

Weighted Sobolev inequalities and Ricci flat manifolds.

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Abstract

In this paper, we prove a weighted Sobolev inequality and a Hardy inequality on manifolds with nonnegative Ricci curvature satisfying a reverse volume doubling condition. It enables us to obtain rigidity results for Ricci flat manifolds.

Introduction

A major question in Riemannian geometry is : what are the topological implications of assumptions on geometric objects, such as the curvature or the volume of balls ? As an example, the classical Bonnet-Myers theorem asserts that a complete Riemannian manifold with Ricci curvature bounded from below by a positive number is compact with finite fundamental group. In the noncompact setting, it is natural to look for geometric properties which ensure the manifold has finite topological type, i.e. is homeomorphic to the interior of a compact manifold with boundary. For instance, a complete flat manifold has finite topological type by Cheeger-Gromoll theorem [CG2] and, indeed, it is sufficient to ask for flatness only outside a compact set [GW]. With this in mind, one can wonder whether an asymptotic flatness is sufficient : has a complete manifold finite topological type as soon as the curvature tends to zero at infinity ? It is far from being true since any manifold carries a complete metric whose curvature tensor R decays quadratically that is, given a point o , there is number C such that : $|R| \leq Cd(o, \cdot)^{-2}$ ([Gro],[LS]). So a quadratic decay (alone) has no topological meaning. A striking fact is the following theorem of Abresch [Abr] : a complete manifold whose curvature decays faster than quadratically, that is $|R| \leq Cd(o, \cdot)^{-2-\delta}$ with $\delta > 0$, has finite topological type. Thus, there is a critical rate of decay, $\mathcal{O}(d(o, \cdot)^{-2})$, around which the situation changes completely. Note it is the rate of decay which is invariant under constant rescalings of the metric.

It is then interesting to understand what happens around the quadratic decay. J. Sha and Z. Shen [SS] showed a complete manifold M^n with quadratic curvature decay has finite topological type if it has nonnegative Ricci curvature and maximal volume growth, that is the volume of balls satisfies $\text{vol} B(o, t) \geq A_o t^n$, with $A_o > 0$. The word "maximal" is explained by Bishop theorem, which asserts : $\text{vol} B(o, t) = \mathcal{O}(t^n)$.

If one assumes such a Euclidian volume growth of balls, requiring quadratic curvature decay is close to requiring the curvature belongs to the Lebesgue space $L^{\frac{n}{2}}$. Many mathematicians studied manifolds with integral bounds on the curvature. In particular, [BKN] proved that Ricci flat n -manifolds with maximal volume growth and curvature in $L^{\frac{n}{2}}$ have faster than quadratic curvature decay and thus finite topological type (and this paper proved a lot more). Another interesting consequence of their methods is the existence of a positive number ϵ for which such a manifold is flat as soon as $\int |R|^{n/2}$ is less than ϵ ; ϵ depends on n and on a lower bound on the volume growth.

Our aim here is to understand how one can generalize such theorems in case the volume growth is not maximal. One result in this direction is the following theorem, by Jeff Cheeger and Gang Tian [CT] : a four-dimensional complete Ricci flat manifold with curvature in L^2 has quadratic curvature decay. Their proof is based on the Gauss-Bonnet-Chern formula and Cheeger-Gromov theory. Our approach is different : unlike J. Cheeger and G. Tian, we still make an assumption on the volume growth and this enables us to generalize previously known rigidity results ; the point is our assumption is much weaker than "maximal volume growth."

The results of [BKN] rely on a Sobolev inequality. Now such an inequality cannot be true if the volume growth is not maximal. But what we will show in our setting is that a weighted

Sobolev inequality happens. Given a point o , we will consider weights involving the function $\rho_o : t \mapsto \frac{t^n}{\text{vol } B(o, t)}$. Note this function appears in Bishop-Gromov theorem : it is nondecreasing when the Ricci curvature is nonnegative. Moreover, if r_o denotes the geodesic distance to the point o ($r_o = d(o, \cdot)$), the quadratic decay is critical for the integral $\int_M |R|^{\frac{n}{2}} \rho_o(r_o) d\text{vol}$: it is finite if R has faster than quadratic decay, but it may be infinite if R only has quadratic decay. Our work leads to the following

Theorem 0.1 (Flatness criterion) *Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. Assume there exists o in M , $\nu > 1$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu.$$

Then there is a constant $\epsilon_1 = \epsilon_1(n, C_o, \nu)$ such that M is flat as soon as

$$\sup_M (|R| r_o^2) < \epsilon_1.$$

If $\nu > 2$, there is also a constant $\epsilon_2 = \epsilon_2(n, C_o, \nu)$ such that M is flat as soon as

$$\int_M |R|^{\frac{n}{2}} \rho_o(r_o) d\text{vol} < \epsilon_2.$$

As a result, in both cases, M is the normal bundle of a compact totally geodesic submanifold, which is finitely covered by a flat torus.

Note the first part of the theorem provides a rigidity phenomenon in the setting of [SS], provided $\text{Ric} = 0$: if $\sup_M (|R| r_o^2)$ is finite, M has finite topological type ([SS]); if it is small enough, we prove M is flat. We can also generalize [BKN] :

Theorem 0.2 (Curvature decay) *Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. Assume there exists o in M , $\nu > 4\frac{n-2}{n-1}$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu.$$

and

$$\int_M |R|^{\frac{n}{2}} \rho_o(r_o) d\text{vol} < +\infty.$$

Then M has faster-than-quadratic curvature decay and thus has finite topological type.

Moreover, we are able to predict a rate of decay. For instance, if the volume of the large balls $B(o, t)$ grows like t^{n-1} , the curvature tensor decays like $r_o^{-(n-1)}$, which is optimal.

The assumption

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu \tag{1}$$

implies the lower bound

$$\forall t \geq 1, \text{vol } B(o, t) \geq C_o \text{vol } B(o, 1) t^\nu \tag{2}$$

and follows from the two-sided estimate

$$\exists A_o, B_o > 0, \forall t \geq 1, A_o t^\nu \leq \text{vol } B(o, t) \leq B_o t^\nu. \tag{3}$$

Note that Bishop theorem ensures $\nu \leq n$. This hypothesis yields the analytical tools we need. Indeed, we prove that on a complete connected manifold M^n , $n \geq 3$, with nonnegative Ricci curvature and satisfying (1), the following weighted Sobolev inequality holds :

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \rho_o(r_o)^{-\frac{2}{n-2}} d\text{vol} \right)^{\frac{n-2}{n}} \leq S \int_M |df|^2 d\text{vol}. \tag{4}$$

In other terms, the completion $H_0^1(M)$ of $C_c^\infty(M)$ for the norm $\|d\cdot\|_{L^2(M, \text{vol})}$ can be continuously injected into $L^{\frac{2n}{n-2}}\left(M, \rho_o(r_o)^{-\frac{2}{n-2}} \text{vol}\right)$. Such a manifold also satisfies the Hardy inequality

$$\forall f \in C_c^\infty(M), \int_M |f| r_o^{-1} d\text{vol} \leq H \int_M |df| d\text{vol}. \quad (5)$$

The constants S and H we find depend only on n , ν and C_o . Now, the curvature of a Ricci flat manifold obeys a nonlinear elliptic equation. When used appropriately, the inequalities (4) and (5) yield estimates on the solutions of this equation, and our theorems follow. In this article, we will give a few other applications of the weighted Sobolev inequality.

The paper is organized as follows.

In the first section, we develop a discretization technique aimed at patching local Sobolev inequalities together. It is based upon ideas and methods of A. Grigor'yan and L. Saloff-Coste [GSC]. Given a convenient covering of a manifold, if we assume some discrete inequality on a graph which is naturally associated to the covering, we are able to deduce a global Sobolev inequality from a local one (theorem 1.8).

In the second section, we explain how to apply this abstract technique in the setting of manifolds with nonnegative Ricci curvature and satisfying (1), so as to obtain a weighted Sobolev inequality and a Hardy inequality. Note we could replace the nonnegativity of the Ricci curvature by two of its consequences : the volume doubling condition and the scaled Poincaré inequality on balls. In [Gril], G. Grillo proves weighted inequalities in the context of homogeneous spaces and indeed, in the case $\nu = n$, the Hardy inequality follows from this work : nevertheless, it should be stressed that our approach is basically different and, in particular, does not require a uniform estimate on the volume of balls ; apart from the volume doubling condition and the scaled Poincaré inequality (which are classical assumptions for such problems), the only measure theoretic assumption we need is the estimate (1) which is some kind of reverse volume doubling condition *around one point*. An important step in our proof could be singled out : the following result gives a sufficient condition for a manifold to satisfies the so called RCA property (Relatively Connected Annuli) and should be compared with proposition 4.5 of [HK] (which, in our context, would require the volume growth to satisfy a uniform Euclidian estimate from below).

Proposition 0.3 (RCA) *Let M be a connected complete Riemannian manifold, satisfying the volume doubling property*

$$\forall x \in M, \forall t > 0, \text{vol} B(x, 2t) \leq C_D \text{vol} B(x, t),$$

the scaled L^p Poincaré inequality centered in some point o in M

$$\forall f \in C_c^\infty(M), \forall t > 0, \int_{B(o,t)} |f - f_{B(o,t)}|^p d\text{vol} \leq C_P t^p \int_{B(o,t)} |df|^p d\text{vol}$$

and the reverse volume doubling condition centered in o

$$\forall t \geq s > 0, \frac{\text{vol} B(o, t)}{\text{vol} B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu$$

with $\nu > p$. Here, $C_D \geq 1$, $p \geq 1$, $C_P > 0$, $C_o > 0$. Then there exists $\kappa_0 > 0$ such that for $t > 0$, if x, y are two points in $S(o, t)$, there is a path from x to y which remains inside $B(o, t) \setminus B(o, \kappa_0^{-1}t)$. Moreover, it is possible to find an explicit constant, in terms of p, C_D, C_P, C_o, ν .

Let us say a few words about this proposition. Cheeger-Gromoll theorem implies that in our setting, M has only one end. A result from [LT] (with [And]) implies that, for large t , the intersection of the only unbounded component of $M \setminus B(o, t)$ with any annulus $A(t, t+s)$, $t > 0$, is connected. But it says nothing about the behaviour of the bounded components of $M \setminus B(o, t)$. What we proved is that, in a sense, these bounded components have at most linear growth. Moreover, we give an explicit estimate of this growth, which is important for our purpose.

In the third section, we investigate the properties of Schrödinger operators $\overline{\Delta} + V$ that can be deduced from our weighted Sobolev inequality. Here, $\overline{\Delta}$ is the Bochner laplacian on some Euclidian

vector bundle and V is a field of symmetric endomorphisms. In particular, we prove that integral assumptions on the potential ensure the kernel is trivial (theorem 3.1). We obtain various technical estimates and also introduce a good space of sections ψ such that the equation $(\overline{\Delta} + V)\sigma = \psi$ has a bounded solution σ (3.10). This section can be seen as a toolbox.

In the fourth and last section, we point out some applications. Let us denote by $S_o(M)$ (resp. $H_o(M)$) the best constant S (resp. H) in (4) (resp. in (5)). We define the "Sobolev-curvature" invariant

$$SC(M^n) := \inf_{o \in M} \left[S_o(M) \left(\int_M |R|^{\frac{n}{2}} \frac{r_o^n}{\text{vol} B(o, r_o)} d\text{vol} \right)^{\frac{2}{n}} \right]$$

and the "Hardy-curvature" invariant

$$\mathcal{HC}(M^n) := \inf_{o \in M} \left[H_o(M) \sup_M (|R| r_o^2) \right],$$

with the convention $0 \cdot \infty = \infty$. First, we generalize the work of G. Carron [Car1] about L^2 -cohomology and obtain in particular the

Theorem 0.4 (L^2 -cohomology) *Let M^n , $n \geq 3$, be a connected complete Riemannian manifold such that $SC(M)$ is finite. Then the L^2 -cohomology of M is finite-dimensional. Moreover, for any integer k , there exists a positive universal constant $\epsilon(n, k)$ such that if $SC(M) < \epsilon(n, k)$, then $\mathcal{H}_{L^2}^k(M) = \{0\}$.*

In case M has nonnegative Ricci curvature and satisfies (3), this means the L^2 -cohomology is finite-dimensional as soon as $\int_M |R|^{\frac{n}{2}} r_o^{n-\nu} d\text{vol} < \infty$; in [Car2], G. Carron required $\int_M |R|^{\frac{\nu}{2}} d\text{vol} < \infty$. These are close assumptions, but ours is a bit weaker. Our work also provides explicit bounds on the dimension of the L^2 -cohomology spaces. Then we study Ricci flat manifolds and prove the following rigidity theorems, which imply the results announced above.

Theorem 0.5 (Flatness criterion) *If M^n , $n \geq 4$, is a connected complete Ricci-flat manifold, there are universal positive constants $\epsilon_1(n)$ and $\epsilon_2(n)$ such that if $SC(M) < \epsilon_1(n)$ or $\mathcal{HC}(M) < \epsilon_2(n)$, then M is flat.*

Theorem 0.6 (Curvature decay) *Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. If $SC(M)$ is finite, then M has quadratic curvature decay. If moreover there exists $\nu > 4 \frac{n-2}{n-1}$ and $A_o > 0$ such that $\text{vol} B(o, t) \geq A_o t^\nu$ for large t , then the curvature decays like $r_o^{-\frac{(\nu-2)(n-1)}{n-3}}$ and M has finite topological type.*

Finally, we give some examples where this rate of decay is the correct one.

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1 Discretization and Sobolev inequalities.

1.1 How to patch local Sobolev inequalities together.

The aim of this paragraph is to explain how to patch local Sobolev inequalities so as to obtain a global one. In [GSC], A. Grigor'yan and L. Saloff-Coste introduce a discretization procedure enabling them to handle Poincaré inequalities. We generalize their ideas in two ways : we use integral inequalities for different measures and we consider general Sobolev-type inequalities.

Here, M is a smooth Riemannian manifold (Lipschitz would be sufficient), and we introduce two Borel measures λ, μ on it. For us, it will be crucial to cope with both of them at the same time. Let us introduce the necessary vocabulary.

Definition 1.1 *Let $A \subset A^\sharp$ be two subsets of M . A family $\mathcal{U} = (U_i, U_i^*, U_i^\sharp)_{i \in I}$ consisting of subsets of M having finite measure with respect to λ and μ is said to be a good covering of A in A^\sharp if the following is true.*

- (i) *There is a Borel subset E of A such that $\lambda(E) = \mu(E) = 0$ and $A \setminus E \subset \bigcup_i U_i \subset \bigcup_i U_i^\sharp \subset A^\sharp$;*
- (ii) *$\forall i \in I, U_i \subset U_i^* \subset U_i^\sharp$;*
- (iii) *$\exists Q_1, \forall i_0 \in I, \text{Card} \left\{ i \in I / U_{i_0}^\sharp \cap U_i^\sharp \neq \emptyset \right\} \leq Q_1$;*
- (iv) *For every $(i, j) \in I^2$ satisfying $\overline{U_i} \cap \overline{U_j} \neq \emptyset$, there is an element $k(i, j)$ such that $U_i \cup U_j$ is a subset of $U_{k(i, j)}^*$;*
- (v) *There exists a constant Q_2 such that for every $(i, j) \in I^2$, if $\overline{U_i} \cap \overline{U_j}$ is not empty then*

$$\lambda(U_{k(i, j)}^*) \leq Q_2 \min(\lambda(U_i), \lambda(U_j)) \quad \text{and} \quad \mu(U_{k(i, j)}^*) \leq Q_2 \min(\mu(U_i), \mu(U_j)).$$

Given a Borel set U with finite and nonzero λ -measure and a λ -integrable function f , we will denote by $f_{U, \lambda}$ the mean value of f on U with respect to the measure λ :

$$f_{U, \lambda} = \frac{1}{\lambda(U)} \int_U f d\lambda.$$

One can associate to every good covering \mathcal{U} a weighted graph (\mathcal{G}, m_λ) : its set of vertices is $\mathcal{V} = I$ and its set of edges is $\mathcal{E} = \{ \{i, j\} \subset \mathcal{V} / i \neq j, \overline{U_i} \cap \overline{U_j} \neq \emptyset \}$; \mathcal{V} and \mathcal{E} are given measures, both of which will be denoted by m_λ , and they are defined by

$$\forall i \in \mathcal{V}, m_\lambda(i) = \lambda(U_i) \quad \text{and} \quad \forall \{i, j\} \in \mathcal{E}, m_\lambda(i, j) = \max(m_\lambda(i), m_\lambda(j)).$$

Remark 1.2 *In what we call a graph, there is at most one edge between two given vertices. So, if there is an edge between two vertices i and j , we will call it $\{i, j\}$. For us, a weighted graph will always consist of a σ -finite measure on the set of vertices \mathcal{V} and of a σ -finite measure on the set of edges \mathcal{E} , which we give the same name m and which are related by $m(i, j) = \max(m(i), m(j))$, for $\{i, j\}$ in \mathcal{E} .*

Now, we introduce three kinds of inequalities : the discrete estimates (the second and third) will enable us to patch the continuous ones (the first) together.

Definition 1.3 *Given k in $]1, \infty]$ and p in $[1, k]$, we will say that a good covering \mathcal{U} satisfies a continuous L^p Sobolev inequality of order k with respect to the pair of measures (λ, μ) if there exists a constant S_c such that for every i in I , one has*

$$\forall f \in C^\infty(U_i^*), \left(\int_{U_i} |f - f_{U_i, \lambda}|^{\frac{pk}{k-p}} d\lambda \right)^{\frac{k-p}{k}} \leq S_c \int_{U_i^*} |df|^p d\mu$$

and

$$\forall f \in C^\infty(U_i^\sharp), \left(\int_{U_i^*} |f - f_{U_i^*, \lambda}|^{\frac{pk}{k-p}} d\lambda \right)^{\frac{k-p}{k}} \leq S_c \int_{U_i^\sharp} |df|^p d\mu.$$

Definition 1.4 Given k in $]1, \infty]$ and p in $[1, k[$, we will say that the weighted graph (\mathcal{G}, m) satisfies a discrete L^p Sobolev-Dirichlet inequality of order k if there exists a constant S_d such that for every $f \in L^p(\mathcal{V}, m)$, one has

$$\left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{pk}{k-p}} m(i) \right)^{\frac{k-p}{k}} \leq S_d \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j).$$

Definition 1.5 Given k in $]1, \infty]$ and p in $[1, k[$, we will say that a finite weighted graph (\mathcal{G}, m) satisfies a discrete L^p Sobolev-Neumann inequality of order k if there exists a constant S_d such that for every $f \in \mathbb{R}^{\mathcal{V}}$, one has

$$\left(\sum_{i \in \mathcal{V}} |f(i) - m(f)|^{\frac{pk}{k-p}} m(i) \right)^{\frac{k-p}{k}} \leq S_d \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j).$$

Remark 1.6 In this terminology, a L^p Poincaré inequality is nothing but a L^p Sobolev inequality of infinite order.

Remark 1.7 Of course, one can say that a good covering \mathcal{U} satisfies a discrete Sobolev inequality, by considering the associated weighted graph (\mathcal{G}, m_λ) .

The following theorem is the crucial tool for us.

