informs
http://pubsonline.informs.org/journal/moor/

MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–13
ISSN 0364-765X (print), ISSN 1526-5471 (online)

Limit Value of Dynamic Zero-Sum Games with Vanishing Stage Duration

Sylvain Sorin[a]

[a] Sorbonne Universités, UPMC Univ Paris 06, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Univ Paris Diderot, Sorbonne Paris Cité, F-75005, Paris, France

Contact: sylvain.sorin@imj-prg.fr (SS)

Received: March 21, 2016
Revised: October 25, 2016
Accepted: January 19, 2017
Published Online in Articles in Advance:

MSC2010 Subject Classification: 91A20, 91A05, 91A15, 91A23, 91A25
OR/MS Subject Classification: Primary: games/group decisions; secondary: noncooperative

https://doi.org/10.1287/moor.2017.0851

Abstract. We consider two-person zero-sum games where the players control, at discrete times $\{t_n\}$ induced by a partition Π of $\mathbb{R}^+$, a continuous time Markov process. We prove that the limit of the values v_Π exist as the mesh of Π goes to 0. The analysis covers the cases of (1) stochastic games (where both players know the state), and (2) games with unknown state and symmetric signals.

The proof is by reduction to deterministic differential games.

Funding: This research was supported, in part, by PGMO 2014-LMG.

Keywords: dynamic games • zero sum • limit value • vanishing stage duration

1. Introduction

Repeated interactions in a stationary environment have been traditionally represented by dynamic games played in stages. An alternative approach is to consider a continuous time process that the players control at discrete times. In the first case, the expected number of interactions increases as the weight θ_n of each stage n goes to zero. In the second case, the number of interactions increases when the duration δ_n of each time interval n vanishes.

In a repeated game framework, one can normalize the model using the evaluation $\theta = \{\theta_n\}$ of the stages ($\theta_n \geq 0$, $\sum \theta_n = 1$), so that stage n is associated to time $t_n = \sum_{j=1}^{n-1} \theta_j$, and then consider the game played on $[0,1]$ (hence time t corresponds to the stage where the fraction t of the total weight has been reached). Each evaluation θ (in the original repeated game) thus induces a partition Π_θ of $[0,1]$, and vanishing mesh corresponds to *vanishing stage weight*. In the two-person zero-sum framework, tools adapted from continuous time models can be used to obtain convergence results, given an ordered set of evaluations, for the corresponding family of values v_θ, see, e.g., for different classes of games, Sorin [32, 33, 34], Vieille [40], Laraki [23], Cardaliaguet et al. [8].

In the alternative approach analyzed here, there is a given evaluation k on $\mathbb{R}^+$. Then, one considers a sequence of partitions $\Pi(m)$ of $\mathbb{R}^+$ with vanishing mesh corresponding to *vanishing stage duration* and the associated sequence of values $v(m)$.

In both cases, for each given partition, the value function exists at the times defined by the partition and the stationarity of the model allows to write a recursive formula (RF). Then, one extends the value function to $[0,1]$ (resp. $\mathbb{R}^+$) by linearity, and one considers the family of values as the mesh of the partition goes to 0. The next two steps in the proof of convergence of the family of values consist in defining, using (RF), a PDE (main equation: ME), and proving

(1) that any accumulation point of the family is a viscosity solution of (ME)
(2) that (ME) has a unique viscosity solution.

Altogether the tools are quite similar to those used in differential games, however, in the current framework, the state is basically a random variable and the players use mixed actions.

Section 2 describes the model. Section 3 presents the main results concerning differential games that are used in the paper. In particular, we define two notions of "mixed extension" for the time discretization of a differential game and prove that the asymptotic behavior of their values is similar. Section 4 is devoted to the framework where both players observe the state variable. Section 5 deals with the situation where the state is unknown but the actions are observed. In both cases, the analysis is done by reduction to an ad hoc differential game.

2. Smooth Continuous Time Games and Discretization

2.1. Discretization of a Continuous Time Process and Associated Game

Consider a continuous time process Z_t, defined on $\mathbb{R}^+ = [0, +\infty)$, with values in a state space Ω and an evaluation given by a probability density $k(t)$ on $\mathbb{R}^+$.

Let $T = \inf\{L \in [0, +\infty]; \int_0^L k(t)dt = 1\}$. A *partition* $\Pi = \{t_n\}$ is an increasing sequence of $\mathbb{R}^+$ with $t_1 = 0$ and $t_n \to T$ as $n \to \infty$. It induces a discrete time game as follows. The time interval $L_n = [t_n, t_{n+1}[$ corresponds to stage n and has duration $\delta_n = t_{n+1} - t_n$. The law of Z_t on L_n is determined by its value at time t_n, $\hat{Z}_n = Z_{t_n}$, and the actions $(i_n, j_n) \in I \times J$ chosen by the players at time t_n, that last for stage n. The payoff at time t in stage n ($t \in L_n$) is defined through a map g from $\Omega \times I \times J$ to $\mathbb{R}$:

$$g_\Pi(t) = g(Z_t, i_n, j_n).$$

(*An alternative choice leading to the same asymptotic results would be* $g_\Pi(t) = g(\hat{Z}_n, i_n, j_n)$).

The outcome along a play is

$$\gamma_{\Pi,k} = \int_0^T g_\Pi(t) k(t)\, dt,$$

and the corresponding value function, defined on Ω, is $v_{\Pi,k}$.

One will study the asymptotics of the family $\{v_{\Pi,k}\}$ as the mesh $\delta = \sup \delta_n$ of the partition Π vanishes.

2.2. Markov Process

From now on, we consider the case where $Z_t, t \in \mathbb{R}^+$ follows a continuous time Markov process controlled by the players: it is specified by a finite state space Ω and a transition rate $\mathbf{q}$ that belongs to the set $\mathcal{M}$ of real bounded maps on $I \times J \times \Omega \times \Omega$ satisfying

$$\mathbf{q}(i,j)[\omega, \omega'] \geq 0 \quad \text{if } \omega' \neq \omega, \qquad \text{and} \qquad \sum_{\omega' \in \Omega} \mathbf{q}(i,j)[\omega, \omega'] = 0, \quad \forall\, i \in I,\ j \in J,\ \omega, \omega' \in \Omega.$$

Let $\mathsf{P}^h(i,j), h \in \mathbb{R}^+$ denote the continuous time Markov chain on Ω generated by the kernel $\mathbf{q}(i,j)$. It satisfies

$$\dot{\mathsf{P}}^h(i,j) = \mathsf{P}^h(i,j)\mathbf{q}(i,j) = \mathbf{q}(i,j)\mathsf{P}^h(i,j),$$

and for $t \geq 0$,

$$\mathsf{P}^{t+h}(i,j) = \mathsf{P}^t(i,j) e^{h\mathbf{q}(i,j)}.$$

In particular, one has for all $z, z' \in \Omega$:

$$\begin{aligned}\mathsf{P}^h(i,j)[z,z'] &= \text{Prob}(Z_{t+h} = z' \mid Z_t = z), \quad \forall\, t \geq 0,\\ &= \mathbf{1}_{\{z\}}(z') + h\mathbf{q}(i,j)[z,z'] + o(h).\end{aligned}$$

2.3. Hypotheses

One assumes from now on the following:

— the evaluation k is continuous on $\mathbb{R}^+$.
— the action sets I, J are compact metric spaces,
— the payoff g and the transition $\mathbf{q}$ are continuous on $I \times J$.

