ON THE ASYMPTOTIC GEOMETRY OF GRAVITATIONAL INSTANTONS.

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Abstract. We investigate the geometry at infinity of the so-called “gravitational instantons”, i.e. asymptotically flat hyperkähler four-manifolds, in relation with their volume growth. In particular, we prove that gravitational instantons with cubic volume growth are ALF, namely asymptotic to a circle fibration over a Euclidean three-space, with fibers of asymptotically constant length.

Titre : Sur la géométrie asymptotique des instantons gravitationnels.

Résumé : Nous étudions la géométrie à l'infini des instantons gravitationnels, i.e. des variétés hyperkähleriennes, asymptotiquement plates et de dimension quatre. En particulier, nous prouvons que les instantons gravitationnels dont la croissance du volume est cubique sont asymptotiques à une fibration en cercles au-dessus d'un espace euclidien à trois dimensions, avec des fibres de longueur asymptotiquement constante ; autrement dit, ils sont ALF (asymptotically locally flat).

Mots-clés : instantons gravitationnels, variétés hyperkähleriennes, variétés asymptotiquement plates.

Keywords : gravitational instantons, hyperkähler manifolds, asymptotically flat manifolds.

MS classification numbers : 53C20, 53C21, 53C23, 53C26, 53C29.

Introduction.

Gravitational instantons are non-compact hyperkähler four-manifolds with decaying curvature at infinity. “Hyperkähler” means the manifold carries three complex structures I, J, K that are parallel with respect to a single Riemannian metric and satisfy the quaternionic relations ($IJ = -JI = K$, etc). In other words, the holonomy group of the metric reduces to $Sp(1) = SU(2)$. As a consequence, hyperkähler four-manifolds are Ricci flat and anti-self-dual [Bes]; the converse is true for simply connected manifolds.

Gravitational instantons were introduced in the late seventies by Stephen Hawking [Haw], as building blocks for his Euclidean quantum gravity theory. Very roughly, the idea consists in modelling gravitation by drawing an analogy with gauge theories, which are so efficient for the other fundamental interactions. The Universe is represented by a *Riemannian* manifold (equivalent in gauge theory: a connection on a principal bundle) which is assumed to be Ricci flat, as a counterpart of the vacuum Einstein equation in Relativity (in gauge theory: the Yang-Mills equation). Curvature decay is a “finite action” assumption: the curvature tensor, which measures the strength of the gravitational field, should typically be in L^2 (we will further discuss this decay issue below). Finally, the jump to “hyperkähler” is explained by the analogy with gauge theory: it can be thought of as an anti-self-duality assumption.

More recently, gravitational instantons also appeared in string theory and it triggered some interest from both mathematicians and physicists (cf. [CK1, CK2, CK3, CH, Hit,

Date: March 29, 2010.

HHM, EH, EJ]...). For instance, their L^2 cohomology was computed ([HHM], [Hit]) so as to test Sen's S-duality conjecture in string theory. New examples were built ([CK1, CK3, CH]) and, from string theory arguments, S. Cherkis and A. Kapustin conjecture a classification scheme [EJ], with four families.

- The first one consists of Asymptotically Locally Euclidean (ALE for short) gravitational instantons. ALE means that, outside a compact set, they are diffeomorphic to the quotient of $\mathbb{R}^4$ (minus a ball) by a finite subgroup of $O(4)$ and the metric is asymptotic to the Euclidean metric $g_{\mathbb{R}^4}$. Indeed, this family is very well understood, since P. Kronheimer ([K1, K2]) classified ALE gravitational instanton in 1989. In particular, he proved the underlying manifold is the minimal resolution of the quotient of $\mathbb{C}^2$ by a finite subgroup of $SU(2)$ (i.e. cyclic, binary dihedral, tetrahedral, octahedral or icosahedral group).
- The second family consists of the so called ALF ("Asymptotically Locally Flat") gravitational instantons: outside a compact set, they are diffeomorphic to the total space of a circle fibration π over $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm \mathrm{id}\}$ (minus a ball); moreover, the fibers have asymptotically constant length and the metric is asymptotic to $\pi^* g_{\mathbb{R}^3} + \eta^2$, where η is a (local) connection one-form on the circle fibration. Some examples are discussed below (section 1.2). A Kronheimer-like classification is conjectured, but involving only cyclic or dihedral groups in $SU(2)$ (see section 1.2 for concrete examples).
- The third and fourth families, called ALG and ALH (by induction !) have a similar fibration structure at infinity. In the ALG case, the fibers are tori and the base is $\mathbb{R}^2$. For ALH gravitational instantons, the fibers are compact orientable flat three-manifolds (there are six possibilities) and the base is $\mathbb{R}$.

A striking feature of this conjectured classification is the quantification it imposes on the volume growth: the volume of a ball of large radius t is of order t^4 in the ALE case, t^3 in the ALF case, etc. Why not $t^{3.5}$? And then, how can one explain this fibration structure at infinity ? The aim of this paper is to answer these questions.

Basically, the volume growth of asymptotically flat manifolds is at most Euclidean : on a complete noncompact Riemannian manifold (M^n, g) whose curvature tensor Rm_g obeys

$$|\mathrm{Rm}|_g = \mathcal{O}(r^{-2-\epsilon}) \quad \text{with } \epsilon > 0 \tag{1}$$

(r is the distance function to some point), there is a constant B such that

$$\forall\, x \in M,\ \forall\, t \geq 1,\ \mathrm{vol}\, B(x,t) \leq B t^n.$$

Note the "faster-than-quadratic" decay rate is not anecdotic. U. Abresh proved such manifolds have finite topological type [A2]: there is a compact subset K of M such that $M \backslash K$ has the *topology* of $\partial K \times \mathbb{R}_+^*$. In contrast, M. Gromov observed any (connected) manifold carries a complete metric with quadratic curvature decay ($|\mathrm{Rm}|_g = \mathcal{O}(r^{-2})$, see [LS]).

A fundamental geometric result was proved by S. Bando, A. Kasue and H. Nakajima [BKN] in 1989: if (M^n, g) satisfies (1) and has maximal volume growth, i.e.

$$\forall\, x \in M,\ \forall\, t \geq 1,\ \mathrm{vol}\, B(x,t) \geq A t^n,$$

then M is indeed ALE: there is a compact set K in M, a ball B in $\mathbb{R}^n$, a finite subgroup G of $O(n)$ and a diffeomorphism ϕ between $\mathbb{R}^n \backslash B$ and $M \backslash K$ such that $\phi^* g$ tends to the standard metric $g_{\mathbb{R}^n}$ at infinity. It is also proved in [BKN] that a complete Ricci flat manifold with maximal volume growth and curvature in $L^{\frac{n}{2}}(dvol)$ is ALE. In particular, gravitational instantons with maximal volume growth are ALE and thus belong to Kronheimer's list. The authors of the paper [BKN] raise the following natural question: can one understand the geometry at infinity of asymptotically flat manifolds whose volume growth is *not* maximal ? No answer has been given since then.

Let us state our main theorem. Here and in the sequel, we will denote by r the distance to some fixed point o, without mentionning it. We will also use the measure $d\mu = \frac{r^n}{\operatorname{vol} B(o,r)} dvol$. It was shown in [Min] that this measure has interesting properties on manifolds with non-negative Ricci curvature. Note that in maximal volume growth, it is equivalent to the Riemannian measure $dvol$.

THEOREM 0.1 — *Let (M^4, g) be a connected complete hyperkähler manifold with curvature in $L^2(d\mu)$. Suppose there are positive constants A and B such that*

$$\forall\, x \in M,\, \forall\, t \geq 1,\; At^{\nu} \leq \operatorname{vol} B(x,t) \leq Bt^{\nu}$$

with $3 \leq \nu < 4$. Then $\nu = 3$ and M is ALF: there is a compact set K in M such that $M \backslash K$ is the total space of a circle fibration π over $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm \operatorname{id}\}$ minus a ball and the metric g can be written

$$g = \pi^* g_{\mathbb{R}^3} + \eta^2 + \mathcal{O}(r^{-\tau}) \quad \text{for any } \tau < 1,$$

where η is a (local) connection one-form for π; moreover, the length of the fibers goes to a finite positive limit at infinity.

Up to a finite covering, the topology at infinity (i.e. modulo a compact set) is therefore either that of $\mathbb{R}^3 \times \mathbb{S}^1$ (trivial fibration over $\mathbb{R}^3$) or that of $\mathbb{R}^4$ (Hopf fibration).

Our integral assumption on the curvature might be surprising at first sight. Its relevance follows from [Min]. Indeed, it turns out to imply $\operatorname{Rm} = \mathcal{O}(r^{-2-\epsilon})$ and even more: a little analysis (cf. appendix A) provides $\nabla^k \operatorname{Rm} = \mathcal{O}(r^{-3-k})$, for any k in $\mathbb{N}$!

Our volume growth assumption is uniform: the constants A and B are assumed to hold at any point x. This is not anecdotic. By looking at flat examples, we will see the importance of this uniformity. This feature is not present in the maximal volume growth case, where the uniform estimate

$$\exists\, A,\; B \in \mathbb{R}_+^*,\, \forall\, x \in M,\, \forall\, t \geq 1,\; At^n \leq \operatorname{vol} B(x,t) \leq Bt^n$$

is equivalent to

$$\exists\, A,\; B \in \mathbb{R}_+^*, \exists\, x \in M,\, \forall\, t \geq 1,\; At^n \leq \operatorname{vol} B(x,t) \leq Bt^n.$$

The idea of the proof is purely Riemannian. The point is the geometry at infinity collapses, the injectivity radius remains bounded while the curvature gets very small, so Cheeger-Fukaya-Gromov theory [CG], [CFG] applies. The fibers of the circle fibration will come from suitable regularizations of short loops based at each point. The hyperkähler assumption will be used to control the holonomy of these short loops, which is crucial in the proof.

The structure of this paper is the following.

In a first section, we will consider examples, with three goals: first, we want to explain our volume growth assumption through the study of flat manifolds; second, these flat examples will also provide some ideas about the techniques we will develop later; third, we will describe the Taub-NUT metric, so as to provide the reader with a concrete example to think of.

In a second section, we will try to analyze some relations between three Riemannian notions: curvature, injectivity radius, volume growth. We will introduce the "fundamental pseudo-group". This object, due to M. Gromov [GLP], encodes the Riemannian geometry at a fixed scale. It is our basic tool and its study will explain for instance the volume growth self-improvement phenomenon in our theorem (from $3 \leq \nu < 4$ to $\nu = 3$).

In the third section, we completely describe the fundamental pseudo-group at a convenient scale, for gravitational instantons. This enables us to build the fibration at infinity, first locally, and then globally. Then we make a number of estimates to obtain the description of the geometry at infinity that we announced in the theorem. This part requires a good control on the covariant derivatives of the curvature tensor and the distance functions. This is provided by the appendices.

Acknowledgements. I wish to thank Gilles Carron for drawing my attention to the geometry of asymptotically flat manifolds and for so many fruitful discussions. I would also like to thank Marc Herzlich for his most valuable advice. This work benefited from the French ANR grant GeomEinstein.

Contents

1. Examples.

1.1. **Flat plane bundles over the circle.** To have a clear picture in mind, it is useful to understand flat manifolds obtained as quotients of the Euclidean space $\mathbb{R}^3$ by the action of a screw operation ρ. Let us suppose this rigid motion is the composition of a rotation of angle θ and of a unit translation along the rotation axis. The quotient manifold is always diffeomorphic to $\mathbb{R}^2\times\mathbb{S}^1$, but its Riemannian structure depends on θ : one obtains a flat plane bundle over the circle whose holonomy is the rotation of angle θ. These very simple examples conceals interesting features, which shed light on the link between injectivity radius, volume growth and holonomy. In this paragraph, we stick to dimension 3 for the sake of simplicity, but what we will observe remains relevant in higher dimension.

When the holonomy is trivial, i.e. $\theta = 0$, the Riemannian manifold is nothing but the standard $\mathbb{R}^2 \times \mathbb{S}^1$. The volume growth is uniformly comparable to that of the Euclidean $\mathbb{R}^2$:

$$\exists\, A,\, B \in \mathbb{R}_+^*,\ \forall\, x \in M,\ \forall\, t \geq 1,\ At^2 \leq \operatorname{vol} B(x,t) \leq Bt^2.$$

The injectivity radius is $1/2$ at each point, because of the lift of the base circle, which is even a closed geodesic; the iterates of these loops yield closed geodesics whose lengths describe all the natural integers, at each point.

Now, consider an angle $\theta = 2\pi \cdot p/q$, for some coprime numbers p, q. A covering of order q brings us back to the trivial case. The volume growth is thus uniformly comparable to that of $\mathbb{R}^2$. What about the injectivity radius ? Because of the cylindric symmetry, it depends only on the distance to the "soul", that is the image of the screw axis: let us denote by $\mathrm{inj}(t)$ the injectivity radius at distance t from the soul. This defines a continuous function admitting uniform upper and lower bounds, but not constant in general. The soul is always a closed geodesic, so that $\mathrm{inj}(0) = 1/2$. But as t increases, it becomes necessary to compare the lengths $l_k(t)$ of the geodesic loops obtained as images of the segments $[x, \rho^k(x)]$, with x at distance t from the axis. We can give a formula:

$$l_k(t) = \sqrt{k^2 + 4t^2 \sin^2(k\theta/2)}. \tag{2}$$

The injectivity radius is given by $2\,\mathrm{inj}(t) = \inf_k l_k(t)$. In a neighbourhood of 0, $2\,\mathrm{inj}$ equals l_1 ; then $2\,\mathrm{inj}$ may coincide with l_k for different indices k. If $k < q$ is fixed, since $\sin\frac{k\theta}{2}$ does not vanish, the function $t \mapsto l_k(t)$ grows linearly and goes to infinity. The function l_q is constant at q and $l_q \leq l_k$ for $k \geq q$. Thus, outside a compact set, the injectivity radius is constant at $q/2$ and it is half the length of a unique geodesic loop which is in fact a closed geodesic. Besides, the other loops are either iterates of this shortest loop, or they are much longer ($l_k(t) \asymp t$).

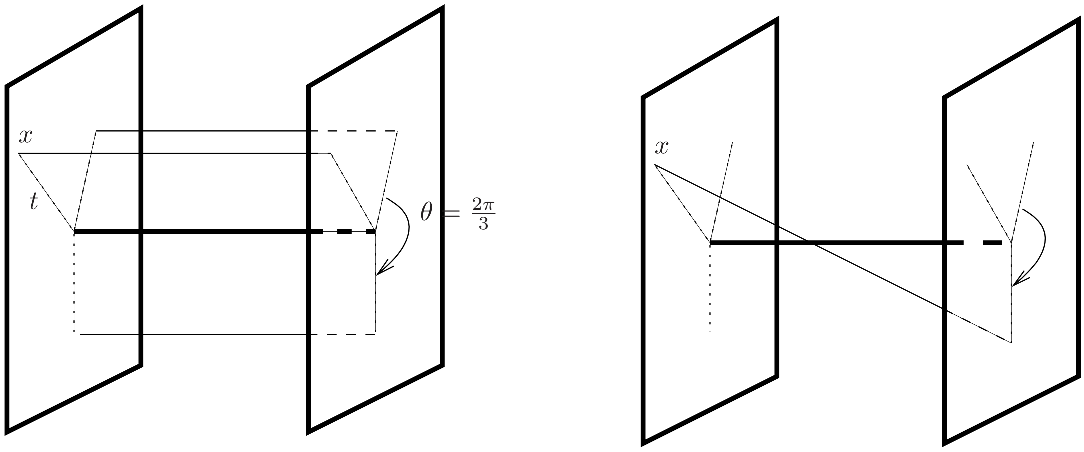


FIGURE 1. The holonomy angle is $\theta = \frac{2\pi}{3}$. On the left, a geodesic loop based at x with length $l_3(t) = 3$. On the right, a geodesic loop based at x with length $l_1(t) = \sqrt{1 + 9t^2}$.

When θ is an irrational multiple of 2π, the picture is much different. In particular, the injectivity radius is never bounded.

PROPOSITION 1.1 — *The injectivity radius is bounded if and only if θ is a rational multiple of 2π.*

Proof. The "only if" part is settled, so we assume the function $t \mapsto \mathrm{inj}(t)$ is bounded by some number C. For every t, there is an integer $k(t)$ such that $2\,\mathrm{inj}(t) = l_{k(t)}$. Formula (2) implies the function $t \mapsto k(t)$ is bounded by C. Since its values are integers, there is a sequence (t_n) going to infinity and an integer k such that $k(t_n) = k$ for every index n. Then (2) yields

$$\forall\, n \in \mathbb{N}, \quad l_k(t_n)^2 = k^2 + 4t_n^2 \sin^2(k\theta/2) \leq C^2.$$

Since t_n goes to infinity, this requires $\sin^2 \frac{k\theta}{2} = 0$: there is an integer m such that $k\theta/2 = m\pi$, i.e. $\theta/2\pi = m/k$. □

What about the volume growth ? The volume of balls centered in some given point grows quadratically:

$$\forall\, x,\, \exists\, B_x,\, \forall\, t \geq 1,\, \operatorname{vol} B(x,t) \leq B_x t^2.$$

In the "rational" case, the estimate is even uniform with respect to the center x of the ball:

$$\exists\, B,\, \forall\, x,\, \forall\, t \geq 1,\, \operatorname{vol} B(x,t) \leq Bt^2. \tag{3}$$

In the "irrational" case, this strictly subeuclidean estimate is never uniform. Why ? The proposition above provides a sequence of points x_n such that $r_n := \operatorname{inj}(x_n)$ goes to infinity. Given a lift $\hat{x}_n$ of x_n in $\mathbb{R}^3$, the ball $B(\hat{x}_n, r_n)$ is the lift of $B(x_n, r_n)$ and its volume is $\frac{4}{3}\pi r_n^3$. If we assume two points v and w of $B(\hat{x}_n, r_n)$ lift the same point y of $B(x_n, r_n)$, there is by definition an integer number k such that $\rho^k(v) = w$; since ρ is an isometry of $\mathbb{R}^3$, we get

$$\left|\rho^k(\hat{x}_n) - \hat{x}_n\right| \leq \left|\rho^k(\hat{x}_n) - \rho^k(v)\right| + \left|\rho^k(v) - \hat{x}_n\right| = |\hat{x}_n - v| + |w - \hat{x}_n| < 2r_n = 2\operatorname{inj}(x_n),$$

which contradicts the definition of $\operatorname{inj}(x_n)$ (the segment $[\rho^k(\hat{x}_n), \hat{x}_n]$ would go down as a too short geodesic loop at x_n). Therefore $B(\hat{x}_n, r_n)$ and $B(x_n, r_n)$ are isometric, hence $\operatorname{vol} B(x_n, r_n) = \frac{4}{3}\pi r_n^3$, which prevents an estimate like (3).

