j}} \wedge \overline{\Omega^{\frac{1}{j}}}, \tag{17}$$

where f has exponential decay on the end $\mathbb{R} \times F_j$. One can either solve directly on $\hat{X}_j$ or find a $\mathbb{Z}_j$-invariant solution on X_j: this amounts to solving the Monge-Ampère equation for cylindrical ends, and we refer to [27, 15, 16]. More specifically the case of X_j is done in [13].

Using the same construction, one can recover the ALH example $\hat{X}_2$ of § 2.1 for $j = 2$, starting from a rational elliptic surface with two singular fibres of type I_0^* with an action of $\mathbb{Z}_2$. In that case, our desingularization procedure of the flat metric $\mathbb{R} \times \mathbb{T}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ gives a good approximation of certain solutions of (17). This flat model is no more available for $j = 3, 4, 6$.

Remark. It might seem disappointing that these non-hyperkähler examples occur as finite quotients of hyperkähler manifolds. It turns out to be a general fact: any Ricci-flat Kähler four-manifold with ALE, ALF, ALG or ALH asymptotics is bound to have a hyperkähler finite cover. To see why, observe that such a manifold M has a flat canonical bundle (because $\mathrm{Ric} = 0$), determined by a representation ρ of $\pi_1(M)$ in $\mathbb{C}$. Building a hyperkähler finite cover amounts to finding a subgroup G of $\pi_1(M)$ of finite index and on which ρ is trivial. Now, since $\mathrm{Ric} = 0$, the Weitzenböck formula ensures the L^2 cohomology vanishes in degree 1. In terms of standard De Rham cohomology, this implies [1] that the image of the natural map $H^1_c(M) \to H^1(M)$ is trivial. Since the complement of a compact set in M is diffeomorphic to $\mathbb{R}_+ \times S$ for some compact 3-manifold S, this means $H^1(M)$ injects into the cohomology space $H^1(S)$ of the 'boundary at infinity' S or in other words $H_1(S, \mathbb{R})$ surjects onto $H_1(M, \mathbb{R})$. In all ALE,F,G,H asymptotics, $H_1(S, \mathbb{R})$ is generated by a finite number of loops γ_i for which some iterate $\gamma_i^{k_i}$ acts trivially on the canonical bundle of M. So the subgroup G of $\pi_1(M)$ generated by the derived subgroup $[\pi_1(M), \pi_1(M)]$ and the $\gamma_i^{k_i}$'s has the required properties.

4. A formula for the Euler characteristic

Let X be an ALF gravitational instanton of dihedral type or cyclic type. Near infinity, one has $X \simeq (A, +\infty) \times S$, where S has a circle fibration over $\Sigma = \mathbb{R}P^2$ (dihedral case) or $\Sigma = \mathbb{S}^2$ (cyclic case). Moreover the metric g has the following asymptotics:

$$g \simeq dr^2 + r^2\gamma + \theta^2,$$

where θ is a connection 1-form on the circle bundle (or its double covering in the dihedral case), and γ is the horizontal metric lifted from the standard metric on Σ. We have the following behavior for the second fundamental form $\mathbb{I}$ and the curvature R:

$$|\mathbb{I}| = \mathcal{O}(\tfrac{1}{r}), \quad |R| = \mathcal{O}(\tfrac{1}{r^3}). \tag{18}$$

There are well known formulas giving the Euler characteristic and signature of X in terms of the integrals of characteristic classes on a large domain $D_\rho = \{r \leq \rho\} \subset X$

and boundary terms: for a gravitational instanton, there remains only

$$\chi = \frac{1}{8\pi^2}\int_{D_\rho} |W_-|^2 + \frac{1}{12\pi^2}\int_{\partial D_\rho} \mathfrak{T}(\mathbb{I}\wedge(\mathbb{I}\wedge\mathbb{I}+3R)),$$
$$\tau = \frac{1}{12\pi^2}\int_{D_\rho} -|W_-|^2 + \frac{1}{12\pi^2}\int_{\partial D_\rho} \mathfrak{S}(\mathbb{I}(\cdot, R(\cdot,\cdot)n)) + \eta(\partial D_\rho).$$