2.4. Notations

If F is a bounded measurable function defined on $I \times J$ with values in a convex set, $F(x,y)$ denotes its multilinear extension to $X \times Y$ with $X = \Delta(I)$ (resp. $Y = \Delta(J)$), set of regular Borel probabilities on I (resp. J). (This applies, in particular, to g and $\mathbf{q}$).

For $\zeta \in \Delta(\Omega)$ and $\mu \in \mathbb{R}^{\Omega^2}$, we define

$$\zeta * \mu(z) = \zeta(\cdot) * \mu[\cdot, z] = \sum_{\omega \in \Omega} \zeta(\omega)\mu[\omega, z], \quad \forall\, z \in \Omega.$$

(When g is a map from Ω to itself and μ its graph: $\mu[\omega, z] = \mathbf{1}_{\{g(\omega)=z\}}$, $\zeta * \mu$ is the usual image measure of ζ by g).

In particular, if $\zeta_t \in \Delta(\Omega)$ is the law of Z_t, one has, if (i,j) is played on $[t, t+h]$,

$$\zeta_{t+h} = \zeta_t * \mathsf{P}^h(i,j),$$

and

$$\dot{\zeta}_t = \zeta_t * \mathbf{q}(i,j).$$

Similarly, we use the following notation for a transition probability or a transition rate μ operating on a real function f on Ω:

$$\mu[z,\cdot] \circ f(\cdot) = \sum_{z' \in \Omega} \mu[z,z'] f(z') = \mu \circ f[z].$$

3. Discretization and Mixed Extension of Differential Games

We study here a continuous time game by introducing a time discretization Π and analyzing the limit behavior of the associated family of values v_Π as the mesh of the partition vanishes. This approach was initiated in Fleming [14, 15, 16] and developed in Friedman [17, 18], Eliott and Kalton [12], and see also Scarf [30].

A differential game γ is defined through the following components: $Z \subset \mathbb{R}^n$ is the state space, I and J are the action sets of player 1 (maximizer) and 2, f from $Z \times I \times J$ to $\mathbb{R}^n$ is the dynamics kernel, g from $Z \times I \times J$ to $\mathbb{R}$ is the payoff-flow function, and k from $\mathbb{R}^+$ to $\mathbb{R}^+$ determines the evaluation.
Formally, the dynamics is defined on $[0,+\infty) \times Z$ by

$$\dot{z}_t = f(z_t, i_t, j_t), \tag{1}$$

and the total payoff is

$$\int_0^T g(z_s, i_s, j_s) k(s)\, ds.$$

We assume in this section:
— I and J metric compact sets,
— f and g bounded, continuous, and uniformly Lipschitz in z,
— k continuous with $\int_0^{+\infty} k(s)\, ds = 1$.
— $\Phi^h(z; i, j)$ denotes the value at time $t+h$ of the solution of (1) starting at time t from z and with play $\{i_s = i, j_s = j\}$ on $[t, t+h]$.

To define the strategies, we have to specify the information: we assume that the players know the initial state z_0, and at time t, the previous play $\{i_s, j_s; 0 \le s < t\}$, hence the trajectory of the state $\{z_s; 0 \le s \le t\}$.
The analysis below will show that Markov strategies (i.e., depending only, at time t, on t and z_t) will suffice.

3.1. Deterministic Analysis

Given a partition Π, we consider the associated discrete time game γ_Π, where on each interval $[t_n, t_{n+1})$ players use constant moves (i_n, j_n) in $I \times J$. This defines the dynamics on the state. At time t_{n+1}, (i_n, j_n) is announced, hence the corresponding value of the state, $z_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i_n, j_n)$ is known.

The associated maxmin w_Π^- satisfies, at each point of the partition, the RF

$$w_\Pi^-(t_n, z_{t_n}) = \sup_I \inf_J \left[\int_{t_n}^{t_{n+1}} g(z_s, i, j) k(s)\, ds + w_\Pi^-(t_{n+1}, z_{t_{n+1}}) \right]. \tag{2}$$

The function $w_\Pi^-(\cdot, z)$ is extended by linearity to $[0, T)$ (and 0 on $[T, +\infty)$), and note that there exists L: $]0, +\infty[\to \mathbb{R}^+$ such that

$$\forall\, \varepsilon > 0, \quad t \ge L(\varepsilon) \implies |w_\Pi^-(t, \cdot)| \le \varepsilon. \tag{3}$$

Denote by $\mathcal{F}$ the set of bounded functions satisfying (3). All "value" functions that we will consider here will belong to $\mathcal{F}$.

The next four results follow from the analysis in Evans and Souganidis [13], see also Barron et al. [3], Souganidis [38], and the presentation in Bardi and Capuzzo-Dolcetta [2], Chapter VII, Section 3.2.

Proposition 3.1. *The family $\{w_\Pi^-(t,z)\}$ is uniformly equicontinuous in both variables.*

Hence the set U of accumulation points of the family $\{w_\Pi^-\}$ (for the uniform convergence on compact subsets of $\mathbb{R}^+ \times Z$), as the mesh δ of Π goes to zero, is nonempty.

We first recall the notion of viscosity solution, see Crandall and Lions [10].

Definition 3.1. Given a Hamiltonian H from $\mathbb{R}^+ \times Z \times \mathbb{R}^n$ to $\mathbb{R}$, a continuous real function u on $\mathbb{R}^+ \times Z$ is a viscosity solution of

$$0 = \frac{d}{dt}u(t,z) + H(t,z,\nabla u(t,z)) \tag{4}$$

if for any real function ψ, $\mathscr{C}^1$ on $\mathbb{R}^+ \times Z$ with $u - \psi$ having a strict local maximum at $(\bar{t},\bar{z}) \in [0,T[\times Z$

$$0 \leq \frac{d}{dt}\psi(\bar{t},\bar{z}) + H(t,z,\nabla\psi(t,z)),$$

and the dual condition holds: for any real function ψ, $\mathscr{C}^1$ on $\mathbb{R}^+ \times Z$ with $u - \psi$ having a strict local minimum at $(\bar{t},\bar{z}) \in [0,T[\times Z$

$$0 \geq \frac{d}{dt}\psi(\bar{t},\bar{z}) + H(t,z,\nabla\psi(t,z)).$$

We can now introduce the Hamilton-Jacobi-Isaacs (HJI) equation that follows from (2), corresponding to the Hamiltonian:

$$h^-(t,z,p) = \sup_I \inf_J [g(z,i,j)k(t) + \langle f(z,i,j), p\rangle]. \tag{5}$$

Proposition 3.2. *Any function $u \in U$ belongs to $\mathscr{F}$ and is a viscosity solution of*

$$0 = \frac{d}{dt}u(t,z) + \sup_I \inf_J [g(z,i,j)k(t) + \langle f(z,i,j), \nabla u(t,z)\rangle]. \tag{6}$$

Note that in the discounted case, $k(t) = \rho e^{-\rho t}$, with the change of variable $u(t,z) = e^{-\rho t}\phi(z)$, one obtains

$$\rho\phi(z) = \sup_I \inf_J [\rho g(z,i,j) + \langle f(z,i,j), \nabla\phi(z)\rangle]. \tag{7}$$

The main property is in the following:

Proposition 3.3. *There exists a unique function in $\mathscr{F}$, which is a viscosity solution of* (6).