Remark 1. *What about the injectivity radius growth in the irrational case ? Using the explicit formula for $l_k(t)$ and the pigeonhole principle, one can always bound* $\operatorname{inj}(t)$ *by a constant times $\sqrt{t}$, for t large. This is optimal : Roth theorem in diophantine approximation theory shows that, if $\theta/(2\pi)$ is an irrational algebraic number and if $\alpha \in]0, 1/2[$, then* $\operatorname{inj}(t)$ *is bounded from below by a constant times t^α. When $\theta/(2\pi)$ admits good rational approximations, an almost rational behaviour can be recovered, with a slowly growing injectivity radius. For instance, if $\theta/(2\pi)$ is the Liouville number $\sum_{n=1}^{\infty} 10^{-n!}$, then* $\liminf_{t\longrightarrow\infty} (t^{-a} \operatorname{inj}(t)) = 0$ *for every $a > 0$.*

1.2. **The Taub-NUT metric.**

The Taub-NUT metric is the basic non trivial example of ALF gravitational instanton. This Riemannian metric over $\mathbb{R}^4$ was introduced by Stephen Hawking in [Haw]. A very detailed description can be found in [Leb].

Thanks to the Hopf fibration, we can see $\mathbb{R}^4 \backslash \{0\} = \mathbb{R}_+^* \times \mathbb{S}^3$ as the total space of a principal circle bundle π over $\mathbb{R}_+^* \times \mathbb{S}^2 = \mathbb{R}^3 \backslash \{0\}$. If $x = (x_1, x_2, x_3)$ denotes the coordinates on $\mathbb{R}^3$, we let V be the harmonic function given on $\mathbb{R}^3 \backslash \{0\}$ by $V = 1 + \frac{1}{2|x|}$ and η be a connection one-form on the circle bundle whose curvature $d\eta$ is the pull back of $*_{\mathbb{R}^3} dV$ (η is essentially the standard contact form on $\mathbb{S}^3$). In what follows, we denote lifts by hats. On $\mathbb{R}^4 \backslash \{0\}$, the Taub-NUT metric is given by the formula

$$g = \hat{V} d\hat{x}^2 + \frac{1}{\hat{V}}\eta^2$$

and one can check (cf. [Leb]) that this can be extended as a complete metric on $\mathbb{R}^4$. By construction, the metric is $\mathbb{S}^1$-invariant and the length of the fibers goes to a (nonzero) constant at infinity, while the induced metric on the base is asymptotically Euclidean (it is at distance $\mathcal{O}(|x|^{-1})$ from the Euclidean metric). Thus there are positive constants A and B such that Taub-NUT balls satisfy

$$\forall R \geq 1,\ AR^3 \leq \operatorname{vol} B(z,R) \leq BR^3.$$

Moreover, the Taub-NUT metric is hyperkähler. Indeed, an almost complex structure J_1 can be defined by requiring the following action on the cotangent bundle:

$$J_1\left(\sqrt{\hat{V}}d\hat{x}_1\right) = \frac{1}{\sqrt{\hat{V}}}\eta \quad \text{and} \quad J_1\left(\sqrt{\hat{V}}d\hat{x}_2\right) = \sqrt{V}d\hat{x}_3.$$

Then (g, J_1) is a Kähler structure (cf. [Leb]). A permutation of the coordinates x_1, x_2, x_3 yields three Kähler structures (g, J_1), (g, J_2), (g, J_3) satisfying the quaternionic relations, hence the hyperkähler structure. In fact, it turns out these complex structures are biholomorphic to that of $\mathbb{C}^2$ [Leb]. Using [Unn], it is possible to compute the curvature of the Taub-NUT metric. It decays at a cubic rate: $|\mathrm{Rm}| = \mathcal{O}(r^{-3})$.

This ansatz produces a whole family of examples: the "multi-Taub-NUT" metrics or A_k ALF instantons [Haw, Leb]. These are obtained as total spaces of a circle bundle π over $\mathbb{R}^3$ minus some points $p_1, \ldots, p_N$, endowed with the metric $\hat{V}d\hat{x}^2 + \frac{1}{\hat{V}}\eta^2$, where V is the function defined on $\mathbb{R}^3 \setminus \{p_1, \cdots, p_N\}$ by $V(x) = 1 + \sum_{i=1}^N \frac{1}{2|x-p_i|}$ and where η is the one-form of a connection with curvature $*_{\mathbb{R}^3} dV$. As above, a completion by N points is possible. The circle bundle restricts on large spheres as a circle bundle of Chern number $-N$. The metric is again hyperkähler and has cubic curvature decay. The underlying manifold is a minimal resolution of $\mathbb{C}^2/\mathbb{Z}_N$. The geometry at infinity is that of the Taub-NUT metric, modulo an action of $\mathbb{Z}_N$, which is the fundamental group of the end.

Other examples are built in [CK1, CH]: the geometry at infinity of these D_k ALF gravitational instantons is essentially that of a quotient of a multi-Taub-NUT metric by the action of a reflection on the base.

2. Injectivity radius and volume growth.

2.1. **An upper bound on the injectivity radius.**

Proposition 2.1 (Upper bound on the injectivity radius) — *There is a universal constant $C(n)$ such that on any complete Riemannian manifold (M^n, g) satisfying*

$$\inf_{t>0} \limsup_{x \longrightarrow \infty} \frac{\operatorname{vol} B(x,t)}{t^n} < C(n), \tag{4}$$

the injectivity radius is bounded from above, outside a compact set.

The assumption (4) means there is a positive number T and a compact subset K of M such that:

$$\forall x \in M \backslash K,\ \operatorname{vol} B(x,T) < C(n)T^n. \tag{5}$$

We think of a situation where there is a function ω going to zero at infinity and such that for any point x, $\operatorname{vol} B(x,t) \leq \omega(t)t^n$. The point is we require a *uniform* strictly subeuclidean volume growth. Even in the flat case, we have seen that a uniform estimate moderates the geometry much more than a centered strictly subeuclidean volume growth.

Proof. The constant $C(n)$ is given by Croke inequality [Cro]:

$$\forall t \leq \operatorname{inj}(x),\ \forall x \in M,\ \operatorname{vol} B(x,t) \geq C(n)t^n. \tag{6}$$

Let x be a point outside the compact K given by (5). If $\mathrm{inj}(x)$ is greater than the number T in (5), (6) yields: $C(n)T^n \leq \mathrm{vol}\, B(x,T) < C(n)T^n$, which is absurd. The injectivity radius at x is thus bounded from above by T. □

Cheeger-Fukaya-Gromov theory applies naturally in this setting: it describes the geometry of Riemannian manifolds with small curvature and injectivity radius bounded from above [CG]. Let us quote the

Corollary 2.2 — *Let (M^n, g) be a complete Riemannian manifold whose curvature goes to zero at infinity and satisfying (4). Outside a compact set, M carries a F-structure of positive rank whose orbits have bounded diameter.*

It means we already know there is some kind of structure at infinity on these manifolds. Our aim is to make it more precise, under additional assumptions.

2.2. **The fundamental pseudo-group.** The notion of "fundamental pseudo-group" was introduced by M. Gromov in the outstanding [GLP]. It is a very natural tool in the study of manifolds with small curvature and bounded injectivity radius. Let us give some details.

Let M be a complete Riemannian manifold and let x be a point in M. We assume the curvature is bounded by Λ^2 ($\Lambda \geq 0$) on the ball $B(x, 2\rho)$, with $\Lambda\rho < \pi/4$. In particular, the exponential map in x is a local diffeomorphism on the ball $\hat{B}(0, 2\rho)$ centered in 0 and of radius 2ρ in T_xM. The metric g on $B(x, 2\rho)$ thus lifts as a metric $\hat{g} := \exp_x^* g$ on $\hat{B}(0, 2\rho)$. We will denote by Exp the exponential map corresponding to $\hat{g}$.

An important fact is proved in [GLP] (8.19): any two points in $\hat{B}(0, 2\rho)$ are connected by a unique geodesic which is therefore minimizing; moreover, balls are strictly convex in this domain.

When the injectivity radius at x is greater than 2ρ, the Riemannian manifolds $(B(x, \rho), g)$ and $(\hat{B}(0, \rho), \hat{g})$ are isometric. But if it is small, there are short geodesic loops based at x and x admits differents lifts in $\hat{B}(0, \rho)$. The fundamental pseudo-group $\Gamma(x, \rho)$ in x and at scale ρ measures the injectivity defect of the exponential map over $\hat{B}(0, \rho)$ [GLP] : $\Gamma(x, \rho)$ is the pseudo-group consisting of all the continuous maps τ from $\hat{B}(0, \rho)$ to T_xM which satisfy

$$\exp_x \circ \tau = \exp_x \quad \text{and} \quad \tau(0) \in \hat{B}(0, \rho).$$

In particular, the elements of $\Gamma(x, \rho)$ map geodesics onto geodesics, so they are isometries.

Given a lift v of x in $\hat{B}(0, \rho)$ (i.e. $\exp_x(v) = p$), consider the map $\tau_v := \mathrm{Exp}_v \circ (T_v \exp_x)^{-1}$, whose action is described in figure 2. Then τ_v defines an element of $\Gamma(x, \rho)$.

It is also easy to see that any element $\tau \in \Gamma(x, \rho)$ mapping 0 to v has to be τ_v. So there is a one-to-one correspondence between elements of $\Gamma(x, \rho)$ and oriented geodesic loops based at x with length bounded by ρ. Since $\exp_x$ is a local diffeomorphism, $\Gamma(x, \rho)$ is in particular finite. Thus $(\Gamma(x, \rho))_{0<\rho<\pi/(4\Lambda)}$ is a nondecreasing family of finite pseudo-groups .

Example 1. *Consider a flat plane bundle over $\mathbb{S}^1$, with rational holonomy ρ (cf. section 1): the screw angle θ is 2π times p/q, with coprime p and q. For large ρ and x farther than $\rho/\sin(\pi/q)$ from the soul (when $q = 1$, there is no condition), the fundamental pseudo-group $\Gamma(x, \rho)$ is generated by the unique geodesic loop with length q. It therefore consists of translations only. In particular, it does not contain ρ, except in the trivial case $\rho = \mathrm{id}$. In general, many geodesic loops are forgotten, for they are too long.*

Every nontrivial element of $\Gamma(x, \rho)$ acts without fixed points. To see this, let us assume a point w is fixed by some τ_v in $\Gamma(x, \rho)$ and introduce the geodesics $\gamma_1 : t \mapsto tw$ and $\gamma_2 : t \mapsto \tau_v(tw)$. Then $\gamma_1(1) = \gamma_2(1) = w$ and, differentiating at $t = 1$ the identity

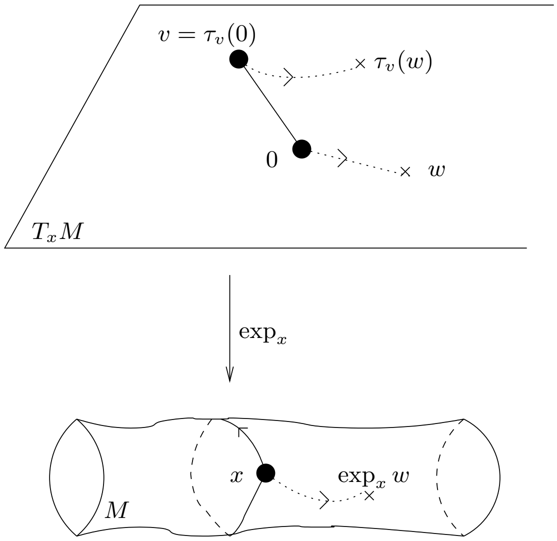


FIGURE 2. $\tau_v(w)$ is obtained in the following way. Push the segment $[0, w]$ from T_xM to M thanks to $\exp_x$ anf lift the resulting geodesic from v to obtain a new geodesic in T_xM whose tip is $\tau_v(w)$.

$\exp_x \circ \gamma_1(t) = \exp_x \circ \gamma_2(t)$, one gets $\gamma_1'(1) = \gamma_2'(1)$. The geodesics γ_1 and γ_2 must then coincide, hence $0 = \gamma_1(0) = \gamma_2(0) = v$ and $\tau_v = \mathrm{id}$.

It is also useful to observe that every element of $\Gamma(x, \rho)$ has a well-defined inverse : it is given by $(\tau_v)^{-1} = \tau_{-\sigma'(1)}$ where $\sigma(t) := \exp_x tv$.

Given a geodesic loop σ with length bounded by ρ, let us call "sub-pseudo-group generated by σ in $\Gamma(x, \rho)$" the pseudo-group $\Gamma_\sigma(x, \rho)$ which we describe now : it contains an element τ_v of $\Gamma(x, \rho)$ if and only if v is the tip of a piecewise geodesic segment staying in $\hat{B}(0, \rho)$ and obtained by lifting several times σ from 0. If τ is an element of $\Gamma(x, \rho)$ which corresponds to a loop σ, we will also write $\Gamma_\tau(x, \rho)$ for the sub-pseudogroup generated by τ in $\Gamma(x, \rho)$. If k is the largest integer such that $\tau^i(0)$ belongs to the ball $\hat{B}(0, \rho)$ for every natural number $i \leq k$, then:

$$\Gamma_\tau(x, \rho) = \Gamma_\sigma(x, \rho) = \left\{ \tau^i / -k \leq i \leq k \right\}.$$

If $2\rho \leq \rho' < \frac{\pi}{4\Lambda}$, then the orbit space of the points of the ball $\hat{B}(0, \rho)$ under the action of $\Gamma(x, \rho')$, $\hat{B}(0, \rho)/\Gamma(x, \rho')$, is isometric to $B(x, \rho)$, through the factorization of $\exp_x$. The only thing to check is the injectivity. Given two lifts $w_1, w_2 \in \hat{B}(0, \rho)$ of the same point $y \in B(x, \rho)$, let us prove they are in the same orbit for $\Gamma(x, \rho')$. Consider the unique geodesic γ_1 from w_1 to 0, push it by $\exp_x$ and lift the resulting geodesic from w_2 to obtain a geodesic γ_2, from w_2 to some point v (cf. figure 3). Then v is a lift of x in $\hat{B}(0, \rho')$ (by triangle inequality) and τ_v maps w_1 to w_2, hence the result.

We will need to estimate the number $N_x(y, \rho)$ of lifts of a given point y in the ball $\hat{B}(0, \rho)$ of T_xM. Lifting one shortest geodesic loop from $0 =: v_0$, we arrive at some point v_1. Lifting the same loop from v_1, we arrive at a new point v_2, etc. This construction yields a sequence of lifts v_k of x which eventually goes out of $\hat{B}(0, \rho)$: otherwise, since there cannot exist an accumulation point, the sequence would be periodic; τ_{v_1} would then fix the centre of the unique ball with minimal radius which contains all the points v_k, which is not possible, since τ_{v_1} is nontrivial hence has no fixed point (the uniqueness of the ball stems from the strict

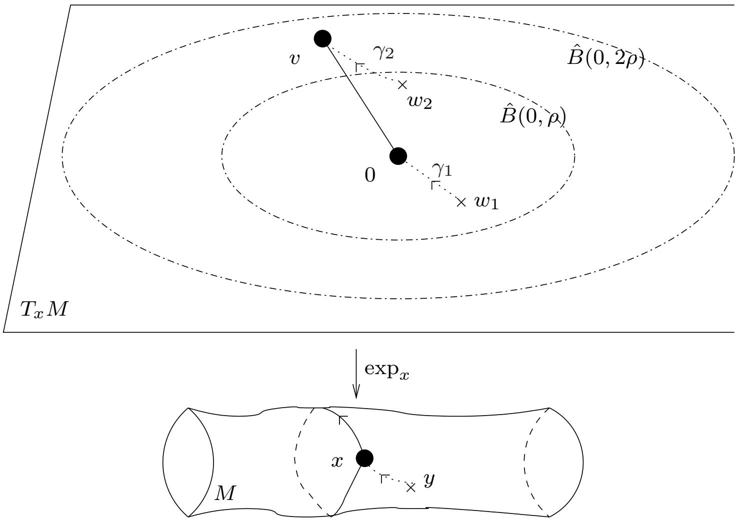


FIGURE 3. $\tau_v(w_1) = w_2$.

convexity of the balls, cf. [G1], 8.16, p. 379-380). Of course, one can do the same thing with the reverse orientation of the same loop. Since the distance between two points v_k is at least $2\,\mathrm{inj}(x)$, this yields at least $\rho/\,\mathrm{inj}(x)$ lifts of x in $\hat{B}(0, \rho)$:

$$|\Gamma(x, \rho)| = N_x(x, \rho) \geq \rho/\,\mathrm{inj}(x).$$

Lifting one shortest geodesic between x and some point y from the lifts of x and estimating the distance between the tip and 0 with the triangle inequality (cf. figure 4), we get:

$$N_x(y, \rho) \geq N_x(x, \rho - d(x, y)) = |\Gamma(x, \rho - d(x, y))| \geq \frac{\rho - d(x, y)}{\mathrm{inj}(x)}. \tag{7}$$

For $d(x, y) \leq \rho/2$, this yields:

$$\frac{\rho}{2\,\mathrm{inj}(x)} \mathrm{vol}\, B(x, \rho/2) \leq |\Gamma(x, \rho/2)| \,\mathrm{vol}\, B(x, \rho/2) \leq \mathrm{vol}\, \hat{B}(0, \rho). \tag{8}$$

For $\rho \leq \rho' < \frac{\pi}{4\Lambda}$, the set

$$\mathcal{F}(x, \rho, \rho') := \left\{ w \in \hat{B}(0, \rho) \Big/ \forall\, \gamma \in \Gamma(x, \rho'),\ d(0, \gamma(w)) \geq d(0, w) \right\}$$

is a fundamental domain for the action of $\Gamma(x, \rho')$ on the ball $\hat{B}(0, \rho)$. Finiteness ensures each orbit intersects $\mathcal{F}$. Furthermore, if τ belongs to $\Gamma(x, \rho')$, the set $\mathcal{F}(x, \rho, \rho') \cap \tau(\mathcal{F}(x, \rho, \rho'))$ consists of points whose distances to 0 and $\tau(0)$ are equal, hence has zero measure: by finiteness again, up to a set with zero measure, $\mathcal{F}(x, \rho, \rho')$ contains a unique element of each orbit. For the same reason, if τ belongs to $\Gamma(x, \rho')$, the set

$$\mathcal{F}_\tau(x, \rho, \rho') := \left\{ w \in \hat{B}(0, \rho) \Big/ \forall\, \gamma \in \Gamma_\tau(x, \rho'),\ d(0, \gamma(w)) \geq d(0, w) \right\}$$

is a fundamental domain for the action of the sub-pseudo-group $\Gamma_\tau(x, \rho')$. From our discussion follows an important fact: if $2\rho \leq \rho' < \frac{\pi}{4\Lambda}$, then $\mathrm{vol}\, \mathcal{F}(x, \rho, \rho') = \mathrm{vol}\, B(x, \rho)$. We will need to control the shape of these fundamental domains.

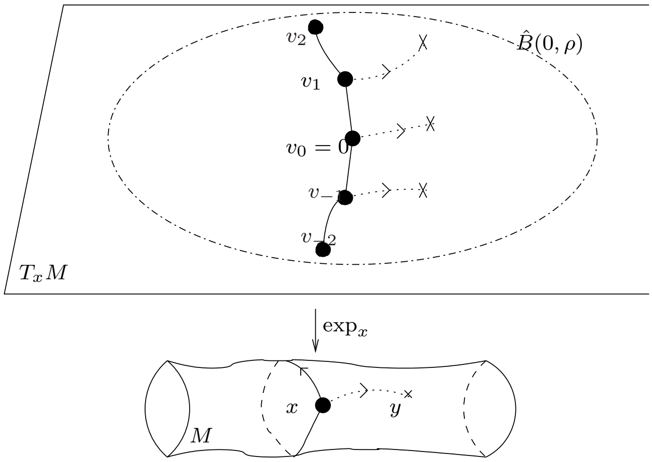


FIGURE 4. Take a minimal geodesic between x and y and lift it from every point in the fiber of x to obtain points in the fiber of y.