Theorem 1.8 Fix k in $]1, \infty]$ and p in $[1, k[$. If a good covering \mathcal{U} of A in A^\sharp satisfies the continuous L^p Sobolev inequality of order k (1.3) and the discrete L^p Sobolev-Dirichlet of order ∞ (1.4), then the following Sobolev-Dirichlet inequality is true :

$$\forall f \in C_c^\infty(A), \int_A \left(|f|^{\frac{pk}{k-p}} d\lambda \right)^{\frac{k-p}{k}} \leq S \int_{A^\sharp} |df|^p d\mu.$$

Moreover, one can choose $S = S_c Q_1 2^{p-1 + \frac{p}{k}} (1 + S_d Q_2 (2^p Q_1^2)^{\frac{k-p}{k-p}})^{\frac{k-p}{k}}$.

Remark 1.9 The case where $\lambda = \mu$, $k = \infty$ and $p = 2$ was proved by A. Grigor'yan and L. Saloff-Coste in [GSC].

Proof :

We set $q := \frac{pk}{k-p}$ and consider $f \in C_c^\infty(A)$. Thanks to a little convexity, we can write

$$\begin{aligned} \int_A |f|^q d\lambda &\leq \sum_{i \in \mathcal{V}} \int_{U_i} |f|^q d\lambda \\ &\leq 2^{q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f - f_{U_i, \lambda}|^q d\lambda + 2^{q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f_{U_i, \lambda}|^q d\lambda \\ &= 2^{q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f - f_{U_i, \lambda}|^q d\lambda + 2^{q-1} \sum_{i \in \mathcal{V}} |f_{U_i, \lambda}|^q \lambda(U_i). \end{aligned}$$

The continuous Sobolev inequality gives an upper bound for the first term ; noticing that $q \geq p$ and remembering the assumptions on the covering, we find

$$\begin{aligned} \sum_{i \in \mathcal{V}} \int_{U_i} |f - f_{U_i, \lambda}|^q d\lambda &\leq S_c^{q/p} \sum_{i \in \mathcal{V}} \left(\int_{U_i^*} |df|^p d\mu \right)^{q/p} \\ &\leq S_c^{q/p} \left(\sum_{i \in \mathcal{V}} \int_{U_i^*} |df|^p d\mu \right)^{q/p} \\ &\leq S_c^{q/p} Q_1^{q/p} \left(\int_{A^\sharp} |df|^p d\mu \right)^{q/p}. \end{aligned}$$

To estimate the second term, we use the discrete Sobolev inequality :

$$\sum_{i \in \mathcal{V}} |f_{U_i, \lambda}|^q \lambda(U_i) \leq S_d \sum_{\{i, j\} \in \mathcal{E}} |f_{U_i, \lambda} - f_{U_j, \lambda}|^q \max(\lambda(U_i), \lambda(U_j)).$$

For $\{i, j\} \in \mathcal{E}$, a Hölder inequality and the fact that we have a good covering lead to :

$$\begin{aligned} |f_{U_i, \lambda} - f_{U_j, \lambda}|^q \max(\lambda(U_i), \lambda(U_j)) &= \frac{\max(\lambda(U_i), \lambda(U_j))}{\lambda(U_j)^q \lambda(U_i)^q} \left| \int_{U_i} \int_{U_j} (f(x) - f(y)) d\lambda(x) d\lambda(y) \right|^q \\ &\leq \frac{\max(\lambda(U_i), \lambda(U_j))}{\lambda(U_i) \lambda(U_j)} \int_{U_i} \int_{U_j} |f(x) - f(y)|^q d\lambda(x) d\lambda(y) \\ &\leq Q_2 \frac{1}{\lambda(U_{k(i,j)}^*)} \int_{U_{k(i,j)}^*} \int_{U_{k(i,j)}^*} |f(x) - f(y)|^q d\lambda(x) d\lambda(y). \end{aligned}$$

Now, if X is a Borel set with finite and nonzero λ -measure and if g is a function in $L^q(X, \lambda)$,

$$\begin{aligned} \frac{1}{\lambda(X)} \int_X \int_X |g(x) - g(y)|^q d\lambda(x) d\lambda(y) &\leq \frac{1}{\lambda(X)} \int_X \int_X 2^{q-1} (|g(x)|^q + |g(y)|^q) d\lambda(x) d\lambda(y) \\ &\leq 2^q \int_X |g(x)|^q d\lambda(x). \end{aligned}$$

Let us apply this to $f - f_{U_{k(i,j), \lambda}^*}$, on $U_{k(i,j)}^*$:

$$|f_{U_i, \lambda} - f_{U_j, \lambda}|^q \max(\lambda(U_i), \lambda(U_j)) \leq Q_2 2^q \int_{U_{k(i,j)}^*} |f - f_{U_{k(i,j), \lambda}^*}|^q d\lambda.$$

The continuous Sobolev inequality yields

$$|f_{U_i, \lambda} - f_{U_j, \lambda}|^q \max(\lambda(U_i), \lambda(U_j)) \leq Q_2 2^q S_c^{q/p} \left(\int_{U_{k(i,j)}^\#} |df|^p d\mu \right)^{\frac{q}{p}}.$$

Therefore :

$$\sum_{i \in \mathcal{V}} |f_{U_i, \lambda}|^q \lambda(U_i) \leq S_d \sum_{\{i, j\} \in \mathcal{E}} Q_2 2^q S_c^{q/p} \left(\int_{U_{k(i,j)}^\#} |df|^p d\mu \right)^{\frac{q}{p}}.$$

As q is greater or equal to p ,

$$\sum_{i \in \mathcal{V}} |f_{U_i, \lambda}|^q \lambda(U_i) \leq S_d Q_2 2^q S_c^{q/p} \left(\sum_{\{i, j\} \in \mathcal{E}} \int_{U_{k(i,j)}^\#} |df|^p d\mu \right)^{\frac{q}{p}}.$$

By using twice the fact that we have a good covering, we see that :

$$\sum_{\{i, j\} \in \mathcal{E}} \int_{U_{k(i,j)}^\#} |df|^p d\mu \leq Q_1^2 \sum_{i \in \mathcal{V}} \int_{U_i^\#} |df|^p d\mu \leq Q_1^3 \int_{A^\#} |df|^p d\mu.$$

Hence :

$$\sum_{i \in \mathcal{V}} |f_{U_i, \lambda}|^q \lambda(U_i) \leq S_d Q_2 2^q S_c^{q/p} Q_1^{3q/p} \left(\int_{A^\#} |df|^p d\mu \right)^{\frac{q}{p}}.$$

Eventually, we get :

$$\int_A |f|^q d\lambda \leq 2^{q-1} (S_c^{q/p} Q_1^{q/p} + S_d Q_2 2^q S_c^{q/p} Q_1^{3q/p}) \left(\int_{A^\#} |df|^p d\mu \right)^{\frac{q}{p}}.$$

And this is what we wanted to prove. ■

There is also a "Neumann" version of this result.

Theorem 1.10 Fix k in $]1, \infty]$ and p in $[1, k[$. If a finite good covering \mathcal{U} of A in A^\sharp satisfies the continuous L^p Sobolev inequality of order k (1.3) and the discrete L^p Sobolev-Neumann inequality of order ∞ (1.5), the following Sobolev-Neumann inequality is true :

$$\forall f \in C^\infty(A), \int_A \left(|f - f_{A,\lambda}|^{\frac{pk}{k-p}} d\lambda \right)^{\frac{k-p}{k}} \leq S \int_{A^\sharp} |df|^p d\mu.$$

And one can choose $S = S_c Q_1 2^{2p-1+\frac{p}{k}} (1 + S_d Q_2 (2^p Q_1^2)^{\frac{k-p}{k}})^{\frac{k-p}{k}}$.

Proof :

Again, set $q := \frac{pk}{k-p}$ and fix $f \in C_c^\infty(A)$. First, note the inequality

$$\|f - f_{A,\lambda}\|_{L^q(A,\lambda)} \leq 2 \inf_{c \in \mathbb{R}} \|f - c\|_{L^q(A,\lambda)}.$$

Indeed, if c is a real number, we can write

$$\begin{aligned} \|f - f_{A,\lambda}\|_{L^q(A,\lambda)} &\leq \|f - c\|_{L^q(A,\lambda)} + \|c - f_{A,\lambda}\|_{L^q(A,\lambda)} \\ &= \|f - c\|_{L^q(A,\lambda)} + |f_{A,\lambda} - c| \lambda(A)^{\frac{1}{q}} \\ &= \|f - c\|_{L^q(A,\lambda)} + \left| \int_A (f - c) d\lambda \right| \lambda(A)^{\frac{1}{q}-1} \end{aligned}$$

and by Hölder inequality,

$$\begin{aligned} \|f - f_{A,\lambda}\|_{L^q(A,\lambda)} &\leq \|f - c\|_{L^q(A,\lambda)} + \left(\int_A |f - c|^q d\lambda \right)^{\frac{1}{q}} \lambda(A)^{1-\frac{1}{q}} \lambda(A)^{\frac{1}{q}-1} \\ &= 2 \|f - c\|_{L^q(A,\lambda)}. \end{aligned}$$

As this is true for each $c \in \mathbb{R}$, this proves the statement. In particular, for

$$c := m_\lambda(f_{U,\lambda}) = \frac{\sum_{i \in \mathcal{V}} f_{U_i,\lambda} \lambda(U_i)}{\sum_{i \in \mathcal{V}} \lambda(U_i)},$$

we can write

$$\begin{aligned} \int_A |f - f_{A,\lambda}|^q d\lambda &\leq 2^q \int_A |f - c|^q d\lambda \\ &\leq \sum_{i \in \mathcal{V}} \int_{U_i} |f - c|^q d\lambda \\ &\leq 2^{2q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f - f_{U_i,\lambda}|^q d\lambda + 2^{2q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f_{U_i,\lambda} - c|^q d\lambda \\ &= 2^{2q-1} \sum_{i \in \mathcal{V}} \int_{U_i} |f - f_{U_i,\lambda}|^q d\lambda + 2^{2q-1} \sum_{i \in \mathcal{V}} |f_{U_i,\lambda} - c|^q \lambda(U_i). \end{aligned}$$

We then estimate both terms as in the proof of theorem 1.8 : for the second, it is made possible by our choice of c . ■

Remark 1.11 In fact, our argument leads to more general theorems. We will not use them but let us phrase the "Dirichlet" version. Suppose $1 \leq p \leq r \leq q \leq \infty$ and set $k = \frac{qp}{q-p}$. If a good covering \mathcal{U} of A in A^\sharp satisfies the continuous L^p Sobolev-Neumann inequality of order k (with constant S_c), the discrete L^r Sobolev-Dirichlet inequality of order $\frac{rq}{q-r}$ (with constant S_d), and the continuous L^p Sobolev-Neumann inequality of order $\frac{pr}{r-p}$ (with constant S'_c), M satisfies the following L^p Sobolev-Dirichlet inequality of order k :

$$\forall f \in C_c^\infty(A), \left(\int_A |f|^q d\lambda \right)^{p/q} \leq S \int_{A^\sharp} |df|^p d\mu,$$

with $S = 2^{p-p/q} \left((Q_1 S_c)^{q/p} + (S_d Q_2 2^r (S'_c)^{r/p})^{q/r} Q_1^{3q/p} \right)^{p/q}$. For instance, this kind of result could be used to patch local Sobolev and Poincaré inequalities together.

1.2 Sobolev and isoperimetric inequalities on graphs.

Now, we know that discrete inequalities on appropriate graphs make it possible to patch local Sobolev inequalities together. The problem is : how can we show such discrete inequalities ? Our first purpose here is to clarify the link between Sobolev inequalities of the same order on weighted graphs. We explain why, as in the continuous case, the L^1 inequality of order k ($1 < k \leq \infty$) imply the L^p inequalities for $1 \leq p < k$.

Proposition 1.12 *We consider an infinite weighted graph $(\mathcal{V}, \mathcal{E}, m)$ (see remark 1.2). We assume that the degree of each vertex is bounded by some integer d and also that there exists a number $C \geq 1$ such that for $\{i, j\}$ in \mathcal{E} , $C^{-1}m(i) \leq m(j) \leq Cm(i)$. Then given k in $]1, \infty]$, the L^1 Sobolev inequality of order k*

$$\forall f \in L^1(\mathcal{V}, m), \left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{k-1}{k-1}} m(i) \right)^{\frac{k-1}{k}} \leq S \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)| m(i, j) \quad (6)$$

implies the L^p Sobolev inequality of order k

$$\forall f \in L^p(\mathcal{V}, m), \left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{pk}{k-p}} m(i) \right)^{\frac{k-p}{pk}} \leq S' \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j) \right)^{\frac{1}{p}}, \quad (7)$$

provided p belongs to $[1, k[$. Moreover, one can choose $S' = 2p \frac{k-1}{k-p} S(dC)^{1-\frac{1}{p}}$.

Proof :

Let f be an element of $\mathbb{R}^{\mathcal{V}}$ with finite support. We apply (6) to $|f|^\gamma$ where $\gamma \geq 1$ is a parameter that we will fix later :

$$\left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{\gamma k}{k-1}} m(i) \right)^{\frac{k-1}{k}} \leq S \sum_{\{i,j\} \in \mathcal{E}} \left| |f(i)|^\gamma - |f(j)|^\gamma \right| m(i, j).$$

If a, b are real numbers, the following is true :

$$\left| |a|^\gamma - |b|^\gamma \right| \leq \gamma \max(|a|, |b|)^{\gamma-1} |a| - |b| \leq \gamma |a - b| (|a|^{\gamma-1} + |b|^{\gamma-1}).$$

Consequently,

$$\left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{\gamma k}{k-1}} m(i) \right)^{\frac{k-1}{k}} \leq \gamma S \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)| (|f(i)|^{\gamma-1} + |f(j)|^{\gamma-1}) m(i, j).$$

Hölder inequality bounds the right hand side by

$$2\gamma S \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j) \right)^{\frac{1}{p}} \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i)|^{(\gamma-1)\frac{p}{p-1}} m(i, j) \right)^{1-\frac{1}{p}}$$

and our assumptions on the graph bound this by

$$2\gamma S(dC)^{1-\frac{1}{p}} \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j) \right)^{\frac{1}{p}} \left(\sum_{i \in \mathcal{V}} |f(i)|^{(\gamma-1)\frac{p}{p-1}} m(i) \right)^{1-\frac{1}{p}}.$$

Set $\gamma := p \frac{k-1}{k-p} \geq 1$ to conclude the proof. ■

Now, let us explain why inequalities like (6) stem from isoperimetric inequalities on the graph.

Definition 1.13 *Let $(\mathcal{V}, \mathcal{E})$ be a graph. We define the boundary $\partial\Omega$ of a subset Ω of \mathcal{V} as*

$$\partial\Omega := \{\{i, j\} \in \mathcal{E}, \{i, j\} \cap \Omega \neq \emptyset \text{ and } \{i, j\} \cap (\mathcal{V} \setminus \Omega) \neq \emptyset\}.$$

Proposition 1.14 *Let $(\mathcal{V}, \mathcal{E}, m)$ be an infinite weighted graph and fix k in $]1, \infty]$. Then the isoperimetric inequality of order k*

$$\forall \Omega \subset \mathcal{V} \text{ with } m(\Omega) < \infty, \frac{m(\Omega)^{\frac{k-1}{k}}}{m(\partial\Omega)} \leq I \quad (8)$$

is equivalent to the L^1 Sobolev inequality of order k

$$\forall f \in L^1(\mathcal{V}, m), \left(\sum_{i \in \mathcal{V}} |f(i)|^{\frac{k}{k-1}} m(i) \right)^{\frac{k-1}{k}} \leq I \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)| m(i, j).$$

Proof :

By considering characteristic functions of subsets of \mathcal{V} , one easily sees that the Sobolev inequality implies the isoperimetric inequality. To prove the converse, set $q = \frac{k}{k-1}$ and let f be a function on \mathcal{V} , with finite support. For every i in \mathcal{V} , we write

$$f(i) = \int_0^{f(i)} dt = \int_0^\infty 1_{t < f(i)} dt.$$

Thus,

$$\|f\|_{L^q(\mathcal{V}, m)} \leq \int_0^\infty \|1_{t < f(\cdot)}\|_{L^q(\mathcal{V}, m)} dt = \int_0^\infty \left(\sum_{\{i \in \mathcal{V}, f(i) > t\}} m(i) \right)^{\frac{1}{q}} dt.$$

If the isoperimetric inequality is true, we find

$$\begin{aligned} \|f\|_{L^q(\mathcal{V}, m)} &\leq I \int_0^\infty m(\partial \{i \in \mathcal{V}, f(i) > t\}) dt \\ &= I \int_0^\infty \sum_{\{\{i,j\} \in \mathcal{E}, f(j) \leq t < f(i)\}} m(i, j) dt \\ &= I \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)| m(i, j). \end{aligned}$$

■

This paragraph shows that if the graph obtained by discretization (as explained above) satisfies an isoperimetric inequality, it will satisfy a convenient Sobolev inequality, so that we will be able to implement our patching process. It is time to turn to geometry so as to obtain concrete inequalities.

2 Inequalities on manifolds with nonnegative Ricci curvature.

Sobolev inequalities are a major tool of global analysis. Unfortunately, they are not always available. It is known that on manifolds with nonnegative Ricci curvature and maximal volume growth, they actually occur ([Cro]), providing a lot of analytical, geometrical and topological information (see [BKN], for instance). As soon as the volume growth is not maximal, the Sobolev inequality cannot be true. Our aim here is to show that, even if the volume growth is not maximal, a weighted Sobolev inequality still occurs.

2.1 Geometric preliminaries.

We would like to emphasize here some features of complete manifolds with nonnegative Ricci curvature. These are the typical manifolds where the discretization scheme applies.

Let us fix some notations. Recall we denote by $B(x, t)$ the geodesic ball centered in x and of radius t . In the same way, $S(x, t)$ will be the corresponding geodesic sphere $\partial B(x, t)$. $V(x, t)$ will be the volume of $B(x, t)$. We will constantly distinguish a point o in the manifold : when the center of the ball considered is this point o , we will often omit it and write $B(t)$ or $V(t)$. Finally, we will work with annuli $A(s, t) := B(t) \setminus B(s)$.

First, the Bishop-Gromov comparison theorem says that, in manifolds with nonnegative Ricci curvature, the volume growth of balls is "subEuclidian" in a very strong way.

Theorem 2.1 (Bishop-Gromov) *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Then for every x in M , the function ρ_x defined for $t \geq 0$ by*

$$\rho_x(t) = \frac{t^n}{\text{vol } B(x, t)}$$

is a nondecreasing function. It implies that for $0 < s < t$,

$$\forall x \in M, \frac{\text{vol } B(x, t)}{\text{vol } B(x, s)} \leq \left(\frac{t}{s}\right)^n. \quad (9)$$

And a useful corollary is that for $x, y \in M$ and $0 < s < t + d(x, y)$:

$$\frac{\text{vol } B(y, t)}{\text{vol } B(x, s)} \leq \frac{\text{vol } B(x, t + d(x, y))}{\text{vol } B(x, s)} \leq \left(\frac{t + d(x, y)}{s}\right)^n. \quad (10)$$

For a proof, see [Cha]. Note the following simple consequence. The argument of the proof will constantly be used in the sequel.