Recall that this notion and this kind of results are due, in a general framework, to Crandall and Lions [10], for more properties, see Crandall et al. [11].

The uniqueness of accumulation point implies:

Corollary 3.1. *The family $\{w_\Pi^-\}$ converges to some w^-.*

An alternative approach is to consider the game γ defined in normal form on $\mathbb{R}^+$. Let w_∞^- be the maxmin (lower value) of the continuous time differential game played using nonanticipative strategies with delay. Then, from Evans and Souganidis [13], extended in Cardaliaguet [6], Chapter 3, one obtains:

Proposition 3.4. (1) *w_∞^- belongs to $\mathscr{F}$ and is a viscosity solution of* (6).

(2) *Hence*

$$w_\infty^- = w^-.$$

Obviously, similar properties hold for the minmax w_Π^+ and w_∞^+.

Finally, define Isaacs's condition on $I \times J$ by

$$\sup_I \inf_J [g(z,i,j)k(t) + \langle f(z,i,j), p\rangle] = \inf_J \sup_I [g(z,i,j)k(t) + \langle f(z,i,j), p\rangle], \quad \forall\, t \in \mathbb{R}^+, \forall\, z \in Z, \forall\, p \in \mathbb{R}^n, \tag{8}$$

which, with the notation (5), corresponds to

$$h^-(t,z,p) = h^+(t,z,p).$$

Proposition 3.5. *Assume condition* (8).
Then, the limit value exists, in the sense that

$$w^- = w^+ (= w_\infty^- = w_\infty^+).$$

Note that the same analysis holds if the players use strategies that, at time t_n, depend only on t_n and z_{t_n}.

3.2. Mixed Extension

We define two mixed extensions of the game γ as follows: for each partition Π, we introduce two discrete time games associated to γ_Π and played on $X = \Delta(I)$ and $Y = \Delta(Y)$ (set of probabilities on I and J, respectively). We will then prove that the asymptotic properties of their values coincide.

3.2.1. Deterministic Actions. The first game Γ^I is defined as in Subsection 3.1 where X and Y are now the sets of actions (this corresponds to "relaxed controls") replacing I and J.

The main point is that the dynamics f (hence the flow) and the payoff g are extended to $X \times Y$ by taking the expectation w.r.t. x and y

$$f(z,x,y) = \int_{I\times J} f(z,i,j)\,x(di)y(dj), \quad \dot{z}_t = f(z_t, x_t, y_t), \quad g(z,x,y) = \int_{I\times J} g(z,i,j)\,x(di)y(dj). \tag{9}$$

Γ^I_Π is the associated discrete time game where on each interval $[t_n, t_{n+1})$ players use constant actions (x_n, y_n) in $X \times Y$. This defines the dynamics: $\bar{\Phi}^h(z; x, y)$ denotes the value at time $t+h$ of the solution of (9) starting at time t from z and with play $\{x_s = x, y_s = y\}$ on $[t, t+h]$. Note that $\bar{\Phi}^h(z; x, y)$ is *not* the bilinear extension of $\Phi^h(z; i, j)$. At time t_{n+1}, (x_n, y_n) is announced, hence the current value of the state, $z_{t_{n+1}} = \bar{\Phi}^{\delta_n}(z_{t_n}; x_n, y_n)$ is known.

The maxmin W^-_Π satisfies the RF

$$W^-_\Pi(t_n, z_{t_n}) = \sup_X \inf_Y \left[\int_{t_n}^{t_{n+1}} g(z_s, x, y)k(s)\,ds + W^-_\Pi(t_{n+1}, z_{t_{n+1}})\right].$$

The analysis of the previous paragraph applies, leading to:

Proposition 3.6. *The family $\{W^-_\Pi(t,z)\}$ is uniformly equicontinuous in both variables.*

The HJI equation corresponds here to the Hamiltonian:

$$H^-(t,z,p) = \sup_X \inf_Y [g(z,x,y)k(t) + \langle f(z,x,y), p\rangle]. \tag{10}$$

Proposition 3.7. (1) *Any accumulation point of the family $\{W^-_\Pi\}$, as the mesh δ of Π goes to zero, belongs to $\mathcal{F}$ and is a viscosity solution of*

$$0 = \frac{d}{dt}W^-(t,z) + \sup_X \inf_Y [g(z,x,y)k(t) + \langle f(z,x,y), \nabla W^-(t,z)\rangle]. \tag{11}$$

(2) *The family $\{W^-_\Pi\}$ converges to W^-, unique viscosity solution of* (11) *in $\mathcal{F}$.*

Finally, let W^-_∞ be the maxmin of the differential game Γ^I played (on $X \times Y$) using nonanticipative strategies with delay. Then,

Proposition 3.8. (1) *W^-_∞ is a viscosity solution of* (11). (2) *$W^-_\infty = W^-$.*

As above, similar properties hold for W^+_Π and W^+_∞.

Due to the bilinear extension, Isaacs's condition on $X \times Y$, which is, with the notation (10):

$$H^-(t,z,p) = H^+(t,z,p) \quad \forall\, t \in \mathbb{R}^+, \forall\, z \in Z, \forall\, p \in \mathbb{R}^n, \tag{12}$$

holds here. Thus one obtains:

Proposition 3.9. *The limit value W exists:*

$$W = W^- = W^+,$$

and is also the value of the differential game played on $X \times Y$.

It is the unique viscosity solution in $\mathcal{F}$ of

$$0 = \frac{d}{dt}W(t,z) + \mathtt{val}_{X\times Y}[g(z,x,y)k(t) + \langle f(z,x,y), \nabla W(t,z)\rangle]. \tag{13}$$

3.2.2. Random Actions. We define now another game Γ_Π^{II}, where the actions $(i_n, j_n) \in I \times J$ are chosen at time t_n, according to $x_n \in X$ and $y_n \in Y$, then constant on $[t_n, t_{n+1})$ and announced at time t_{n+1}. The new state is thus, if $(i_n, j_n) = (i, j)$, $z^{ij}_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i, j)$.
It is clear, see, e.g., Mertens et al. [24], Chapter 4, that the next dynamic programming property holds.