LEMMA 2.3 — *Fix $\rho \le \rho' < \frac{\pi}{4\Lambda}$ and consider a nontrivial element τ in $\Gamma(x,\rho')$. Denote by $\mathcal{I}_\tau(x,\rho)$ the set of points w in $\hat{B}(0,\rho)$ such that*

$$\max\left\{g_x\left(w,\tau(0)\right), g_x\left(w,\tau^{-1}(0)\right)\right\} \le \frac{|\tau(0)|^2}{2} + \frac{\Lambda^2\rho^2\,|\tau(0)|^2}{2}.$$

Then $\mathcal{F}_\tau(x,\rho,\rho')$ is a subset of $\mathcal{I}_\tau(x,\rho)$.

Figure 5 provides a picture, in the plane containing 0, $\tau(0)$ and $\tau^{-1}(0)$).

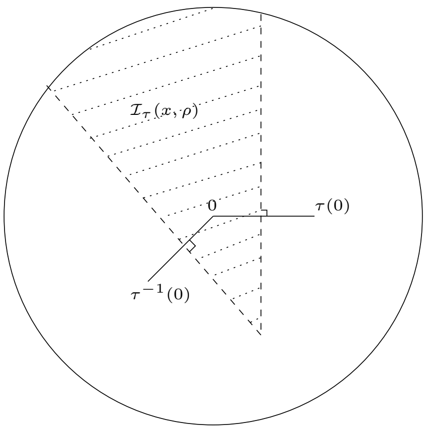


FIGURE 5. The domain $\mathcal{I}_\tau(x,\rho)$.

Proof. Consider a point w in $\mathcal{F}_\tau(x,\rho,\rho')$, set $v = \tau(0)$ and denote by $\theta \in [0,\pi]$ the angle between v and w. We first assume $g_x(w,v) > 0$, that is $\theta < \pi/2$. Since any two points in $\hat{B}(0,\rho)$ are connected by a unique geodesic which is therefore minimizing, we can apply Toponogov theorem to all triangles. In particular, in the triangle $0vw$, we find

$$\cosh(\Lambda d(v,w)) \le \cosh(\Lambda\,|w|)\cosh(\Lambda\,|v|) - \sinh(\Lambda\,|w|)\sinh(\Lambda\,|v|)\cos\theta.$$

Observing $|w| = d(0,w) \leq d(0,\tau^{-1}(w)) = d(v,w)$, we get

$$\cosh(\Lambda\,|w|) \leq \cosh(\Lambda d(v,w)) \leq \cosh(\Lambda\,|w|)\cosh(\Lambda\,|v|) - \sinh(\Lambda\,|w|)\sinh(\Lambda\,|v|)\cos\theta,$$

hence

$$\tanh(\Lambda\,|w|)\cos\theta \leq \frac{\cosh(\Lambda\,|v|)-1}{\sinh(\Lambda\,|v|)}$$

and

$$\frac{g_x(v,w)}{|v|^2} \leq \frac{\Lambda\,|w|}{\tanh\Lambda\,|w|}\frac{\cosh(\Lambda\,|v|)-1}{\Lambda\,|v|\sinh(\Lambda\,|v|)} \leq \frac{1}{2} + \frac{\Lambda^2\rho^2}{2}$$

(Taylor formulas). Assuming $g_x(w,\tau^{-1}(0)) > 0$, we can work in the same way (with $v = \tau^{-1}(0)$) so as to complete the proof. □

To understand the action of the elements in the fundamental pseudo-group, the following lemma is useful: it approximates them by affine transformations.

Lemma 2.4 — *Consider a complete Riemannian manifold (M,g) and a point x in M such that the curvature is bounded by Λ^2, $\Lambda \geq 0$, on the ball $B(x,\rho)$, $\rho > 0$, with $\Lambda\rho < \pi/4$. Let v be a lift of x in $\hat{B}(0,\rho) \subset T_xM$. Define*

- *the translation t_v with vector v in the affine space T_xM,*
- *the parallel transport p_v along $t \mapsto \exp_x tv$, from $t=0$ to $t=1$.*
- *the map $\tau_v = \mathrm{Exp}_v \circ (T_v \exp_x)^{-1}$,*

where Exp *denotes the exponential map of $(T_xM, \exp_x^* g)$. Then for every point w in $\hat{B}(0,\rho - |v|)$,*

$$d(\tau_v(w), t_v \circ p_v^{-1}(w)) \leq \Lambda^2\,|v|\,|w|\,(|v|+|w|).$$

Proof. Proposition 6.6 of [BK] yields the following comparaison result: if V is defined by $\mathrm{Exp}_0 V = v$ and if W belongs to T_0T_xM, then

$$d(\mathrm{Exp}_v \circ \hat{p}_v(W), \mathrm{Exp}_0(V+W)) \leq \frac{1}{3}\Lambda\,|V|\,|W|\sinh(\Lambda(|V|+|W|))\sin\angle(V,W), \tag{9}$$

where $\hat{p}_v$ is the parallel transport along $t \mapsto \mathrm{Exp}_0\, tV$, from $t=0$ to $t=1$. Set $w = \mathrm{Exp}_0 W$. We stress the fact that $Exp_0 = T_0 \exp_x$ is nothing but the canonical identification between the tangent space T_0T_xM to the vector space T_xM and the vector space itself, T_xM. In particular, $\mathrm{Exp}_0(V+W) = v + w = t_v(w)$. Since $\exp_x$ is a local isometry, we have

$$\hat{p}_v = (T_v \exp_x)^{-1} \circ p_v \circ T_0 \exp_x,$$

so that $\mathrm{Exp}_v \circ \hat{p}_v(W) = \tau_v \circ p_v(w)$. With $\sinh(\Lambda(|V|+|W|)) \leq 3\Lambda(|V|+|W|)$, it follows from (9) that: $d(\tau_v \circ p_v(w), t_v(w)) \leq \Lambda^2\,|v|\,|w|\,(|v|+|w|)$. Changing w into $p_v^{-1}(w)$, we obtain the result. □

2.3. **Fundamental pseudo-group and volume.**

2.3.1. *Back to the injectivity radius.* Our discussion of the fundamental pseudo-group enables us to recover a result of [CGT].

Proposition 2.5 (Lower bound for the injectivity radius.) — *Let (M^n,g) be a complete Riemannian manifold. Assume the existence of $\Lambda \geq 0$ and $V > 0$ such that for every point x in M, $|\mathrm{Rm}_x| \leq \Lambda^2$ and $\mathrm{vol}\, B(x,1) \geq V$. Then the injectivity radius admits a positive lower bound $I = I(n,\Lambda,V)$.*

Proof. Set $\rho = \min(1, \frac{\pi}{8\Lambda})$ and assume there is a point x in M and a geodesic loop based at x with length bounded by ρ. Apply (8) to find

$$\frac{\rho}{2\,\mathrm{inj}(x)} \operatorname{vol} B(x, \rho/2) \leq \operatorname{vol} \hat{B}(0, \rho).$$

Bishop theorem estimates the right-hand side by $\omega_n \cosh(\Lambda\rho)^{n-1}\rho^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We thus obtain $\mathrm{inj}(x) \geq C(n, \Lambda) \operatorname{vol} B(x, \rho/2)$ for some $C(n, \Lambda) > 0$. Since Bishop theorem also yields a constant $C'(n, \Lambda) > 0$ such that

$$\operatorname{vol} B(x, 1) \leq C'(n, \Lambda)^{-1} \operatorname{vol} B(x, \rho/2),$$

we are left with $\mathrm{inj}(x) \geq C(n, \Lambda)C'(n, \Lambda) \operatorname{vol} B(x, 1) \geq C(n, \Lambda)C'(n, \Lambda)V$. □

Combining propositions 2.1 and 2.5, we obtain

Corollary 2.6 (Injectivity radius pinching.) — *Let (M^n, g) be a complete Riemannian manifold with bounded curvature. Suppose:*

$$\forall\, x \in M,\ V \leq \operatorname{vol} B(x, t) \leq \omega(t)t^n$$

for some positive number V and some function ω going to zero at infinity. Then there are positive numbers I_1, I_2 such that for any point x in M:

$$I_1 \leq \mathrm{inj}(x) \leq I_2.$$

2.3.2. *Self-improvement of volume estimates.* Here and in the sequel, we will distinguish a point o in our complete non-compact Riemannian manifolds, which will always be smooth and connected. The distance function to o will be denoted by r_o or r.

Proposition 2.7 — *Let (M^n, g) be a complete non-compact Riemannian manifold with faster than quadratic curvature decay, i.e. $|\mathrm{Rm}| = \mathcal{O}(r^{-2-\epsilon})$ for some $\epsilon > 0$. If there exists a function ω which goes to zero at infinity and satisfies*

$$\forall\, x \in M,\ \forall t \geq 1,\ \operatorname{vol} B(x, t) \leq \omega(t)t^n,$$

then there is in fact a number B such that

$$\forall\, x \in M,\ \forall t \geq 1,\ \operatorname{vol} B(x, t) \leq Bt^{n-1}.$$

Proof. Proposition 2.1 yields an upper bound I_2 on the injectivity radius. Our assumption on the curvature implies that, given a point x in $M \backslash B(o, R_0)$, with large enough R_0, one can apply (8) with $2I_2 \leq \rho = 2t \leq r(x)/2$:

$$\frac{t}{\mathrm{inj}(x)} \operatorname{vol} B(x, t) \leq \operatorname{vol} \hat{B}(0, 2t).$$

Thanks to the curvature decay, if R_0 is large enough, Bishop theorem bounds the right-hand side by $\omega_n \cosh(1)^{n-1}(2t)^n$; with proposition 2.1, it follows that for $I_2 \leq t \leq r(x)/2$:

$$\operatorname{vol} B(x, t) \leq \omega_n \cosh(1)^{n-1} 2^n I_2 t^{n-1}.$$

We have found some number B_1 such that for every x outside the ball $B(o, R_0)$ and for every t in $[I_2, r(x)/2]$,

$$\operatorname{vol} B(x, t) \leq B_1 t^{n-1}. \tag{10}$$

From lemma 3.6 in [LT], which refers to the construction in the fourth paragraph of [A2], we can find a number N such that for any natural number k, setting $R_k = R_0 2^k$, the annulus $A_k := B(o, 2R_k) \backslash B(o, R_k)$ is covered by a family of balls $(B(x_{k,i}, R_k/2))_{1 \leq i \leq N}$ centered in

A_k. Since the volume of the balls $B(x_{k,i}, R_k/2)$ is bounded by $B_1(R_k/2)^{n-1}$, we deduce the existence of a constant B_2 such that for every $t \geq I_2$,

$$\operatorname{vol} B(o,t) \leq B_2 \sum_{k=0}^{\lceil \log_2(t/R_0) \rceil} (2^k)^{n-1},$$

and thus, for some new constant B_3, we have

$$\forall\, t \geq I_2,\ \operatorname{vol} B(o,t) \leq B_3\, t^{n-1}. \tag{11}$$

Now, for every x in $M \backslash B(o, R_0)$ and every $t \geq r(x)/4$, we can write

$$\operatorname{vol} B(x,t) \leq \operatorname{vol} B(o, t + r(x)) \operatorname{vol} B(o, 5t) \leq 5^{n-1} B_3 t^{n-1}.$$

And when x belongs to $B(o, R_0)$, for $t \geq I_2$, we observe

$$\operatorname{vol} B(x,t) \leq \operatorname{vol} B(o, t + R_0) \leq \operatorname{vol} B(o, (1 + R_0/2)t) \leq B_3(1 + R_0/2)^{n-1} t^{n-1}.$$

Therefore there is a constant B such that for every x in M and every $t \geq I_2$, the volume of the ball $B(x,t)$ is bounded by Bt^{n-1}. The result immediately follows. □

When the Ricci curvature is nonnegative, the assumption on the curvature can be relaxed.

Proposition 2.8 — *Let (M^n, g) be a complete non-compact Riemannian manifold with nonnegative Ricci curvature and quadratic curvature decay, i.e.* $|\mathrm{Rm}| = \mathcal{O}(r^{-2})$. *If there exists a function ω which goes to zero at infinity and satisfies*

$$\forall\, x \in M,\ \forall t \geq 1,\ \operatorname{vol} B(x,t) \leq \omega(t) t^n,$$

then there is in fact a number B such that

$$\forall\, x \in M,\ \forall t \geq 1,\ \operatorname{vol} B(x,t) \leq B t^{n-1}.$$

Proof. The previous proof can easily be adapted. (10) holds for $I_2 \leq t \leq \delta r(x)$, with a small $\delta > 0$. The existence of the covering leading to (11) stems from Bishop-Gromov theorem (the $x_{k,i}$ are given by a maximal $R_k/2$-net). □

This threshold effect shows that the first collapsing situation to study is that of a "codimension 1" collapse, where the volume of balls with radius t is (uniformly) comparable to t^{n-1}. This explains the gap between ALE and ALF gravitational instantons, under a uniform upper bound on the volume growth: there is no gravitational instanton with intermediate volume growth, between $\operatorname{vol} B(x,t) \asymp t^3$ and $\operatorname{vol} B(x,t) \asymp t^4$.

3. Collapsing at infinity.

3.1. **Local structure at infinity.**

We turn to codimension 1 collapsing at infinity. It turns out that the holonomy of short geodesic loops plays an important role. In order to obtain a nice structure, we will make a strong assumption on it. The next paragraph will explain why gravitational instantons satisfy this assumption.

Proposition 3.1 (Fundamental pseudo-group structure) — *Let (M^n, g) be a complete Riemannian manifold such that, for some positive numbers A and B,*

$$\forall\, x \in M,\ \forall\, t \geq 1,\ At^{n-1} \leq \operatorname{vol} B(x,t) \leq Bt^{n-1}.$$

We further assume that there is constant $c > 1$ such that $|\mathrm{Rm}| \leq c^2 r^{-2}$ and such that if γ is a geodesic loop based at x and with length $L \leq c^{-1} r(x)$, then the holonomy H of γ satisfies

$$|H - \mathrm{id}| \leq \frac{cL}{r(x)}.$$

Then there exists a compact set K in M such that for every x in $M \backslash K$, there is a unique geodesic loop σ_x of minimal length $2\,\mathrm{inj}(x)$. Besides there are geometric constants L and $\kappa > 0$ such that the fundamental pseudo-group $\Gamma(x, \kappa r(x))$ has at most $Lr(x)$ elements, all of which are obtained by successive lifts of σ.

Definition 3.2 — *σ_x is the "fundamental loop at x".*

Proof. Let us work around a point x far away from o, say with $r(x) > 100 I_2 c$. Recall (2.6) yields constants I_1, I_2 such that $0 < I_1 \leq \mathrm{inj} \leq I_2$. The fundamental pseudo-group $\Gamma := \Gamma(x, \frac{r(x)}{4c})$ contains the sub-pseudo-group $\Gamma_\sigma := \Gamma_\sigma(x, \frac{r(x)}{4c})$ corresponding to the loop σ of minimal length $2\,\mathrm{inj}(x)$. Denote by $\tau = \tau_v$ one of the two elements of Γ that correspond to σ: $|v| = 2\,\mathrm{inj}(x)$. (8) implies for $\rho = \frac{r(x)}{2c}$:

$$|\Gamma| \operatorname{vol} B\left(x, \frac{r(x)}{4c}\right) \leq \operatorname{vol} \hat{B}\left(0, \frac{r(x)}{2c}\right).$$

Bishop theorem bounds (from above) the Riemannian volume of $\hat{B}(0, \frac{r(x)}{2c})$ by $(\cosh c)^n$ times its Euclidean volume. With the lower bound on the volume growth, we thus obtain:

$$|\Gamma|\, A \left(\frac{r(x)}{4c}\right)^{n-1} \leq (\cosh c)^n\, \omega_n \left(\frac{r(x)}{2c}\right)^n,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. We deduce the estimate

$$|\Gamma| \leq Lr(x) \quad \text{with} \quad L := \frac{2^{n-2}\omega_n (\cosh c)^n}{Ac}.$$

Now, consider an oriented geodesic loop γ, based at x and with length inferior to $\frac{r(x)}{4c}$. Name τ_z the corresponding element of $\Gamma := \Gamma(x, \frac{r(x)}{4c})$. H_z will denote the holonomy of the opposite orientation of γ. By assumption,

$$|H_z - \mathrm{id}| \leq \frac{c\,|z|}{r(x)}.$$

The vector $z = \tau_z(0)$ is the initial speed of the geodesic γ, parameterized by $[0,1]$ in the chosen orientation. In the same way, $\tau_z^{-1}(0)$ is the initial speed vector of γ, parameterized by $[0,1]$, but in the opposite orientation. We deduce $-z$ is obtained as the parallel transport of $\tau_z^{-1}(0)$ along γ: $H_z(\tau_z^{-1}(0)) = -z$. From the estimate above stems:

$$\left|\tau_z^{-1}(0) + z\right| \leq \frac{c\,|z|^2}{r(x)}. \tag{12}$$

Given a small λ, say $\lambda = \frac{1}{100c}$, we consider a point w in the domain $\mathcal{I}_{\tau_z}(x, \lambda r(x))$ (see the definition in 2.3). It satisfies

$$g_x(w, \tau_z^{-1}(0)) \leq \frac{|z|^2}{2} + 2c^2\lambda^2\,|z|^2.$$

With

$$g_x(w,z) = -g_x(w, \tau_z^{-1}(0)) + g_x(w, \tau_z^{-1}(0) + z) \geq -g_x(w, \tau_z^{-1}(0)) - |w|\left|\tau_z^{-1}(0) + z\right|,$$

we find

$$g_x(w,z) \geq -\frac{|z|^2}{2} - 2c^2\lambda^2\,|z|^2 - \lambda c\,|z|^2,$$

that is

$$g_x(w,z) \geq -\frac{|z|^2}{2}\left(1 + 4c^2\lambda^2 + 2\lambda c\right).$$

With lemma 2.3, this ensures:

$$\mathcal{F}_{\tau_z}\left(x, \lambda r(x), \frac{r(x)}{4}\right) \subset \left\{ w \in \hat{B}(0, \lambda r(x)) \Big/ \ |g_x(w,z)| \le \frac{|z|^2}{2}\left(1 + 4c^2\lambda^2 + 2\lambda c\right) \right\}.$$

And with $\lambda = \frac{1}{100c}$, this leads to

$$\mathcal{F}_{\tau_z}\left(x, \lambda r(x), \frac{r(x)}{4}\right) \subset \left\{ w \in \hat{B}(0, \lambda r(x)) \Big/ \ |g_x(w,z)| \le \frac{3|z|^2}{4} \right\}. \tag{13}$$

Let τ' be an element of $\Gamma \backslash \Gamma_\sigma$ such that $v' := \tau'(0)$ has minimal norm. Suppose $|v'| < \lambda r(x)$. Then, the minimality of $|v'|$ combined with (13) yields

$$\left|g_x(v', v)\right| \le \frac{3|v|^2}{4}.$$

If $\theta \in [0, \pi]$ is the angle between v and v', this means: $|v'| \, |\cos\theta| \le 0.75 \, |v|$. Since $|v| \le |v'|$, we deduce $|\cos\theta| \le 0.75$, hence $\sin\theta \ge 0.5$. Applying (13) to τ and τ', we also get

$$\begin{aligned} \mathcal{F}\left(x, \lambda r(x), \frac{r(x)}{4c}\right) &\subset \mathcal{F}_{\tau_v}\left(x, \lambda r(x), \frac{r(x)}{4c}\right) \cap \mathcal{F}_{\tau_{v'}}\left(x, \lambda r(x), \frac{r(x)}{4c}\right) \\ &\subset \left\{ w \in \hat{B}(0, \lambda r(x)) \Big/ \ |g_x(w,v)| \le |v|^2, \ \left|g_x(w,v')\right| \le \left|v'\right|^2 \right\}. \end{aligned}$$

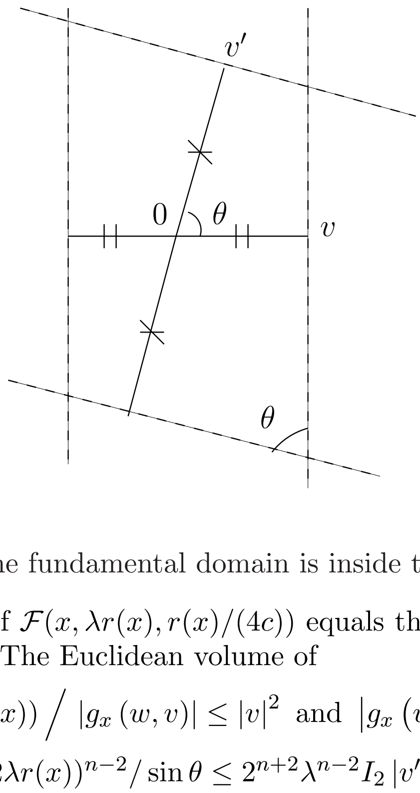


FIGURE 6. The fundamental domain is inside the dotted line.