Corollary 2.2 *Let M^n be a connected complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then for every $\kappa > 1$, there exists a positive constant $C(n, \kappa)$ such that for any $x \in M$ and $t > 0$,*

$$C(n, \kappa)^{-1} \leq \frac{\text{vol}(B(x, \kappa t) \setminus B(x, t))}{\text{vol}(B(x, t) \setminus B(x, \kappa^{-1}t))} \leq C(n, \kappa).$$

Proof :

To prove the lower bound, choose a point y on the sphere $S(x, (\kappa+1)t/2)$ centered in x and of radius $(\kappa+1)t/2$ (such a point exists since M is assumed to be complete, noncompact and connected). Then the ball $B := B(y, (\kappa-1)t/2)$ is contained in $B(x, \kappa t) \setminus B(x, t)$. Therefore

$$\frac{\text{vol}(B(x, t) \setminus B(x, \kappa^{-1}t))}{\text{vol}(B(x, \kappa t) \setminus B(x, t))} \leq \frac{\text{vol } B(x, t)}{\text{vol } B(y, (\kappa-1)t/2)},$$

and (10) yields

$$\frac{\text{vol}(B(x, t) \setminus B(x, \kappa^{-1}t))}{\text{vol}(B(x, \kappa t) \setminus B(x, t))} \leq \left(\frac{t + \frac{(\kappa+1)t}{2}}{\frac{(\kappa-1)t}{2}}\right)^n = \left(\frac{\kappa+3}{\kappa-1}\right)^n.$$

The upper bound is proved likewise. ■

Moreover, starting from the comparison theorem, P. Buser [Bus] showed the following

Theorem 2.3 (Buser) *In a complete noncompact Riemannian manifold with nonnegative Ricci curvature, for any p in $[1, \infty[$, every ball $B(x, t)$ satisfies the L^p Poincaré inequality*

$$\forall f \in C^\infty(B(x, t)), \int_{B(x, t)} |f - f_{B(x, t)}|^p d\text{vol} \leq C(n, p) t^p \int_{B(x, t)} |df|^p d\text{vol}, \quad (11)$$

where $f_{B(x, t)}$ denotes the mean value of f on the ball $B(x, t)$, with respect to the Riemannian measure vol .

This result yields the fundamental inequalities we need. Besides, it will prove useful in the study of the geometry at infinity of manifolds with nonnegative Ricci curvature.

Let us mention the Cheeger-Gromoll theorem ([CG1],[Bes]), which enlightens the structure of manifolds with nonnegative Ricci curvature :

Theorem 2.4 (Cheeger-Gromoll) *A connected complete Riemannian manifold with nonnegative Ricci curvature is always the Riemannian product of the Euclidian space \mathbb{R}^d and a connected complete Riemannian manifold with nonnegative Ricci curvature which possesses no line.*

Corollary 2.5 *A connected complete noncompact Riemannian manifold with nonnegative Ricci curvature possesses exactly one end, unless it is a Riemannian product of \mathbb{R} and a compact manifold.*

Remark 2.6 *In our setting, the volume growth of balls will forbid the particular case, which is therefore irrelevant here.*

In what follows, we will be working on annuli, so that we are interested in understanding their topology/geometry ; connectedness is particularly important for us since it is an obvious necessary condition for a Sobolev or Poincaré inequality on them. In [And], M. Anderson proved that the first Betti number of a connected complete Riemannian manifold with nonnegative Ricci curvature is bounded by its dimension. Now, [LT] points out a consequence of the finiteness of the first Betti number :

Proposition 2.7 *Let M be a connected complete Riemannian manifold with nonnegative Ricci curvature, finite first Betti number and exactly k ends. Let us fix a point $o \in M$ and consider balls and annuli centered in o . Then for large R and any $r > 0$, denoting by M_R the union of all unbounded connected components $M \setminus B(R)$, it is true that $A(R, R+r) \cap M_R$ has exactly k connected components. In particular, if M has exactly one end, for large R and any $r > 0$, the annulus $A(R, R+r)$ possesses one and only one component that can be connected to infinity inside $M \setminus B(R)$.*

Let us give an interpretation in terms of discretization. Consider a manifold M with nonnegative Ricci curvature, possessing one end, and fix a point in M . Let us choose $R > 0$ and $\kappa > 0$. We discretize M in the following manner. We associate a vertex to $B(R)$ and to every connected component of the annuli $A(\kappa^i R, \kappa^{i+1} R)$, $i \in \mathbb{N}$. Let us decide that there is an edge between two given vertices if and only if the closures of the corresponding subsets of M intersect. Then the proposition above says that for large R this graph is a tree and its root is the vertex corresponding to $B(R)$. From another point of view, it says, that even if R is small, outside a finite subset, the graph is a tree.

Now, there is no reason why this tree should not have branches, and for technical reasons (see the proof of lemma 2.15 below), we would like to make them as small as possible. What we need is some kind of control on the size of bounded connected components of the complements of balls in the manifold. This is given by the following proposition, which we state with rather general assumptions.

Proposition 2.8 (RCA) *Let M be a connected complete Riemannian manifold, satisfying the volume doubling property*

$$\forall x \in M, \forall t > 0, \text{vol } B(x, 2t) \leq C_D \text{vol } B(x, t),$$

the scaled L^p Poincaré inequality centered in some point o in M

$$\forall f \in C_c^\infty(M), \forall t > 0, \int_{B(o,t)} |f - f_{B(o,t)}|^p d\text{vol} \leq C_P t^p \int_{B(o,t)} |df|^p d\text{vol}$$

and the reverse volume doubling condition centered in o

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu$$

with $\nu > p$. Here, $C_D \geq 1$, $p \geq 1$, $C_P > 0$, $C_o > 0$. Then there exists $\kappa_0 > 0$ such that for $R > 0$, if x, y are two points in the geodesic sphere $S(o, R)$, there is a path from x to y which remains inside $B(o, R) \setminus B(o, \kappa_0^{-1} R)$. Moreover, it is possible to find an explicit constant κ_0 , in terms of p, C_D, C_P, C_o, ν .

In terms of the discretization we have introduced, this means that for large κ , for every two vertices on the same level of the tree (i.e. corresponding to the same annulus), there exists a vertex of the previous level which is connected to both of them.

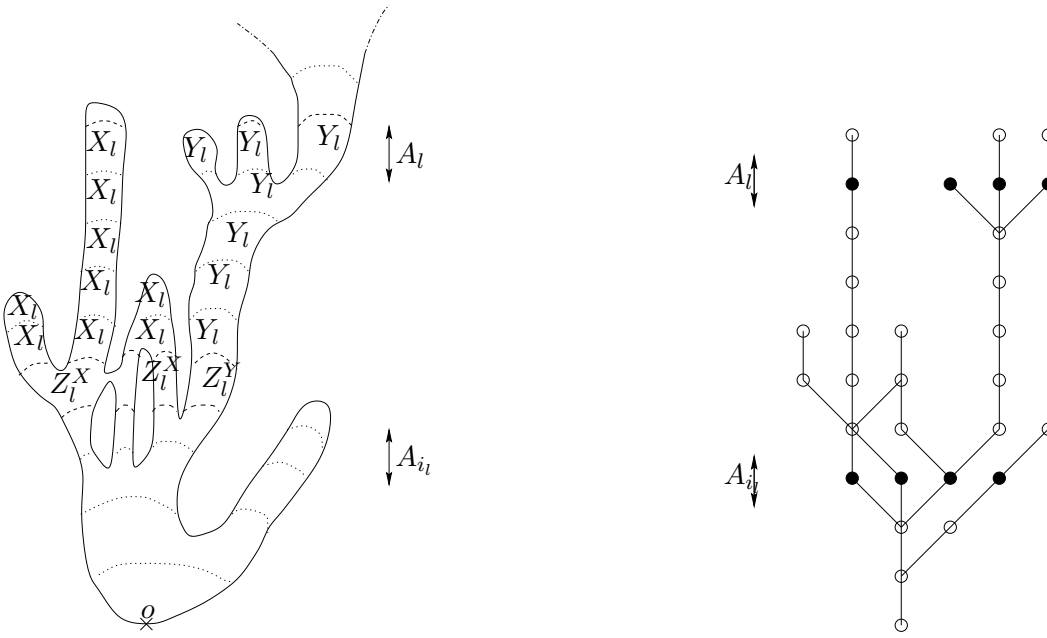


Figure 1: A manifold and its discretization.

Proof :

Consider the graph obtained as above by working with $A_i := A(2^{i-1}R, 2^iR)$, $i \in \mathbb{N}^*$, $R > 0$, plus $B(R) =: A_0$. Set $B_i = B(2^iR)$. We define \mathcal{C} as the bijective map which associates to every vertex of the graph the corresponding component of annulus. Let us write \mathcal{A}_i for $\mathcal{C}^{-1}(A_i)$ and \mathcal{B}_i for $\mathcal{C}^{-1}(B_i)$. Now, fix $l \in \mathbb{N}^*$, consider the nonempty set

$$I_l = \{i \in [0, l], \mathcal{A}_i \text{ is contained in a connected component of } \mathcal{B}_l \setminus \mathcal{B}_{i-1}\}$$

and set $i_l = \max I_l$. Call \mathcal{M}_l the connected component of $\mathcal{B}_l \setminus \mathcal{B}_{i_l-1}$ which contains \mathcal{A}_l . We assume $l - i_l$ is greater than 3 and think of it as a large number. By definition, $\mathcal{M}_l \setminus \mathcal{A}_{i_l}$ is not connected. We choose one of its connected component \mathcal{X}'_l and name \mathcal{Y}'_l the union of the other connected components. We finally define $X'_l := \mathcal{C}^{-1}(\mathcal{X}'_l)$, $Y'_l := \mathcal{C}^{-1}(\mathcal{Y}'_l)$, $X_l := X'_l \setminus A_{i_l+1}$, $Y_l := Y'_l \setminus A_{i_l+1}$, $Z_l^X := X'_l \cap A_{i_l+1}$, $Z_l^Y := Y'_l \cap A_{i_l+1}$ and $Z_l := Z_l^X \cup Z_l^Y$ (see figure 2.1).

Given real numbers a and b , we can define a Lipschitz function f_l on B_l in the following way :

$$f_l = \begin{cases} a & \text{on } X_l, \\ b & \text{on } Y_l, \\ a \frac{r_a - 2^{i_l}R}{2^{i_l}R} & \text{on } Z_l^X, \\ b \frac{r_a - 2^{i_l}R}{2^{i_l}R} & \text{on } Z_l^Y, \\ 0 & \text{everywhere else.} \end{cases}$$

The Poincaré inequality says

$$\int_{B_l} |f_l - (f_l)_{B_l}|^p dvol \leq C_P 2^{lp} R^p \int_{B_l} |df_l|^p dvol. \quad (12)$$

We choose a and b so that the mean value of f_l on $X_l \cup Y_l$ is 0 : $a \text{ vol } X_l + b \text{ vol } Y_l = 0$. With $a := 1$, this means $b = -\frac{\text{vol } X_l}{\text{vol } Y_l}$.

On the one hand,

$$\begin{aligned} \int_{B_l} |f_l - (f_l)_{B_l}|^p dvol &\geq 2^{-p} \frac{\int_{B_l} \int_{B_l} |f_l(x) - f_l(y)|^p dx dy}{\text{vol } B_l} \\ &\geq 2^{-p} \frac{\text{vol } X_l \text{ vol } Y_l |b - a|^p}{\text{vol } B_l} \\ &= 2^{-p} \frac{\text{vol } X_l \text{ vol } Y_l \left(1 + \frac{\text{vol } X_l}{\text{vol } Y_l}\right)^p}{\text{vol } B_l}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{B_l} |df_l|^p d\text{vol} &\leq \left(\text{vol } Z_l^X \left(\frac{a}{2^{i_l} R} \right)^p + \text{vol } Z_l^Y \left(\frac{b}{2^{i_l} R} \right)^p \right) \\ &= \frac{\text{vol } Z_l^X + \text{vol } Z_l^Y \left(\frac{\text{vol } X_l}{\text{vol } Y_l} \right)^p}{2^{i_l p} R^p}. \end{aligned}$$

So

$$\begin{aligned} 2^{-p} \frac{\text{vol } X_l \text{vol } Y_l \left(1 + \frac{\text{vol } X_l}{\text{vol } Y_l} \right)^p}{\text{vol } B_l} &\leq C_P 2^{p(l-i_l)} \left(\text{vol } Z_l^X + \text{vol } Z_l^Y \left(\frac{\text{vol } X_l}{\text{vol } Y_l} \right)^p \right) \\ &\leq C_P 2^{p(l-i_l)} \text{vol } Z_l \left(1 + \left(\frac{\text{vol } X_l}{\text{vol } Y_l} \right)^p \right), \end{aligned}$$

hence

$$1 \leq 2^p C_P 2^{p(l-i_l)} \frac{\text{vol } Z_l \text{vol } B_l}{\text{vol } X_l \text{vol } Y_l}. \quad (13)$$

By definition, the volume of Z_l is bounded by $V(o, 2^{i_l+1}R)$. A lower bound on $\text{vol } X_l$ can be obtained as in the proof of (2.2). Choose a point x_l in $S(o, (2^{l-2} + 2^{l-1})R/2) \cap X_l$ et note that $B(x_l, 2^{l-3}R)$ is contained in X_l : it lies in $A(2^{l-2}R, 2^{l-1}R)$ and it is connected, so it lies in the connected component of its center x_l in $A(2^{l-2}R, 2^{l-1}R)$, hence in X_l . The volume doubling property implies

$$\forall x \in M, \forall t \geq s > 0, V(x, t) \leq C_D (t/s)^{\log_2 C_D} V(x, s),$$

so that

$$\frac{V(o, 2^l R)}{V(x_l, 2^{l-3}R)} \leq C_D \left(\frac{2^l + (2^{l-2} + 2^{l-1})/2}{2^{l-3}} \right)^{\log_2 C_D} = C_D 11^{\log_2 C_D}$$

and

$$\text{vol } X_l \geq V(x_l, 2^{l-3}R) \geq C_D^{-1} 11^{-\log_2 C_D} V(o, 2^l R).$$

As we have the same lower bound for $\text{vol } Y_l$, (13) yields :

$$1 \leq 2^p C_P C_D^2 121^{\log_2 C_D} 2^{p(l-i_l)} \frac{V(o, 2^{i_l+1}R)}{V(o, 2^l R)}.$$

(1) enables us to write :

$$1 \leq 2^p C_P C_D^2 121^{\log_2 C_D} 2^\nu C_o 2^{(l-i_l)(p-\nu)}.$$

Since $\nu > p$, this inequality says that $l - i_l$ is bounded by some constant independent of l : the branches of the tree have a bounded length. (2.9) stems from it easily. ■

Corollary 2.9 *Let M be a connected complete Riemannian manifold with nonnegative Ricci curvature and assume there are o in M , $C_o > 0$ and $\nu > 1$ such that*

$$\forall t > s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu$$

Then there exists $\kappa_0 = \kappa_0(n, \nu, C_o) > 0$ such that for $R > 0$, if x, y are two points in $S(o, R)$, there is a path from x to y which remains inside $B(o, R) \setminus B(o, \kappa_0^{-1}R)$.

2.2 Inequalities on connected components of annuli.

We show here that Poincaré or Sobolev inequalities on balls imply analogous inequalities on connected subsets of annuli.

Lemma 2.10 *Let M^n be a noncompact connected complete Riemannian manifold with nonnegative Ricci curvature. Fix $p \geq 1$, $R > 0$, $\kappa > 1$ and consider a connected Borel subset A of the annulus $B(o, \kappa R) \setminus B(o, R)$, $o \in M$. Then if we let A_δ be the δR -neighbourhood of A , with $0 < \delta < 1$, the following Poincaré inequality is true :*

$$\forall f \in C^\infty(A_\delta), \int_A |f - f_A|^p d\text{vol} \leq C(n, \kappa, \delta, p) R^p \int_{A_\delta} |df|^p d\text{vol}.$$

Proof :

Set $s = \delta R$ and consider a s -lattice $(x_i)_{i \in I}$ of A , i.e. a maximal subset of A such that the distance between any two of its elements is at least s . We set $V_i = B(x_i, s)$, $V_i^* = V_i^\sharp = B(x_i, 3s)$. It is easy to see that $(V_i, V_i^*, V_i^\sharp)_{i \in I}$ is a good covering of A in A_δ (cf. (1.1)), with respect to the Riemannian measure. Indeed, for (iii), we can note that the V_i^* under consideration are contained in $B(x_{i_0}, 9s)$ and use (10) to get $\text{vol}(B(x_{i_0}, 9s)) \leq 30^n \text{vol}(B(x_i, \frac{s}{2}))$; since the balls $B(x_i, \frac{s}{2})$ do not intersect, we see that $Q_1 = 30^n$ is convenient. In (iv), we can choose $k(i, j) = i$. As to (v), (9) yields $\text{vol}(V_i^*) \leq 3^n \text{vol}(V_i)$ and (10) gives $\text{vol}(V_i^*) \leq 5^n \text{vol}(V_j)$, so that we can set $Q_2 = 5^n$.

We intend to apply the theorem 1.10 with $k = \infty$. Buser Theorem (11) yields the continuous inequality, with constant $C(n, p)s^2$. What about the discrete inequality ?

Noticing the balls $B(x_i, \frac{s}{2})$ do not intersect and are contained in the ball $B(o, \kappa R + \frac{s}{2})$, we find that

$$\text{Card}(I) \min_{i \in I} \text{vol}(B(x_i, s/2)) \leq \text{vol}(B(o, \kappa R + s/2)),$$

and with (10), this implies an upper bound on the number of balls in the covering

$$\text{Card}(I) \leq \left(\frac{\kappa R + s/2 + \kappa R}{s/2} \right)^n = (1 + 4\kappa/\delta)^n =: N = N(n, \kappa, \delta).$$

The point is it is independent of R .