Here n is the normal vector, $\mathfrak{T}$ and $\mathfrak{S}$ are linear operations which we do not need to write down explicitly, since from the control (18) and the fact that the volume of ∂D_ρ is $\mathcal{O}(\rho^2)$, we obtain that all boundary integrals go to zero when ρ goes to infinity. Finally this implies the following form of the Hitchin-Thorpe inequality:

$$2\chi + 3\tau = \lim_{\rho\to\infty} \eta(\partial D_\rho). \tag{19}$$

For the gravitational instanton X, if $X \neq \mathbb{R}^3\times\mathbb{S}^1$ we have $b_1(X) = b_3(X) = 0$, and the intersection form is negative definite (see [11], this follows immediately from the fact that the relevant cohomology classes can be represented by L^2 harmonic forms), so it follows that $\tau = -(\chi - 1)$. On the other hand, since the η-invariant is conformally invariant, the limit in (19) is the adiabatic limit:

$$\eta_{\mathrm{ad}}(S) := \lim_{r\to\infty} \eta(\gamma + \frac{1}{r^2}\theta^2).$$

Therefore we obtain the following result:

Theorem 4.1. *For an ALF gravitational instanton $X \neq \mathbb{R}^3\times\mathbb{S}^1$, with boundary S, one has*

$$\chi(X) = 3\Big(1 - \eta_{\mathrm{ad}}(S)\Big). \tag{20}$$

The calculation of the adiabatic limit of the η-invariant is well known, but we can also deduce it from the theorem: in both cyclic and dihedral cases, we have examples obtained by desingularizing the quotient of $\mathbb{C}^2$ with the Taub-NUT metric by the cyclic group A_k (this gives the multi-Taub-NUT metrics), or the dihedral group D_k (the metrics coming from theorem 2.5). This results in a k-dimensional 2-cohomology and therefore $\chi = k + 1$ and

$$\eta_{\mathrm{ad}} = \frac{2-k}{3}. \tag{21}$$

In the dihedral case, the formula extends immediately to the D_2 case, which is Hitchin's metric on the desingularization of $\mathbb{R}^3\times\mathbb{S}^1/\pm$. In this way the values of (21) for $k \geq 2$ give the adiabatic η invariant for all possible boundaries S of an ALF gravitational instanton. Nevertheless observe that the sign of the η-invariant is changed if the orientation of S is changed.

From the theorem 4.1, since one must have $\chi \geq 1$, one deduces the constraint

$$\eta_{\mathrm{ad}}(S) \leq \frac{2}{3}. \tag{22}$$

From the values obtained in (21), we see that the only three cases where the boundary S of a dihedral ALF gravitational instanton, endowed with the opposite orientation, can be filled by another gravitational instanton, are $k = 2$, 3 or 4. Indeed, for $k = 4$, the D_0 gravitational instanton (the Atiyah-Hitchin metric) has the same boundary as the D_4 instanton, but with the opposite orientation; observe that since it retracts on a $\mathbb{R}P^2$ it has $\chi = 1$ and $\eta_{\text{ad}} = \frac{2}{3}$, so the formulas (20) and (21) remain true. For $k = 3$, we have the same phenomenon with the D_1 ALF gravitational instanton (the double cover of the D_0 one) which has the same boundary as the D_3 one up to orientation. Finally for $k = 2$, one has $\eta_{\text{ad}} = 0$ and the opposite orientation is obtained by the same space, since the flat space $\mathbb{R}^3 \times \mathbb{S}^1/\pm$ admits an orientation reversing isometry.

We have proved:

Corollary 4.2. *There is no dihedral ALF gravitational instantons with boundary equal to $\mathbb{S}^3/D_k$ with negative orientation for $k > 4$.*

Let us observe from the ansatz (11) (12) for the Taub-NUT metric that the orientation of the boundary S depends on the sign of the mass m. Specifying the sign of the mass is therefore the same as specifying the orientation of the boundary S. In the cyclic case, all ALF gravitational instantons but $\mathbb{R}^3 \times \mathbb{S}^1$ have positive mass [22, 23]. In the dihedral case, the corollary implies that all ALF gravitational instantons have positive mass, with the only exceptions of D_0 or D_1 asymptotics (negative mass), or D_2 asymptotics (zero mass).

Finally, remind that, if in the cyclic case the ALF gravitational instantons are completely classified [23], the classification is still an open problem in the dihedral case: at least the corollary tells us that there is no possible new class with negative mass in the D_k case for $k > 4$.

Appendix A. Analysis in weighted spaces

Our construction relies on a few facts about the behaviour of the Laplacian on functions in complete non-compact Riemannian manifolds (M, g) with prescribed asymptotics. Let us sum up the theory.