The Riemannian volume of $\mathcal{F}(x, \lambda r(x), r(x)/(4c))$ equals that of $B(x, \lambda r(x))$, so it is not smaller than $A\lambda^{n-1} r(x)^{n-1}$. The Euclidean volume of

$$\left\{ w \in \hat{B}(0, \lambda r(x)) \Big/ \ |g_x(w,v)| \le |v|^2 \ \text{ and } \ \left|g_x(w,v')\right| \le \left|v'\right|^2 \right\}$$

is not greater than $4\,|v|\,|v'|\,(2\lambda r(x))^{n-2}/\sin\theta \le 2^{n+2}\lambda^{n-2} I_2 \, |v'| \, r(x)^{n-2}$. Comparison yields

$$A\lambda^{n-1} r(x)^{n-1} \le 2^{n+2} (\cosh c)^n \lambda^{n-2} I_2 \left|v'\right| r(x)^{n-2},$$

that is

$$\left|v'\right| \ge \frac{\lambda A}{2^{n+2} I_2 (\cosh c)^n} r(x).$$

Given a positive number κ which is smaller than λ and $\frac{\lambda A}{2^{n+2}I_2(\cosh c)^n}$, we conclude that for any x outside some compact set, the pseudo-group $\Gamma(x,\kappa r(x))$ only consists of iterates of τ (in $\Gamma(x,\frac{r(x)}{4c})$).

Suppose there are two geodesic loops with minimal length $2\,\mathrm{inj}(x)$ at x. They correspond to distinct points v and v' in T_xM. Write

$$\begin{aligned}\mathcal{F}\left(x,\lambda r(x),\frac{r(x)}{4}\right) &\subset \mathcal{F}_{\tau_v}\left(x,\lambda r(x),\frac{r(x)}{4}\right)\cap\mathcal{F}_{\tau_{v'}}\left(x,\lambda r(x),\frac{r(x)}{4}\right)\\ &\subset \left\{w\in\hat{B}(0,\lambda r(x))\Big/\ |g_x(w,v)|\le|v|^2,\ |g_x(w,v')|\le|v|^2\right\}.\end{aligned}$$

As above, we find

$$A\lambda^{n-1}r(x)^{n-1}\le 2^n(\cosh c)^n\lambda^{n-2}|v|\,|v'|\,r(x)^{n-2}/\sin\theta,$$

where $\theta\in[0,\pi]$ is the angle between the vectors v and v'. Here, $|v|=|v'|\le 2I_2$, so

$$A\lambda r(x)\sin\theta\le 2^{n+2}I_2^2(\cosh c)^n.$$

The minimality of $|v|$ and distance comparison yield

$$|v|\le d(v,v')\le\cosh(0.02)\,|v-v'|$$

hence $\cos\theta\le 0.51$. For the same reason, we find

$$|v|\le d(\tau_v^{-1}(0),v')\le\cosh(0.02)\left|\tau_v^{-1}(0)-v'\right|.$$

With (12), which gives

$$\left|\tau_v^{-1}(0)+v\right|\le 0.01\,|v|,$$

we deduce

$$|v+v'|\ge\left|\tau_v^{-1}(0)-v'\right|-\left|\tau_v^{-1}(0)+v\right|\ge 0.98\,|v|$$

hence $\cos\theta\ge -0.52$, then $|\cos\theta|\le 0.52$, and $\sin\theta\ge 0.8$. Eventually, we obtain

$$0.8A\lambda r(x)\le 2^{n+2}I_2^2(\cosh c)^n,$$

which cannot hold if x is far enough from o. This proves the uniqueness of the shortest geodesic loop. □

Uniqueness implies smoothness:

Lemma 3.3 — *In the setting of proposition 3.1, there are smooth local parameterizations for the family of loops $(\sigma_x)_x$. More precisely, given an orientation of σ_x, we can lift it to T_xM through $\exp_x$; denoting the tip of the resulting segment by v, if w is in neighborhood of 0 in T_xM, then the fundamental loop at $\exp_x w$ is the image by $\exp_x$ of the unique geodesic connecting w to $\tau_v(w)$.*

Proof. We first prove continuity. Let y be in M (outside the compact set K) and let (y_n) be a sequence converging to y. Let V_n be a sequence of of initial unit speed vector for σ_{y_n}. Compactness ensures V_n can be assumed to converge to V. Let α be the geodesic emanating from y with initial speed V. For every index n, we have $\exp_{y_n}(\mathrm{inj}(y_n)V_n)=\sigma_{y_n}(\mathrm{inj}(y_n))=y_n$. Continuity of the injectivity radius ([GLP]) allows to take a limit: $\alpha(\mathrm{inj}(y))=\exp_y\mathrm{inj}(y)V=y$. Uniqueness implies α parameters σ_y. This yields the continuity of $(\sigma_x)_x$. Now, given w in a neighborhood of 0 in T_xM, consider the $e(w)$ of the lift of $\sigma_{\exp_x w}$. The map e is a continuous section of $\exp_x$ and $e(0)=\tau_v(0)$: $e=\tau_v$. The result follows. □

Now we turn to gravitational instantons: we can control their holonomy and thus apply the previous proposition.

3.2. **Holonomy in gravitational instantons.**

LEMMA 3.4 — *Let* (M^4, g) *be a complete hyperkähler manifold with* $\mathrm{inj}(x) \geq I_1 > 0$ *and* $|\mathrm{Rm}| \leq Qr^{-3}$*. Then there is some positive* $c = c(I_1, Q)$ *such that the holonomy* H *of geodesic loops based at* x *and with length* $L \leq r(x)/c$ *satisfies*

$$|H - \mathrm{id}| \leq \frac{c}{r(x)}.$$

Proof. Consider a point x (far from o) and a geodesic loop based at x, with $L \leq r(x)/4$. Let $\tau_v \in \Gamma(x, r(x)/4)$ be a corresponding element. Thanks to (2.4), we know that for every point w in T_xM such that $|w| \leq r(x)/4$:

$$d(\tau_v(w), t_v \circ p_v^{-1}(w)) \leq \frac{8Q}{r(x)^3}\, |v|\, |w|\, (|v| + |w|)$$

and therefore, with $H = p_v^{-1}$,

$$d(\tau_v(w), Hw + v) \leq \frac{QL}{r(x)}.$$

Since (M, g) is a hyperkähler four-manifold, the holonomy group is included in $SU(2)$, so that in some orthonormal basis of T_xM, seen as a complex two-space, $H = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with an angle θ in $]-\pi, \pi]$. Suppose θ is not zero (otherwise the statement is trivial). The equation $Hw + v = w$ admits a solution:

$$w = \begin{pmatrix} \frac{v_1}{1-e^{i\theta}} \\ \frac{v_2}{1-e^{-i\theta}} \end{pmatrix}$$

where v_1 and v_2 denote the coordinates of v.

If $|w| \leq r(x)/4$, we obtain $d(\tau_v(w), w) \leq \frac{QL}{r(x)}$. Since the lower bound on the injectivity radius yields $d(\tau_v(w), w) \geq 2I_1$, we find $L \geq \frac{2I_1}{Q} r(x)$.

As a consequence, if $L < \frac{2I_1}{Q} r(x)$, then $|w| > \frac{r(x)}{4}$, that is

$$|H - \mathrm{id}| = \left|1 - e^{i\theta}\right| = \frac{L}{|w|} \leq \frac{4L}{r(x)}.$$

□

As a result, we obtain the

PROPOSITION 3.5 — *Let* (M^4, g) *be a complete hyperkähler manifold such that*

$$\int_M |\mathrm{Rm}|^2\, r dvol < \infty \tag{14}$$

and, for some positive numbers A *and* B*,*

$$\forall\, x \in M,\ \forall\, t \geq 1,\ At^3 \leq \mathrm{vol}\, B(x, t) \leq Bt^3. \tag{15}$$

Then there exists a compact set K *in* M *such that for every* x *in* $M \backslash K$*, there is a unique geodesic loop* σ_x *of minimal length* $2\,\mathrm{inj}(x)$*. Besides there are geometric constants* L *and* $\kappa > 0$ *such that the fundamental pseudo-group* $\Gamma(x, \kappa r(x))$ *has at most* $Lr(x)$ *elements, all of which are obtained by successive lifts of* σ*.*

Proof. Since M is hyperkähler, it is Ricci flat. So [Min] applies (see appendix A): $|\mathrm{Rm}| = \mathcal{O}(r^{-3})$. So we use proposition 3.1, thanks to lemma 3.4. □

Remark 2. *From now on, we will remain in the setting of four dimensional hyperkähler manifolds. It should nonetheless be noticed that the only reason for this is lemma 3.4. If the conclusion of this lemma is assumed and if we suppose convenient estimates on the covariant derivatives of the curvature tensor, then we can work in any dimension (see 3.26 below).*

3.3. **An estimate on the holonomy at infinity.** To go on, we will need a better estimate of the holonomy of short loops. This is the goal of this paragraph. First, let us state an easy lemma, adapted from [BK] (6.2).

Lemma 3.6 (Holonomy comparison) — *Let $\gamma : [0, L] \longrightarrow N$ be a curve in a Riemannian manifold N and let $t \mapsto \alpha_t$ be a family of loops, parameterized by $0 \le s \le l$ with $\alpha_t(0) = \alpha_t(l) = \gamma(t)$. We denote by $p_\gamma(t)$ the parallel transportation along γ, from $\gamma(0)$ to $\gamma(t)$. We consider a vector field $(s,t) \mapsto X(s,t)$ along the family α and we suppose it is parallel along each loop α_t ($\nabla_s X(s,t) = 0$) and along γ ($\nabla_t X(0,t) = 0$). Then:*

$$\left|p_\gamma(L)^{-1}X(l,L) - X(l,0)\right| \le \int_0^L \int_0^l |\mathrm{Rm}(\partial_s\sigma_t, \partial_t\sigma_t)X(s,t)|\, dsdt.$$

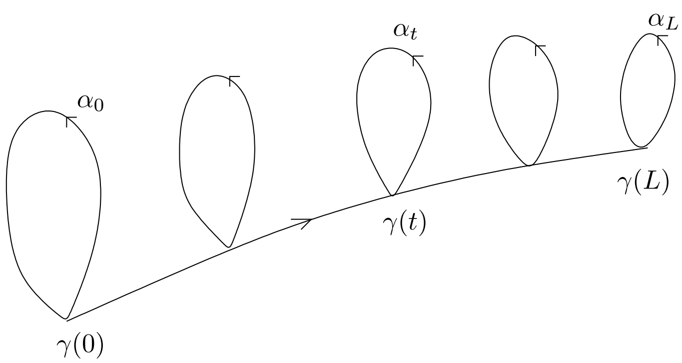


Figure 7. A one parameter family of loops.

We consider a complete hyperkähler manifold (M^4, g) with

$$\int_M |\mathrm{Rm}|^2\, rdvol < \infty \quad \text{or equivalently} \quad |\mathrm{Rm}| = \mathcal{O}(r^{-3})$$

and for some positive constants A, B,

$$\forall\, x \in M,\ \forall\, t \ge 1,\ At^3 \le \mathrm{vol}\, B(x,t) \le Bt^3.$$

We choose a unit ray $\gamma : \mathbb{R}_+ \longrightarrow M$ starting from o and we denote by $p_\gamma(t)$ the parallel transportation along γ, from $\gamma(0)$ to $\gamma(t)$. For large t, we can define the holonomy endomorphism $H_{\gamma(t)}$ of the fundamental loop $\sigma_{\gamma(t)}$: here, there is an implicit choice of orientation for the loops $\sigma_{\gamma(t)}$, which we can assume continuous. This yields an element of $O(T_{\gamma(t)}M)$. Holonomy comparison lemma 3.6 asserts that for large $t_1 \le t_2$:

$$\begin{aligned} &\left|p_\gamma(t_2)^{-1}H_{\gamma(t_2)}p_\gamma(t_2) - p_\gamma(t_1)^{-1}H_{\gamma(t_1)}p_\gamma(t_1)\right| \\ \le\ &\int_{t_1}^{\infty}\int_0^1 |\mathrm{Rm}|\,(c(t,s))\,|\partial_s c(t,s) \wedge \partial_t c(t,s))|\, dsdt, \end{aligned}$$

where, for every fixed t, $c(t,.)$ parameterizes $\sigma_{\gamma(t)}$ by $[0,1]$, at speed $2\,\mathrm{inj}(\gamma(t))$.

Lemma 3.7 — $|\partial_s c \wedge \partial_t c)|$ *is uniformly bounded.*

Proof. The upper bound on the injectivity radius bounds $|\partial_s c|$. We need to bound the composant of $\partial_t c$ that is orthogonal to $\partial_s c$. We concentrate on a neighborhhood of some point x along γ. For convenience, we change the parameterization so that $x = \gamma(0)$. We also lift the problem to $T_x M =: E$, endowed with the lifted metric $\hat{g}$. If $v = \gamma'(0)$, γ lifts as a curve $\hat{\gamma}$ parameterized by $t \mapsto tv$. The lift $\hat{c}$ of c consists of the geodesics $\hat{c}(t,.)$ connecting tv to $\tau(tv)$; τ is the element of the fundamental pseudo-group corresponding to σ_x, for the chosen orientation. Observe $\hat{c}(t,s) = \mathrm{Exp}_{tv}\, sX(t)$ where $X(t) \in T_{tv}E$ is defined by $\mathrm{Exp}_{tv}\, X(t) = \tau(tv)$. The vector field J defined along $\hat{c}(0,.)$ by

$$J(s) = \partial_t \hat{c}(0,s) = \frac{d}{dt}\Big|_{t=0} \mathrm{Exp}_{tv}\, sX(t)$$

is a Jacobi field with initial data $J(0) = v$ ($E = T_0 E$) and $J'(0) = (\nabla_t X)(0)$. Suppose the curvature is bounded by Λ^2, $\Lambda > 0$, in the area under consideration and apply lemma 6.3.7 of [BK]: the part $\tilde{J}$ of J that is orthogonal to $\hat{c}(0,.)$ satisfies

$$\left| \tilde{J}(s) - p(sv)\tilde{J}(0) - sp(sv)\tilde{J}'(0) \right| \leq a(s)$$

where $p(.)$ is the radial parallel transportation and where a solves

$$a'' - \Lambda^2 a = \Lambda^2 \left(\left|\tilde{J}(0)\right| + \left|\tilde{J}'(0)\right| \right)$$

with $a(0) = a'(0) = 0$, i.e. $a(s) = \left(\left|\tilde{J}(0)\right| + \left|\tilde{J}'(0)\right| \right)(\cosh(\Lambda s) - 1)$. Since here $0 \leq s \leq 1$ and $\Lambda << 1$, we only need a bound on $\left|\tilde{J}(0)\right|$ and $\left|\tilde{J}'(0)\right|$ to control $\tilde{J}$ and end the proof. Since $J(0) = v$ has unit length, we are left to bound $\tilde{J}'(0)$.

We consider the family of vectors Y in E that is defined by $X(t) = p(tv)Y(t)$. Then $J'(0) = \nabla_t X(0) = Y'(0)$. Let f be the map from E^2 to E given by $f(w,W) = \mathrm{Exp}_w\, p(w)W$. The equality $f(tv, Y(t)) = \tau(tv)$ can be differentiated into

$$\partial_1 f_{(0,Y(0))} v + \partial_2 f_{(0,Y(0))} Y'(0) = (D\tau)_0 v. \tag{16}$$

Lemma 6.6 in [BK] ensures $\partial_2 f_{(0,Y(0))}$ is Λ^2-close to the identity. Besides, τ is an isometry for $\hat{g}$, so $(D\tau)_0$ is uniformly bounded. Finally, $\partial_1 f_{(0,Y(0))} v$ is the value at time 1 of the Jacobi field K along $s \mapsto sY(0)$ corresponding to the geodesic variation $H(t,s) \mapsto \mathrm{Exp}_{tv}\, sp(tv)Y(0)$. As the initial data for $K(s) = \partial_t H(0,s)$ are $K(0) = v$ and $K'(0) = 0$, we obtain (corollary 6.3.8 of [BK]) a bound on K and thus on $\partial_1 f_{(0,Y(0))} v$. This yields a bound on $Y'(0)$ (thanks to (16)) and we are done. $\square$

This lemma and the curvature decay lead to the estimate

$$\left| p_\gamma(t_2)^{-1} H_{\gamma(t_2)} p_\gamma(t_2) - p_\gamma(t_1)^{-1} H_{\gamma(t_1)} p_\gamma(t_1) \right| \leq C \int_{t_1}^{\infty} t^{-3} dt \leq C\, t_1^{-2}.$$

Now recall the holonomy of the loops under consideration goes to the identity at infinity. Setting $t_1 =: t$ and letting t_2 go to infinity, we find

$$\left| p_\gamma(t)^{-1} H_{\gamma(t)} p_\gamma(t) - \mathrm{id} \right| \leq C\, t^{-2}. \tag{17}$$

Since M has zero (hence nonnegative) Ricci curvature and cubic volume growth, it follows from Cheeger Gromoll theorem that M has only one end. Relying on faster than quadratic curvature decay, [Kas] then ensures large spheres $S(o,t)$ are connected with intrinsic diameter bounded by Cs. Thus every point x in $S(o,t)$ is connected to $\gamma(t)$ by some curve β with length at most Ct et and remaining outside $B(o,t/2)$. Holonomy comparison lemma 3.6 yields:

$$\left|p_\beta^{-1}H_x p_\beta - H_{\gamma(t)}\right| \le Ct^{-2}, \tag{18}$$

where p_β is the parallel transportation along β and H_x is the holonomy endomorphism corresponding to a consistent orientation of σ_x. It follows that

$$|H_x - id| \le Cr(x)^{-2}. \tag{19}$$

So we have managed to improve our estimate on the holonomy of fundamental loops.