Now, every finite connected graph endowed with the counting measure satisfies a Poincaré inequality : this stems from the fact that any two norms on a vector space of finite dimension are equivalent (the connectivity is necessary here to ensure that we indeed compare two norms). As there is only a finite number of such graphs which have at most N vertices, we conclude that every such graph satisfies a Poincaré inequality for some constant $P = P(N, p)$ (see below for an explicit constant). Since (10) implies

$$\forall i, j \in \mathcal{V}, \frac{\text{vol}(V_i)}{\text{vol}(V_j)} \leq (1 + 2\kappa/\delta)^n,$$

there is a number $K = K(n, \kappa, \delta) \geq 1$ such that $K^{-1}m_0 \leq m(i) \leq Km_0$, where m_0 is proportionnal to the counting measure on our graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then for every $f \in \mathbb{R}^{\mathcal{V}}$,

$$\begin{aligned} \left(\sum_{i \in \mathcal{V}} |f(i) - m(f)|^p m(i) \right)^{1/p} &\leq 2 \inf_{c \in \mathbb{R}} \left(\sum_{i \in \mathcal{V}} |f(i) - c|^p m(i) \right)^{1/p} \\ &\leq 2 \left(\sum_{i \in \mathcal{V}} |f(i) - m_0(f)|^p m(i) \right)^{1/p} \\ &\leq 2K \left(\sum_{i \in \mathcal{V}} |f(i) - m_0(f)|^p m_0(i) \right)^{1/p} \\ &\leq 2PK^{1/p} \left(\sum_{\{i, j\} \in \mathcal{E}} |f(i) - f(j)|^p m_0(i, j) \right)^{1/p} \\ &\leq 2PK^{2/p} \left(\sum_{\{i, j\} \in \mathcal{E}} |f(i) - f(j)|^p m(i, j) \right)^{1/p}. \end{aligned}$$

This yields a discrete Poincaré inequality with a constant depending only on n, κ, δ, p and finishes the proof, thanks to (1.10). ■

The same pattern gives an analogous Sobolev inequality. We first recall a theorem of P. Maheux and L. Saloff-Coste ([MSC]) : in a complete Riemannian manifold with nonnegative Ricci curvature, every smooth function f on a ball $B(x, t)$ satisfies the Sobolev inequality

$$\left(\int_{B(x,t)} |f - f_{B(x,t)}|^{\frac{2n}{n-2}} d\text{vol} \right)^{\frac{n-2}{n}} \leq C(n) \frac{t^2}{\text{vol } B(x, t)^{\frac{2}{n}}} \int_{B(x,t)} |df|^2 d\text{vol}, \quad (14)$$

This Sobolev inequality on balls is an automatic consequence of the volume doubling property (9) and Poincaré inequality (11). We deduce the

Lemma 2.11 *Let M^n be a noncompact connected complete Riemannian manifold with nonnegative Ricci curvature and $n \geq 3$. Fix $R > 0$, $\kappa > 1$ and consider a connected Borel subset A of the annulus $B(o, \kappa R) \setminus B(o, R)$, $o \in M$. Then if we let A_δ be the δR -neighbourhood of A , with $0 < \delta < 1$, the following Sobolev inequality is true.*

$$\forall f \in C^\infty(A_\delta), \left(\int_A |f - f_A|^{\frac{2n}{n-2}} d\text{vol} \right)^{\frac{n-2}{n}} \leq C(n, \kappa, \delta) \frac{R^2}{\text{vol } B(o, R)^{\frac{2}{n}}} \int_{A_\delta} |df|^2 d\text{vol}.$$

Proof :

We just explain how to adapt the previous argument, using the same notation. We want to apply (1.10) for $p = 2$ and $k = n$, with the same good covering. Set $q = \frac{2n}{n-2}$. The discrete L^q Poincaré inequality we need is given by the previous proof. Now (10) gives for every i in I :

$$\frac{V(o, R)}{V(x_i, \delta R)} \leq \left(\frac{1 + \kappa}{\delta} \right)^n,$$

hence $V(x_i, 3s) \geq V(x_i, s) \geq C(n, \kappa, \delta)V(o, R)$, so that Saloff-Coste theorem (14) yields a continuous Sobolev-Neumann inequality for the pair of measures (vol, vol) :

$$\forall f \in C^\infty(V_i^*), \left(\int_{V_i} |f - f_{V_i}|^q d\text{vol} \right)^{\frac{2}{q}} \leq C(n, \kappa, \delta) R^2 V(o, R)^{-2/n} \int_{V_i^*} |df|^2 d\text{vol}, \quad (15)$$

and

$$\forall f \in C^\infty(V_i^\sharp), \left(\int_{V_i^*} |f - f_{V_i^*}|^q d\text{vol} \right)^{\frac{2}{q}} \leq C(n, \kappa, \delta) R^2 V(o, R)^{-2/n} \int_{V_i^\sharp} |df|^2 d\text{vol}. \quad (16)$$

(1.10) ends the proof. ■

Let us make a little remark. In the arguments above, we obtained discrete inequalities thanks to a finiteness argument. Indeed, we can make the constants (P in the proof of lemma 2.10) explicit, using the following proposition.

Proposition 2.12 *Consider a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with N_v vertices, endowed with the counting measure. Fix $p \geq 1$. Then for every real function f on \mathcal{V} ,*

$$\sup_{i \in \mathcal{V}} |f(i) - m(f)| \leq N_e^{1-1/p} \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p \right)^{1/p}$$

and in particular,

$$\sum_{i \in \mathcal{V}} |f(i) - m(f)|^p \leq N_v (N_v - 1)^{p-1} \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p.$$

Proof :

First, we can assume the graph is a tree : cutting off edges does not change the left-hand sides and makes the right-hand sides of the inequalities grow. Then, the graph has exactly $N_v - 1$ edges. Now, to each edge e we associate a copy I_e of the segment $[0, 1]$; the ends of I_e (corresponding to

0 and 1) can be viewed as two vertices in the graph \mathcal{G} . We then build a space X by gluing all I_e , in the natural way, that is, we decide that the ends of segments corresponding to the same vertex in \mathcal{G} give rise to one point in X . X is endowed with a natural topology and a natural Borel measure, stemming from those of $[0, 1]$. Note that the complement \tilde{X} of the points where two segments are glued together even possesses a natural differential structure, and a Riemannian metric. Given a function g which is continuous on X , C^1 on \tilde{X} and vanishes somewhere, we claim the following inequality is true :

$$\|g\|_{L^\infty(X)} \leq (N_v - 1)^{1-1/p} \|g'\|_{L^p(X)}. \quad (17)$$

Let us prove it. We choose x_0 such that $g(x_0) = 0$. Then, given a point x in the arcwise connected space X , we can find a unit speed path γ from x_0 to x which runs along each segment at most once. We can write $g(x) = \int_\gamma g'$ and use Hölder inequality :

$$|g(x)| \leq \text{length}(\gamma)^{1-1/p} \left(\int_\gamma |g'|^p \right)^{1/p} \leq (N_v - 1)^{1-1/p} \|g'\|_{L^p(X)}.$$

Given f in $\mathbb{R}^\mathcal{V}$, we can define a continuous function g on X in the following manner : g is linear on each segment I_e and its values at the ends of segments are simply those of f . Let e be an edge of the graph, between the vertices i and j , that we identify (respectively) with 0 and 1 in $[0, 1]$. The restriction of g on I_e can be identified with a function g_e defined on $[0, 1]$ by the formula

$$g_e(t) = f(i) + t(f(j) - f(i)).$$

Such a function g has a derivative g' which is defined outside the vertices and constant on the (image in X of the) interior of each I_e : $g'_e = f(j) - f(i)$. If f has zero mean value, we can apply (17) to the corresponding function, since g takes every value in the convex hull of the values of f and thus vanishes somewhere. Observing

$$\|g\|_{L^\infty(X)} = \|f\|_{L^\infty(\mathcal{V})}$$

and

$$\|g'\|_{L^p(X)} = \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p \right)^{1/p},$$

(17) yields the result. ■

Remark 2.13 *It is possible to give a discrete proof of this result. For instance, observing that for any real number c*

$$\left(\sum_{i \in \mathcal{V}} |f(i) - m(f)|^p \right)^{1/p} \leq 2 \left(\sum_{i \in \mathcal{V}} |f(i) - c|^p \right)^{1/p}$$

we can choose c so that $f - c$ vanishes at some vertex. It is then easy to adapt the argument above, keeping it completely discrete. But the constant we find that way is twice the one in the proposition.

2.3 The weighted Sobolev inequality.

In this paragraph, M is a connected complete Riemannian manifold, with dimension $n \geq 3$, non-negative Ricci curvature and satisfying (1) for some point o . We want to prove a weighted Sobolev inequality on M , by applying the theorem (1.8) for $p = 2$ and $k = n$ with a good covering that we design now.

2.3.1 A good covering

We fix some large κ , so as to be sure that, for any $R > 0$, any two connected components of $A(R, \kappa R)$ are contained in one connected component of $A(\kappa^{-1}R, \kappa R)$: this is made possible by (2.9). Recall κ can be chosen so that it depends only on n, C_o and ν . We also choose a ray starting from o and call it γ . We will sometimes use the notation $R_i := \kappa^i, i \in \mathbb{Z}$.

For every integer i , we denote by $U'_{i,a}$, $0 \leq a \leq h'_i$ the connected components of $A(R_{i-1}, R_i)$, $U'_{i,0}$ being the one which intersects γ . As in the proofs of 2.10 and 2.11, (10) provides a bound $h = h(n, \kappa) < \infty$ on the various h'_i , $i \in \mathbb{Z}$.

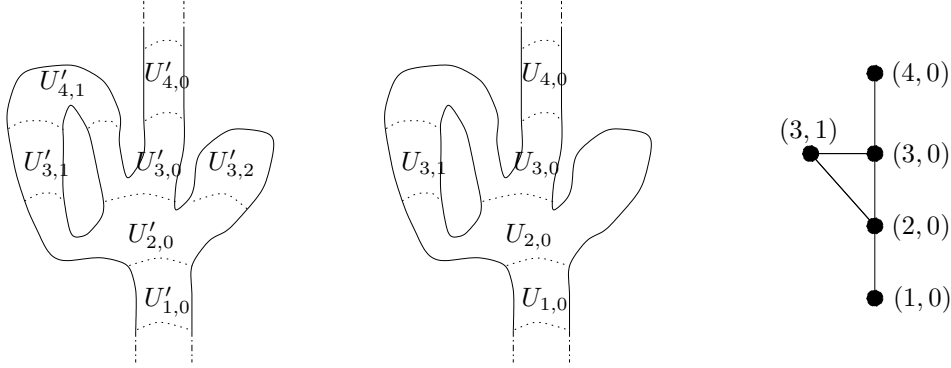
A priori, this will not yield a good covering because some of the $U'_{i,a}$ may be small compared to their neighbours, contradicting (v) in 1.1. This is the reason why we need to modify the covering slightly : we will glue every small component on the level i to a large one on the level $i - 1$. Let us explain what we mean precisely.

We proceed in two steps.

- First, we set $U_{i,a} = U'_{i,a}$ for every i in \mathbb{Z} and $1 \leq a \leq h'_i$ such that $\overline{U'_{i,a}}$ intersects $A(R_i, R_{i+1})$; every such $U_{i,a}$ contains a point x on the sphere $S((R_{i-1} + R_i)/2)$ and thus a ball centered in x and with radius R_{i-2} , whose volume is comparable to $V(R_i)$ (with (10)).
- Then we consider every (i, a) such that $\overline{U'_{i,a}} \cap A(R_i, R_{i+1})$ is empty. There is b in $[0, h'_{i-1}]$ such that $U'_{i,a} \cup U'_{i-1,b}$ is connected : we enlarge $U_{i-1,b}$ by adding $U'_{i,a}$ to it.

After deleting the indices which are not used any more, this yields a covering $(U_{i,a})$ of $M \setminus \{o\}$, indexed by $i \in \mathbb{Z}$ and $a \in [0, h_i]$, $h_i \leq h'_i$, with $U_{i,a} \subset A(R_{i-1}, R_{i+1})$ and $\text{vol } U_{i,a} \approx V(R_i)$.

Figure 2.3.1 gives an example : on the left, different connected components of annuli $A(R_{i-1}, R_i)$; in the center, the modified covering ; on the right, the associated graph.



For i in \mathbb{Z} and $0 \leq a \leq h_i$, we furthermore let $U^*_{i,a}$ be the union of all the $U_{j,b}$, $j \in \mathbb{Z}$, $0 \leq b \leq h_j$, whose closure intersects $\overline{U_{i,a}}$. And likewise, let $U^\sharp_{i,a}$ be the union of all the $U^*_{j,b}$, $j \in \mathbb{Z}$, $0 \leq b \leq h_j$, whose closure intersects $\overline{U^*_{i,a}}$.

We introduce the measure $d\mu_\rho = \rho(r)^{-\frac{2}{n-2}} d\text{vol}$, where $r = r_o = d(o, \cdot)$ and $\rho(t) = \rho_o(t)$ is defined for $t \geq 0$ by $\rho(t) = \frac{t^n}{V(t)}$. Bishop-Gromov theorem says it is a nondecreasing function and indeed, for $0 < s \leq t$, the following inequality is true :

$$1 \leq \frac{\rho(t)}{\rho(s)} \leq \left(\frac{t}{s}\right)^n. \quad (18)$$

Besides, $\rho(0) = \omega_n^{-1}$, where ω_n denotes the volume of the unit sphere in \mathbb{R}^n .

It is easy to see that $\mathcal{U} = (U_i, U^*_i, U^\sharp_i)$ is a good covering of M in M with respect to (μ_ρ, vol) : (v) is again a consequence of (10).

Let us prove the continuous and discrete Sobolev inequalities we need.

2.3.2 The continuous Sobolev inequality.

Lemma 2.14 *For every i in \mathbb{Z} and $0 \leq a \leq h_i$, each smooth function f on $U^\sharp_{i,a}$ satisfies*

$$\left(\int_{U_{i,a}} |f - f_{U_{i,a}}|^{\frac{2n}{n-2}} d\mu_\rho \right)^{\frac{n-2}{n}} \leq S_c \int_{U_{i,a}} |df|^2 d\text{vol}$$

and

$$\left(\int_{U^*_{i,a}} |f - f_{U^*_{i,a}}|^{\frac{2n}{n-2}} d\mu_\rho \right)^{\frac{n-2}{n}} \leq S_c \int_{U^*_{i,a}} |df|^2 d\text{vol},$$

with $S_c = S_c(n, \kappa)$.

Proof :

Set $q = \frac{2n}{n-2}$. For f in $C^\infty(U_{i,a}^\sharp)$ and i in \mathbb{Z} , we can write

$$\begin{aligned} \int_{U_{i,a}} |f - f_{U_{i,a},\mu_\rho}|^q d\mu_\rho &\leq 2^q \inf_{c \in \mathbb{R}} \int_{U_{i,a}} |f - c|^q d\mu_\rho \\ &\leq 2^q \int_{U_{i,a}} |f - f_{U_{i,a},\text{vol}}|^q d\mu_\rho, \end{aligned}$$

so that (2.11) (with some small $\delta : 0 < \delta < 1 - \kappa^{-1}$) and (18) imply

$$\begin{aligned} \left(\int_{U_{i,a}} |f - f_{U_{i,a}}|^q d\mu_\rho \right)^{2/q} &\leq \rho(R_{i-1})^{-2/n} C(n, \kappa) \rho(R_{i+1})^{2/n} \int_{U_{i,a}^*} |df|^2 d\text{vol} \\ &\leq C(n, \kappa) \kappa^{2n} \int_{U_{i,a}^*} |df|^2 d\text{vol} \\ &\leq C(n, \kappa) \int_{U_{i,a}^*} |df|^2 d\text{vol}. \end{aligned}$$

And such estimates with the pairs $(U_{i,a}^*, U_{i,a}^\sharp)$ also hold for the same reason. ■

2.3.3 The discrete Sobolev inequality.

We consider the weighted graph $(\mathcal{V}, \mathcal{E}, m_\rho)$ associated to the good covering \mathcal{U} of M in M , with respect to (μ_ρ, vol) (to simplify the notation, we write m_ρ instead of m_{μ_ρ}). What about the structure of the graph? Proposition 2.7, plus the fact that the geometry near o is quasi-Euclidian, implies the associated graph, outside a finite subset, consists of two trunks, corresponding to neighbourhoods of o and of infinity; moreover, thanks to the bound $h(n, \kappa)$ on the h_i , the degrees of the vertices admit an upper bound in terms of n and κ .

The measure m_ρ is defined as follows: for each i in \mathbb{Z} and a in $[0, h_i]$,

$$m_\rho(i, a) = \int_{U_{i,a}} \rho(r)^{-\frac{2}{n-2}} d\text{vol},$$

so that we can estimate:

$$\text{vol}(U_{i,a}) \rho(R_{i+1})^{-\frac{2}{n-2}} \leq m_\rho(i, a) \leq \text{vol}(U_{i,a}) \rho(R_{i-1})^{-\frac{2}{n-2}};$$

using (10) and (18), this yields

$$C(n, \kappa)^{-1} V(R_i) \rho(R_i)^{-\frac{2}{n-2}} \leq m_\rho(i, a) \leq C(n, \kappa) V(R_i) \rho(R_i)^{-\frac{2}{n-2}}. \quad (19)$$

In particular, again with (10) and (18), this allows us to apply proposition 1.12: we are left to show that an isoperimetric inequality (8) actually occurs.

Let Ω be a finite subset of \mathcal{V} . Set $l := \max\{i \in \mathbb{Z}, \exists a \in [0, h_i], (i, a) \in \Omega\}$. First, we choose a convenient edge in $\partial\Omega$.

- If $(l, 0)$ belongs to Ω , the edge $e := ((l, 0), (l+1, 0))$ is in $\partial\Omega$.
- Otherwise, we choose (l, b) in Ω . Our choice of κ ensures there is a sequence of edges staying on the levels l and $l-1$ and which connects (l, b) to $(l, 0)$. Among these, there is necessarily an edge which connects a vertex in Ω to a vertex outside Ω and we call it e : it belongs to $\partial\Omega$.

Then we can write

$$\frac{m_\rho(\Omega)}{m_\rho(\partial\Omega)} \leq \frac{\sum_{i=-\infty}^l \sum_{a=0}^{h_i} m_\rho(i, a)}{m_\rho(e)} \leq C(n, \kappa) \sum_{i=-\infty}^l \frac{\sum_{a=0}^{h_i} m_\rho(i, a)}{m_\rho(l, 0)}.$$

With (18), we find

$$\begin{aligned} \frac{m_\rho(\Omega)}{m_\rho(\partial\Omega)} &\leq C(n, \kappa) \sum_{i=-\infty}^l \frac{V(R_i)\rho(R_i)^{-\frac{2}{n-2}}}{V(R_i)\rho(R_i)^{-\frac{2}{n-2}}} \\ &= C(n, \kappa) \sum_{i=-\infty}^l \left[\frac{V(R_i)}{V(R_l)} \left(\frac{R_i}{R_l} \right)^{-2} \right]^{\frac{n}{n-2}} \end{aligned}$$

so that (1) gives

$$\begin{aligned} \frac{m_\rho(\Omega)}{m_\rho(\partial\Omega)} &\leq C(n, \kappa) C_o^{-\frac{n}{n-2}} \sum_{i=-\infty}^l \left(\frac{R_i}{R_l} \right)^{\frac{n(\nu-2)}{n-2}} \\ &= C(n, \kappa) C_o^{-\frac{n}{n-2}} \sum_{j=0}^{\infty} \kappa^{-j \frac{n(\nu-2)}{n-2}} \\ &= \frac{C(n, \kappa) C_o^{-\frac{n}{n-2}}}{1 - \kappa^{-\frac{n(\nu-2)}{n-2}}}, \end{aligned}$$

since $\nu > 2$. Then (1.12) and (1.14), with $k = \infty$, lead to the

Lemma 2.15 *For any $1 \leq p < \infty$, there exists a constant S_d , depending on p, κ, n, C_o, ν , such that for every real function f with finite support in \mathcal{V} :*

$$\left(\sum_{v \in \mathcal{V}} |f(v)|^p m_\rho(v) \right)^{\frac{1}{p}} \leq S_d \left(\sum_{(v,w) \in \mathcal{E}} |f(v) - f(w)|^p m_\rho(v,w) \right)^{\frac{1}{p}}.$$

2.3.4 Conclusion.

Theorem 2.16 (Weighted Sobolev inequality) *Let M^n , $n \geq 3$, be a connected complete Riemannian manifold with nonnegative Ricci curvature. Assume that there exists o in M , $\nu > 2$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu.$$

Then M satisfies the weighted Sobolev inequality

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \rho_o(r_o)^{-\frac{2}{n-2}} d\text{vol} \right)^{1-\frac{2}{n}} \leq S \int_M |df|^2 d\text{vol}.$$

Here, S can be chosen to depend only on n, C_o, ν .