Basically, we assume here the existence of a compact domain K in M such that $M\backslash K$ has finitely many connected components which, up to a finite covering, are diffeomorphic to the complement of the unit ball in $\mathbb{R}^m \times \mathbb{T}^{4-m}$, for $m = 1, 2, 3, 4$. We will further assume that the metric g coincides with the standard flat metric $g_0 = g_{\mathbb{R}^m} + g_{\mathbb{T}^{4-m}}$ at infinity in each end. The notation $g_{\mathbb{T}^{4-m}}$ is for the flat metric obtained as a quotient of $\mathbb{R}^{4-m}$ by any lattice. The case $m = 3$ will include slightly more sophisticated situations, like in [22]. Basically, the Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ can be extended radially into $\pi : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$ and we may assume that $M\backslash K$ is the total space of (a restriction of) this circle fibration. Then we define the model metric at infinity to be $g_0 := \pi^* g_{\mathbb{R}^3} + \eta^2$, where η is any constant multiple of the standard contact form on the three-sphere ([22]). Note also that all we will say will remain true if g is only asymptotic to g_0, thanks to perturbation arguments (cf. [22] for instance). The analysis on such spaces is somehow understood, so we will drop the proofs. The reader interested in the details of this analytical material is referred to [21, 11] for the Mazzeo-Melrose approach or to [20, 22] for softer arguments.

We will denote by r the Euclidean distance to the origin in $\mathbb{R}^m$. In what follows, we will always write A_R for the “annulus” defined by $R \leq r \leq 2R$ and A_R^κ for

$2^{-\kappa}R \leq r \leq 2^{\kappa+1}R$ ($\kappa \geq 0$). Similarly, the "balls" $K \cup \{r \leq R\}$ will be denoted by B_R.

A.1. **The Sobolev theory.** Given a real number δ and a subset Ω of M, we first define the weighted Lebesgue space $L^2_\delta(\Omega)$ as the set of functions $u \in L^2_{loc}(\Omega)$ such that the following norm is finite:

$$\|u\|_{L^2_\delta(\Omega)} := \left(\int_{\Omega \cap K} u^2 + \int_{\Omega \backslash K} u^2 r^{-2\delta} \right)^{\frac{1}{2}}.$$

We will often write L^2_δ for $L^2_\delta(M)$. The following should be kept in mind:

$$r^a \in L^2_\delta(M \backslash K) \Leftrightarrow \delta > \frac{m}{2} + a.$$

Any function u on $M\backslash K$ can be written $u = \Pi_0 u + \Pi_\perp u$ where $\Pi_0 u$ is obtained by computing the mean value of u along $\mathbb{T}^{4-m}$. In other words, $\Pi_0 u$ is the part in the kernel of the Laplacian on $\mathbb{T}^{4-m}$ while $\Pi_\perp u$ lies in the positive eigenspaces of this operator. The point is these projector commute with the Laplacian and elliptic estimates will be different for $\Pi_0 u$ and $\Pi_\perp u$. We therefore introduce the Hilbert space $L^2_{\delta,\epsilon}(\Omega)$ of functions $u \in L^2_{loc}(\Omega)$ such that $\|\Pi_0 u\|_{L^2_\delta(\Omega\backslash K)}$ and $\|\Pi_\perp u\|_{L^2_\epsilon(\Omega\backslash K)}$ are finite. The good Sobolev space for us is the Hilbert space H^2_δ of functions $u \in H^2_{loc}$ such that $\nabla^k \Pi_0 u \in L^2_{\delta-k}$ and $\nabla^k \Pi_\perp u \in L^2_{\delta-2}$ for $k = 0, 1, 2$.

To state the main a priori estimate, we need a definition. We will say that the exponent δ is *critical* if $r^{\delta - \frac{m}{2}}$ is the (pointwise) order of growth of an harmonic function on $\mathbb{R}^m \backslash \{0\}$. More precisely, the critical values correspond to $\delta - 2 \in \mathbb{Z}\backslash\{-1\}$ when $m = 4$, $\delta - \frac{3}{2} \in \mathbb{Z}$ when $m = 3$, $\delta - 1 \in \mathbb{Z}$ when $m = 2$, $\delta - \frac{1}{2} = 0$ or 1 when $m = 1$. When $m = 2$, the value $\delta = 1$ is doubly critical, owing to the constants and the harmonic function $\log r$. When $m = 1$, there are only two critical values because the Laplacian on $\mathbb{R}$ is also (minus) the Hessian, so that harmonic functions are affine ; in this case, exponential weights are usually used, but we will not really need them and we prefer to give a general framework including faster than linear volume growths. Note also that when one of the ends of M is a non-trivial finite quotient of the model, some critical values (as defined above) may turn irrelevant: for instance, there is no harmonic function with exactly linear growth on $\mathbb{R}^2/\pm$, which makes $\delta = 0$ and $\delta = 2$ non-critical.