Lemma 3.8 — *Let (M^4,g) be a complete hyperkähler manifold satisfying (14) and (15). Then the holonomy H_x of the fundamental loops σ_x satisfies*

$$|H_x - \mathrm{id}| \le Cr(x)^{-2}.$$

3.4. **Local Gromov-Hausdorff approximations.** The following lemma ensures the elements of the fundamental group are almost translations.

Lemma 3.9 — *Let (M^4,g) be a complete hyperkähler manifold satisfying (14) and (15). Then there exist a compact set K in M and geometric constants J, L, $\kappa > 0$ such that for every point x in $M\backslash K$ and every τ in $\Gamma(x,\kappa r(x))$, one has*

$$\forall\, w \in \hat{B}(0,\kappa r(x)),\ |\tau(w) - t_{kv_x}(w)| \le J$$

where v_x is a lift of the tip of σ_x and k is a natural number bounded by $Lr(x)$.

Proof. Proposition 3.1 asserts we can write $\tau = \tau_{v_x}^k$, where v_x is a lift of a tip of σ_x and k is a natural number bounded by $Lr(x)$. Lemma 2.4 ensures that for every w in $\hat{B}(0,r(x)/4)$:

$$\left|\tau_{v_x}(w) - v_x - p_{v_x}^{-1}(w)\right| \le Cr(x)^{-3}\,|v_x|\,|w|\,(|v_x| + |w|).$$

Thanks to cubic curvature decay (appendix A), (19) yields: $\left|p_{v_x}^{-1}(w) - w\right| \le Cr(x)^{-2}\,|w|$. Combining these estimates, we obtain:

$$|\tau_{v_x}(w) - t_{v_x}(w)| = |\tau_{v_x}(w) - v_x - w| \le Cr(x)^{-2}\,|w|.$$

For $i \le k$, let us set $e_i = \tau_{v_x}^i - t_{v_x}^i$. Then $e_{i+1} - e_i = e_1 \circ \tau_{v_x}^i$. With

$$\left|\tau_{v_x}^i(w)\right| = d(\tau_{v_x}^i(w),0) = d(\tau_{v_x}^{-i}(0),w) \le \left|\tau_{v_x}^{-i}(0)\right| + |w|,$$

we find that for every w in $\hat{B}(0,\kappa r(x))$:

$$|e_{i+1}(w) - e_i(w)| \le Cr(x)^{-2}\left|\tau_{v_x}^i(w)\right| \le Cr(x)^{-1}.$$

By induction, it follows that $|e_k(w)| \le Ckr(x)^{-1}$ and since $k \le Lr(x)$, we are lead to:

$$|\tau(w) - t_{kv_x}(w)| = \left|\tau_{v_x}^k(w) - kv_x - w\right| = |e_k(w)| \le C.$$

□

Proposition 3.10 (Gromov-Hausdorff approximation) — *Let (M^4,g) be a complete hyperkähler manifold satisfying (14) and (15). Then there exists a compact set K in M and geometric constants I, $\kappa > 0$ such that every point x in $M\backslash K$ has a neighborhood Ω whose Gromov-Hausdorff distance to the ball of radius $\kappa r(x)$ in $\mathbb{R}^3$ is bounded by I.*

Proof. Choose a lift of σ_x in T_xM and denote by v_x its tip. We call H the hyperplane orthogonal to v_x and write $v \mapsto v_H$ for the Euclidean orthogonal projection onto H (for g_x).

If y is a point in $B(x, \kappa r(x)/2)$, we can define $h(y)$ as affine center of mass of the points v_H obtained from lifts v of y in $\hat{B}(0, \kappa r(x)/2)$. This defines a map h from $B(x, \kappa r(x)/2)$ to $H \cong \mathbb{R}^3$ (we endow H of the Euclidean structure induced by $g_x = |.|^2$).

We consider the ball B centered in 0 and with radius $0.1\kappa r(x)$ in H: 0.1κ will be the κ of the statement. Let us set $\Omega := h^{-1}(B)$. We want to see that $h : \Omega \longrightarrow B$ is the promised Gromov-Hausdorff approximation. We need to check that this map h has I-dense image and that for all points y and z in Ω: $|d(y,z) - |h(y) - h(z)|| \leq I$.

Firstly, since v is in B, lemma 3.9 ensures that for every $\tau = \tau_{v_x}^k$ in $\Gamma(x, \kappa r(x))$, we have $|\tau(v) - v - kv_x| \leq J$ and thus, with Pythagore theorem, $|\tau(v)_H - v| \leq J$. Passing to the center of mass, we get $|h(\exp_x v) - v| \leq J$. If $d(v, H\backslash B) > J$, this proves $h(\exp_x v)$ belongs to B and therefore $\exp_x v$ belongs to Ω; as a result, $d(v, h(\Omega)) \leq J$. As $\{v \in B \,/\, d(v, H\backslash B) > J\}$ is J-dense in B, we have shown that $h(\Omega)$ is $2J$-dense in B.

Secondly, consider two points y and z in Ω. Lift them into v and w ($\in B(x, \kappa r(x)/2)$) with $d(v,w) = d(y,z)$. As above, we get $|h(y) - v_H| \leq J$ and $|h(z) - w_H| \leq J$, hence

$$|h(y) - h(z)| \leq |v_H - w_H| + 2J \leq |v - w| + 2J.$$

Since comparison yields $|v - w| \leq \left(1 + Cr(x)^{-1}\right) d(v,w) \leq d(v,w) + C$ for some constant C (changing from line to line), we deduce

$$|h(y) - h(z)| \leq d(v,w) + C = d(y,z) + C. \tag{20}$$

Now, consider lifts v' and w' at minimal distance from H and observe lemma 2.3 yields: $|v' - v'_H| \leq C$ and $|w' - w'_H| \leq C$. We deduce: $|v' - w'| \leq |v'_H - w'_H| + C$. The distance between y and z is the infimum of the distances between their lifts, so $d(y,z) \leq d(v', w')$. As above, comparison ensures: $d(v', w') \leq |v' - w'| + C$. These three inequalities give alltogether:

$$d(y,z) \leq \left|v'_H - w'_H\right| + C$$

And since $||h(y) - h(z)| - |v'_H - w'_H|| \leq C$, we get

$$d(y,z) \leq |h(y) - h(z)| + C. \tag{21}$$

The combination of (20) and (21) yields $|d(y,z) - |h(y) - h(z)|| \leq C$, hence the result. □

The following step consists in regularizing local Gromov-Hausdorff approximations to obtain local fibrations which accurately describe the local geometry at infinity.

3.5. **Local fibrations.** The local Gromov-Hausdorff approximation that we built above has no reason to be regular. We will now smooth it into a fibration. The technical device is simply a convolution, as in [Fuk] and [CFG]. We basically need theorem 2.6 in [CFG]. The trouble is this general result will have to be refined, by using fully the cubic decay of the curvature and the symmetry properties of the special Gromov-Hausdorff approximation we smooth. This technique requires a control on the covariant derivatives of the curvature, but it is heartening to know that this is given for free on gravitational instantons (see theorem A.4 in appendix A).

We say f is a C-almost-Riemannian submersion if f is a submersion such that for every horizontal vector v (i.e. orthogonal to fibers), $e^{-C}|v| \leq |df_x(v)| \leq e^{C}|v|$.

Proposition 3.11 (Local fibrations) — *Let (M^4, g) be a complete hyperkähler manifold satisfying (14) and (15). Then there exists a compact set K in M and geometric constants $\kappa > 0$, $C > 0$ such that for every point x in $M\backslash K$, there is a circle fibration $f_x : \Omega_x \longrightarrow B_x$*

defined on a neighborhood Ω_x *of* x *and with values in the Euclidean ball* B_x *with radius* $\kappa r(x)$ *in* $\mathbb{R}^3$*. Moreover,*

- f_x *is a* $Cr(x)^{-1}$*-almost-Riemannian submersion,*
- *its fibers are submanifolds diffeomorphic to* $\mathbb{S}^1$*, with length pinched between* C^{-1} *and* C*,*
- $\left|\nabla^2 f_x\right| \leq Cr(x)^{-2}$*,*
- $\forall i \geq 3, \left|\nabla^i f_x\right| = \mathcal{O}(r(x)^{1-i})$*.*

Proof. In the proof of 3.10, we introduced a function h from the ball $B(x, \kappa r(x))$ to the hyperplane H, orthogonal to the tip v_x of a lift of σ_x in T_xM; this hyperplane H is identified to the Euclidean space $\mathbb{R}^3$ through the metric induced by g_x.

Let us choose a smooth nonincreasing function χ from $\mathbb{R}_+$ to $\mathbb{R}_+$, equal to 1 on $[0, 1/3]$ and 0 beyond $2/3$. We also fix a scale $\epsilon := 0.1\kappa r(x)$ and set $\chi_\epsilon(t) = \chi(2t/\epsilon^2)$. Note the estimates:

$$\left|\chi_\epsilon^{(k)}\right| \leq C_k \epsilon^{-2k}. \tag{22}$$

We consider the function defined on $B(x, \kappa r(x))$ by:

$$f(y) := \frac{\int_{T_yM} h(\exp_y v)\chi_\epsilon(d(0,v)^2/2)dvol(v)}{\int_{T_yM} \chi_\epsilon(d(0,v)^2/2)dvol(v)}.$$

Here, $dvol$ and d are taken with respect to $\exp_y^* g$. If w is a lift of y in T_xM, we can change variables thanks to the isometry $\tau_w := \mathrm{Exp}_w \circ (T_w \exp_x)^{-1}$ between $(T_yM, \exp_y^* g)$ and $(T_xM, \exp_x^* g)$. For every point v in T_xM, we introduce the function $\rho_v := \frac{d(v,.)^2}{2}$ and set $\hat{f} := f \circ \exp_x$, $\hat{h} := h \circ \exp_x$. We then get the formula:

$$f(y) = \hat{f}(w) = \frac{\int_{T_xM} \hat{h}(v)\chi_\epsilon(\rho_v(w))dvol(v)}{\int_{T_xM} \chi_\epsilon(\rho_v(w))dvol(v)}.$$

The point is we can now work on a fixed Euclidean space, (T_xM, g_x). The Riemannian measure $dvol$ can be compared to Lebesgue measure dv: on a scale ϵ, if the curvature is bounded by Λ^2, we have $\left(\frac{\sin \Lambda\epsilon}{\Lambda\epsilon}\right)^4 dv \leq dvol \leq \left(\frac{\sinh \Lambda\epsilon}{\Lambda\epsilon}\right)^4 dv$. Cubic curvature decay implies Λ is of order $\epsilon^{-\frac{3}{2}}$, so that we find

$$-C\epsilon^{-1}dv \leq dvol - dv \leq C\epsilon^{-1}dv. \tag{23}$$

Distance comparison yields in the same way: $|d(v,w) - |v-w|| \leq C\Lambda^2\epsilon^2 d(v,w) \leq C$, hence

$$\left|\rho_v(w) - |v-w|^2/2\right| \leq C\epsilon. \tag{24}$$

Eventually, the proof of 3.10 shows $\hat{h}$ is close to a Euclidean projection onto H:

$$\left|\hat{h}(v) - v_H\right| \leq C. \tag{25}$$

We can write

$$\begin{aligned}\int \hat{h}(v)\chi_\epsilon(\rho_v(w))dvol(v) &= \int \hat{h}(v)\chi_\epsilon(\rho_v(w))(dvol(v)-dv)\\ &+ \int \hat{h}(v)\left(\chi_\epsilon(\rho_v(w)) - \chi_\epsilon(|v-w|^2/2)\right)dv\\ &+ \int (\hat{h}(v)-v_H)\chi_\epsilon(|v-w|^2/2)dv\\ &+ \int v_H\chi_\epsilon(|v-w|^2/2)dv.\end{aligned}$$

The support of $v \mapsto \chi_\epsilon(\rho_v(w))$ is included in a ball whose radius is of order ϵ: $\hat{h}$ will therefore take its values in a ball with radius of order ϵ. With (23), we can then bound the first term of the right-hand side by $C\epsilon \cdot \epsilon^{-1} \cdot \epsilon^4 = C\epsilon^4$. With (22) and (24), we bound the second term by $C\epsilon \cdot \epsilon^{-2} \cdot \epsilon \cdot \epsilon^4 = C\epsilon^4$. Eventually, (25) controls the third term by $C\epsilon^4$. We get:

$$\int \hat{h}(v)\chi_\epsilon(\rho_v(w))dvol(v) = \int v_H\chi_\epsilon(|v-w|^2/2)dv + \mathcal{O}(\epsilon^4),$$

where $\mathcal{O}(\epsilon^4)$ stands for an error term of magnitude ϵ^4.

Thanks to (23), (22) and (24), we obtain in the same way:

$$\int_{T_xM} \chi_\epsilon(\rho_v(w))dvol(v) = \int \chi_\epsilon(|v-w|^2/2)dv + \mathcal{O}(\epsilon^3).$$

Observing

$$\int v_H\chi_\epsilon(|v-w|^2/2)dv = \mathcal{O}(\epsilon^5)$$

and

$$C^{-1}\epsilon^4 \leq \int \chi_\epsilon(|v-w|^2/2)dv \leq C\epsilon^4,$$

we deduce

$$\hat{f}(w) = \frac{\int v_H\chi_\epsilon(|v-w|^2/2)dv}{\int \chi_\epsilon(|v-w|^2/2)dv} + \mathcal{O}(1).$$

The change of variables $z = v - w$ yields:

$$\hat{f}(w) - w_H = \underbrace{\frac{\int z_H\chi_\epsilon(|z|^2/2)dz}{\int \chi_\epsilon(|z|^2/2)dz}}_{=0 \text{ by parity}} + \mathcal{O}(1),$$

hence

$$\hat{f}(w) = w_H + \mathcal{O}(1). \tag{26}$$

The differential of $\hat{f}$ reads

$$d\hat{f}_w = \frac{\int (\hat{h}(v) - \hat{f}(w))\chi_\epsilon'(\rho_v(w))(d\rho_v)_w dvol(v)}{\int \chi_\epsilon(\rho_v(w))dvol(v)}.$$

The same kind of approximations, based on (8), (22), (24), (B.3), (25) and (26) imply

$$d\hat{f}_w = -\frac{\int z_H\chi_\epsilon'(|z|^2/2)(z,.)dz}{\int \chi_\epsilon(|z|^2/2)dz} + \mathcal{O}(\epsilon^{-1}). \tag{27}$$

Let us choose an orthonormal basis $(e_1,\dots,e_4)$ of T_xM, with $e_4\perp H$. If $i\neq j$, parity shows

$$\int z_i\chi_\epsilon'(|z|^2/2)z_j dz = 0.$$

On the contrary, an integration by parts ensures that for every $\alpha\geq 0$:

$$\int_{-\infty}^{\infty} z_i^2\chi_\epsilon'(z_i^2/2+\alpha)dz_i = -\int_{-\infty}^{\infty}\chi_\epsilon(z_i^2/2+\alpha)dz_i,$$

so that

$$-\int z_i^2\chi_\epsilon'(|z|^2/2)dz = \int \chi_\epsilon(|z|^2/2)dz.$$

This means precisely:

$$-\frac{\int z_H\chi_\epsilon'(|z|^2/2)(z,.)dz}{\int\chi_\epsilon(|z|^2/2)dz} = \sum_{i=1}^3 e_i\otimes(e_i,.).$$

And one can recognize the Euclidean projection onto H. We deduce

$$d\hat{f}_w = \sum_{i=1}^3 e_i\otimes(e_i,.)+\mathcal{O}(\epsilon^{-1})$$

With (40), this proves $\hat{f}$ is a $C\epsilon^{-1}$-almost-Riemannian submersion. Since exp is a local isometry, f is also a $C\epsilon^{-1}$-almost-Riemannian submersion.

The Hessian reads:

$$\begin{aligned}\nabla^2\hat{f}_w &= \frac{\int(\hat{h}(v)-\hat{f}(w))\left(\chi_\epsilon''(\rho_v(w))(d\rho_v)_w\otimes(d\rho_v)_w+\chi_\epsilon'(\rho_v(w))(\nabla^2\rho_v)_w\right)dvol(v)}{\int\chi_\epsilon(\rho_v(w))dvol(v)}\\ &- 2d\hat{f}_w\otimes\frac{\int\chi_\epsilon'(\rho_v(w))(d\rho_v)_w)dvol(v)}{\int\chi_\epsilon(\rho_v(w))dvol(v)}.\end{aligned}$$

Again, with (8), (22), (24), (B.3), (25) and (26), we arrive at

$$\begin{aligned}\nabla^2\hat{f}_w &= \frac{\int z_H\left(\chi_\epsilon''(|z|^2/2)(z,.)\otimes(z,.)+\chi_\epsilon'(|z|^2/2)(.,.)\right)dz}{\int\chi_\epsilon(|z|^2/2)dz}\\ &- 2d\hat{f}_w\otimes\frac{\int\chi_\epsilon'(|z|^2/2)(z,.)dz}{\int\chi_\epsilon(|z|^2/2)dz}+\mathcal{O}(\epsilon^{-2}).\end{aligned}$$

To begin with, parity ensures

$$\int\chi_\epsilon'(|z|^2/2)(z,.)dz = 0 \quad\text{and}\quad \int z_H\chi_\epsilon'(|z|^2/2)(.,.)dz = 0.$$

The ith component of the integral $\int z_H\chi_\epsilon''(|z|^2/2)(z,.)\otimes(z,.)$ can be written as a sum of terms

$$\left(\int z_iz_jz_k\chi_\epsilon''(|z|^2/2)dz_1\dots dz_4\right)(e_j,.)\otimes(e_k,.)$$

which vanish for a parity reason. Therefore: $\nabla^2\hat{f}_w=\mathcal{O}(\epsilon^{-2})$.

The proof of theorem 2.6 in [CFG] yields the remaining properties of $f_x:=f$. Essentially, f is a fibration because it is C^1-close to a fibration. The connectedness of the fibers follows from the bound on the Hessian of f. The length of the fibers is controlled by the assumption on the volume growth (since f is a almost-Riemannian submersion). □

We will need to relate neighboring fibrations (this somewhat corresponds to proposition 5.6 in [CFG]).

Lemma 3.12 (Closeness of local fibrations I) — *The setting is the same as in proposition 3.11. Given two points x and x' in $M\backslash K$, with $d(x,x') \leq \kappa r(x)$, if $\Omega_{x,x'} = \Omega_x \cap \Omega_{x'}$ (notations in 3.11), then there is a $Cr(x)^{-1}$-almost-isometry $\phi_{x,x'}$ between $f_{x'}(\Omega_{x,x'})$ and $f_x(\Omega_{x,x'})$, for which moreover*

- $\left|f_x - \phi_{x,x'} \circ f_{x'}\right| \leq C$,
- $\left|Df_x - D\phi_{x,x'} \circ Df_{x'}\right| \leq Cr(x)^{-1}$,
- $\left|D^2\phi_{x,x'}\right| \leq Cr(x)^{-2}$,
- $\forall\, i \geq 3, \left|D^i\phi_{x,x'}\right| = \mathcal{O}(r(x)^{1-i})$.