Proof :

We just use 1.8, 2.14 and 2.15. ■

Remark 2.17 *Note 2.16 implies there is a constant \tilde{S} such that*

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \underline{r}_o^{-\frac{2(n-\nu)}{n-2}} d\text{vol} \right)^{1-\frac{2}{n}} \leq \tilde{S} \int_M |df|^2 d\text{vol},$$

where \underline{r}_o is the function which is equal to 1 inside $B(o, 1)$ and to r_o outside this ball (just use (3)). Observe we cannot write r_o instead of \underline{r}_o , unless $\nu = n$. The obstruction to do this for the Sobolev inequality is that locally the weight would not fit : the corresponding inequality is false on \mathbb{R}^n , hence on any Riemannian manifold (use the family of functions $\max(1 - r_o/\epsilon, 0)$, $\epsilon > 0$). Note also that \tilde{S} depends on n, C_o, ν and $V(o, 1)$.

Let us introduce some notation for the best constant in our inequality.

Definition 2.18 Let M^n be a connected complete Riemannian manifold, $n \geq 3$. For every o in M , we define the Riemannian invariant

$$S_o(M) := \sup_{f \in C_c^\infty(M) \setminus \{0\}} \frac{\left(\int_M |f|^{\frac{2n}{n-2}} \rho_o(r_o)^{-\frac{2}{n-2}} dvol \right)^{1-\frac{2}{n}}}{\int_M |df|^2 dvol}.$$

The same method gives the

Theorem 2.19 Let M^n , $n \geq 3$, be a connected noncompact complete Riemannian manifold with nonnegative Ricci curvature. Assume that there exists $o \in M$, $\nu > 1$ and $C_o > 0$ such that

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu.$$

Then, if $\beta > -\frac{\nu-2}{n-\nu}$, M satisfies the weighted Sobolev inequality

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \rho_o(r_o)^{\frac{n\beta-2}{n-2}} dvol \right)^{\frac{n-2}{n}} \leq S_\beta \int_M |df|^2 \rho_o(r_o)^\beta dvol,$$

with $S_\beta = S_\beta(n, C_o, \nu, \beta)$.

Proof :

We wish to apply (1.8) to the measures $\rho_o(r_o)^{\frac{n\beta-2}{n-2}} dvol$ and $\rho_o(r_o)^\beta dvol$ and the same good covering. Our choice of weights ensures the continuous Sobolev inequality, as in (2.14) : for i in \mathbb{Z} , a in $[0, h_i]$ and f in $C^\infty(U_i^*)$, (2.11) yields

$$\left(\int_{U_{i,a}} |f - f_{U_{i,a}}|^{\frac{2n}{n-2}} \rho_o(r_o)^{\frac{n\beta-2}{n-2}} dvol \right)^{1-2/n} \leq C(n, \kappa) \rho_o(R_i)^{\frac{n\beta-2}{n-2}} \rho_o(R_i)^{\frac{2}{n}} \int_{U_{i,a}^*} |df|^2 dvol$$

and the right hand side can be bounded by

$$C(n, \kappa) \rho_o(R_i)^{\frac{n\beta-2}{n-2}} \rho_o(R_i)^{\frac{2}{n}} \rho_o(R_i)^{-\beta} \int_{U_{i,a}^*} |df|^2 \rho_o(r_o)^\beta dvol = C(n, \kappa) \int_{U_{i,a}^*} |df|^2 \rho_o(r_o)^\beta dvol.$$

As for the discrete inequality, we proceed as in the proof of 2.15. Essentially, using the same notations as in this proof, we obtain

$$\frac{m(\Omega)}{m(\partial\Omega)} \leq C(n, \kappa) \sum_{i=-\infty}^l \frac{V(R_i) \rho(R_i)^{\frac{n\beta-2}{n-2}}}{V(R_l) \rho(R_l)^{\frac{n\beta-2}{n-2}}} = C(n, \kappa) \sum_{i=-\infty}^l \left[\left(\frac{V(R_i)}{V(R_l)} \right)^{1-\beta} \left(\frac{R_i}{R_l} \right)^{n\beta-2} \right]^{\frac{n}{n-2}}$$

so that (1) gives

$$\frac{m(\Omega)}{m(\partial\Omega)} \leq C(n, \kappa) C_o^{-\frac{n(1-\beta)}{n-2}} \sum_{j=0}^{\infty} \kappa^{-j \frac{n(\nu-2+\beta(n-\nu))}{n-2}}$$

which is finite thanks to our assumption on β . ■

Remark 2.20 If $\beta = 1$, the inequality reads

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \frac{r_o^n}{\text{vol } B(o, r)} dvol \right)^{\frac{n-2}{n}} \leq S \int_M |df|^2 \frac{r_o^n}{\text{vol } B(o, r)} dvol$$

The picture is the following : the volume growth of balls is in general not Euclidian (i.e. it does not behave like r^n) and therefore we cannot hope to find a nonweighted Sobolev inequality (cf. next paragraph) ; nevertheless, by radially modifying the Riemannian measure so that it has Euclidian growth, we manage to obtain a Sobolev inequality.

2.3.5 Weighted Sobolev inequality and volume growth.

The aim of this short paragraph is to explain what the weighted inequality (in the form given by remark 2.17) implies on the volume growth of balls. The following proposition generalizes a well known feature of the standard Sobolev inequality (without weight).

Proposition 2.21 *Let M^n , $n \geq 3$, be a connected noncompact complete Riemannian manifold with nonnegative Ricci curvature. Assume that there exists o in M , $\alpha \geq 0$ and $S > 0$ such that*

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} r_o^{-\alpha} dvol \right)^{\frac{n-2}{n}} \leq S \int_M |df|^2 dvol.$$

Then there is a constant $A_o > 0$ such that for $t \geq 1$, $\text{vol} B(o, t) \geq A_o t^\nu$, where ν is the real number defined by $\alpha = 2\frac{n-\nu}{n-2}$.

Remark 2.22 *In case there is no weight ($\alpha = 0$), we recover the fact that the usual Sobolev inequality requires a volume growth of balls which is at least Euclidian.*

Proof :

As usual, we set $q = 2n/(n-2) > 2$. Then we fix $R \geq 2$ and $0 < t \leq R/2$ and consider the Lipschitz function $f := \max(t - d(\cdot, S(o, R)), 0) : f = t$ on the sphere $S(o, R)$, $f = 0$ outside some t -neighbourhood of this sphere and, on this t -neighborhood, it decreases radially at unit speed. Thus

$$\int_M |f|^q r^{-\alpha} dvol \geq (t/2)^q (R+t)^{-\alpha} \text{vol}(A(R-t/2, R+t/2))$$

and

$$\int_M |df|^2 dvol \leq \text{vol}(A(R-t, R+t)).$$

The Sobolev inequality yields :

$$(t/2)^2 (R+t)^{-2\alpha/q} \text{vol}(A(R-t/2, R+t/2))^{2/q} \leq S \text{vol}(A(R-t, R+t)).$$

For $i \in \mathbb{N}^*$, we apply this to $t = 2^{-i}R$. With $V_i := \text{vol}(A(R(1-2^{-i}), R(1+2^{-i})))$,

$$R^2 4^{-i-1} ((1+2^{-i})R)^{-2\alpha/q} V_{i+1}^{2/q} \leq S V_i.$$

By induction, there is a constant C which does not depend on R such that for every $i \geq 1$

$$\text{vol}(B(2R)) \geq V_1 \geq \left(C R^{2-2\alpha/q} \right)^{\sum_{j=0}^{i-1} (2/q)^j} \left(\prod_{j=0}^{i-1} (4^{-j})^{(2/q)^j} \right) V_i.$$

As a Riemannian manifold is locally quasi-Euclidian,

$$\liminf_{i \rightarrow \infty} V_i^{(2/q)^i} \geq \liminf_{i \rightarrow \infty} (\omega_n (2^{-i}R)^n)^{(2/q)^i} = 1.$$

Eventually,

$$\text{vol}(B(2R)) \geq C^{\frac{1}{1-2/q}} R^{\frac{2-2\alpha/q}{1-2/q}} \prod_{j=0}^{\infty} (4^{-j})^{(2/q)^j}.$$

And indeed, $\nu = \frac{2-2\alpha/q}{1-2/q}$ is the same as $\alpha = 2\frac{n-\nu}{n-2}$. ■

2.4 The Hardy inequality.

With 1.8, we can also patch local Poincaré inequalities together. Working under the same assumptions as above, the global inequality we find is a Hardy inequality.

Theorem 2.23 (Hardy inequality) *Let M^n , $n \geq 3$, be a connected noncompact complete Riemannian manifold with nonnegative Ricci curvature. Fix some $p \geq 1$. Assume that there exists o in M , $\nu > p$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol} B(o, t)}{\text{vol} B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu.$$

Then M satisfies the Hardy inequality

$$\forall f \in C_c^\infty(M), \int_M |f|^p r_o^{-p} d\text{vol} \leq H \int_M |df|^p d\text{vol},$$

with a constant H depending only on n, C_o, ν, p .

Proof :

The proof consists in applying 1.8 with $k = \infty$. We will use the same "good" covering \mathcal{U} as in paragraph 2.3.1, noticing it is also "good" for the pair of measures $(r^{-p} d\text{vol}, d\text{vol})$. We need a continuous Poincaré inequality. Indeed, as for 2.14, if we choose i in \mathbb{Z} and a in $[0, h_i]$, each smooth function f on $U_{i,a}^\sharp$ satisfies

$$\begin{aligned} \int_{U_{i,a}} |f - f_{U_{i,a}, \mu_\alpha}|^p r^{-p} d\text{vol} &\leq 2^p \inf_{c \in \mathbb{R}} \int_{U_{i,a}} |f - c|^p r^{-p} d\text{vol} \\ &\leq 2^p \int_{U_{i,a}} |f - f_{U_{i,a}}|^p r^{-p} d\text{vol} \end{aligned}$$

so that, with 2.10,

$$\begin{aligned} \int_{U_{i,a}} |f - f_{U_{i,a}, \mu_\alpha}|^p r^{-p} d\text{vol} &\leq 2^p C(n, \kappa) R_{i-1}^{-p} R_{i+1}^p \int_{U_{i,a}^*} |df|^p d\text{vol} \\ &\leq C(n, \kappa, p) \int_{U_{i,a}^*} |df|^p d\text{vol}. \end{aligned}$$

And the same argument works with the pairs $(U_{i,a}^*, U_{i,a}^\sharp)$.

The discrete inequality required in 1.8 follows from the argument of 2.15 ; here, we estimate the discrete isoperimetric quotient by

$$C(n, \kappa) \sum_{i=-\infty}^l \frac{V(R_i)}{V(R_l)} \left(\frac{R_i}{R_l} \right)^{-p}$$

which is bounded by

$$C(n, \kappa) C_o \sum_{j=0}^{\infty} \kappa^{-j(\nu-p)} < \infty$$

thanks to our assumption on the volume growth of balls. ■

For convenience, we give a name to the best constant in the L^1 Hardy inequality.

Definition 2.24 *Let M be a connected complete Riemannian manifold. Given o in M and $r_o := d(o, \cdot)$, we define the Riemannian invariant*

$$H_o(M) := \sup_{f \in C_c^\infty(M) \setminus \{0\}} \frac{\int_M |f| r_o^{-1} d\text{vol}}{\int_M |df| d\text{vol}}.$$

3 Weighted Sobolev inequalities and Schrödinger operators.

In this section, we explain a few analytical consequences of the weighted Sobolev inequality. They will find geometric applications in the last section. We assume here that M^n is a connected

complete noncompact manifold with nonnegative Ricci curvature and such that for some point o in M and $S > 0$, the following weighted Sobolev inequality is true :

$$\forall f \in C_c^\infty(M), \left(\int_M |f|^{\frac{2n}{n-2}} \rho_o(r_o)^{-\frac{2}{n-2}} dvol \right)^{1-\frac{2}{n}} \leq S \int_M |df|^2 dvol.$$

As previously, we will often write $\rho(r)$ for $\rho_o(r_o) = \frac{r_o^n}{V(r_o)}$, but also $d\mu_\rho = \rho(r)^{-\frac{2}{n-2}} dvol$ and $q = \frac{2n}{n-2}$. We only need the assumption $\text{Ric} \geq 0$ in 3.4 and 3.6 (to ensure the volume doubling property).

We consider a smooth Euclidian vector bundle $E \rightarrow M$, endowed with a compatible connection ∇ . We will always denote by (\cdot) the pointwise scalar product on a Euclidian vector bundle, by $|\cdot|$ the pointwise norm, by $\overline{\Delta} = \nabla^* \nabla$ the Bochner laplacian (or "rough laplacian"). Our interest lies in Schrödinger operators $\overline{\Delta} + V$, where V is a continuous field of symmetric endomorphisms of E . We decompose V as $V = V_+ - V_-$, where V_+ and V_- are fields of positive symmetric endomorphisms of E . We describe here some consequences of the weighted Sobolev inequality on these operators.

3.1 A vanishing theorem.

The following theorem is a generalization of [Car1]. It says there are no nontrivial sections which are small at infinity and subharmonic for Schrödinger operators with nearly nonnegative potential.

Theorem 3.1 (Vanishing theorem) *Fix $m > 1$ and assume the potential V satisfies*

$$S \left(\int_M |V_-|^{\frac{n}{2}} \rho(r) dvol \right)^{\frac{2}{n}} < \frac{2}{m} \left(2 - \frac{2}{m} \right)$$

Then every locally Lipschitz section σ of E such that $(\overline{\Delta}\sigma + V\sigma, \sigma) \leq 0$ and

$$\int_{A(R/2, R)} |\sigma|^m dvol = o(R^2)$$

is identically zero.

Remark 3.2 *In this statement, the distribution $(\overline{\Delta}\sigma, \sigma)$ is defined by :*

$$\forall \phi \in C_c^\infty(M), (\overline{\Delta}\sigma, \sigma), \phi > = \int_M (\nabla\sigma, \nabla(\phi\sigma)) dvol.$$

Proof :

Let R be a positive number and let χ be a smooth function which is equal to 1 on $B(R)$, to 0 on $M \setminus B(2R)$, takes its values in $[0, 1]$ and satisfies $|d\chi| \leq 2/R$. We apply the weighted Sobolev inequality to the locally lipschitz function $\chi u_\epsilon^{m/2}$, where $u_\epsilon = \sqrt{|\sigma|^2 + \epsilon}$, $\epsilon > 0$ (we omit the riemannian measure in the next formulas so as to make them easier to read) :

$$\begin{aligned} \frac{1}{S} \left(\int_M \chi^q u_\epsilon^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} &\leq \int_M |d(\chi u_\epsilon^{m/2})|^2 \\ &= \int_M |d\chi|^2 u_\epsilon^m + \int_M \chi^2 |d(u_\epsilon^{m/2})|^2 + 2 \int_M (u_\epsilon^{m/2} d\chi, \chi d(u_\epsilon^{m/2})) \\ &\leq (1 + 1/b) \int_M |d\chi|^2 u_\epsilon^m + (1 + b) \int_M \chi^2 |d(u_\epsilon^{m/2})|^2 \end{aligned}$$

for any $b > 0$. Integration by parts yields

$$\begin{aligned}
\int_M \chi^2 \left| d(u_\epsilon^{m/2}) \right|^2 &= \int_M (\chi^2 d(u_\epsilon^{m/2}), d(u_\epsilon^{m/2})) \\
&= \int_M 2\chi(u_\epsilon^{m/2} d\chi, d(u_\epsilon^{m/2})) + \int_M \chi^2 u_\epsilon^{m/2} \Delta(u_\epsilon^{m/2}) \\
&= 2 \int_M (u_\epsilon^{m/2} d\chi, \chi d(u_\epsilon^{m/2})) + \frac{m}{2} \int_M \chi^2 u_\epsilon^{m-1} \Delta u_\epsilon \\
&\quad + \left(\frac{2}{m} - 1 \right) \int_M \chi^2 \left| d(u_\epsilon^{m/2}) \right|^2.
\end{aligned}$$

So, if $a > 0$, we find

$$\int_M \chi^2 \left| d(u_\epsilon^{m/2}) \right|^2 \leq \left(\frac{2}{m} - 1 + a \right) \int_M \chi^2 \left| d(u_\epsilon^{m/2}) \right|^2 + \frac{m}{2} \int_M \chi^2 u_\epsilon^{m-1} \Delta u_\epsilon + \frac{1}{a} \int_M |d\chi|^2 u_\epsilon^m.$$

If moreover $a < 2 - 2/m$, we obtain

$$\int_M \chi^2 \left| d(u_\epsilon^{m/2}) \right|^2 \leq \left(2 - \frac{2}{m} - a \right)^{-1} \left(\frac{m}{2} \int_M \chi^2 u_\epsilon^{m-1} \Delta u_\epsilon + \frac{1}{a} \int_M |d\chi|^2 u_\epsilon^m \right).$$

Hence :

$$\frac{1}{S} \left(\int_M \chi^q u_\epsilon^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C(m, a, b) \int_M |d\chi|^2 u_\epsilon^m + D(m, a, b) \int_M \chi^2 u_\epsilon^{m-1} \Delta u_\epsilon$$

where

$$C(m, a, b) = 1 + 1/b + \frac{1+b}{a(2-2/m-a)}$$

and

$$D(m, a, b) = \frac{(1+b)m}{2(2-2/m-a)}.$$

We compute

$$u_\epsilon \Delta u_\epsilon = (\sigma, \overline{\Delta \sigma}) - \frac{\epsilon |\nabla \sigma|^2}{u_\epsilon^2} - \frac{|\sigma|^2 |\nabla \sigma|^2 - (\sigma, \nabla \sigma)^2}{u_\epsilon^2},$$

to ensure

$$u_\epsilon \Delta u_\epsilon \leq (\sigma, \overline{\Delta \sigma}) \leq (V_- \sigma, \sigma) \leq |V_-| u_\epsilon^2.$$

Therefore, we can write

$$\frac{1}{S} \left(\int_M \chi^q u_\epsilon^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C(m, a, b) \int_M |d\chi|^2 u_\epsilon^m + D(m, a, b) \int_M \chi^2 |V_-| u_\epsilon^m$$

so that, when ϵ goes to zero, we find

$$\frac{1}{S} \left(\int_M \chi^q |\sigma|^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C(m, a, b) \int_M |d\chi|^2 |\sigma|^m + D(m, a, b) \int_M \chi^2 |V_-| |\sigma|^m.$$

Hölder inequality implies

$$\begin{aligned}
\int_M \chi^2 |V_-| |\sigma|^m &\leq \int_M \chi^2 |\sigma|^m \rho(r)^{-\frac{2}{n}} |V_-| \rho(r)^{\frac{2}{n}} \\
&\leq \left(\int_M \chi^q |\sigma|^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \underbrace{\left(\int_M |V_-|^{\frac{n}{2}} \rho(r) \right)^{\frac{2}{n}}}_{N_V}.
\end{aligned}$$

All in all, we find

$$(1 - SN_V D(m, a, b)) \left(\int_M \chi^q |\sigma|^{\frac{mq}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq \frac{4SC(m, a, b)}{R^2} \int_{A(R, 2R)} |\sigma|^2$$

and the assumption on the potential allows us to write

$$\left(\int_{B(R)} |\sigma|^{mq/2} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq \frac{1}{1 - SN_V D(m, a, b)} \frac{4SC(m, a, b)}{R^2} \int_{A(R, 2R)} |\sigma|^m,$$

providing

$$N_V < \frac{1}{SD(m, a, b)} = \frac{2}{mS} \frac{1}{1+b} (2 - 2/m - a),$$

which, under our assumption on V , can always be achieved by choosing sufficiently small a and b . Letting R go to infinity, we obtain $\sigma = 0$. ■

3.2 Some general decay estimates.

Now what can we say if we only have $\int_M |V_-|^{\frac{n}{2}} \rho(r) dvol < \infty$? Adapting a technique developed in [BKN], we can prove some decay estimates on the sections σ such that $\bar{\Delta}\sigma + V\sigma \leq 0$. We prove three general lemmas and we will see later (cf. 4.2) how to apply them in a geometric setting, where the potential and the section σ are related. The idea is to implement a Moser iteration : this is the third lemma. But this lemma only works under a technical assumption on the potential, which can be ensured by the first lemma. Finally, the second lemma is a key to a "self-improvement" of the decay estimate we will find.