Proposition 3.10. *The game Γ_Π^{II} has a value W_Π, which satisfies the RF*

$$\mathsf{W}_\Pi(t_n, z_{t_n}) = \mathtt{val}_{X\times Y}\mathsf{E}_{x,y}\left[\int_{t_n}^{t_{n+1}} g(z_s, i, j)k(s)\,ds + \mathsf{W}_\Pi(t_{n+1}, z^{ij}_{t_{n+1}})\right], \tag{14}$$

and given the hypothesis, one obtains as above:

Proposition 3.11. *The family $\{\mathsf{W}_\Pi(t,z)\}$ is equicontinuous in both variables.*

Note that we are not dealing with the discretization of a deterministic differential game, nevertheless, one has:

Proposition 3.12. (1) *Any accumulation point U of the family $\{\mathsf{W}_\Pi\}$, as the mesh δ of Π goes to zero, belongs to $\mathcal{F}$ and is a viscosity solution of the previous Equation* (13).
(2) *The family $\{\mathsf{W}_\Pi\}$ converges to W, unique solution in $\mathcal{F}$ of* (13).

Proof. (1) Let $\psi(t,z)$ be a $\mathscr{C}^1$ test function such that $\mathsf{U} - \psi$ has a strict local maximum at $(\bar t, \bar z)$. Consider a sequence $W_m = \mathsf{W}_{\Pi(m)}$ converging uniformy locally to U as $m \to \infty$, and let $(t^*(m), z(m))$ be a maximizing sequence for $(W_m - \psi)(t,z)$, $t \in \Pi(m)$. In particular, $(t^*(m), z(m))$ converges to $(\bar t, \bar z)$ as $m \to \infty$. Given $x^*(m)$ optimal in (14), one has with $t^*(m) = t_n \in \Pi(m)$

$$W_m(t_n, z(m)) \le \mathsf{E}_{x^*(m),y}\left[\int_{t_n}^{t_{n+1}} g(z_s, i, j)k(s)\,ds + W_m(t_{n+1}, z^{ij}_{t_{n+1}})\right], \quad \forall\, y \in Y.$$

The choice of $(t^*(m), z(m))$ implies

$$\psi(t_n, z(m)) - W_m(t_n, z(m)) \le \psi(t_{n+1}, z^{ij}_{t_n+1}) - W_m(t_{n+1}, z^{ij}_{t_{n+1}}).$$

Hence using the continuity of k and ψ being $\mathscr{C}^1$, one obtains

$$\begin{aligned}\psi(t_n, z(m)) \le\,& \mathsf{E}_{z,x^*(m),y}\left[\int_{t_n}^{t_{n+1}} g(z_s, i, j)k(s)\,ds + \psi(t_{n+1}, z^{ij}_{t_n+1})\right]\\ \le\,& \delta_n k(t_n) g(z(m), x^*(m), y) + \psi(t_{n+1}, z(m))\\ &+ \delta_n \mathsf{E}_{x^*(m)y}\langle f(z(m), i, j), \nabla\psi(t_{n+1}, z(m))\rangle + o(\delta_n).\end{aligned}$$

This gives

$$0 \le \delta_n \frac{d}{dt}\psi(t_n, z(m)) + \delta_n k(t_n) g(z(m), x^*(m), y) + \delta_n \mathsf{E}_{x^*(m)y}\langle f(z(m), i, j), \nabla\psi(t_n, z(m))\rangle + o(\delta_n),$$

hence dividing by δ_n and taking the limit as $m \to \infty$, one obtains, for some accumulation point $x^* \in \Delta(I)$ (we use again the continuity of k and ψ being $\mathscr{C}^1$)

$$0 \le \frac{d}{dt}\psi(\bar t, \bar z) + k(\bar t) g(\bar z, x^*, y) + \mathsf{E}_{x^* y}\langle f(\bar z, i, j), \nabla\psi(\bar t, \bar z)\rangle, \qquad \forall y \in Y. \tag{15}$$

Thus U is a viscosity solution of

$$0 = \frac{d}{dt}u(t,z) + \mathtt{val}_{X\times Y}\int_{I\times J}[g(z,i,j)k(t) + \langle f(z,i,j), \nabla u(t,z)\rangle]\,x(di)y(dj),$$

which by linearity, reduces to (13).
(2) The proof of uniqueness follows from Proposition 3.9. □

Note again that the same analysis holds if the players use strategies that depend only, at time t_n, on t_n and z_{t_n}.

3.2.3. Comments. Games (Γ^I and Γ^{II}) lead to the same limit PDE (13) but with different sequences of approximations:

In the first case (Γ^I), the evolution is deterministic and the state (or (x,y)) is announced. Moreover, the value may not exist along the sequence.

In the second case (Γ^{II}), the evolution is random and the state (or the actions) is announced (the knowledge of (x,y) would not be enough).

The fact that both games have the same limit value is a justification for playing distribution or mixed actions as pure actions in continuous time and for assuming that the distributions are observed, see Neyman [25].

Remark also that the same analysis holds if f and g depend in addition continuously on t.

A related study of differential games with mixed actions, but concerned with the analysis through strategies can be found in Buckdahn et al. [4, 5], Jimenez et al. [22].

The advantage of working with discretization is to have a well defined and simple set of strategies, hence the RF is immediate to check for the associated maxmin or minmax $W^{\pm}_{\Pi}$. On the other hand the main equation (HJI) is satisfied by accumulation points.

The use of mixed actions in extensions of type II allows to have existence of a value in the associated discrete-time game.

4. State Controlled and Publicly Observed

This section is devoted to the case were the process Z_t is controlled by both players and observed by both (there are no assumptions on the signals on the actions). At stage n (time t_n), both players know Z_{t_n}. This corresponds to a stochastic game G in continuous time analyzed through a time discretization along Π, G_Π.

Previous related papers to stochastic games in continuous time include Zachrisson [42], Tanaka and Wakuta [39], Guo and Hernandez-Lerma [19, 20], Neyman [25].

The approach via time discretization is related to similar procedures in differential games, see the previous Section 3 and Neyman [26].

4.1. General Case

Consider a general evaluation k. Since k is fixed during the analysis, we will write v_Π for $v_{\Pi,k}$, defined on $\mathbb{R}^+ \times \Omega$.

4.1.1. Recursive Formula. The hypothesis on the data implies that v_Π exists, see, e.g., Mertens et al. [24]; Chapters IV, VII; Neyman and Sorin [27]; and in the current framework, the RF takes the following form:

Proposition 4.1. *The game G_Π has a value v_Π satisfying the recursive equation*

$$v_\Pi(t_n, Z_{t_n}) = \mathtt{val}_{X\times Y} \mathsf{E}_{z,x,y}\left[\int_{t_n}^{t_{n+1}} g(Z_s,i,j)k(s)\,ds + v_\Pi(t_{n+1}, Z_{t_{n+1}})\right]$$
$$= \mathtt{val}_{X\times Y}\left\{\mathsf{E}_{z,x,y}\left[\int_{t_n}^{t_{n+1}} g(Z_s,i,j)k(s)\,ds\right] + \mathsf{P}^{\delta_n}(x,y)[Z_{t_n},\cdot]\circ v_\Pi(t_{n+1},\cdot)\right\}. \tag{16}$$

Proof. This is the basic RF for the stochastic game with state-space Ω, action sets I and J, and transition kernel $\mathsf{P}^{\delta_n}(i,j)$, going back to Shapley [31]. □

The value $v_\Pi(\cdot, z)$, defined at times $t_n \in \Pi$, is extended by linearity to $[0,T[$ and 0 on $[T,+\infty[$.