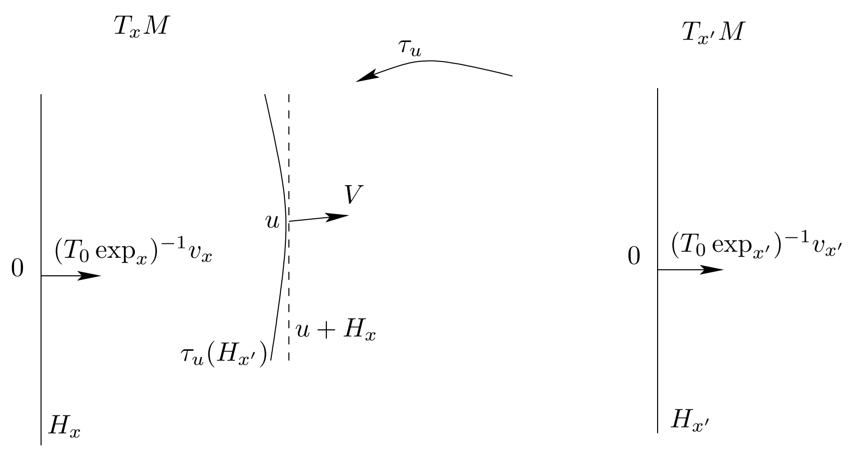
Proof. We use the same notations as in the previous proof, adding subscripts to precise the point under consideration, and we work in T_xM. Choose a lift u of y at minimal distance from o and set $\tau_u := \mathrm{Exp}_u \circ (T_u \exp_x)^{-1}$ the corresponding isometry (between large balls in $T_{x'}M$ and T_xM). We consider the map

$$\phi_{x,x'} := f_x \circ \exp_{x'} |_{f_{x'}(\Omega_{x,x'})}.$$

In order to bring everything back into T_xM, we write $\phi_{x,x'} \circ f_{x'} \circ \exp_x = f_x \circ \exp_{x'} \circ f_{x'} \circ \exp_x$. The relation $\exp_x \circ \tau_u = \exp_{x'}$ leads to the reformulation

$$\phi_{x,x'} \circ f_{x'} \circ \exp_x = \hat{f}_x \circ \tilde{f}_{x'} \tag{28}$$

with $\hat{f}_x = f_x \circ \exp_x$ and $\tilde{f}_{x'} = \tau_u \circ f_{x'} \circ \exp_{x'} \circ \tau_u^{-1}$. We need to understand this lattest map.



Since τ_u is an isometry between the metrics $\exp_{x'}^* g$ and $\exp_x^* g$ and since $H_{x'}$ is the union of all the geodesics starting from 0 and with a unit speed orthogonal to $(T_0 \exp_{x'})^{-1}(v_{x'})$, $\tau_u(H_{x'})$ is the hypersurface generated by the geodesics starting from u with a unit speed orthogonal to $V := (d\tau_u)_0 \circ (T_0 \exp_{x'})^{-1}(v_{x'})$. $v_{x'}$ is by definition one of the lifts of x' by $\exp_{x'}$ which is not 0 but at minimal distance from 0 (in $T_{x'}M$). So $\tau_u(v_{x'})$ is one of the two lifts of x' by $\exp_x$ which is not $\tau_u(0) = u$ but at minimal distance from $\tau_u(0) = u$ (in T_xM). We have seen in lemma 3.3 that such a point $\tau_u(v_x')$ is $\tau_{v_x}(u)$ or $\tau_{v_x}^{-1}(u)$. To fix ideas, assume we are in the first case: $\tau_u(v_x') = \tau_{v_x}(u)$.

The exponential map of $T_{x'}M$ (at 0) maps $(T_0 \exp_{x'})^{-1}(v_{x'})$ to $v_{x'}$, so $V = (d\tau_u)_0 \circ (T_0 \exp_{x'})^{-1}(v_{x'})$ is the vector which is mapped by the exponential map of T_xM (at $\tau_u(0) = u$) to $\tau_u(v_x') = \tau_{v_x}(u)$: $\mathrm{Exp}_u V = \tau_{v_x}(u)$. Consider the geodesic $\gamma(t) := \mathrm{Exp}_u tV$. Taylor formula

$\gamma(1)-\gamma(0)-\dot\gamma(0)=\int_0^1(1-t)\ddot\gamma(t)dt$ and the estimate $|\ddot\gamma|\le Cr(x)^{-2}|V|^2\le Cr(x)^{-2}$, steming from lemma B.2 and the bound on the injectivity radius (2.1), together imply

$$|\tau_{v_x}(u)-u-V|\le Cr(x)^{-2}.$$

With the estimate $|\tau_{v_x}(u)-u-v_x|\le Cr(x)^{-1}$, we deduce

$$|V-v_x|\le Cr(x)^{-1} \tag{29}$$

so that, with $\hat{\mathcal{U}}:=\hat B(0,\kappa r(x))\cap\hat B(u,\kappa r(x'))$, the affine hyperplanes pieces $(u+V^\perp)\cap\hat{\mathcal{U}}$ and $(u+H_x)\cap\hat{\mathcal{U}}$ remain at bounded distance. Considering the geodesic $\gamma(t)=\mathrm{Exp}_u\, tW$, with $W\perp V$ and $|W|\le Cr(x)$, we obtain in the same way (thanks to lemma B.2):

$$|\mathrm{Exp}_u W-u-W|\le Cr(x)^{-2}r(x)^2=C.$$

This means the affine hyperplane piece $(u+V^\perp)\cap\hat{\mathcal{U}}$ and the hypersurface piece $\tau_u(B_{x'})\cap\hat{\mathcal{U}}=\mathrm{Exp}_u V^\perp\cap\hat{\mathcal{U}}$ remain at bounded distance. And we conclude $\tau_u(B_{x'})\cap\hat{\mathcal{U}}$ and $(u+H_x)\cap\hat{\mathcal{U}}$ remain C-close. In the previous proof, we saw that $f_{x'}\circ\exp_{x'}$ was C-close to the orthogonal projection (for $g_{x'}$) onto $H_{x'}$. Now, τ_u is an isometry between the metrics $\exp_{x'}^* g$ and $\exp_x^* g$, which are respectively $Cr(x)^{-1}$-close to $g_{x'}$ and g_x. It follows from all this that in the area under consideration, $\tilde f_{x'}$ is C-close to the projection onto $(u+H_x)\cap\hat{\mathcal{U}}$. And since $w_{u+H_x}=w_{H_x}+(u-u_{H_x})$, we deduce

$$\left|\tilde f_{x'}(w)-w_{H_x}-(u-u_{H_x})\right|\le C,\quad\text{hence}\quad\left|\tilde f_{x'}(w)-\hat f_x(w)-(u-u_{H_x})\right|\le C.$$

Composing with $\hat f_x$, we find $\left|\hat f_x\circ\tilde f_{x'}(w)-\hat f_x(w)\right|\le C$. Recalling formula (28), we obtain $\left|\phi_{x,x'}\circ f_{x'}\circ\exp_x-f_x\circ\exp_x\right|\le C$, and, with the surjectivity of $\exp_x$, this yields

$$\left|\phi_{x,x'}\circ f_{x'}-f_x\right|\le C.$$

Relation (28) also implies

$$D(\phi_{x,x'}\circ f_{x'}\exp_x)=D\hat f_x\circ D\tilde f_{x'}. \tag{30}$$

Let z be a point in $\hat{\mathcal{U}}$ and set $z'=\tau_u^{-1}(z)\in T_{x'}M$. The previous proof has shown that $D_z\hat f_x$ is $Cr(x)^{-1}$-close to the orthogonal projection in the direction of H_x. In the same way, $D_{z'}(f_{x'}\exp_{x'})$ is $Cr(x)^{-1}$-close to the orthogonal projection in the direction of $H_{x'}$, i.e. in the direction orthogonal to $v_{x'}$. Conjugating by $D\tau_u$, we find that $D_z\tilde f_{x'}$ is $Cr(x)^{-1}$ close to the projection in the direction orthogonal to $D_z\tau_u(v_{x'})$.

Let Z' be the initial speed of the geodesic connecting z' to $\tau_{v_{x'}}(z')$ in unit time. The argument leading to (29) yields $|Z'-v_{x'}|\le Cr(x)^{-1}$. If we set $Z:=D_z\tau_u Z'$, we thus have $|Z-D_z\tau_u(v_{x'})|\le Cr(x)^{-1}$. Since Z is the initial speed of the geodesic connecting z to $\tau_{v_x}(z)$ (or $\tau_{v_x}^{-1}(z)$) in unit time, we also find $|Z-v_x|\le Cr(x)^{-1}$, so that we can deduce:

$$|v_x-D_z\tau_u(v_{x'})|\le Cr(x)^{-1}.$$

Finally, $D_z\tilde f_{x'}$ is $Cr(x)^{-1}$-close to the projection in the direction of the hyperplane H_x, orthogonal to v_x: $\left|D(\phi_{x,x'}\circ f_{x'}\circ\exp_x)-D\hat f_x\right|\le Cr(x)^{-1}$. Hence:

$$\left|D\phi_{x,x'}\circ Df_{x'}-Df_x\right|\le Cr(x)^{-1}. \tag{31}$$

Let W be a vector tangent to $f_{x'}(\Omega_{x,x'})$ and let W' be its horizontal lift for $f_{x'}$: $Df_{x'}W'=W$. As $D\tilde f_{x'}$ and $D\hat f_x$ are $Cr(x)^{-1}$-close, an horizontal vector for $f_{x'}$ is $Cr(x)^{-1}$-close to a horizontal vector for f_x. And since f_x and $f_{x'}$ are $Cr(x)^{-1}$-almost-Riemannian submersions, we get

$$\left||Df_x(W')| - |W'|\right| \leq Cr(x)^{-1}|W'| \quad \text{and} \quad \left||W| - |W'|\right| \leq Cr(x)^{-1}|W'|.$$

Writing

$$\left||D\phi_{x,x'}W| - |W|\right| \leq |D\phi_{x,x'}(Df_{x'}W') - Df_xW'| + \left||Df_xW'| - |W'|\right| + \left||W'| - |W|\right|$$

and using (31), we obtain $\left||D\phi_{x,x'}(W)| - |W|\right| \leq Cr(x)^{-1}|W|$, which proves $\phi_{x,x'}$ is a $Cr(x)^{-1}$-quasi-isometry.

Higher order estimates stem from those on f_x and $f_{x'}$, thanks to formula (28). □

We will also need the following lemma. Indeed, it stems from the previous one.

Lemma 3.13 (Local fibration closeness II) — *The setting is the same as in lemma 3.12. We consider three points x, x' and x'' in $M\backslash K$, whose respective distances are bounded by $\kappa r(x)$. Then, wherever it makes sense, we have*

- $|\phi_{x,x''} - \phi_{x,x'} \circ \phi_{x',x''}| \leq C$,
- $|D\phi_{x,x''} - D\phi_{x,x'} \circ D\phi_{x',x''}| \leq Cr(x)^{-1}$.

Proof. On the intersection of Ω_x, $\Omega_{x'}$ and $\Omega_{x''}$, we can write

$$|f_x - \phi_{x,x'} \circ f_{x'}| \leq C \quad \text{and} \quad |f_{x'} - \phi_{x',x''} \circ f_{x''}| \leq C.$$

Since $\phi_{x,x'}$ is a quasi-isometry, it follows that:

$$|f_x - \phi_{x,x'} \circ \phi_{x',x''} \circ f_{x''}| \leq |f_x - \phi_{x,x'} \circ f_{x'}| + |\phi_{x,x'} \circ (f_{x'} - \phi_{x',x''} \circ f_{x''})| \leq C.$$

Using the estimate $|f_x - \phi_{x,x''} \circ f_{x''}| \leq C$, we obtain by triangle inequality:

$$|(\phi_{x,x''} - \phi_{x,x'} \circ \phi_{x',x''}) \circ f_{x''}| \leq C.$$

The first part of the statement then follows from the surjectivity of $f_{x''}$. Since $f_{x''}$ is a submersion, the same argument applies to the differentials, yielding the second part. □

3.6. **Local fibration gluing.** Now, we need to adjust the local fibrations so as to make them compatible. The technical device is essentially the same as in [CFG]. The following lemma will be widely used in this process.

Lemma 3.14 (Local fibration adjustment I) — *The setting is that of lemma 3.12. Given two points x and x' in $M\backslash K$ with $\alpha r(x) \leq d(x,x') \leq \beta r(x)$ for some real numbers $0 < \alpha < \beta < 1$. We assume that on $B(x,\gamma r(x))$ and $B(x',\gamma r(x'))$, some fibrations f_x and $f_{x'}$ as in 3.11 are defined, that $B(x,\delta r(x))$ and $B(x',\delta r(x'))$ have nonempty intersection, with $0 < \delta < \gamma$, and that a map $\phi_{x',x}$ as in 3.12 is defined. We can then build a fibration $\tilde{f}_{x'}$ on $B(x',\delta r(x'))$, with the same properties as $f_{x'}$, plus:*

$$\tilde{f}_{x'} = \phi_{x',x} \circ f_x$$

on $B(x,\delta r(x)) \cap B(x',\delta r(x'))$. Moreover, this new fibration coincides with the old $f_{x'}$ on $B(x,\gamma r(x))$ and wherever we already had $f_{x'} = \phi_{x',x} \circ f_x$.

Proof. We set $\tilde{f}_{x'}(y) = \lambda(y)\phi_{x',x}(f_x(y)) + (1-\lambda(y))f_{x'}(y)$ with $\lambda(y) = \theta\left(\frac{f_x(y)}{r(x)}\right)$ where $\theta : \mathbb{R}^3 \longrightarrow [0,1]$ is a truncature function equal to 1 on the ball centered in 0 and with radius δ, equal to 0 outside the ball centered in 0 and with radius γ. Using the bounds on f_x, we find $|\nabla^k\lambda| \leq C_k r(x)^{-k}$, and the announced estimates can be obtained by differentiating the equation $\tilde{f}_{x'}(y) - f_{x'}(y) = \lambda(y)\left(\phi_{x',x} \circ f_x(y)) - f_{x'}(y)\right)$. □

Lemma 3.15 (Local fibration adjustment II) — *The setting is that of lemma 3.13. Given three points* x,x' *and* x'' *in* $M\backslash K$ *with* $\alpha r(x) \leq d(x,x'), d(x',x''), d(x,x'') \leq \beta r(x)$ *for some real numbers* $0 < \alpha < \beta < 1$*. We assume that on* $B(x, \gamma r(x))$*,* $B(x', \gamma r(x'))$ *and* $B(x'', \gamma r(x''))$*, some fibrations* f_x*,* $f_{x'}$ *and* $f_{x''}$ *as in 3.11 are defined, that the intersection of* $B(x, \delta r(x))$*,* $B(x', \delta r(x'))$ *and* $B(x'', \delta r(x''))$ *is nonempty for some* $0 < \delta < \gamma$ *and that maps* $\phi_{x',x}$*,* $\phi_{x,x''}$ *and* $\phi_{x',x''}$ *as in 3.13 are defined. We can then build a new diffeomorphism* $\tilde{\phi}_{x',x''}$*, with the same properties as* $\phi_{x',x''}$*, plus:*

$$\tilde{\phi}_{x',x''} = \phi_{x',x} \circ \phi_{x,x''}$$

on $f_{x''}(B(x, \delta r(x)) \cap B(x', \delta r(x')) \cap B(x'', \delta r(x'')))$*. Moreover, this new diffeomorphism coincides with* $\phi_{x',x''}$ *on* $B(x'', \gamma r(x''))$ *and wherever we already had* $\phi_{x',x''} = \phi_{x',x} \circ \phi_{x,x''}$*.*

Proof. We simply set $\tilde{\phi}_{x',x''}(v) = \lambda(v)\phi_{x',x} \circ \phi_{x,x''}(v) + (1-\lambda(v))\phi_{x',x''}(v)$ with $\lambda(v) = \theta\left(\frac{|v|^2}{r(x)^2}\right)$ where θ is the same function as in the previous proof. □

Theorem 3.16 (Global fibration) — *Let* (M^4, g) *be a complete hyperkähler manifold such that*

$$\int_M |\mathrm{Rm}|^2 \, r dvol < \infty \quad \textit{and} \quad \forall\, x \in M, \forall\, t \geq 1, \, At^3 \leq \mathrm{vol}\, B(x,t) \leq Bt^3$$

with $0 < A \leq B$*. Then there exists a compact set* K *in* M *such that* $M\backslash K$ *is endowed with a smooth circle fibration* π *over a smooth open manifold* X*. Besides, there is a geometric positive constant* C *such that fibers have length pinched between* C^{-1} *and* C *and second fundamental form bounded by* Cr^{-2}*.*

Remark 3. The proof will show that for any point x *in* $M\backslash K$*, there is a diffeomorphism* ψ_x *between a neighborhood of* $\pi(x)$ *in* X *and a ball in* $\mathbb{R}^3$ *such that* $\psi_x \circ \pi$ *is a fibration satisfying estimates as in proposition 3.11.*

Proof. We take a maximal set of points x_i, $i \in I$, such that for all indices $i \neq j$, $d(x_i, x_j) \geq \kappa r(x_i)/8$. This provides a uniformly locally finite covering of M by the balls $B(x_i, \kappa r(x_i)/2)$. For every index i, we let f_i be the local fibration given by 3.11. We will work with the minimal saturated (for f_i) sets $\Omega_i(\alpha)$ containing the balls $B(x_i, \alpha r(x_i))$, where α is a parameter inferior to κ. As in [CFG], we divide I into packs S_1, ..., S_N such that any two distinct points x_i, x_j whose indices are in the same pack are far from each other:

$$\exists\, a \in [1,N], \; \{i,j\} \subset S_a \Rightarrow d(x_i, x_j) \geq 100\kappa \min(r(x_i), r(x_j)).$$

In particular, $\Omega_i(\alpha)$ and $\Omega_j(\alpha)$ have empty intersection if i and j are in different packs; in this case, if the number of the pack of i is greater than for j, one denotes by $\phi_{i,j}$ the diffeomorphism given by 3.12 and by $\phi_{j,i}$ its inverse.

In order to improve the approximations $f_i \approx \phi_{i,j} f_j$ into equalities $f_i = \phi_{i,j} f_j$, we set up an adjustment campaign in the following way. The idea consists in giving priority to packs with small number. To do so, given an area where several fibrations are defined, we will modify them so that they all fit with the fibration with smallest number among them. The order of implementation is important. We will distinguish several stages, indexed by subsets $\mathcal{A} := \{a_1 < \cdots < a_k\}$ of $[1,N]$. We implement these 2^N stages by increasing order of a_1, then decreasing order of k, then increasing order of a_2, then increasing order of a_3, etc. To rephrase it, we have

$$\{a_1 < \cdots < a_k\} \prec \{b_1 < \cdots < b_l\}$$

if one of these exclusive conditions is realized:

- $a_1 < b_1$;

- $a_1 = b_1$ and $k > l$;
- $a_i = b_i$ for $i \leq i_0$ and $k = l$ and $a_{i_0} < b_{i_0}$.

We denote by $m_{\mathcal{A}}$ the rank of $\mathcal{A}$ in this order and set $\alpha_m := \kappa \cdot \left(\frac{1}{2}\right)^{\frac{m}{2^N}}$. Along the campaign, the fibration domains $\Omega_i(\alpha)$ will be shrinked: $\alpha_{m_{\mathcal{A}}}$ will be the domain size at stage $\mathcal{A}$.