Lemma 3.3 (Initiation) *We assume V_- belongs to $L^{n/2}(E, \rho(r) dvol)$ and we consider a locally Lipschitz section σ of E such that $(\sigma, \bar{\Delta}\sigma + V\sigma) \leq 0$ and for some $m > 1$:*

$$\int_{A(R, 2R)} |\sigma|^m dvol = o(R^2).$$

Then for large R :

$$\left(\int_{M \setminus B(2R)} |\sigma|^{\frac{mq}{2}} d\mu_\rho \right)^{\frac{2}{q}} \leq \frac{C}{R^2} \int_{A(R, 2R)} |\sigma|^m dvol.$$

Proof :

Proceeding as in the proof of the vanishing theorem, we find for $u := |\sigma|^{m/2}$ and χ in $C_c^\infty(M)$:

$$\left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C \left(\int_M \chi^2 u^2 |V_-| + \int_M |d\chi|^2 u^2 \right).$$

Using Hölder inequality, this yields :

$$\left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C \left(\int_{\text{supp } \chi} |V_-|^{\frac{n}{2}} \rho(r) \right)^{\frac{2}{n}} \left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} + C \int_M |d\chi|^2 u^2.$$

Now we set $R \gg 1$, $R' > 2R$ and we choose χ with values in $[0, 1]$ and support in $A(R, 2R')$, with value 1 on $[2R, R']$, satisfying $|d\chi| \leq \frac{2}{R}$ on $A(R, 2R)$ and $|d\chi| \leq \frac{2}{R'}$ on $A(R', 2R')$. Thus :

$$\begin{aligned} \left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} &\leq C \left(\int_{A(R, 2R')} |V_-|^{\frac{n}{2}} \rho(r) \right)^{\frac{2}{n}} \left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \\ &\quad + \frac{C}{R^2} \int_{A(R, 2R)} u^2 + \frac{C}{R'^2} \int_{A(R', 2R')} u^2. \end{aligned}$$

By assumption, the integral $\int_M |V_-|^{n/2} \rho(r)$ is finite : we can make $\int_{B(R)^c} |V_-|^{n/2} \rho(r)$ as small as we like by choosing a large R , so that we obtain the estimate :

$$\left(\int_{A(2R, R')} u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq \frac{C}{R^2} \int_{A(R, 2R)} u^2 + \frac{C}{R'^2} \int_{A(R', 2R')} u^2.$$

Letting R' go to infinity, we find :

$$\left(\int_{M \setminus B(2R)} u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq \frac{C}{R^2} \int_{A(R, 2R)} u^2.$$

■

Lemma 3.4 (Key to the self-improvement) *We assume V_- belongs to $L^{n/2}(E, \rho(r) d\text{vol})$ and we consider a locally Lipschitz section σ of E which belongs to $L^m(E, d\mu_\rho)$ for some $m > q/2$ and satisfies $(\sigma, \bar{\Delta}\sigma + V\sigma) \leq 0$. Then for large R :*

$$\int_{M \setminus B(2R)} |\sigma|^m d\mu_\rho \leq C \int_{A(R, 2R)} |\sigma|^m d\mu_\rho.$$

As a consequence, there exists a positive number a such that

$$\int_{M \setminus B(R)} |\sigma|^m d\mu_\rho = \mathcal{O}(R^{-a}).$$

Remark 3.5 *The proof will show that a can be chosen so that it depends continuously on m .*

Proof :

Set $m' = 2m/q$. The preceding proof says that for large R , with the same truncature function χ and $u := |\sigma|^{m'/2}$:

$$\left(\int_M \chi^q u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C \int_M |d\chi|^2 u^2.$$

We use Hölder inequality to obtain :

$$\left(\int_{A(2R, R')} u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C \left(\int_M |d\chi|^n \rho(r) \right)^{\frac{2}{n}} \left(\int_{\text{supp } d\chi} u^q \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}}.$$

From the definition of ρ and the volume doubling property, we see that

$$\int_{A(R, 2R)} |d\chi|^n \rho(r) \leq CR^{-n} \rho(2R) \text{vol } A(R, 2R) \leq C$$

and

$$\int_{A(R', 2R')} |d\chi|^n \rho(r) \leq C,$$

so that we obtain

$$\left(\int_{A(2R, R')} |\sigma|^m \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C \left(\int_{A(R, 2R) \cup A(R', 2R')} |\sigma|^m \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}}.$$

Letting R' go to infinity, we find the first part of the claim :

$$\int_{M \setminus B(2R)} |\sigma|^m \rho(r)^{-\frac{2}{n-2}} \leq C \int_{A(R, 2R)} |\sigma|^m \rho(r)^{-\frac{2}{n-2}}.$$

Now, if we consider the function $I : R \rightarrow \int_{M \setminus B(R)} |\sigma|^m \rho(r)^{-\frac{2}{n-2}}$, we see that it satisfies the inequality $I(2R) \leq C(I(R) - I(2R))$, i.e.

$$I(2R) \leq \frac{C}{C+1} I(R). \tag{20}$$

Fix a large R_1 and denote by k_R the integer such that $\log_2 R/R_1 \leq k_R < \log_2 2R/R_1$. Iterating the inequality (20), we find

$$I(R) \leq \left(\frac{C}{C+1} \right)^{k_R} I(R/2^{k_R}) \leq \left(\frac{C}{C+1} \right)^{k_R} \|\sigma\|_{L^m(E, \mu_\rho)}^m,$$

hence :

$$I(R) \leq C \left(\frac{C}{C+1} \right)^{\log_2 R} = CR^{\log_2(\frac{C}{C+1})}.$$

Since $\frac{C}{C+1} < 1$, the second statement follows. ■

Lemma 3.6 (Moser iteration) *We assume the potential V satisfies for some $x > n/2$:*

$$\left(\int_{A(R, 2R)} |V_-|^x \rho(r)^{\frac{x-1}{n/2-1}} d\text{vol} \right)^{\frac{1}{x-n/2}} = O\left(\rho(R)^{\frac{2}{n-2}} R^{-2}\right)$$

Consider a locally Lipschitz section σ which belongs to $L^m(M, \mu_\rho)$ for some $m > 1$ and satisfy $(\sigma, \bar{\Delta}\sigma + V\sigma) \leq 0$. Then the following estimate holds for large R :

$$\sup_{A(R, 2R)} |\sigma| \leq C \left(\rho(R)^{\frac{2}{n-2}} R^{-2} \right)^{\frac{n}{2m}} \left(\int_{A(R/2, 5R/2)} |\sigma|^m d\mu_\rho \right)^{1/m}.$$

Proof :

Fix $\beta \geq m$. In this proof, C will denote a constant which does not depend on β . Again with the same technique, one sees that if χ belongs to $C_c^\infty(M)$, the following estimate holds :

$$\left(\int_M \chi^q |\sigma|^{\frac{q\beta}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} \leq C\beta \int_M \chi^2 |\sigma|^\beta |V_-| + C \int_M |d\chi|^2 |\sigma|^\beta. \quad (21)$$

If the numbers t and s are defined by

$$\frac{1}{x} + \frac{1}{s} + \frac{1}{t} = 1 \quad \text{and} \quad \frac{q}{2s} + \frac{1}{t} = 1, \quad (22)$$

Hölder inequality gives :

$$\begin{aligned} & \beta \int_M \chi^2 |\sigma|^\beta |V_-| \\ & \leq \beta \left(\int_{\text{supp } \chi} |V_-|^x \rho(r)^{\frac{x-1}{n/2-1}} \right)^{\frac{1}{x}} \left(\int_M \chi^q |\sigma|^{\frac{q\beta}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{1}{s}} \left(\int_M \chi^2 |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{1}{t}}. \end{aligned}$$

Note $t = \frac{x}{x-n/2}$. Given $\epsilon > 0$, Young inequality and (22) yield a constant C_ϵ such that

$$\begin{aligned} & \beta \int_M \chi^2 |\sigma|^\beta |V_-| \\ & \leq \epsilon \left(\int_M \chi^q |\sigma|^{\frac{q\beta}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} + C_\epsilon \beta^t \left(\int_{\text{supp } \chi} |V_-|^x \rho(r)^{\frac{x-1}{n/2-1}} \right)^{\frac{1}{x}} \left(\int_M \chi^2 |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}} \right). \end{aligned}$$

Consequently, for small ϵ (regardless of β), we obtain in (21) :

$$\begin{aligned} \left(\int_M \chi^q |\sigma|^{\frac{q\beta}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} & \leq C\beta^t \left(\int_{\text{supp } \chi} |V_-|^x \rho(r)^{\frac{x-1}{n/2-1}} \right)^{\frac{1}{x}} \left(\int_M \chi^2 |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}} \right) \\ & + C \int_M |d\chi|^2 |\sigma|^\beta. \end{aligned}$$

Now, given large $R_1 < R_2 < 5R_1$ and $0 < \delta \leq R_1/2$, we consider a truncature function χ with the following properties : it takes its values in $[0, 1]$, it is equal to 1 on $A(R_1, R_2)$, it is equal to 0 outside $A(R_1 - \delta, R_2 + \delta)$ and its differential is bounded by $2/\delta$. Our assumption, thanks to (18), implies

$$\left(\int_{A(R_1-\delta, R_2+\delta)} |V_-|^x \rho(r)^{\frac{x-1}{n/2-1}} \right)^{\frac{1}{x}} \leq C \rho(R_1 - \delta)^{\frac{2}{n-2}} (R_1 - \delta)^{-2}$$

With this in mind, our estimate gives

$$\begin{aligned} \left(\int_{A(R_1, R_2)} |\sigma|^{\frac{q\beta}{2}} \rho(r)^{-\frac{2}{n-2}} \right)^{\frac{2}{q}} &\leq C \beta^t \rho(R_1 - \delta)^{\frac{2}{n-2}} (R_1 - \delta)^{-2} \int_{A(R_1-\delta, R_2+\delta)} |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}} \\ &+ C \rho(R_2 + \delta)^{\frac{2}{n-2}} \delta^{-2} \int_{A(R_1-\delta, R_2+\delta)} |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}} \\ &\leq C \beta^t \rho(R_2)^{\frac{2}{n-2}} \delta^{-2} \int_{A(R_1-\delta, R_2+\delta)} |\sigma|^\beta \rho(r)^{-\frac{2}{n-2}}, \end{aligned}$$

so that, with respect to the measure μ_ρ ,

$$\|\sigma\|_{L^{\beta q/2}(A(R_1, R_2))} \leq \left(C \beta^t \rho(R_2)^{\frac{2}{n-2}} \delta^{-2} \right)^{1/\beta} \|\sigma\|_{L^\beta(A(R_1-\delta, R_2+\delta))}. \quad (23)$$

Given some large $R > 0$, we set for every non negative integer k :

$$\beta_k := m \left(\frac{q}{2} \right)^k, \quad \delta_k := 2^{-k-1} R, \quad R_{1,k} := R - \sum_{i=1}^k \delta_i, \quad R_{2,k} := 2R + \sum_{i=1}^k \delta_i.$$

Iterating (23), we find $\|\sigma\|_{L^{\beta_k}(A(R, 2R))} \leq C_k \|\sigma\|_{L^{\beta_0}(A(R_{1,k}, R_{2,k}))}$ where the constant C_k is estimated by

$$C_k \leq \prod_{i=0}^{k-1} \left(C \beta_i^t \rho(R)^{\frac{2}{n-2}} R^{-2} 4^i \right)^{1/\beta_i} \leq \left(C \rho(R)^{\frac{2}{n-2}} R^{-2} \right)^{\sum_{i=0}^{k-1} 1/\beta_i} (4(q/2)^t)^{\sum_{i=0}^{k-1} i/\beta_i}.$$

Since $\sum_{i=0}^{\infty} \frac{1}{\beta_i} = \frac{n}{2m}$ et $\sum_{i=0}^{\infty} \frac{i}{\beta_i} < \infty$, this implies $\limsup_{k \rightarrow \infty} C_k \leq C \left(\rho(R)^{\frac{2}{n-2}} R^{-2} \right)^{\frac{n}{2m}}$, so that

$$\sup_{A(R, 2R)} |\sigma| = \lim_{k \rightarrow \infty} \|\sigma\|_{L^{\beta_k}(A(R, 2R))} \leq C \left(\rho(R)^{\frac{2}{n-2}} R^{-2} \right)^{\frac{n}{2m}} \|\sigma\|_{L^m(A(R/2, 5R/2))}.$$

■

Let us carry on our study of general Schrödinger operators. We wish to point out a Gagliardo-Nirenberg type inequality, which will prove useful later.

3.3 The inversion of Schrödinger operators.

Our purpose is to solve $(\bar{\Delta} + V)\sigma = \tau$ with a convenient τ and to obtain bounded solutions. First, the weighted Sobolev inequality easily yields the

Lemma 3.7 *For $s \geq \frac{2n}{n+2}$, there exists a constant $C(n, s)$ such that for every section σ in $C_c^\infty(E)$,*

$$\|\sigma\|_{L^{\frac{ns}{n-2s}}(E, \mu_\rho)} \leq C(n, s) S \|\bar{\Delta}\sigma\|_{L^s(E, \rho(r)^{\frac{s-1}{n/2-1}} dvol)}.$$

Proof :

Set $k = \frac{s}{n-2s} \frac{n-2}{2} \geq 1$ and fix σ in $C_c^\infty(E)$. The weighted Sobolev inequality gives

$$\frac{1}{S} \|\sigma\|_{L^{\frac{2k}{n-2s}}(E, \mu_\rho)}^{2k} \leq \int_M |d(|\sigma|^k)|^2 = \int_M |\sigma|^k \Delta(|\sigma|^k) \leq k \int_M |\sigma|^{2k-1} \Delta|\sigma|.$$

Kato inequality then implies

$$\frac{1}{S} \|\sigma\|_{L^{\frac{2k}{n-2s}}(E, \mu_\rho)}^{2k} \leq k \int_M |\sigma|^{2k-1} |\overline{\Delta}\sigma| = k \int_M |\sigma|^{\frac{n(s-1)}{n-2s}} |\overline{\Delta}\sigma|$$

and Hölder inequality yields

$$\frac{1}{S} \|\sigma\|_{L^{\frac{2k}{n-2s}}(E, \mu_\rho)}^{2k} \leq k \left(\int_M |\overline{\Delta}\sigma|^s \rho(r)^{\frac{s-1}{n/2-1}} d\text{vol} \right)^{1/s} \left(\int_M |\sigma|^{\frac{n s}{n-2s}} d\mu_\rho \right)^{1-1/s}$$

so that eventually

$$\frac{1}{S} \|\sigma\|_{L^{\frac{n s}{n-2s}}(E, \mu_\rho)} \leq k \left(\int_M |\overline{\Delta}\sigma|^s \rho(r)^{\frac{s-1}{n/2-1}} d\text{vol} \right)^{1/s},$$

which is indeed the claim. ■

We now use a Moser iteration to obtain a L^∞ estimate.

Lemma 3.8 *For every $t > n/2$ and $x \geq 1$, there exists a constant $C(n, x, t)$ such that for every section σ in $C_c^\infty(E)$,*

$$\|\sigma\|_{L^\infty(E)} \leq C(n, x, t) \left(S^{\frac{tn}{2t-n}} \|\overline{\Delta}\sigma\|_{L^t(E, \rho(r)^{\frac{t-1}{n/2-1}} d\text{vol})}^{\frac{tn}{2t-n}} \|\sigma\|_{L^x(E, \mu_\rho)}^x \right)^{\frac{1}{2t-n+x}}.$$

Proof :

As above, for every σ in $C_c^\infty(E)$ and every $k \geq 1$:

$$\left(\int_M |\sigma|^{kq} d\mu_\rho \right)^{2/q} \leq kS \int_M |\sigma|^{2k-1} |\overline{\Delta}\sigma| d\text{vol}.$$

Using Hölder inequality, we deduce:

$$\left(\int_M |\sigma|^{kq} d\mu_\rho \right)^{2/q} \leq kS \left(\int_M |\overline{\Delta}\sigma|^t \rho(r)^{\frac{t-1}{n/2-1}} d\text{vol} \right)^{1/t} \left(\int_M |\sigma|^{\frac{(2k-1)t}{t-1}} d\mu_\rho \right)^{1-1/t}.$$

Define the sequence (β_i) by $\beta_0 = x$ and $\beta_{i+1} = \frac{q}{2} \left(\frac{t-1}{t} \beta_i + 1 \right)$. For every nonnegative integer i , the following holds :

$$\|\sigma\|_{L^{\beta_{i+1}}(E, \mu_\rho)}^{\beta_{i+1}} \leq (q^{-1} \beta_{i+1} N_t)^{\frac{q}{2}} \left(\|\sigma\|_{L^{\beta_i}(E, \mu_\rho)}^{\beta_i} \right)^\zeta,$$

where

$$N_t = S \left(\int_M |\overline{\Delta}\sigma|^t \rho(r)^{\frac{t-1}{n/2-1}} d\text{vol} \right)^{1/t} \quad \text{and} \quad \zeta = \frac{q(t-1)}{2t} > 1.$$

Iterating this, we see that for every nonnegative integer i ,

$$\|\sigma\|_{L^{\beta_i}(E, \mu_\rho)}^{\beta_i} \leq (q^{-1} N_t)^{\frac{q}{2} \sum_{j=0}^{i-1} \zeta^j} \left(\prod_{j=1}^i \beta_j^{\zeta^{i-j}} \right)^{q/2} \left(\|\sigma\|_{L^{\beta_0}(E, \mu_\rho)}^{\beta_0} \right)^{\zeta^i}.$$

Thus

$$\|\sigma\|_{L^{\beta_i}(E, \mu_\rho)} \leq (q^{-1} N_t)^{\frac{q}{2\beta_i} \frac{\zeta^i - 1}{\zeta - 1}} \left(\prod_{j=1}^i \beta_j^{\zeta^{-j}} \right)^{\frac{q\zeta^i}{2\beta_i}} \left(\|\sigma\|_{L^{\beta_0}(E, \mu_\rho)}^{\beta_0} \right)^{\frac{\zeta^i}{\beta_i}}.$$

Using $\beta_i = \zeta^i \left(\beta_0 + \frac{q}{2(\zeta-1)} \right) - \frac{q}{2(\zeta-1)}$, we see that $\frac{\zeta^i}{\beta_i}$ tends to $\frac{1}{\beta_0 + \frac{q}{2(\zeta-1)}}$, as i tends to infinity.