4.1.2. Main Equation. The first property is standard in this framework.

Proposition 4.2. *The family of values $\{v_\Pi\}$ is uniformly equicontinuous w.r.t. $t \in \mathbb{R}^+$.*

Denote thus by $\mathbf{V}$ the (nonempty) set of accumulation points of the family $\{v_\Pi\}$ (for the uniform convergence on compact subsets of $\mathbb{R}^+ \times \Omega$) as the mesh δ vanishes.

Definition 4.1. A continuous real function u on $\mathbb{R}^+ \times \Omega$ is a viscosity solution of

$$0 = \frac{d}{dt}u(t,z) + \mathtt{val}_{X\times Y}\{g(z,x,y)k(t) + \mathbf{q}(x,y)[z,\cdot]\circ u(t,\cdot)\}, \tag{17}$$

if for any real function ψ, $\mathscr{C}^1$ on $\mathbb{R}^+ \times \Omega$ with $u-\psi$ having a strict maximum at $(\bar t,\bar z) \in [0,T[\times\Omega$:

$$0 \le \frac{d}{dt}\psi(\bar t,\bar z) + \mathtt{val}_{X\times Y}\{g(\bar z,x,y)k(\bar t) + \mathbf{q}(x,y)[\bar z,\cdot]\circ\psi(\bar t,\cdot)\},$$

and the dual condition:
if for any real function ψ, $\mathscr{C}^1$ on $\mathbb{R}^+ \times \Omega$ with $u - \psi$ having a strict minimum at $(\bar{t}, \bar{z}) \in [0, T[\times \Omega$:

$$0 \geq \frac{d}{dt}\psi(\bar{t}, \bar{z}) + \mathtt{val}_{X\times Y}\{g(\bar{z}, x, y)k(\bar{t}) + \mathbf{q}(x, y)[\bar{z}, \cdot] \circ \psi(\bar{t}, \cdot)\}.$$

Proposition 4.3. *Any $u \in \mathbf{V}$ is a viscosity solution of* (17).

Proof. Let $\psi(t, z)$ be a $\mathscr{C}^1$ function such that $u - \psi$ has a strict maximum at $(\bar{t}, \bar{z})$. Consider a sequence $V_m = v_{\Pi(m)}$ converging uniformy locally to u as $m \to \infty$ and let $(t^*(m), z(m))$ be a maxmizing sequence for $\{(V_m - \psi)(t, z), t \in \Pi(m)\}$. In particular, $(t^*(m), z(m))$ converges to $(\bar{t}, \bar{z})$ as $m \to \infty$. Given x^*_m optimal for $V_m(t^*(m), z(m))$ in (16), one obtains, with $t^*(m) = t_n \in \Pi_m$

$$V_m(t_n, z(m)) \leq \mathsf{E}_{z(m), x^*_m, y}\left[\int_{t_n}^{t_{n+1}} g(Z_s, i, j)k(s)\, ds\right] + \mathsf{P}^{\delta_n}(x^*_m, y)[z(m), \cdot] \circ V_m(t_{n+1}, \cdot), \quad \forall\, y \in Y,$$

so that by the choice of $(t^*(m), z(m))$:

$$\begin{aligned}\psi(t_n, z(m)) &\leq \mathsf{E}_{z(m), x^*_m, y}\left[\int_{t_n}^{t_{n+1}} g(Z_s, i, j)k(s)\, ds\right] + \mathsf{P}^{\delta_n}(x^*_m, y)[z(m), \cdot] \circ \psi(t_{n+1}, \cdot) \\ &\leq \delta_n k(t_n) g(z(m), x^*_m, y) + \psi(t_{n+1}, z(m)) + \delta_n \mathbf{q}(x^*_m, y)[z(m), \cdot] \circ \psi(t_{n+1}, \cdot) + o(\delta_n).\end{aligned}$$

This implies

$$0 \leq \delta_n k(t_n) g(z(m), x^*_m, y) + \delta_n \frac{d}{dt}\psi(t_n, z(m)) + \delta_n \mathbf{q}(x^*_m, y)[z(m), \cdot] \circ \psi(t_{n+1}, \cdot) + o(\delta_n).$$

hence dividing by δ_n and taking the limit as $m \to \infty$, one obtains, for some accumulation point x^* in the compact set $\Delta(I)$,

$$0 \leq k(\bar{t}) g(\bar{z}, x^*, y) + \frac{d}{dt}\psi(\bar{t}, \bar{z}) + \mathbf{q}(x^*, y)[\bar{z}, \cdot] \circ \psi(\bar{t}, \cdot), \quad \forall\, y \in Y, \tag{18}$$

so that

$$0 \leq \frac{d}{dt}\psi(\bar{t}, \bar{z}) + \mathtt{val}_{X\times Y}\{g(\bar{z}, x, y)k(\bar{t}) + \mathbf{q}(x, y)[\bar{z}, \cdot] \circ \psi(\bar{t}, \cdot)\}. \quad \square$$

4.1.3. Convergence. A first proof of the convergence of the family $\{v_\Pi\}_\Pi$ would follow from the property (P): Equation (17) has a unique viscosity solution in $\mathscr{F}$.

An alternative approach is to relate the game to a differential game on an extended state-space $\Delta(\Omega)$. Define V_Π on $\mathbb{R}^+ \times \Delta(\Omega)$ as the expectation of v_Π, namely,

$$V_\Pi(t, \zeta) = \sum_{\omega \in \Omega} \zeta(\omega) v_\Pi(t, \omega),$$

and denote $\mathbf{X} = X^\Omega$ and $\mathbf{Y} = Y^\Omega$.

Proposition 4.4. *V_Π satisfies*

$$V_\Pi(t_n, \zeta_{t_n}) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\left[\sum_\omega \zeta_{t_n}(\omega) \mathsf{E}_{\omega, \mathbf{x}(\omega), \mathbf{y}(\omega)}\left\{\int_{t_n}^{t_{n+1}} g(Z_s, i, j)k(s)\, ds\right\} + V_\Pi(t_{n+1}, \zeta_{t_{n+1}})\right], \tag{19}$$

where $\zeta_{t_{n+1}}(z) = \sum_\omega \zeta_{t_n}(\omega) \mathsf{P}^{\delta_n}(\mathbf{x}(\omega), \mathbf{y}(\omega))(\omega, z)$.