We are interested in the unbounded operator

$$\begin{array}{rccc} P_\delta : & \mathcal{D}(P_\delta) & \longrightarrow & L^2_{\delta-2,\delta-2} \\ & u & \mapsto & \Delta u \end{array}$$

whose domain $\mathcal{D}(P_\delta)$ is the dense subset of $L^2_{\delta,\delta-2}$ whose elements u have their Laplacian in $L^2_{\delta-2,\delta-2}$. The usual L^2 pairing identifies the topological dual space of $L^2_{\delta,\delta-2}$ (resp. $L^2_{\delta-2,\delta-2}$) with $L^2_{-\delta,2-\delta}$ (resp. $L^2_{2-\delta,2-\delta}$). For this identification, the adjoint P^*_δ of P_δ is

$$\begin{array}{rccc} P^*_\delta : & \mathcal{D}(P^*_\delta) & \longrightarrow & L^2_{-\delta,2-\delta} \\ & u & \mapsto & \Delta u \end{array}$$

where the domain $\mathcal{D}(P_\delta^*)$ is the dense subset of $L^2_{2-\delta}$ whose elements u have their Laplacian in $L^2_{-\delta,2-\delta}$. The following proposition can be proved for instance along the lines of Proposition 1 in [22].

Proposition A.1. *If δ is non-critical, then P_δ is Fredholm and its cokernel is the kernel of P_δ^*.*

The following property is classical in this context and makes it possible to understand precisely the growth of solutions to our equations (cf. Lemma 5 in [22]).

Proposition A.2. *Suppose $\Delta u = f$ with u in $L^2_\delta(B^c_{R_0})$ and f in $L^2_{\delta'-2}(B^c_{R_0})$ for non-critical exponents $\delta > \delta'$ and a large number R_0. Then in each end of M, we may write $u = h + v$, where h is a harmonic function on $\mathbb{R}^m \setminus \{0\}$ and v is in $L^2_{\delta',\delta'-2}$.*

For instance, if $m = 1$ and f is a smooth and compactly supported function, we obtain that, in each end of M, v lies in L^2_δ for every δ. Since $\Delta v = f$, we can use standard elliptic estimates such as Lemma 24 (below) to see that $v = \mathcal{O}(r^{-a})$ for every a (together with its derivatives, indeed). We will abbreviate this by $v = \mathcal{O}(r^{-\infty})$. So a solution u of $\Delta u = f$ behaves in each end like an affine function on $\mathbb{R}$, up to $\mathcal{O}(r^{-\infty})$.

This proposition also implies that P_δ is injective as soon as $\delta - \frac{m}{2} < 0$ and, by duality, surjective as soon as $\delta - \frac{m}{2} > 2 - m$ (cf Corollary 2 in [22]).

As a consequence, when $m \geq 3$ and $2 - m < \delta - \frac{m}{2} < 0$, P_δ is an isomorphism and, if f is in $L^2_{\delta-2}$, we can find a solution u to the equation $\Delta u = f$ with the expected asymptotic behviour, i.e. $u \in L^2_\delta$.

When $m = 1$ or 2, there is no such value of δ. In practice, this can be easily circumvented in the following way. Assume f is in $L^2_{\delta-2}$ with $\delta - \frac{m}{2} \leq 2 - m$. Define δ_m by $\delta_m - \frac{m}{2} = 2 - m + \frac{1}{2}$. Then there is a function u in $\mathcal{D}(P_{\delta_m})$ such that $P_{\delta_m} u = f$. Proposition A.2 then ensures that the solution can be written as $u = \tilde{h} + v$, where v is in L^2_δ and $\tilde{h}$ is a smooth function which is harmonic outside a compact set and belongs to $L^2_{\delta_m} \setminus L^2_\delta$. In fact, such a function $\tilde{h}$ can be chosen in a finite dimensional space depending only on δ so we still get some control on the asymptotics of the solution. We refer to Lemma 1.1 for a concrete example.