Writing

$$\log \left(\prod_{j=1}^i \beta_j^{\zeta^{-j}} \right) = \sum_{j=1}^i j \zeta^{-j} \log \zeta + \sum_{j=1}^i \zeta^{-j} \log \frac{\beta_j}{\zeta^j},$$

we see that this expression has a limit when i goes to infinity. Eventually :

$$\|\sigma\|_{L^\infty(E, \mu_\rho)} \leq (q^{-1} N_t)^{\frac{\frac{q}{2}(\zeta-1)}{\beta_0 + \frac{q}{2}(\zeta-1)}} \left(\prod_{j=1}^{\infty} \beta_j^{\zeta-j} \right)^{\frac{\frac{q}{2}}{\beta_0 + \frac{q}{2}(\zeta-1)}} \|\sigma\|_{L^{\beta_0}(E, \mu_\rho)}^{\frac{\beta_0}{\beta_0 + \frac{q}{2}(\zeta-1)}}.$$

As $\frac{q}{2(\zeta-1)} = \frac{tn}{2t-n}$ and $\beta_0 = x$, this is what we claimed. ■

These facts lead to the

Theorem 3.9 (Inversion of the Bochner laplacian) *Fix an element s in $[\frac{2n}{n+2}, \frac{n}{2}]$ and a number $t > \frac{n}{2}$. Let Ω be an open set with smooth boundary. Then we can define a continuous operator*

$$\overline{\Delta}^{-1} : L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol) \cap L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol) \longrightarrow L^\infty(E_\Omega)$$

which is an inverse for the Bochner laplacian over Ω , with Dirichlet boundary condition. Moreover, every section σ in $C_c^\infty(E_\Omega)$ obeys the estimate

$$\|\sigma\|_{L^\infty(E_\Omega)} \leq C(n, s, t) S \left(\|\overline{\Delta}\sigma\|_{L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol)}^{\frac{s}{n-2s}} \|\overline{\Delta}\sigma\|_{L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol)}^{\frac{t}{2t-n}} \right)^{\frac{1}{\frac{s}{n-2s} + \frac{t}{2t-n}}}.$$

Proof :

The estimate is simply obtained by combining (3.7) and (3.8). Given ψ in $C_c^\infty(E_\Omega)$, the classical L^2 theory yields a smooth solution σ_R to the equation $\overline{\Delta}\sigma_R = \psi$ on $\Omega \cap B(R)$, with Dirichlet boundary condition. We extend it into a continuous function on Ω by deciding it is zero outside $B(R)$. The L^∞ -estimate (which is easily seen to hold for σ_R , by looking at the proofs above) gives

$$\|\sigma_R\|_{L^\infty(E_\Omega)} \leq C(n, s, t) S \left(\|\psi\|_{L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol)}^{\frac{s}{n-2s}} \|\psi\|_{L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol)}^{\frac{t}{2t-n}} \right)^{\frac{1}{\frac{s}{n-2s} + \frac{t}{2t-n}}}.$$

For every compact set K , there is an R_K such that the family $(\sigma_R|_K, R \geq R_K)$ is uniformly bounded in $C^\infty(E_K)$ (by elliptic regularity), so that Ascoli yields a sequence converging in $C^\infty(E_K)$. By diagonal extraction, we find a sequence (σ_{R_i}) which converges to σ in $C_c^0(E_\Omega)$. σ is easily seen to be a weak solution of $\overline{\Delta}\sigma = \psi$, it is therefore smooth and thus a strong solution. For every compact set K , we can write $\|\sigma\|_{L^\infty(E_K)} = \lim_{i \rightarrow \infty} \|\sigma_{R_i}\|_{L^\infty(E_K)}$, hence

$$\|\sigma\|_{L^\infty(E_K)} \leq C(n, s, t) S \left(\|\psi\|_{L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol)}^{\frac{s}{n-2s}} \|\psi\|_{L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol)}^{\frac{t}{2t-n}} \right)^{\frac{1}{\frac{s}{n-2s} + \frac{t}{2t-n}}},$$

so that, by taking the supremum over K , we obtain a L^∞ -estimate on Ω . We can thus define an operator $\overline{\Delta}^{-1}$ on $C_c^\infty(E_\Omega)$ which is continuous for the expected norms. We then extend it by continuity. ■

By a perturbation technique, we deduce an analogous result for Schrödinger operators.

Theorem 3.10 (Inversion of Schrödinger operators) *Set $\frac{2n}{n+2} \leq s < \frac{n}{2}$ and $t > \frac{n}{2}$. Then there exists a positive number $\eta(n, s, t, S)$ such that, given an open set with smooth boundary Ω and a potential V satisfying*

$$\max \left(\|V_-\|_{L^s(\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol)}, \|V_-\|_{L^t(\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol)} \right) < \frac{\eta(n, s, t)}{S},$$

there is a continuous operator

$$(\overline{\Delta} + V)^{-1} : L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol) \cap L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol) \longrightarrow L^\infty(E_\Omega).$$

Proof :

First, the previous analysis works for $H := \bar{\Delta} + V_+$ as well as for $\bar{\Delta}$. Then define $\eta(n, s, t)$ to be S divided by the norm of

$$H^{-1} : L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol) \cap L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol) \longrightarrow L^\infty(E_\Omega),$$

so that, under our assumption,

$$V_- : L^\infty(E_\Omega) \longrightarrow L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol) \cap L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol),$$

is a continuous operator whose norm is strictly inferior to $\eta(n, s, t)/S$ and $H^{-1}V_-$ is a continuous endomorphism of $L^\infty(E_\Omega)$, with norm strictly inferior to 1. The operator $\text{Id} + H^{-1}V_-$ is then an automorphism of $L^\infty(E_\Omega)$. So we can define the continuous operator $(\text{Id} + H^{-1}V_-)^{-1}H^{-1} = (\bar{\Delta} + V)^{-1}$, from $L^s(E_\Omega, \rho(r)^{\frac{s-1}{n/2-1}} dvol) \cap L^t(E_\Omega, \rho(r)^{\frac{t-1}{n/2-1}} dvol)$ to $L^\infty(E_\Omega)$. ■

4 Applications.

4.1 L^2 -cohomology.

Our study of Schrödinger operators gives geometric information as soon as the potential depends only on the curvature tensor. For instance, if the weighted Sobolev inequality is true, the vanishing theorem (3.1) forces the kernel of such "geometric operators" to be trivial, under integral assumptions on the curvature. We discuss here the case of the Hodge laplacian $\Delta = dd^* + d^*d$. It is well known that this operator, when acting on k -forms, admits the Weitzenböck decomposition $\Delta^k = \bar{\Delta} + \mathcal{R}^k$, where \mathcal{R}^k is a field of symmetric endomorphisms of the vector bundle of k -exterior forms, depending only on the curvature. In particular, $\mathcal{R}^1 = \text{Ric}$. Our results apply and we can obtain information on the (reduced) L^2 -cohomology $\mathcal{H}_{L^2}(M)$. We refer to [Car1] for the definitions. The point is that $\mathcal{H}_{L^2}^k(M)$ can be identified with the kernel of Δ^k , seen as an unbounded operator on L^2 k -forms. We can indeed generalize G. Carron's results in [Car1]. Before stating our theorem, we need to introduce the following decreasing function, derived from the Euler Γ function ($q = 2n/(n-2)$):

$$\iota_q : x \mapsto \frac{2}{x} \left(\frac{\Gamma\left(\frac{x+q}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} \right)^{2/q}.$$

Theorem 4.1 (L^2 -cohomology) *Let M^n , $n \geq 3$, be a connected complete Riemannian manifold such that for some point o in M , $S_o(M)$ is finite. Then for every k in \mathbb{N} , the following holds :*

- if $\|\mathcal{R}_-^k\|_{L^{\frac{n}{2}}(\rho_o(r_o)dvol)} < \infty$ then $\dim \mathcal{H}_{L^2}^k(M) < \infty$;
- if $\|\mathcal{R}_-^k\|_{L^{\frac{n}{2}}(\rho_o(r_o)dvol)} < S_o(M)^{-1}$ then $\mathcal{H}_{L^2}^k(M) = \{0\}$;
- if $\|\mathcal{R}_-^k\|_{L^{\frac{n}{2}}(\rho_o(r_o)dvol)} \leq S_o(M)^{-1} \frac{\iota_q(k)}{\iota_q(N_0)}$ for some integer $N_0 \geq \binom{n}{k}$ then $\dim \mathcal{H}_{L^2}^k(M) \leq N_0$;
- given $\frac{2n}{n+2} \leq s < n/2 < t$, there exists a constant $C = C(n, s, t)$ such that the dimension of $\mathcal{H}_{L^2}^k(M)$ is bounded by

$$\binom{n}{k} \max \left(1, C \|S_o(M)\mathcal{R}_-^k\|_{L^t(E, \rho_o(r_o)^{\frac{t-1}{n/2-1}} dvol)}^t \|S_o(M)\mathcal{R}_-^k\|_{L^s(E, \rho_o(r_o)^{\frac{s-1}{n/2-1}} dvol)}^{\frac{s(2x-n)}{n-2s}} \right).$$

Corollary 4.2 *Let M^n , $n \geq 3$, be a connected complete Riemannian manifold with nonnegative Ricci curvature. Assume there exists o in M , $\nu > 2$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol} B(o, t)}{\text{vol} B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu$$

and the Riemann curvature tensor satisfies $\int_M |R|^{\frac{n}{2}} \rho_o(r_o) dvol < \infty$. Then the L^2 -cohomology of M is finite dimensional.

We omit the proof, which consists in using the weighted Sobolev inequality (2.16), in order to make the techniques of [Car1] work. The vanishing results stem from 3.1, of course.

4.2 Ricci flat manifolds.

4.2.1 Flatness criterions.

We want to explain here why the weighted Sobolev inequality and the Hardy inequality help understanding Ricci flat manifolds. We are interested in rigidity properties of these manifolds, under volume growth assumptions. In this paragraph, we will show that if their curvature is small, in some sense, then they are actually flat. The key tool is a property of the Weyl tensor W of a Ricci-flat manifold with dimension $n \geq 4$ (note this tensor is nothing but the Riemann curvature tensor R , since $\text{Ric} = 0$) : it obeys the nonlinear equation

$$\overline{\Delta}W = W * W,$$

where the right-hand side is a quadratic expression in the Weyl curvature [Bes]. In particular, W is either identically zero, or vanishes only on a set of zero measure. So, outside a set of zero measure, $|W|$ is smooth and satisfies the estimate

$$|\Delta |W|| \leq c(n) |W|^2,$$

where $c(n)$ is a universal constant, depending only on the dimension n . Now, for every $k \geq 1$, we can write

$$\Delta |W|^k = k |W|^{k-1} \Delta |W| - k(k-1) |W|^{k-2} |d|W||^2 \leq kc(n) |W|^{k+1}.$$

It turns out that this inequality is still true for some $k < 1$. This is made possible by the refined Kato inequality ([BKN], [CGH]), which says that the Weyl tensor W of a Ricci-flat n -manifold satisfies almost everywhere

$$|d|W||^2 \leq \frac{n-1}{n+1} |\nabla W|^2.$$

From this, one can deduce that almost everywhere,

$$\Delta |W|^\gamma \leq c(n)\gamma |W|^{\gamma+1},$$

with $\gamma := \frac{n-3}{n-1}$. Indeed, note $\frac{n-1}{n+1} = \frac{1}{2-\gamma}$ and write

$$\begin{aligned} \Delta |W|^\gamma &= \gamma |W|^{\gamma-1} \Delta |W| + \gamma(1-\gamma) |W|^{\gamma-2} |d|W||^2 \\ &= \gamma |W|^{\gamma-2} \left(\frac{1}{2} \Delta |W|^2 + |d|W||^2 \right) + \gamma(1-\gamma) |W|^{\gamma-2} |d|W||^2 \\ &= \gamma |W|^{\gamma-2} \left((W, \overline{\Delta}W) - |\nabla W|^2 \right) + \gamma(2-\gamma) |W|^{\gamma-2} |d|W||^2 \\ &\leq c(n)\gamma |W|^{\gamma+1} - \gamma |W|^{\gamma-2} |\nabla W|^2 + \gamma |W|^{\gamma-2} |\nabla W|^2 \\ &= c(n)\gamma |W|^{\gamma+1}. \end{aligned}$$

Now, given $k \geq \gamma$, we can write $k = \gamma l$, $l \geq 1$, and then

$$\begin{aligned} \Delta |W|^k &= \Delta (|W|^\gamma)^l \\ &= l(|W|^\gamma)^{l-1} \Delta (|W|^\gamma) - l(l-1) (|W|^\gamma)^{l-2} |d(|W|^\gamma)|^2 \\ &\leq l(|W|^\gamma)^{l-1} c(n)\gamma |W|^{\gamma+1} \\ &= kc(n) |W|^{k+1}. \end{aligned}$$

Therefore, for any $k \geq \gamma = \frac{n-3}{n-1}$, the following is true :

$$\Delta |W|^k \leq c(n)k |W|^{k+1}. \quad (24)$$

With this differential inequality in hand, we can prove flatness and curvature decay results. To express them, we need the

Definition 4.3 The "Sobolev-curvature" invariant of a connected complete manifold M^n is defined by

$$\mathcal{SC}(M) := \inf_{o \in M} \left[S_o(M) \left(\int_M |R|^{\frac{n}{2}} \rho_o(r_o) d\text{vol} \right)^{\frac{2}{n}} \right],$$

where R is the Riemann curvature tensor. We also define a "Hardy-curvature" invariant :

$$\mathcal{HC}(M) := \inf_{o \in M} \left[H_o(M)^2 \sup_M (|R| r_o^2) \right],$$

We use the convention $0 \cdot \infty = \infty$.

Now, let us phrase our first rigidity result.

Theorem 4.4 (Flatness criterion (1)) We consider a connected complete Ricci-flat manifold M^n , with $n \geq 4$. There exists $\epsilon(n) > 0$ such that if $\mathcal{SC}(M) < \epsilon(n)$, then M is flat.

Proof :

Set $\epsilon(n) = \frac{4}{nc(n)} \left(2 - \frac{4}{n}\right)$. The theorem is a consequence of the vanishing theorem 3.1, applied to the operator $\Delta - c(n)|W|$ and the section $|W|$, thanks to our weighted Sobolev inequality. Setting $m = \frac{n}{2}$ in (3.1), we obtain $W = 0$, and as $\text{Ric} = 0$, M is flat. ■

Corollary 4.5 Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. Assume there exists o in M , $\nu > 2$ and $C_o > 0$ such that

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu.$$

Then there is a constant $\epsilon(n, C_o, \nu)$ such that M is flat as soon as

$$\int_M |W|^{\frac{n}{2}} \rho_o(r_o) d\text{vol} < \epsilon(n, C_o, \nu).$$

There is also a flatness criterion based on the Hardy inequality (2.23). An easy adaptation of the previous argument would work if we used the L^2 Hardy inequality. But we want to use the L^1 version of this inequality : indeed, it will allow quite small volume growth in corollary 4.7 (by doing this, we pass from $\nu > 2$ to $\nu > 1$ in the assumptions of this corollary). It will just need a little more work.

Theorem 4.6 (Flatness criterion (2)) We consider a connected complete Ricci-flat manifold M^n , with $n \geq 4$. There exists $\epsilon(n) > 0$ such that if $\mathcal{HC}(M) < \epsilon(n)$, then M is flat.

Proof :

We set $\epsilon(n) = c(n)^{-1}(n+2)^{-3}$ and choose o in M such that $H_o(M)^2 \sup_M (|W| r_o^2) < \epsilon(n)$. Set $H = H_o(M)$, $K = \sup_M (|W| r_o^2)$ and $k = \frac{n+2}{4}$ (our choices will become clear at the end of the proof). We consider, for large R , a smooth function χ which is equal to 1 on $B(R)$, equal to 0 on $M \setminus B(2R)$, has values in $[0, 1]$ and satisfies $|d\chi| \leq \frac{2}{R}$. The Hardy inequality says that

$$\int_M \chi^2 |W|^{2k} r^{-1} \leq H \int_M \left| d(\chi^2 |W|^{2k}) \right|.$$

The right hand side can be bounded via triangle and Cauchy-Schwarz inequalities :

$$\int_M \left| d(\chi^2 |W|^{2k}) \right| \leq 2 \int_M \chi |d\chi| |W|^{2k} + 2 \left(\int_M \chi^2 |W|^{2k} r^{-1} \right)^{1/2} \left(\int_M \chi^2 \left| d(|W|^k) \right|^2 r \right)^{1/2}.$$

Set $k' := k - 1/4$. So as to perform integration by parts, we kill the r in the lattest integral :

$$\int_M \chi^2 \left| d(|W|^k) \right|^2 r = (k/k')^2 \int_M \chi^2 \left| d(|W|^{k'}) \right|^2 |W|^{1/2} r \leq k^2 K^{1/2} \int_M \chi^2 \left| d(|W|^{k'}) \right|^2.$$

We can write

$$\int_M \chi^2 \left| d(|W|^{k'}) \right|^2 = \int_M \left| d(\chi |W|^{k'}) - |W|^{k'} d\chi \right|^2 \leq 2 \int_M \left| d(\chi |W|^{k'}) \right|^2 + 2 \int_M |W|^{2k'} |d\chi|^2.$$

Integration by parts and (24) yield :

$$\begin{aligned} \int_M \chi^2 \left| d(|W|^{k'}) \right|^2 &\leq 2 \int_M \chi^2 |W|^{k'} \Delta |W|^{k'} + 4 \int_M |W|^{2k'} |d\chi|^2 \\ &\leq 2k'c(n) \int_M \chi^2 |W|^{2k'+1} + 4 \int_M |W|^{2k'} |d\chi|^2 \\ &\leq 2kc(n)\sqrt{K} \int_M \chi^2 |W|^{2k} r^{-1} + 4 \int_M |W|^{2k-1/2} |d\chi|^2. \end{aligned}$$

Hence the estimate :

$$\begin{aligned} \int_M \chi^2 |W|^{2k} r^{-1} &\leq 2H \int_M \chi |d\chi| |W|^{2k} \\ &\quad + \sqrt{8H^2 K c(n) k^3} \int_M \chi^2 |W|^{2k} r^{-1} \\ &\quad + 4HK^{1/4} k \left(\int_M \chi^2 |W|^{2k} r^{-1} \right)^{1/2} \left(\int_M |W|^{2k-1/2} |d\chi|^2 \right)^{1/2}. \end{aligned}$$

Our choice of $\epsilon(n)$ and k ensures $\sqrt{8H^2 K c(n) k^3}$ is strictly less than 1, so that we obtain

$$\begin{aligned} \left(1 - \sqrt{8H^2 K c(n) k^3}\right) \int_{B(R)} |W|^{2k} r^{-1} &\leq \frac{4H}{R} \int_M |W|^{2k} \\ &\quad + \frac{8HK^{1/4} k}{R} \left(\int_M |W|^{2k} r^{-1} \right)^{1/2} \left(\int_M |W|^{2k-1/2} \right)^{1/2}. \end{aligned}$$

Recall W has quadratic decay and the volume growth is at most Euclidian : our choice of k ensures the integrals on the right-hand side are finite. Letting R go to infinity, we find $W = 0$, and since the Ricci tensor vanishes, M is flat. ■

Corollary 4.7 *Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. Assume there exists o in M , $\nu > 1$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu.$$

Then there is a constant $\epsilon(n, C_o, \nu)$ such that M is flat as soon as

$$\sup_M (|W| r_o^2) < \epsilon(n, C_o, \nu).$$

4.2.2 Curvature decay.

In the preceding paragraph, we have seen that when $\mathcal{SC}(M)$ is small, the curvature vanishes. Now using the decay lemmas of 3.2, we can show that if $\mathcal{SC}(M)$ is only finite, then the curvature decays at infinity. We will first show quadratic curvature decay, and then we will try to improve the rate of decay.