Proof. Equation (19) follows from (16), the definition of V_Π and the formula expressing $\zeta_{t_{n+1}}$. By independence, the optimization in X at each ω can be replaced by optimization in $\mathbf{X}$ and one uses the linearity in the transition. $\square$

Equation (16) corresponds to the usual approach following the trajectory of the process on the original state space. Equation (19) expresses the dynamics of the law ζ of the process, where the players act differently at different states ω. (This corresponds to the RF written on the space of entrance laws, see Mertens et al. [24] IV.3).

4.1.4. Related Differential Game. We will prove that the recursive equation (19) is satisfied by the value of the time discretization along Π of the mixed extension of a deterministic differential game $\mathcal{G}$ (see Section 3) on $\mathbb{R}^+$, defined as follows:

(1) the state space is $\Delta(\Omega)$,
(2) the action sets are $\mathbf{I} = I^\Omega$ and $\mathbf{J} = J^\Omega$,
(3) the dynamics on $\Delta(\Omega)\times\mathbb{R}^+$ is

$$\dot{\zeta}_t = f(\zeta_t, \mathbf{i}, \mathbf{j})$$

with

$$f(\zeta, \mathbf{i}, \mathbf{j})(z) = \sum_{\omega\in\Omega} \zeta_t(\omega)\mathbf{q}(\mathbf{i}(\omega), \mathbf{j}(\omega))[\omega, z].$$

(4) the flow payoff function is given by

$$\langle \zeta, g(\cdot, \mathbf{i}(\cdot), \mathbf{j}(\cdot))\rangle = \sum_{\omega\in\Omega} \zeta(\omega) g(\omega, \mathbf{i}(\omega), \mathbf{j}(\omega)).$$

(5) the global outcome is

$$\int_0^T g_t k(t)\,dt,$$

where g_t is the payoff at time t.
In $\mathcal{G}_\Pi$, the state is deterministic, and at each time t_n, the players know ζ_{t_n} and choose $\mathbf{i}_n$ (resp. $\mathbf{j}_n$).
Consider now the mixed extension $\mathcal{G}_\Pi^{II}$ (Section 3) and let $\mathcal{V}_\Pi(t, \zeta)$ be the associated value.

Proposition 4.5. *The value $\mathcal{V}_\Pi(t, \zeta)$ satisfies the recursive equation* (19).

Proof. The mixed action set for player 1 is $\tilde{\mathbf{X}} = \Delta(\mathbf{I}) = \Delta(I^\Omega)$, but due to the separability property in ω, one can work with $\mathbf{X} = \Delta(I)^\Omega$. Then it is easy to see that Equation (14) corresponds to (19). □

The analysis in Section 3 thus implies that
—any accumulation point U of the sequence $\mathcal{V}_\Pi$ belongs to $\mathcal{F}$ and is a viscosity solution of

$$0 = \frac{d}{dt}U(t, \zeta) + \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}[\langle \zeta, g(\cdot, \mathbf{x}(\cdot), \mathbf{y}(\cdot))\rangle k(t) + \langle f(\zeta, \mathbf{x}, \mathbf{y}), \nabla U(t, \zeta)\rangle]. \tag{20}$$

—Equation (20) has a unique viscosity solution in $\mathcal{F}$.
This leads to the convergence property as follows.

Corollary 4.1. *Both families V_Π and v_Π converge to some V and v with*

$$V(t, \zeta) = \sum_\omega \zeta(\omega) v(t, \omega).$$

— V is the viscosity solution of (20).
— v is a viscosity solution of (17).

Proof. One has $\nabla \mathcal{V}(t, \zeta) = \{v(t, \cdot)\}$, hence

$$\begin{aligned} 0 &= \left\langle \zeta(\cdot), \frac{d}{dt}v(t, \cdot)\right\rangle + \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\left[\langle \zeta(\cdot), g(\cdot, \mathbf{x}(\cdot), \mathbf{y}(\cdot))\rangle k(t) + \sum_z \left[\sum_\omega \zeta(\omega)\mathbf{q}(\mathbf{x}(\omega), \mathbf{y}(\omega))[\omega, z]v(t, z)\right]\right] \\ &= \left\langle \zeta(\cdot), \frac{d}{dt}v(t, \cdot)\right\rangle + \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\left[\langle \zeta(\cdot), g(\cdot, \mathbf{x}(\cdot), \mathbf{y}(\cdot))\rangle k(t) + \sum_\omega \zeta(\omega)\left[\sum_z \mathbf{q}(\mathbf{x}(\omega), \mathbf{y}(\omega))[\omega, z]v(t, z)\right]\right]. \end{aligned}$$

This gives

$$0 = \left\langle \zeta(\cdot), \frac{d}{dt}v(t, \cdot) + \mathtt{val}_{X\times Y}[g(\cdot, x, y)k(t) + \mathbf{q}(x, y)[\cdot, \bullet] \circ v(t, \bullet)]\right\rangle$$

(where $\bullet$ is the variable for the operator $\circ$), which is equivalent to

$$0 = \frac{d}{dt}v(t, z) + \mathtt{val}_{X\times Y}[g(z, x, y)k(t) + \mathbf{q}(x, y)[z, \cdot] \circ v(t, \cdot)],$$

and this is (17). □

4.2. Stationary Case

We consider the case $k(t)=\rho e^{-\rho t}$ and again the game along the partition Π.

4.2.1. Recursive Formula. The general RF (16) now takes the following form.

Proposition 4.6.

$$\begin{aligned} v_{\Pi,\rho}(t_n,Z_{t_n}) &= \mathtt{val}_{X\times Y}\mathsf{E}_{z,x,y}\left[\int_{t_n}^{t_{n+1}} g(Z_s,i,j)\rho e^{-\rho}\,ds + v_{\Pi,\rho}(t_{n+1},Z_{t_{n+1}})\right]\\ &= \mathtt{val}_{X\times Y}\left[\mathsf{E}_{z,x,y}\left(\int_{t_n}^{t_{n+1}} g(Z_s,i,j)\rho e^{-\rho}\,ds\right) + \mathsf{P}^{\delta_n}(x,y)[Z_{t_n},\cdot]\circ v_{\Pi,\rho}(t_{n+1},\cdot)\right], \end{aligned} \tag{21}$$

and if Π *is uniform with mesh* δ, $v_{\Pi,\rho}(t,z)=e^{-\rho t}\nu_{\delta,\rho}(z)$ *with*

$$\nu_{\delta,\rho}(Z_0)=\mathtt{val}_{X\times Y}\left[\mathsf{E}_{z,x,y}\left(\int_0^{\delta} g(Z_s,x,y)\rho e^{-\rho}\,ds\right)+e^{-\rho\delta}\,\mathsf{P}^{\delta}(x,y)[Z_0,\cdot]\circ\nu_{\delta,\rho}(\cdot)\right]. \tag{22}$$

4.2.2. Main Equation. The next result is standard, see, e.g., Neyman [26]; Prieto-Rumeau and Hernandez-Lerma [28], p. 235. We provide a short proof for convenience.