At stage $\mathcal{A} := \{a_1 < \cdots < a_k\}$, we consider all elements $\mathcal{I} = (i_1, \cdots, i_k)$ of $S_{a_1} \times \cdots \times S_{a_k}$: to each such element corresponds one step. At step $\mathcal{I}$, we are interested in $\Omega_{\mathcal{I}} := \Omega_{i_1}(\alpha_{m_{\mathcal{A}}+1}) \cap \cdots \cap \Omega_{i_k}(\alpha_{m_{\mathcal{A}}+1})$. One should notice that our choice of packing ensures all the intersections $\Omega_{i_1}(\alpha_{m_{\mathcal{A}}}) \cap \cdots \cap \Omega_{i_k}(\alpha_{m_{\mathcal{A}}})$ treated at the same stage are away from each other, so that the following modifications are independent (during the stage). Essentially, the fibration f_{i_1} will overrule its neighbour on $\Omega_{\mathcal{I}}$. Given $2 \leq p \leq k$, we build $\tilde{f}_{i_p}$ on $\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1})$, from f_{i_1} and f_{i_p}, as in 3.14, so as to obtain

- $\tilde{f}_{i_p} = \phi_{i_p,i_1} f_{i_1}$ sur $\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1}) \cap \Omega_{i_1}(\alpha_{m_{\mathcal{A}}+1})$,
- $\tilde{f}_{i_p} = f_{i_p}$ sur $\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1}) \backslash \Omega_{i_1}(\alpha_{m_{\mathcal{A}}})$.

We also build, for $2 \leq p < q \leq k$, $\tilde{\phi}_{i_p,i_q}$ on $\tilde{f}_{i_q}(\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1}) \cap \Omega_{i_q}(\alpha_{m_{\mathcal{A}}+1}))$ from $\phi_{i_p,i_1}\phi_{i_1,i_q}$ and ϕ_{i_p,i_q}, as in 3.15, so that

- $\tilde{\phi}_{i_p,i_q} = \phi_{i_p,i_1}\phi_{i_1,i_q}$ sur $\tilde{f}_{i_q}(\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1}) \cap \Omega_{i_q}(\alpha_{m_{\mathcal{A}}+1}) \cap \Omega_{i_1}(\alpha_{m_{\mathcal{A}}}))$,
- $\tilde{\phi}_{i_p,i_q} = \phi_{i_p,i_q}$ sur $\tilde{f}_{i_q}(\Omega_{i_p}(\alpha_{m_{\mathcal{A}}+1}) \cap \Omega_{i_q}(\alpha_{m_{\mathcal{A}}+1}) \backslash \Omega_{i_1}(\alpha_{m_{\mathcal{A}}}))$.

After this, we can add that wherever it makes sense, we have for every $\{p, q\} \subset [2, k]$:

$$\tilde{\phi}_{i_q,i_p}\tilde{f}_{i_p} = \phi_{i_q,i_1}\phi_{i_1,i_p}\phi_{i_p,i_1} f_{i_1} = \phi_{i_q,i_1} f_{i_1} = \tilde{f}_{i_q}.$$

Now forget the tildes. We have just ensured that on $\Omega_{\mathcal{I}}$, for all relevant indices i, j, one has $f_i = \phi_{i,j} f_j$.

We proceed, independently, for all possible $\mathcal{I}$ at this stage, then we go on with the next stage, following the chosen order.

At the moment we pass from a stage $\{a_1 < \cdots\}$ to a stage $\{b_1 < \cdots\}$, with $a_1 \neq b_1$, we can notice the fibrations f_i and the diffeomorphisms $\phi_{i,j}$ are definitively fixed on the sets with number in the pack S_{a_1}: indeed, the device of 3.14 and 3.15 does not modify the fibrations which are already consistent. Afterwards, on these areas, we have definitively ensured the *equalities* $f_i = \phi_{i,j} f_j$.

For the same reason, at the moment we pass from a stage $\{a_1 < \cdots < a_k\}$ to a stage $\{a_1 < \cdots < b_{k-1}\}$, the fibrations f_i and the diffeomorphisms $\phi_{i,j}$ are definitively fixed on the sets $\Omega_{\mathcal{I}}$, where $\mathcal{I}$ is a k-tuple beginning with an element of S_{a_1}. Therefore, on these intersections of order k, we have definitively ensured the *equalities* $f_i = \phi_{i,j} f_j$ and all that is done afterwards on intersections of order $k - 1$ will not perturb it.

After this adjustment campaign, we have local fibrations f_i on the sets $\Omega_i := \Omega_i(\kappa/2)$ and diffeomorphisms $\phi_{i,j}$ such that $\phi_{i,j} \circ f_j = f_i$ on $\Omega_i \cap \Omega_j$. The initial estimates still hold, with different constants.

Let us define an equivalence relation: x and y are considered equivalent if there is an index i such that x and y belong to Ω_i and $f_i(x) = f_i(y)$. Denote by X the quotient topological space and by π the corresponding projection. Maps f_i induce homeomorphisms (from their domain to their image) $\check{f}_i$, which endow X with a structure of smooth 3-manifold: for every (relevant) pair i, j, $\check{f}_i \check{f}_j^{-1} = \phi_{i,j}$ is a diffeomorphism between open sets in $\mathbb{R}^3$. By construction, π is then a smooth fibration. □

3.7. **The circle fibration geometry.** In this whole paragraph, the setting is a complete hyperkähler manifold (M^4, g) satisfying (14) and (15). We have built a circle fibration

$\pi : M\backslash K \longrightarrow X$. The vectors that are tangent to the fibers will be called "vertical" whereas vectors orthogonal to the fibers will be called "horizontal". Let us average the metric g along the fibers of this fibration. Given a point x in $M\backslash K$, we can choose a unit vector field V, defined on a saturated neighborhood of x and vertical (there are two choices of sign). Let ϕ_t be the flow of V. Denote by l_x the length of the fiber $\pi^{-1}(\pi(x))$. We define a scalar product on T_xM by the formula

$$h_x := \frac{1}{l_x}\int_0^{l_x} \phi_t^* g \, dt.$$

This definition does not depend on the choice of V. We thus obtain a Riemannian metric h on $M\backslash K$ and the flows ϕ_t are isometries for h. To estimate the closeness of h to g, we proceed to a few estimations.

Lemma 3.17 — *The covariant derivatives of V can be estimated by $\nabla V = \mathcal{O}(r^{-2})$ and $\forall k \geq 2$, $\nabla^k V = \mathcal{O}(r^{-k})$.*

Proof. Let $f : \Omega \longrightarrow \mathbb{R}^3$ be one of the local fibrations. By construction, we have $df(V) = 0$. Differentiation yields:

$$\nabla^2 f(V, .) = -df(\nabla V). \tag{32}$$

Since V has constant norm, one has

$$(\nabla V, V) = 0 \tag{33}$$

so, with (3.11): $|\nabla V| \leq C\left|\nabla^2 f\right| \leq Cr^{-2}$. We then implement an induction, assuming the result up to order $k-1$. Differentiating $k-1$ times (32), we get a formula that looks like

$$df(\nabla^k V) = \sum_{i=1}^{k-1} \nabla^{1+k-i} f * \nabla^i V + \sum_{i=0}^{k-1} \nabla^{1+k-i} f * \nabla^i V,$$

which enables us to bound the horizontal part of $\nabla^k V$ by

$$\left|\nabla^k V^{\perp}\right| \leq C_k \sum_{i=1}^{k-1} \left|\nabla^{1+k-i} f\right| \left|\nabla^i V\right| + C_k \sum_{i=0}^{k-1} \left|\nabla^{1+k-i} f\right| \left|\nabla^i V\right| \leq C_k r^{-k}$$

(from induction assumption and (3.11)). Differentiating (33), we also get

$$\left|(\nabla^k V, V)\right| \leq C_k \sum_{i=1}^{k-1} \left|\nabla^{k-i} V\right| \left|\nabla^i V\right| \leq C_k r^{-k},$$

All in all: $\left|\nabla^k V\right| \leq C_k r^{-k}$. □

Lemma 3.18 — *The Lie derivative of g along V satisfies $L_V g = \mathcal{O}(r^{-2})$ and its derivatives obey $\forall\, k \geq 1$, $\nabla^k L_V g = \mathcal{O}(r^{-1-k})$.*

Proof. The formula $L_V g(X,Y) = (\nabla_X V, Y) + (\nabla_Y V, X)$ ensures $\left|\nabla^k L_V g\right|$ is estimated by $\left|\nabla^{k+1} V\right|$ so we can apply lemma 3.17. □

If ϕ^t is the flow V, we are interested in the family of metrics $g_t := \phi^{t*} g$, with Levi-Civita connection ∇^t and curvature Rm^t. First, a nice formula.

Lemma 3.19 — *For every vector fields X and Y, $\dfrac{d}{dt}\nabla^t_X Y = \mathrm{Rm}^t(X,V)Y - \nabla^{t,2}_{X,Y} V$.*

Proof. The connection ∇^t is obtained by transporting ∇ thanks to the isometry ϕ^t:

$$\phi^t_* \nabla^t_X Y = \nabla_{\phi^t_* X} \phi^t_* Y. \tag{34}$$

Let us differentiate with respect to t: $\phi^t_*[V, \nabla^t_X Y] + \phi^t_* \frac{d}{dt} \nabla^t_X Y = \nabla_{[V,\phi^t_* X]} \phi^t_* Y + \nabla_{\phi^t_* X}[V, \phi^t_* Y]$. Thanks to (34) and the invariance of V under its flow, this simplifies into

$$\frac{d}{dt} \nabla^t_X Y = \nabla^t_{[V,X]} Y + \nabla^t_X [V, Y] - [V, \nabla^t_X Y].$$

It then suffices to use the symmetric connection ∇ to expand the brackets and then simplify to get the formula. □

This formula gives a control on the covariant derivatives of g_t (with respect to g).

Lemma 3.20 — *For every t, g_t satisfies $g_t = g + \mathcal{O}(r^{-2})$ and $\forall\, k \in \mathbb{N}^*$, $\nabla^k g_t = \mathcal{O}(r^{-1-k})$.*

Proof. Let X be a vector field. By definition, one has $\frac{d}{dt} g_t(X, X) = (\phi^{t*} L_V g)(X, X)$. Integrating and using the bound on $|L_V g|$ given by lemma 3.18, we get $g(X, X) e^{-Cr^{-2}} \leq g_t(X, X) \leq g(X, X) e^{Cr^{-2}}$, hence the first estimate.

Given three vector fields X, Y, Z, we have

$$(\nabla^t_X g_t)(Y, Z) = 0 = X \cdot g_t(Y, Z) - g_t(\nabla^t_X Y, Z) - g_t(Y, \nabla^t_X Z),$$
$$(\nabla_X g_t)(Y, Z) = X \cdot g_t(Y, Z) - g_t(\nabla_X Y, Z) - g_t(Y, \nabla_X Z),$$

so, if $A^t := \nabla^t - \nabla$, we arrive at $(\nabla_X g_t)(Y, Z) = g_t(A^t(X, Y), Z) + g_t(Y, A^t(X, Z))$, which we write

$$\nabla g_t = g_t * A^t. \tag{35}$$

Lemma 3.19 implies $A^t = \displaystyle\int_0^t (\mathrm{Rm}^s(., V) - \nabla^{s,2} V) ds$. Since the curvature is invariant under isometries, we find

$$\mathrm{Rm}^t = \phi^{t*} \mathrm{Rm} \tag{36}$$

and, thanks to (34) and the invariance of V under the flow,

$$\nabla^{t,2} V = \phi^{t*} \nabla^2 V. \tag{37}$$

This leads to $\left|\mathrm{Rm}^t\right| \leq Cr^{-3}$, $\left|\nabla^{t,2} V\right| \leq Cr^{-2}$ and $\left|A^t\right| \leq Cr^{-2}$. Let us then assume (by induction) that for some $k \geq 1$:

$$\forall\, t,\, \forall\, i \in [0, k-1],\ \left|\nabla^i (g_t - g)\right| \leq Cr^{-1-i}, \quad \left|\nabla^i \mathrm{Rm}^t\right| \leq Cr^{-2-i}, \quad \left|\nabla^i \nabla^{t,2} V\right| \leq Cr^{-2-i}.$$

In particular, this implies

$$\forall\, t,\, \forall\, i \in [0, k-1],\ \left|\nabla^i A^t\right| \leq Cr^{-2-i}.$$

Fixing t, we differentiate (35) and use the induction assumption to get

$$\nabla^k g_t = \sum_{i=0}^{k-1} \nabla^{k-1-i} g_t * \nabla^i A^t = \mathcal{O}(r^{-1-k}).$$

To go on, we need to estimate $\left|\nabla^{t,i} A^t\right|$, $i \leq k-1$. To do this, we write $\nabla^t = \nabla + A^t$ and observe that $\left|\nabla^{t,i} A^t\right|$ can be controlled by a sum of a bounded number of terms like $\left(\displaystyle\prod_{\alpha=0}^{i-1} \left|\nabla^\alpha A^t\right|^{m_\alpha}\right) \left|\nabla^\beta A^t\right|$ with natural numbers m_α, β satisfying $\displaystyle\sum_{\alpha=0}^{i-1} (1+\alpha) m_\alpha + \beta = i$. Induction assumption implies each of these terms is $\mathcal{O}(r^{-(2+\alpha)m_\alpha - 2 - \beta}) = \mathcal{O}(r^{-2-i})$, so

$\nabla^{t,i}A^t = \mathcal{O}(r^{-2-i})$. Then, writing $\nabla = \nabla^t - A^t$, we estimate $\left|\nabla^k \mathrm{Rm}^t\right|$ by a sum of a bounded number of terms like $\left(\prod_{\alpha=0}^{k-1} \left|\nabla^{t,\alpha}A^t\right|^{m_\alpha}\right)\left|\nabla^{t,\beta}\mathrm{Rm}^t\right|$ with natural numbers m_α, β satisfying $\sum_{\alpha=0}^{k-1}(1+\alpha)m_\alpha + \beta = k$. With (36) and (34), we bound $\left|\nabla^{t,\beta}\mathrm{Rm}^t\right|$ by $\left|\nabla^\beta \mathrm{Rm}\right|$ and thus by $r^{-2-\beta}$. Eventually, we find $\left|\nabla^k \mathrm{Rm}^t\right| = \mathcal{O}(r^{-2-k})$. In the same way, we get $\left|\nabla^k\nabla^{t,2}V\right| = \mathcal{O}(r^{-2-k})$ and conclude by induction. □

LEMMA 3.21 — *The length l of the fibers is controlled by the estimates : $dl = \mathcal{O}(r^{-2})$ and $\forall k \geq 2$, $\nabla^k l = \mathcal{O}(r^{-k})$. As a consequence, l goes a finite limit at infinity.*

Proof. By construction, we have the identity $\phi^{l(x)}(x) = x$, at every point x in $M\backslash K$. Differentiation yields $dl\otimes V + T\phi^l = id$. Taking the scalar product with V, we obtain $dl = (g-g_l)(V,.)$. Differentiating this leads to $\nabla^k l = \sum_{i=0}^{k-1}\nabla^i(g-g_l) * \nabla^{k-1-i}V$. Now we use (3.20), (3.17) and the bound on l: $\left|\nabla^k l\right| \leq Cr^{-k}$. The existence of a finite limit at infinity follows from Cauchy criterion. □

We can finally control the metric h, obtained by averaging g along the fibers.

PROPOSITION 3.22 — *The averaged metric h obeys the estimates $h = g + \mathcal{O}(r^{-2})$ and $\forall k \geq 1$, $\nabla^k h = \mathcal{O}(r^{-1-k})$.*

Proof. The definition of h can be written $h - g = \frac{1}{l}\int_0^l (g_t - g)dt$, hence the first estimate (cf. (3.20)). This formula differentiates into: $\nabla^k h = \sum_{i=1}^{k}\binom{k}{i}\frac{\nabla^i l}{l}\otimes\nabla^{k-i}(g_l - h) + \frac{1}{l}\int_0^l \nabla^k g_t dt$ and an induction (with (3.20) and (3.21)) finally yields $\nabla^k h = \mathcal{O}(r^{-1-k})$. □

Since g has cubic curvature decay, we deduce the

COROLLARY 3.23 — *The curvature of h has cubic decay.*

Now, let us push h down into a Riemannian metric $\check{h}$ on X: for every point y in X, for every vector w in T_yX, we choose a lift x of y ($\pi(x) = y$) and we set $\check{h}_y(w,w) = h_x(v,v)$ where v is the horizontal lift of w in T_xM; this definition makes sense because the flow ϕ_t is isometric for h.

PROPOSITION 3.24 — *The manifold X is diffeomorphic to the complementary set of a ball in $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm\,\mathrm{id}\}$. Moreover, $\check{h} = g_{\mathbb{R}^3} + \mathcal{O}(r^{-\tau})$ for every $\tau < 1$.*

Proof. Observe the volume of a ball of radius t in $(X^3, \check{h})$ is comparable to t^3. To estimate the curvature on the base, we use O'Neill formula ([Bes]), which asserts that if Y and Z are orthogonal unit horizontal vector fields on $M\backslash K$, then

$$\mathrm{Sect}_{\check{h}}(\pi_*Y\wedge\pi_*Z) = \mathrm{Sect}_h(Y\wedge Z) + \frac{3}{4}h([Y,Z],V)^2.$$

The first term decays at a cubic rate by 3.23. Moreover, lemma 3.17 and corollary 3.22 yield $h([Y,Z],V) = -(\nabla_Y h)(Z,V) - h(Z,\nabla_Y V) + (\nabla_Z h)(Y,V) + h(Y,\nabla_Z V) = \mathcal{O}(r^{-2})$, hence $\mathrm{Sect}_{\check{h}}(\pi_*Y\wedge\pi_*Z) = \mathcal{O}(r^{-3})$. This cubic curvature decay, combined with Euclidean volume growth, enables us to apply the main theorem of [BKN]. □

3.8. **What have we proved ?** We have proved the following theorem, based on 3.16, 3.21, 3.22, 3.24. Recall $d\mu = \frac{r^n}{\operatorname{vol} B(o,r)} dvol_g$ in dimension n.

Theorem 3.25 — *Let (M^4, g) be a complete hyperkähler manifold with curvature in $L^2(d\mu)$ and such that, for some positive numbers A and B,*

$$\forall\, x \in M,\, \forall\, t \geq 1,\, At^\nu \leq \operatorname{vol} B(x,t) \leq Bt^\nu.$$

with $3 \leq \nu < 4$. Then there is compact set K in M, a ball B in $\mathbb{R}^3$ and a circle fibration $\pi : M \backslash K \longrightarrow \mathbb{R}^3 \backslash B$ or $(\mathbb{R}^3 \backslash B)/\{\pm \mathrm{id}\}$. Moreover, the fibers have asymptotically constant length and the metric g obeys $g = \pi^* \tilde{g} + \eta^2 + \mathcal{O}(r^{-2})$, where η is a (local) connection one-form and $\tilde{g}$ is an ALE metric of order 1^-.