Proposition 4.8 *Consider a connected complete Ricci-flat manifold M^n , $n \geq 4$, such that $\mathcal{SC}(M)$ is finite. Then for any point o in M , $\sup_{S(o, t)} |W| = o(t^{-2})$.*

Corollary 4.9 *Consider a connected complete Ricci-flat manifold M^n with $n \geq 4$. Assume there exists o in M , $\nu > 2$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s} \right)^\nu$$

and the curvature tensor belongs to $L^{\frac{n}{2}}(M, \rho_o(r_o) d\text{vol})$. Then $\sup_{S(o, t)} |W| = o(t^{-2})$.

Remark 4.10 This should be compared with the result of [CT] : $\sup_{S(o,t)} |W| = \mathcal{O}(t^{-2})$ as soon as $n = 4$, $\text{Ric} = 0$ and W belongs to $L^2(M, d\text{vol})$.

Remark 4.11 If we assume W behaves like $r_o^{-\sigma}$, the assumption $\int_M |W|^{\frac{n}{2}} \rho_o(r_o) d\text{vol} < +\infty$ is equivalent to $\sigma > 2$: the above result therefore turns an integral estimate into the pointwise estimate we can hope. The next theorem will point out an automatic improvement of the decay : it is another rigidity phenomenon.

Proof :

As $(\Delta - c(n)|W|)|W| \leq 0$, we want to apply the lemma 3.6 with the operator $\Delta - c(n)|W|$. To do this, we need an estimate on the potential. Set $x = \frac{n^2}{2(n-2)}$, so that in particular $x - n/2 = \frac{n}{n-2}$. For large R , Lemma 3.3 (with $m = n/2$) implies :

$$\left(\int_{M \setminus B(R)} |W|^x d\mu_\rho \right)^{\frac{1}{x-n/2}} \leq \frac{C}{R^2} \int_{A(R/2, R)} |W|^{\frac{n}{2}} d\text{vol}.$$

Since $\rho(R) \int_{A(R/2, R)} |W|^{\frac{n}{2}} d\text{vol}$ is uniformly bounded, we obtain the following estimate :

$$\begin{aligned} \left(\int_{A(R, 2R)} |W|^x \rho(r)^{\frac{x-1}{n/2-1}} d\text{vol} \right)^{\frac{1}{x-n/2}} &\leq CR^{-2} \rho(R)^{-1 + \frac{\frac{x-1}{n/2-1} + \frac{2}{n-2}}{x-n/2}} \\ &= CR^{-2} \rho(R)^{\frac{2}{n-2}}. \end{aligned}$$

It allows us to use lemma 3.6 with $m = n/2$:

$$\begin{aligned} \sup_{S(R)} |W| &\leq C \rho(R)^{\frac{2}{n-2}} R^{-2} \left(\int_{A(R/2, 5R/2)} |W|^{n/2} d\mu_\rho \right)^{\frac{2}{n}} \\ &\leq CR^{-2} \left(\int_{A(R/2, 5R/2)} |W|^{n/2} \rho(r) d\text{vol} \right)^{\frac{2}{n}}. \end{aligned}$$

As the latest integral tends to zero when R goes to infinity, the result follows. ■

In general, such a quadratic curvature decay is not so meaningful : actually, any smooth connected noncompact manifold admits a metric with quadratic curvature decay ([Gro],[LS]). Note however that a Riemannian manifold with nonnegative Ricci curvature, maximal volume growth ($\nu = n$) and quadratic curvature decay has finite topological type [SS]. In case the volume growth is not maximal, such a strong topological consequence is not known.

We would like to point out a consequence of the quadratic curvature decay. Applying one of the results of [LS], it yields the

Corollary 4.12 Let M^n , $n \geq 4$, be a connected complete Ricci-flat manifold. Suppose there exists o in M , $2 < \nu < n$ and $B_o \geq A_o > 0$ such that

$$\forall t \geq 1, A_o t^\nu \leq \text{vol} B(o, t) \leq B_o t^\nu,$$

and the curvature tensor belongs to $L^{\frac{n}{2}}(M, \rho_o(r_o) d\text{vol})$. Then the integral of the Chern-Gauss-Bonnet form is an integer.

Remark 4.13 On a four-manifold, this means $\frac{1}{8\pi^2} \int_M |W|^2 d\text{vol}$ is an integer. In particular, if $\int_M |W|^2 d\text{vol} < 8\pi^2$, M has to be flat. We recover a flatness criterion (in dimension 4).

Now, it is well known that manifolds with faster than quadratic curvature decay enjoy nice properties [Abr]. This motivates our quest for a better estimate on the curvature. The key is the refined Kato inequality.

Theorem 4.14 (Curvature decay (1)) *We consider a connected complete Ricci-flat manifold M^n , $n \geq 4$, such that $\mathcal{SC}(M)$ is finite. Fix a point o in M and assume there exists $\nu > 2$ and $A_o > 0$ such that $\forall t \geq 1$, $\text{vol} B(o, t) \geq A_o t^\nu$. Then*

$$\sup_{S(o, t)} |W| = \mathcal{O}(t^{-b}) \quad \text{for } b = 2 \text{ and every } b < \frac{\nu - 2}{\gamma} = \frac{(\nu - 2)(n - 1)}{n - 3}.$$

Proof :

Set $w = |W|^\gamma$ and $b_0 = \sup \left\{ b > 0 / w = \mathcal{O} \left([r^2/V(r)]^b \right) \right\}$. We know, from the previous proposition, that $w = \mathcal{O}(r^{-2\gamma})$; since $V(r) \leq \omega_n r^n$ (Bishop), this implies $w = \mathcal{O}(V(r)^{-2\gamma/n}) = \mathcal{O}([r^2/V(r)]^{2\gamma/n})$, so that b_0 is a positive number. Suppose $b_0 < 1$. We can choose $b_1 > 0$, $m > 0$ such that

$$m > \frac{n}{b_1(n-2)} > \frac{n}{b_0(n-2)} > \frac{n}{n-2}.$$

Since $b_1 < b_0$, $w = \mathcal{O} \left([r^2/V(r)]^{b_1} \right)$, so that for any $R > 0$,

$$\begin{aligned} \int_{A(R, 2R)} |w|^m d\mu_\rho &\leq C [R^2/V(R)]^{mb_1} \rho(R)^{-\frac{2}{n-2}} V(R) \\ &= C [R^2/V(R)]^{mb_1 - \frac{n}{n-2}} \\ &\leq CR^{-(\nu-2)(mb_1 - \frac{n}{n-2})}. \end{aligned}$$

This implies $\int_M |w|^m d\mu_\rho < +\infty$. Now, recall that almost everywhere $(\Delta - \gamma c(n) |W|)w \leq 0$. We intend to apply lemma 3.4 to the function w , which is unfortunately not locally Lipschitz. To overcome this, once again, we consider $u_\epsilon := \sqrt{|W|^2 + \epsilon}$, $\epsilon > 0$. Direct computation yields almost everywhere

$$\begin{aligned} u_\epsilon^\gamma \Delta u_\epsilon^\gamma &= \gamma u_\epsilon^{2\gamma-2} \left(|W| \Delta |W| - \epsilon u_\epsilon^{-2} |d|W||^2 \right) + \gamma(1-\gamma) u_\epsilon^{2\gamma-4} |W|^2 |d|W||^2 \\ &\leq \gamma u_\epsilon^{2\gamma-2} \left(|W| \Delta |W| + (1-\gamma) |d|W||^2 \right) \end{aligned}$$

and, using the refined Kato inequality as in the proof of (24), we find (everywhere)

$$u_\epsilon^\gamma \Delta u_\epsilon^\gamma \leq \gamma u_\epsilon^{2\gamma} (W, \overline{\Delta} W).$$

As in the proof of (3.1), by making ϵ go to zero, we are able to obtain the first inequality in the proof of lemma 3.4 ($m > \frac{n}{n-2}$). Eventually, we find :

$$\int_{M \setminus B(R)} |w|^m d\mu_\rho = \mathcal{O}(R^{-a}),$$

for some $a > 0$ which is independent of the choice of m in a neighbourhood of $\frac{n}{b_0(n-2)}$. Now, applying the lemma 3.6 (again, one must adapt the proof because w is not locally Lipschitz) with this m , we find for large R :

$$\begin{aligned} \sup_{S(R)} w &\leq C \left(\rho(R)^{\frac{2}{n-2}} R^{-2} \right)^{\frac{n}{2m}} R^{-a/m} \\ &= C [R^2/V(R)]^{\frac{n}{m(n-2)}} R^{-a/m} \\ &\leq C [R^2/V(R)]^{\frac{n}{m(n-2)} + \frac{a}{nm}}, \end{aligned}$$

where we again used the Euclidian upper bound on the volume growth of balls. When m goes to $\frac{n}{b_0(n-2)}$, the exponent tends to $b_o + \frac{b_o(n-2)a}{n^2}$: if we choose m sufficiently close to $\frac{n}{b_0(n-2)}$, we obtain a contradiction to the definition of b_o . So $b_o \geq 1$ and, with the lower bound on the volume growth, we are done. ■

Corollary 4.15 *We consider a connected complete Ricci-flat manifold M^n with $n \geq 4$. Assume there exists o in M , $\nu > 2$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu$$

and the curvature tensor belongs to $L^{\frac{n}{2}}(M, \rho_o(r_o)d\text{vol})$. Then

$$\sup_{S(o, R)} |W| = \mathcal{O}(R^{-b}) \quad \text{for } b = 2 \text{ and every } b < \frac{\nu - 2}{\gamma} = \frac{(\nu - 2)(n - 1)}{n - 3}.$$

Let us point out the topological consequence we were expecting.

Corollary 4.16 (Finite topology) *Let M^n be a connected complete Ricci-flat manifold, $n \geq 4$, for which there exists a point o , $\nu > 4\frac{n-2}{n-1}$, $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu$$

and whose curvature tensor belongs to $L^{\frac{n}{2}}(M, \rho_o(r_o)d\text{vol})$. Then M is homeomorphic to the interior of a compact manifold with boundary.

Proof :

The previous theorem implies M has faster than quadratic curvature decay so that the statement follows from [Abr]. ■

One can wonder whether the limiting decay exponent in 4.14 is indeed attained. Actually, this is true.

Theorem 4.17 (Curvature decay (2)) *We consider a connected complete Ricci-flat manifold M^n , $n \geq 4$, such that $SC(M)$ is finite. Fix a point o in M and assume there exists $\nu > 4\frac{n-2}{n-1}$ and $A_o > 0$ such that $\forall t \geq 1$, $\text{vol } B(o, t) \geq A_o t^\nu$. Then*

$$\sup_{S(o, R)} |W| = O\left(r^{-\frac{(\nu-2)(n-1)}{n-3}}\right).$$

Proof :

In [Gur], Gursky introduced the following operator :

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} \text{Scal}_g - \gamma c(n) |W|_g.$$

It turns out that this operator is conformally invariant in the following sense : if ϕ is a smooth positive function,

$$L_{\phi^{\frac{4}{n-2}}g} = \phi^{-\frac{n+2}{n-2}} L_g(\phi). \quad (25)$$

We intend to use this property to find in the conformal class of g a new metric \tilde{g} such that outside a compact set

$$L_{\tilde{g}} = \Delta_{\tilde{g}},$$

i.e.

$$\frac{n-2}{4(n-1)} \text{Scal}_{\tilde{g}} - \gamma c(n) |W|_{\tilde{g}} = 0.$$

We seek \tilde{g} in the form of $\tilde{g} = (1+u)^{\frac{4}{n-2}}g$, where g is our Ricci-flat metric and u is a smooth function to determine. Applying (25) to the constant function 1, we find

$$L_{\tilde{g}}(1) = L_{(1+u)^{\frac{4}{n-2}}g}(1) = (1+u)^{-\frac{n+2}{n-2}} L_g(1+u),$$

so that, since $\text{Scal}_g = 0$,

$$\frac{n-2}{4(n-1)} \text{Scal}_{\tilde{g}} - \gamma c(n) |W|_{\tilde{g}} = (1+u)^{-\frac{n+2}{n-2}} (\Delta_g u - \gamma c(n) |W|_g) (1+u).$$

We thus have to solve

$$\Delta_g u - \gamma c(n) |W|_g u = \gamma c(n) |W|_g. \quad (26)$$

Let us solve it on $M \setminus B_g(o, R)$, with large R (let us assume $S_g(o, R)$ is smooth, this not a problem). We would like to use the inversion theorem 3.10 with $\Delta_g - \gamma c(n) |W|_g$. The assumption $\nu > 4\frac{n-2}{n-1}$ ensures $\frac{\nu-2}{\gamma} > 2$: theorem 4.14 says that $|W| = \mathcal{O}(r^{-b})$ for some $b > 2$. In particular, using Bishop's upper bound on the volume growth, one sees that for small $\delta > 0$,

$$\int_M |W|^{n/2 \pm \delta} \rho(r)^{\frac{n/2 \pm \delta - 1}{n/2 - 1}} d\text{vol} < \infty.$$

Choosing R sufficiently large, we ensure

$$S_o(M) \left(\int_{M \setminus B_g(o, R)} |W|^{n/2 \pm \delta} \rho(r)^{\frac{n/2 \pm \delta - 1}{n/2 - 1}} d\text{vol} \right)^{\frac{1}{n/2 \pm \delta}} < \eta(n/2, n/2 - \delta, n/2 + \delta).$$

So 3.10 yields a bounded solution u of (26) on $M \setminus B_g(o, R)$, and by enlarging R if necessary, we can even assume $\|u\|_{L^\infty} < 1$. Extending u to the whole M in a convenient way, we obtain a metric \tilde{g} which is conformally quasi-isometric to g and such that its Gursky operator and its laplacian coincide outside some ball. Note that the Hölder elliptic regularity implies u is C^2 (since the coefficients of the equation are Lipschitz) and this is what we need.

Next, we observe that, as soon as $|W|_g$ is positive, $|W_g|_g^\gamma$ is smooth and

$$L_g |W_g|_g^\gamma = \Delta_g |W_g|_g^\gamma - \gamma c(n) |W|_g |W_g|_g^\gamma \leq 0,$$

so that, with (25),

$$L_{\tilde{g}}((1+u)^{-1} |W_g|_g^\gamma) \leq 0,$$

which means

$$\Delta_{\tilde{g}}((1+u)^{-1} |W_g|_g^\gamma) \leq 0,$$

outside a compact set.

Now, since (M, \tilde{g}) is quasi-isometric to (M, g) , it satisfies the volume doubling property as well as the scaled Poincaré inequality. These properties are equivalent to the following two-sided gaussian estimate on the heat kernel $p_t(\cdot, \cdot)$: for every x, y in M , for every $t > 0$,

$$\frac{c}{V(x, \sqrt{t})} \exp\left(-\frac{Cd(x, y)^2}{t}\right) \leq p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{cd(x, y)^2}{t}\right)$$

(see [SC], [Grig]). As for large R , $V_{\tilde{g}}(o, R) \geq \tilde{A}_o R^\nu$, $\nu > 2$, this in turn implies the existence of a positive Green function $G(\cdot, \cdot)$, which is simply $\int_0^\infty p_t(\cdot, \cdot) dt$ [LY]. Using this formula and the upper bound on the heat kernel, we see that :

$$G(o, x) = \mathcal{O}(r_o(x)^{2-\nu})$$

when $r_o(x)$ goes to infinity. The maximum principle implies that for every point x in $M \setminus B_g(o, R)$,

$$(1+u)^{-1} |W_g|_g^\gamma(x) \leq \frac{\max_{S(o, R)} (1+u)^{-1} |W_g|_g^\gamma}{\min_{S(o, R)} G(o, \cdot)} G(o, x).$$

We deduce

$$\sup_{S(o, R)} |W| = \mathcal{O}(R^{\frac{2-\nu}{\gamma}}).$$

■

Corollary 4.18 *We consider a connected complete Ricci-flat manifold M^n , with $n \geq 4$. Assume there exists o in M , $\nu > 4\frac{n-2}{n-1}$ and $C_o > 0$ such that*

$$\forall t \geq s > 0, \frac{\text{vol } B(o, t)}{\text{vol } B(o, s)} \geq C_o \left(\frac{t}{s}\right)^\nu$$

and the curvature tensor belongs to $L^{\frac{n}{2}}(M, \rho_o(r_o)d\text{vol})$. Then

$$\sup_{S(o, R)} |W| = \mathcal{O}\left(r^{-\frac{(\nu-2)(n-1)}{n-3}}\right).$$

Remark 4.19 *When $\nu = n = 4$, we obtain the same decay as [BKN].*

Example 4.20 *The Taub-NUT metric is a Riemannian metric on \mathbb{R}^4 introduced by Stephen Hawking in [Haw] (see [Leb] for a mathematical point of view). This is a Hyperkähler hence Ricci-flat metric with curvature decaying like r^{-3} and volume growth like r^3 . In this example, our theorem predicts the exact decay of the curvature.*

Example 4.21 *Let us give another example, inspired from the famous Schwarzschild metric. We consider $\mathbb{R}^2 \times \mathbb{S}^{n-2}$, $n \geq 4$, endowed with the metric*

$$g = dr^2 + F(r)^2 dt^2 + G(r)^2 d\sigma^2.$$

r, t are polar coordinates on the \mathbb{R}^2 factor, $d\sigma^2$ is the standard metric on \mathbb{S}^{n-2} , F and G are smooth functions. Using the symmetries of this metric (see [Bes], [Pet]), it is easy to obtain formulas for the curvature. And one sees that g has vanishing Ricci tensor if and only if for some positive parameter γ , G satisfies

$$\begin{cases} G'(r) = \sqrt{1 - \left(\frac{\gamma}{G}\right)^{n-3}} \\ G(0) = \gamma \\ G'(0) = 0 \end{cases}$$

and

$$F(r) = \frac{2\gamma}{n-3} \sqrt{1 - \left(\frac{\gamma}{G}\right)^{n-3}}.$$

G increases from γ to ∞ and $G \sim r$ at infinity ; F increases from 0 to $\frac{2\gamma}{n-3}$ and $F \sim r$ near 0. In particular, g is C^0 -close to the flat metric on $\mathbb{R}^{n-1} \times \mathbb{S}^1$ at infinity (the radius of the circles at infinity are proportionnal to γ) and the distance to a fixed point in this manifold behaves like the coordinate r at infinity. Eventually, this provides on $\mathbb{R}^2 \times \mathbb{S}^{n-2}$, $n \geq 4$, a complete Riemannian metric which is Ricci flat, has volume growing like r^{n-1} and curvature decreasing like $r^{-(n-1)}$. This is what our theorem predicted.

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