Proposition 4.7. (1) *For any* $R\in\mathcal{M}$ *and any* $\rho\in(0,1]$, *the equation with variable* φ *from* Ω *to* $\mathbb{R}$

$$\rho\varphi(z)=\mathtt{val}_{X\times Y}[\rho g(z,x,y)+R(x,y)[z,\cdot]\circ\varphi(\cdot)] \tag{23}$$

has a unique solution, denoted W_ρ.

(2) For any $\delta\in(0,1]$ such that $\|\delta R/(1-\delta\rho)\|\le 1$, the solution of (23) is the value of the repeated stochastic game with payoff g, transition $P=I+\delta R/(1-\delta\rho)$, and discounted factor $\delta\rho$.

Proof. Recall from Shapley [31], that the value $W_{\rho\delta}$ of a repeated stochastic game with payoff $\mathbf{g}$ and discounted factor $\delta\rho$ satisfies

$$W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho g(z,x,y)+(1-\delta\rho)\mathsf{E}_{z,x,y}\{W_{\rho\delta}(\cdot)\}]. \tag{24}$$

Assume the transition to be of the form $P=I+\delta q$ with $q\in\mathcal{M}$. One obtains

$$W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho g(z,x,y)+(1-\delta\rho)\{W_{\rho\delta}(z)+\delta q(x,y)[z,\cdot]\circ W_{\rho\delta}(\cdot)\}], \tag{25}$$

which gives

$$\delta\rho W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\delta\rho g(z,x,y)+\delta(1-\delta\rho)q(x,y)[z,\cdot]\circ W_{\rho\delta}(\cdot)], \tag{26}$$

so that

$$\rho W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\rho g(z,x,y)+(1-\delta\rho)q(xy)[z,\cdot]\circ W_{\rho\delta}(\cdot)]. \tag{27}$$

Hence with $q=R/(1-\delta\rho)$, one obtains

$$\rho W_{\rho\delta}(z)=\mathtt{val}_{X\times Y}[\rho g(z,x,y)+R(x,y)[z,\cdot]\circ W_{\rho\delta}(\cdot)].\quad\square \tag{28}$$

4.2.3. Convergence. Again, the following result can be found in Neyman [26], Theorem 1; see also Guo and Hernandez-Lerma [19, 20].

Proposition 4.8. *As the mesh* δ *of the partition* Π *goes to 0,* $v_{\Pi,\rho}$ *converges to the solution* W_ρ *of* (23) *with* $R=\mathbf{q}$

$$\rho W_\rho(z)=\mathtt{val}_{X\times Y}[\rho g(z,x,y)+\mathbf{q}(x,y)[z,\cdot]\circ W_\rho(\cdot)]. \tag{29}$$

Proof. Consider the strategy σ of player 1 in G_Π defined as follows: at state z, use an optimal strategy $x\in X=\Delta(I)$, for $W_\rho(z)$ given by (29). Let us evaluate, given τ, strategy of player 2, the following amount:

$$A_1=\mathsf{E}_{\sigma,\tau}\left[\int_{t_1}^{t_2} g_\Pi(s)\rho e^{-\rho s}\,ds+e^{-\rho t_2}W_\rho(Z_{t_2})\right].$$

Let x_1 the mixed move of player 1 at stage one given $Z_0=\hat{Z}_1$. Then, if y_1 is induced by τ, there exists a constant L such that

$$\begin{aligned} A_1 &\ge \delta_1\rho g(\hat{Z}_1,x_1,y_1)+(1-\delta_1\rho)[W_\rho(\hat{Z}_1)+\delta_1\mathbf{q}(x_1,y_1)[\hat{Z}_1,\cdot]\circ W_\rho(\cdot)]-\delta_1 L\delta\\ &\ge \delta_1\rho g(\hat{Z}_1,x_1,y_1)-\delta_1\rho W_\rho(\hat{Z}_1)+\delta_1\mathbf{q}(x_1,y_1)[\hat{Z}_1,\cdot]\circ W_\rho(\cdot)+W_\rho(\hat{Z}_1)-2\delta_1 L\delta\\ &\ge W_\rho(\hat{Z}_1)-2\delta_1 L\delta. \end{aligned}$$

Similarly, let

$$A_n = \mathsf{E}_{\sigma,\tau}\left[\int_{t_n}^{t_{n+1}} g_\Pi(s)\rho e^{-\rho s}\, ds ds + e^{-\rho t_{n+1}} W_\rho(\hat{Z}_{n+1}) \mid h_n\right],$$

where $h_n = (\hat{Z}_1, i_1, j_1, \ldots, i_{n-1}, j_{n-1}, \hat{Z}_n)$.
Then, with obvious notations

$$\begin{aligned} A_n &\geq e^{-\rho t_n}[\delta_n \rho g(\hat{Z}_n, x_n, y_n) + (1-\delta_n\rho)[W_\rho(\hat{Z}_n) + \delta_n \mathbf{q}(x_n, y_n)[\hat{Z}_n, \cdot] \circ W_\rho(\cdot)] - \delta_n L\delta] \\ &\geq e^{-\rho t_n}[\delta_n \rho g(\hat{Z}_n, x_n, y_n) - \delta_n \rho W_\rho(\hat{Z}_n) + \delta_n \mathbf{q}(x_n, y_n)[\hat{Z}_n, \cdot] \circ W_\rho(\cdot) + W_\rho(\hat{Z}_n) - 2\delta_n L\delta] \\ &\geq e^{-\rho t_n}[W_\rho(\hat{Z}_n) - 2\delta_n L\delta]. \end{aligned}$$

Taking the sum and the expectation, one obtains that the payoff induced by (σ, τ) in G_Π satisfies

$$\mathsf{E}_{\sigma,\tau}\left[\int_0^{+\infty} g_\Pi(s) k(s)\, ds\right] \geq W_\rho(\hat{Z}_1) - 2\left(\sum_n \delta_n e^{-\rho t_n}\right) L\delta,$$

and $(\sum_n \delta_n e^{-\rho t_n}) L\delta \to 0$ as $\delta \to 0$. □

Comments: The proof in Neyman [26] is done, in the finite case, for a uniform partition, but shows the robustness with respect to the parameters (converging family of games).

This procedure of proof is reminiscent of the "direct approach" for differential games, introduced by Isaacs [21]. To show convergence of the family of values v_Π: (i) one identifies a tentative limit value v and a RF $RF(v)$ and (ii) one shows that to play in the discretized game G_Π, an optimal strategy in $RF(v)$ gives an amount close to v for δ small enough.

For an alternative approach and proof, based on properties of the Shapley operator, see Sorin and Vigeral [37].

Remark that if $k(t) = \rho e^{-\rho t}$, $v(t,z) = e^{-\rho t}\nu(z)$ satisfies (17), iff $\nu(z)$ satisfies (29).

5. State Controlled and Not Observed

This section studies the game G, where the process Z_t is controlled by both players but not observed. However, the past actions are known: this defines a symmetric framework where the new state variable is the law of Z_t, $\zeta_t \in \Delta(\Omega)$. Even in the stationary case, there is no explicit smooth solution to the ME, hence a direct approach for proving convergence, as in the previous Section 4.2, is not feasible.