Let us describe the topology at infinity, namely the topology of the connected space $E = M \backslash K$, which, according to theorem 3.25, is a circle bundle over $X = \mathbb{R}^3 \backslash B$ or $(\mathbb{R}^3 \backslash B)/\{\pm \mathrm{id}\}$. We can get rid of the $\mathbb{Z}_2$-action: just pull back the fibration π into a circle fibration $\bar{\pi} : \bar{E} \longrightarrow \bar{X}$ between two-fold coverings of E and X. Then $\bar{X} = \mathbb{R}^3 \backslash B$ has the homotopy type of $\mathbb{S}^2$, so that we can classify its circle fibrations. Moreover, the homotopy groups of $\bar{E}$ can be computed thanks to the long exact homotopy sequence associated to $\bar{\pi}$. In this way, we obtain essentially two cases, which are distinguished by the homotopy groups at infinity (those of $M \backslash K$).

If the fundamental group at infinity is finite, then a finite covering of $M \backslash K$ is $\mathbb{R}^4 \backslash \mathbb{B}^4$ and the circle fibration is the Hopf fibration, up to a finite group action. In this case, the π_2 at infinity is trivial. This is typically the "Taub-NUT" situation.

If the fundamental group at infinity is infinite, then, up to a two-fold covering, $M \backslash K$ is $\mathbb{R}^3 \backslash \mathbb{B}^3 \times \mathbb{S}^1$ and the circle fibration comes from the trivial one. The π_2 at infinity is then $\mathbb{Z}$.

It is also easy to adapt the arguments above in order to obtain the following result.

Theorem 3.26 — *Let (M^n, g) be a complete manifold satisfying*

$$\forall k \in \mathbb{N},\, \left| \nabla^k \operatorname{Rm} \right| = \mathcal{O}(r^{-3-k}).$$

Suppose there is a positive number A and a function $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ going to zero at infinity such that

$$\forall\, x \in M,\, \forall\, t \geq 1,\, At^{n-1} \leq \operatorname{vol} B(x,t) \leq \omega(t) t^n.$$

Further assume there is a number $c \geq 1$ such that the holonomy H of any geodesic loop based at x and with length $L \leq r(x)/c$ satisfies

$$|H - \mathrm{id}| \leq \frac{cL}{r(x)}.$$

Then there is compact set K in M, a ball B in $\mathbb{R}^{n-1}$, a finite subgroup G of $O(n-1)$ and a circle fibration $\pi : M \backslash K \longrightarrow (\mathbb{R}^{n-1} \backslash B)/G$. Moreover, the fibers have asymptotically constant length and the metric g obeys

$$g = \pi^* \tilde{g} + \eta^2 + \mathcal{O}(r^{-2}),$$

where η is a (local) connection one-form and $\tilde{g}$ is an ALE metric of order 1^- (1 if $n \geq 5$).

Remark 4. The required estimates on the curvature are satisfied on a Ricci flat manifold with cubic curvature decay. This allows one to englobe the Schwarzschild metrics ([Min] *for instance) in this setting. Note a little topology ensures the fibration is trivial if $n \geq 5$.*

Appendix A. Curvature decay.

In this appendix, we wish to sharpen some results from [Min] : we want to obtain pointwise bounds on the derivatives of the curvature of a Ricci flat manifold, starting from an integral bound on the curvature tensor. To do this, we need a technical inequality.

Lemma A.1 (Moser iteration with source term) — *Let (M^n, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $E \longrightarrow M$ be a smooth Euclidean vector bundle, endowed with a compatible connection ∇. We denote by $\overline{\Delta} = \nabla^*\nabla$ the Bochner Laplacian and suppose V is a continuous field of symmetric endomorphisms of E whose negative part satisfies $|V_-| = \mathcal{O}(r^{-2})$. Given a locally bounded section ϕ and a locally Lipschitz section σ such that $(\sigma, \overline{\Delta}\sigma + V\sigma) \leq (\sigma, \phi)$, the following estimate holds for large R:*

$$\sup_{A(R,2R)} |\sigma| \leq \frac{C}{\operatorname{vol} B(o,R)^{\frac{1}{2}}} \|\sigma\|_{L^2(A(R/2,5R/2))} + CR^2 \|\phi\|_{L^\infty(A(R,2R))} .$$

Proof. Set $u := |\sigma| + F$, with $F := R^2 \|\phi\|_{L^\infty(A(R,2R))}$. The case $\phi = 0$ is treated in [Min]. Actually, in [Min], the estimation is written assuming a global weighted Sobolev inequality. But since we work at a fixed scale R, there is no need for such a global inequality: the local Sobolev inequality of L. Saloff-Coste [SC], with controlled constant, is sufficient for our purpose; and its validity only requires $\operatorname{Ric} \geq 0$. Therefore we assume $F \neq 0$.

To avoid troubles on the zero set of σ, let us consider the regularizations $v_\epsilon := \sqrt{|\sigma|^2 + \epsilon}$ and $u_\epsilon := v_\epsilon + F$. Observing the inequalities

$$v_\epsilon \Delta v_\epsilon \leq (\sigma, \overline{\Delta}\sigma) \leq |\sigma| \left(|V_-| |\sigma| + |\phi| |\sigma|\right) \leq v_\epsilon(|V_-| |\sigma| + |\phi| |\sigma|),$$

we deduce $\Delta v_\epsilon \leq |V_-| v_\epsilon + |\phi|$ and thus find

$$\Delta u_\epsilon \leq |V_-| u_\epsilon + |\phi| \leq \left(|V_-| + \frac{|\phi|}{F}\right) u_\epsilon.$$

Our choice of F enables us to use the estimate without source term in [Min]:

$$\sup_{A(R,2R)} u_\epsilon \leq \frac{C}{\operatorname{vol} B(o,R)^{\frac{1}{m}}} \|u_\epsilon\|_{L^m(A(R/2,5R/2))}$$

Let ϵ go to zero, so as to obtain

$$\sup_{A(R,2R)} |\sigma| \leq \sup_{A(R,2R)} u \leq \frac{C}{\operatorname{vol} B(o,R)^{\frac{1}{m}}} \|\sigma\|_{L^m(A(R/2,5R/2))} + CF,$$

which is what we want. □

Let us use it to prove that on a Ricci flat manifold, if the curvature decays at infinity, then the covariant derivatives of the curvature also decay.

Proposition A.2 — *Let (M^n, g) be a complete noncompact Ricci flat manifold. If $a \geq 2$, the estimate $|\operatorname{Rm}| = \mathcal{O}(r^{-a})$ implies : $\left|\nabla^i \operatorname{Rm}\right| = \mathcal{O}(r^{-a-i})$, $i \in \mathbb{N}$.*

Proof. Since M is Ricci flat, its curvature tensor obeys an elliptic equation $\overline{\Delta} \operatorname{Rm} = \operatorname{Rm} * \operatorname{Rm}$ [BKN], which implies [TV]:

$$\forall\, k \in \mathbb{N},\ \overline{\Delta}\nabla^k \operatorname{Rm} = \sum_{i=0}^{k} \nabla^i \operatorname{Rm} * \nabla^{k-i} \operatorname{Rm} . \tag{38}$$

Let us prove the result by induction on i. The case $i = 0$ is contained in the assumptions. Suppose that the result is established for $i \leq k$. Formula (38) can be written

$$(\overline{\Delta} - \operatorname{Rm} *)\nabla^{k+1} \operatorname{Rm} = \sum_{i=1}^{k} \nabla^i \operatorname{Rm} * \nabla^{k+1-i} \operatorname{Rm}.$$

Since the right-hand side is bounded by $C_{k+1} r^{-2a-k-1}$, lemma A.1 yields:

$$\sup_{A(R,2R)} \left|\nabla^{k+1} \operatorname{Rm}\right| \leq \frac{C_{k+1}}{\operatorname{vol} B(o,R)^{\frac{1}{2}}} \left\|\nabla^{k+1} \operatorname{Rm}\right\|_{L^2(A(R/2,5R/2))} + C_{k+1} R^{1-2a-k}. \tag{39}$$

Let χ be a positive smooth function equal to 1 on $A(R/2, 5R/2)$, 0 on $A(R/3, 3R)^c$ and with gradient bounded by $10/R$. Then we can write

$$\int_{A(R/2,5R/2)} \left|\nabla^{k+1} \operatorname{Rm}\right|^2 \leq \int_{A(R/3,3R)} \left|\nabla \left(\chi \nabla^k \operatorname{Rm}\right)\right|^2$$

and, after integration by parts, we find

$$\int_{A(R/2,5R/2)} \left|\nabla^{k+1} \operatorname{Rm}\right|^2 \leq \int_{A(R/3,3R)} |d\chi|^2 \left|\nabla^k \operatorname{Rm}\right|^2 + \int_{A(R/3,3R)} \chi^2 (\nabla^k \operatorname{Rm}, \overline{\Delta} \nabla^k \operatorname{Rm}).$$

With (38), we obtain the upper bound

$$\begin{aligned}\int_{A(R/2,5R/2)} \left|\nabla^{k+1} \operatorname{Rm}\right|^2 &\leq \frac{100}{R^2} \int_{A(R/3,3R)} \left|\nabla^k \operatorname{Rm}\right|^2 \\ &+ C_{k+1} \sum_{i=0}^{k} \int_{A(R/3,3R)} \left|\nabla^k \operatorname{Rm}\right| \left|\nabla^i \operatorname{Rm}\right| \left|\nabla^{k-i} \operatorname{Rm}\right|.\end{aligned}$$

Using $a \geq 2$, we estimate this by

$$\begin{aligned}\int_{A(R/2,5R/2)} \left|\nabla^{k+1} \operatorname{Rm}\right|^2 &\leq C_{k+1} \operatorname{vol} B(o,R) \left(R^{-2-2a-2k} + R^{-3a-2k}\right) \\ &\leq C_{k+1} \operatorname{vol} B(o,R) R^{-2-2a-2k}.\end{aligned}$$

As a result, (39) implies

$$\sup_{A(R/2,5R/2)} \left|\nabla^{k+1} \operatorname{Rm}\right| \leq C_{k+1} \left(R^{-1-a-k} + R^{1-2a-k}\right) \leq C_{k+1} R^{-1-a-k},$$

hence $\left|\nabla^{k+1} \operatorname{Rm}\right| \leq C_{k+1} r^{-a-(k+1)}$. □

This proposition, together with [Min], leads to the two following results. Recall we always distinguish a point o in our manifolds. We will use the measure μ defined by $d\mu = \frac{r^n}{\operatorname{vol} B(o,r)} dvol$ and assume $\int |\operatorname{Rm}|^{\frac{n}{2}} d\mu < \infty$, which is weaker than $|\operatorname{Rm}| = \mathcal{O}(r^{-2-\epsilon})$ for some positive ϵ.

Theorem A.3 — *Let (M^n, g) be a complete Ricci flat manifold such that for some numbers $\nu > 2$ and $C > 0$:*

$$\forall t \geq s > 0, \ \frac{\operatorname{vol} B(o,t)}{\operatorname{vol} B(o,s)} \geq C \left(\frac{t}{s}\right)^{\nu}.$$

Then the integral bound $\int_M |\operatorname{Rm}|^{\frac{n}{2}} d\mu < \infty$ implies for every k in $\mathbb{N}$:

$$\left|\nabla^k \operatorname{Rm}\right| = \mathcal{O}(r^{-a(n,\nu)-k}) \quad \textit{with} \quad a(n,\nu) = \max\left(2, \frac{(\nu-2)(n-1)}{n-3}\right).$$

Corollary A.4 — *Let (M^n, g) be a complete Ricci flat manifold, with $n \geq 4$. Assume there are positive numbers A and B such that $At^{n-1} \leq \operatorname{vol} B(o,t) \leq Bt^{n-1}$ for every $t \geq 1$. Then the integral bound $\int_M |\mathrm{Rm}|^{\frac{n}{2}} d\mu < \infty$ implies for every k in $\mathbb{N}$:*

$$\left|\nabla^k \mathrm{Rm}\right| = \mathcal{O}(r^{-(n-1)-k}).$$

Appendix B. Distance and curvature.

The following lemma sums up some comparison estimates on the distance function. Up to order two, it is quite classical. Higher order estimates do not seem to be proved in the litterature, so we include a proof.

Lemma B.1 — *Consider a complete Riemannian manifold (M, g), a point x in M and a number $a \geq 2$ such that $\operatorname{inj}(x) > \epsilon \geq 1$ and $\forall\, i \in [0, k]$, $\left|\nabla^i \mathrm{Rm}\right| \leq c\epsilon^{-a-i}$ on the ball $B(x, \epsilon)$. Then there is a constant C such that on this ball, the function $\rho = d(x, .)^2/2$ satisfies:*

- $|d\rho| \leq \epsilon$;
- $\left|\nabla^2 \rho - g\right| \leq C\epsilon^{2-a}$;
- *for* $3 \leq i \leq k$, $\left|\nabla^i \rho\right| \leq C\epsilon^{4-a-i}$.

Proof. The first estimate is obvious and the second follows from [BK]. Let us turn to higher order estimates. We consider the gradient N of $r := d(x, .)$ and use the Riccati equation for the second fundamental form $S := \nabla N$ of geodesic spheres:

$$\nabla_N S = -S^2 - \mathrm{Rm}(N, .)N.$$

Identifying quadratic forms to symmetric endomorphisms, we can write the endomorphism $E := \nabla^2\rho - \mathrm{Id}$ as $E = dr \otimes N + rS - \mathrm{Id}$ and, setting $V = \operatorname{grad}\rho = rN$, we obtain the equation

$$\nabla_V E = -E - E^2 - \mathrm{Rm}(V, .)V.$$

Since $\nabla V = \mathrm{Id} + E$ and $\nabla_V \nabla E = \nabla\nabla_V E - \nabla_{\nabla V} E + \mathrm{Rm}(V, .)E$, it follows that

$$\nabla_V \nabla E = -2\nabla E + E * \nabla E + \nabla \mathrm{Rm} * V * V + \mathrm{Rm} * \nabla V * V + \mathrm{Rm} * V.$$

Observing that for $k \geq 2$, we have $\nabla^k V = \nabla^{k-1} E$, an induction yields:

$$\begin{aligned}\nabla_V \nabla^k E = \; & - \; (k+1)\nabla^k E + \sum_{i+j=k} \nabla^i E * \nabla^j E \\ & + \sum_{i+j+l=k} \nabla^i \mathrm{Rm} * \nabla^j V * \nabla^l V + \sum_{i+j=k-1} \nabla^i \mathrm{Rm} * \nabla^j V,\end{aligned}$$

for every k. We then set $F_k = r^{k+1}\nabla^k E$ and $G = E/r$, so that $\nabla_N F_k = G * F_k + H_k$, where

$$H_k = r^{-2}\sum_{i=1}^{k-1} F_i * F_{k-i} + r^k \left(\sum_{i+j+l=k} \nabla^i \mathrm{Rm} * \nabla^{j+1}\rho * \nabla^{l+1}\rho + \sum_{i+j=k-1} \nabla^i \mathrm{Rm} * \nabla^{j+1}\rho\right)$$

Along a geodesic starting from x, we find $\partial_r |F_k| \leq C_k |F_k| |G| + |H_k|$ and since the order two estimate ensures $r|G|$ is small, we can bound $|F_k|$ by $r \sup |H_k|$, up to a constant. We will prove by induction the estimate

$$|F_k| \leq C_k r^{k+1} \epsilon^{2-a-k}$$

which will ensure $\left|\nabla^k E\right| \leq C_k \epsilon^{2-a-k}$, hence $\left|\nabla^{k+2}\rho\right| \leq C_k \epsilon^{4-a-(k+2)}$. It will conclude the proof. Initialization ($k=0$) follows from the order two estimate on ρ. Assume the estimates up to order $k-1$. It implies:

$$|H_k| \leq C_k r^k \left(\epsilon^{4-2a-k} + \epsilon^{4-2a-k+2-a} + \epsilon^{4-2a-k}\right).$$

With $a \geq 2$, we find $|H_k| \leq C_k r^k \epsilon^{4-2a-k} \leq C_k r^k \epsilon^{2-a-k}$ and therefore we get the promised estimate $|F_k| \leq C_k r^{k+1} \epsilon^{2-a-k}$, hence the result. □

H. Kaul [Kau] proved a control on Christoffel coefficients in the exponential chart, given bounds on Rm and ∇ Rm. We need the following

PROPOSITION B.2 — *Consider a complete Riemannian manifold (M,g), a point x in M and a number $a \geq 2$ such that $|\mathrm{Rm}| \leq c\epsilon^{-a}$ and $|\nabla \mathrm{Rm}| \leq c\epsilon^{-a-1}$ on the ball $B(x,\epsilon)$, with $\epsilon \geq 1$. Then there is a constant C such that, on the ball $\hat{B}(0,\epsilon)$ in T_xM, the connection $\nabla^{\hat{g}}$ of the metric $\hat{g} = \exp_x^* g$ and the flat connection ∇^0 are related by*

$$\left|\nabla^{\hat{g}} - \nabla^0\right| \leq C\epsilon^{1-a}.$$

A better control on the distance function stems from this.

LEMMA B.3 — *Consider a complete Riemannian manifold (M,g), a point x in M and a number $a \geq 2$ such that $|\mathrm{Rm}| \leq c\epsilon^{-a}$ and $|\nabla \mathrm{Rm}| \leq c\epsilon^{-a-1}$ on the ball $B(x,\epsilon)$, with $\epsilon \geq 1$. Then there is a constant C such that if v and w belong to $\hat{B}(0, C^{-1}\epsilon)$, endowed with $\hat{g}$, then*

$$|(d\rho_v)_w - g_x(w-v,.)| \leq C\epsilon^{3-a}.$$

Proof. First, choose a sufficiently large C to ensure the convexity of the ball under consideration. Observe the expression $(d\rho_v)_w = -\hat{g}_w(\mathrm{Exp}_w^{-1} v,.)$, where Exp is the exponential map of $\hat{g}$. Comparison yields

$$|\hat{g}_w - g_x| \leq C\epsilon^{-a}\epsilon^2 = C\epsilon^{2-a}. \tag{40}$$

Suppose γ parameterizes the geodesic connecting w to v in unit time. The geodesic equation $\nabla^{\hat{g}}_{\dot{\gamma}}\dot{\gamma} = 0$ can be written $\ddot{\gamma} + \left(\nabla^{\hat{g}}_{\dot{\gamma}} - \nabla^0\right)\dot{\gamma} = 0$. With (B.2), we obtain $|\ddot{\gamma}| \leq C\epsilon^{1-a}\epsilon^2 = C\epsilon^{3-a}$. Taylor formula $\gamma(1) - \gamma(0) - \dot{\gamma}(0) = \int_0^1 (1-t)\ddot{\gamma}(t)dt$ then yields $\left|v - w - \mathrm{Exp}_w^{-1} v\right| \leq C\epsilon^{3-a}$. To conclude, we write

$$\begin{aligned}
|(d\rho_v)_w - g_x(w-v,.)| &= \left|\hat{g}_w(\mathrm{Exp}_w^{-1} v,.) - g_x(v-w,.)\right| \\
&\leq \left|(\hat{g}_w - g_x)(\mathrm{Exp}_w^{-1} v,.)\right| + \left|g_x(\mathrm{Exp}_w^{-1} v,.) - g_x(v-w,.)\right| \\
&\leq C\epsilon^{2-a}\epsilon + C\epsilon^{3-a} \\
&\leq C\epsilon^{3-a}.
\end{aligned}$$

□

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