Finally, as a by-product of the theory (cf. Lemma 4 in [22]), we are given, for every (large) number R_0, and every non-critical $\delta < \frac{m}{2}$, a bounded operator

$$G_{R_0} \,:\, L^2_{\delta-2}(B^c_{R_0}) \longrightarrow H^2_{0,\delta}(B^c_{R_0}) \tag{23}$$

which is an inverse for the Laplacian. Its domain $H^2_{0,\delta}$ is the space of functions $u \in H^2_\delta$ such that $\Pi_\perp u$ vanishes along ∂B_{R_0}. On $\ker \Pi_0$, G_{R_0} is defined by first solving the equation on the domains $B_R \setminus B_{R_0}$ with Dirichlet boundary condition and then letting R go to infinity. On $\ker \Pi_\perp$, it is given by an explicit formula. For instance, when $m = 1$, we set for each $f \in \ker \Pi_\perp$:

$$G_{R_0} f := \int_{R_0}^r (\rho - r) f(\rho) d\rho.$$

A.2. **From integral to pointwise bounds.** In view of handling (weighted) Hölder norms, more adapted to nonlinear analysis, the following Moser inequality is useful:

$$\|u\|_{L^\infty(A_R)} \leq c\left(\frac{1}{\sqrt{R^m}}\|u\|_{L^2(A^1_R)} + R^2\|\Delta u\|_{L^\infty(A^1_R)}\right) \tag{24}$$

A way to obtain this consists in lifting the problem to a square-like domain of size R in $\mathbb{R}^4$ and applying the standard elliptic estimate on $\mathbb{R}^4$; the behaviour of the constants with respect to R follows from scaling and counting fundamental domains. As a consequence of this inequality, the inverse G_{R_0} for the Laplacian on exterior domains (cf. 23) obeys an L^∞ estimate. The proof relies on an idea that can be found in [20, 3].

Lemma A.3. *Given positive numbers R_0 and a, there is a constant $c = c(R_0, a)$ such that for every continuous function f on $B_{R_0^c}$ with $f = \mathcal{O}(r^{-a-2})$,*

$$\|r^a G_{R_0} f\|_{L^\infty} \leq c\left\|r^{a+2}f\right\|_{L^\infty}.$$

Proof. First, write $f = \Pi_0 f + \Pi_\perp f$ and observe that $\Pi_0 f$ is obtained as an integral along the $\mathbb{T}^{4-m}$ factor, so that the sup norms of both terms can be estimated by the sup norms of f. We may therefore tackle them separately. The case $f = \Pi_0 f$ consists in using the explicit formula used to define G_{R_0} on $\ker \Pi_\perp$, so we assume $f = \Pi_\perp f$. Then $G_{R_0} f$ vanishes along ∂B_{R_0}. Let us put $R_i := 2^i R_0$. Using a partition of unity, we may write $f = \sum_i f_i$ with $\operatorname{supp} f_i \subset A_{R_i}$ and $|f_i| \leq |f|$. Then 24 yields:

$$R_i^a\|G_{R_0}f_j\|_{L^\infty(A_i)} \leq c\, R_i^{a+2}\|f_j\|_{L^\infty(A^1_{R_i})} + c\, R_i^{-\delta_a}\|G_{R_0}f_j\|_{L^2(A^1_{R_i})},$$

where $\delta_a = \frac{m}{2} - a$ (note that $A^1_{R_0}$ should be understood as $B_{4R_0}\backslash B_{R_0}$ and that the corresponding Moser-type estimate near the boundary is standard). Picking any δ close to δ_a, we get

$$\begin{aligned} R_i^{-\delta_a}\|G_{R_0}f_j\|_{L^2(A^1_{R_i})} \leq c\, R_i^{\delta-\delta_a}\|G_{R_0}f_j\|_{L^2_\delta(A^1_{R_i})} &\leq c\, R_i^{\delta-\delta_a}\|f_j\|_{L^2_{\delta-2}} \\ &\leq c\left(\frac{R_i}{R_j}\right)^{\delta-\delta_a}\left\|r^{a+2}f_j\right\|_{L^\infty}. \end{aligned}$$

Now, given i and j, we choose δ so that $\delta - \delta_a$ is ϵ times the sign of $j - i$ for some small positive number ϵ (and zero if $i = j$). Then we find

$$R_i^a\|G_{R_0}f_j\|_{L^\infty(A_{R_i})} \leq c\, 2^{-\epsilon|j-i|}\left\|r^{a+2}f\right\|_{L^\infty}.$$

Summing over j leads to:

$$R_i^a\|G_{R_0}f\|_{L^\infty(A_i)} \leq c\left\|r^{a+2}f\right\|_{L^\infty}$$

and the result follows at once. □

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