Here, also the analysis will be done through the connection to a differential game $\bar{\mathscr{G}}$ on $\Delta(\Omega)$ but different from the previous one $\mathscr{G}$, introduced in Section 4.

Given a partition Π denoted by G_Π, the associated game and again, since k is fixed during the analysis, we will write V_Π for its value $\mathsf{V}_{\Pi,k}$ defined on $\mathbb{R}^+ \times \Delta(\Omega)$.

Recall that given the initial law ζ_{t_n} and the actions $(i_{t_n}, j_{t_n}) = (i, j)$, one has

$$\zeta_{t_{n+1}}^{ij} = \zeta_{t_n} * \mathsf{P}^{\delta_n}(i, j), \tag{30}$$

and that this parameter is known by both players.

Extend $g(\cdot, x, y)$ from Ω to $\Delta(\Omega)$ by linearity: $g(\zeta, x, y) = \sum \zeta(z) g(z, x, y)$.

5.1. Recursive Formula

In this framework, the recursive structure leads to:

Proposition 5.1. *The value* V_Π *satisfies the following RF*:

$$\mathsf{V}_\Pi(t_n, \zeta_{t_n}) = \mathtt{val}_{X\times Y}\mathsf{E}_{\zeta,x,y}\left[\int_{t_n}^{t_{n+1}} g(\zeta_s, i, j) k(s)\, ds + \mathsf{V}_\Pi(t_{n+1}, \zeta_{t_{n+1}}^{ij})\right]. \tag{31}$$

Proof. Standard, since G_Π is basically a stochastic game with parameter ζ. □

5.2. Main Equation

Consider the differential game $\overline{\mathcal{G}}$ on $\Delta(\Omega)$ with actions sets I and J, dynamics on $\Delta(\Omega)\times\mathbb{R}^+$ given by

$$\dot{\zeta}_t = \zeta_t * \mathbf{q}(i,j),$$

current payoff $g(\zeta, i, j)$ and evaluation k.

As in Section 3, define the discretized mixed extension $\bar{\mathcal{G}}_\Pi^{II}$ to $X\times Y$ and let $\bar{\mathcal{V}}_\Pi$ be its value.

Proposition 5.2. *$\bar{\mathcal{V}}_\Pi$ satisfies* (31).

Proof. $\bar{\mathcal{V}}_\Pi$ satisfies (14) which is, using (30), equivalent to (31). □

The analysis in Section 3, Proposition 3.12 thus implies:

Proposition 5.3. *The family of values* V_Π *converges to* V *unique viscosity solution in* $\mathcal{F}$ *of*

$$0 = \frac{d}{dt}u(t,\zeta) + \mathtt{val}_{X\times Y}[g(\zeta,x,y)k(t) + \langle \zeta * \mathbf{q}(x,y), \nabla u(t,\zeta)]. \tag{32}$$

5.3. Stationary Case

Assume $k(t) = \rho e^{-\rho t}$. In this case, one has $\mathsf{V}(\zeta,t) = e^{-\rho t}\mathsf{v}(\zeta)$, hence (32) becomes

$$\rho \mathsf{v}(\zeta) = \mathtt{val}_{X\times Y}[\rho g(\zeta,x,y) + \langle \zeta * \mathbf{q}(x,y), \nabla \mathsf{v}(\zeta)\rangle]. \tag{33}$$

5.4. Comments

A differential game similar to $\bar{\mathcal{G}}$, where the state space is the set of probabilities on some set Ω has been studied in full generality by Cardaliaguet and Quincampoix [7], see also As Soulaimani [1]. Equation (33) is satisfied by the value of the nonrevealing game in the framework analyzed by Cardaliaguet et al. [9], see Section 6.

6. Concluding Comments and Extensions

This research is part of the analysis of asymptotic properties of dynamic games through their recursive structure: operator approach (Shapley [31], Rosenberg and Sorin [29]).

Recall that the asymptotic study for repeated games may lead to nonconvergence, in the framework of Section 4 with compact action spaces, see Vigeral [41], or in the framework of Section 5 even with finite action spaces, see Ziliotto [43] (for an overview of similar phenomena, see Sorin and Vigeral [36]).

The approach in terms of vanishing duration of a continuous-time process allows, via the extension of the state space from Ω to $\Delta(\Omega)$, to obtain smooth transition and nice limit behavior as the mesh of the partition goes to 0.

A similar procedure has been analyzed by Neyman [26], in the finite action case, for more general classes of approximating games and developed in Sorin and Vigeral [37].

The case of private information on the state variable has been treated by Cardaliaguet et al. [9] in the stationary finite framework: the viscosity solution corresponding to (ME) involves a geometric aspect due to the revelation of information that makes the analysis much more difficult. The (ME) obtained here in Section 4 corresponds to the nonrevealing value that players can obtain without revealing their private information.

Let us finally mention three directions of research:

—the study of the general symmetric case, i.e., a framework between Sections 4 and 5 where the players receive partially revealing symmetric signals on the state, Sorin [35],

—the asymptotic properties when both the evaluation tends to $+\infty$ and the mesh goes to 0: in the stationary case, this means ρ and δ vanishes. In the framework of Section 4, with finite actions spaces, this was done by Neyman [26] using the algebraic property of Equation (29), see also related results in Sorin and Vigeral [37],

—the construction of optimal strategies based at time t on the current state z_t and the instantaneous discount rate $k(t)/\int_t^{+\infty} k(s)\,ds$.

Acknowledgments

Dedicated to the memory of L. S. Shapley. A first version of this work was presented at the conference *Mathematical Aspects of Game Theory and Applications, Roscoff, June 30–July 4, 2014*. The author thanks the referees and the associate editor for very constructive comments.

References

[1] As Soulaimani S (2008) Viability with probabilistic knowledge of initial condition, application to optimal control. *Set-Valued Anal.* 16(7–8):1037–1060.

[2] Bardi M, Capuzzo-Dolcetta I (1996) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations* (Birkhauser, Boston).

[3] Barron EN, Evans LC, Jensen R (1984) Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls. *J. Differential Equations* 53(2):213–233.

[4] Buckdahn R, Li J, Quincampoix M (2013) Value function of differential games without Isaacs' conditions. An approach with nonanticipative mixed strategies. *Internat. J. Game Theory* 42(4):989–1020.

[5] Buckdahn R, Quincampoix M, Rainer C, Xu Y (2016) Differential games with asymmetric information and without Isaacs' condition. *Internat. J. Game Theory* 45(4):795–816.

[6] Cardaliaguet P (2010) *Introduction to Differential Games*. Unpublished lecture notes.

[7] Cardaliaguet P, Quincampoix M (2008) Deterministic differential games under probability knowledge of initial condition. *Internat. Game Theory Rev.* 10(1):1–16.

[8] Cardaliaguet P, Laraki R, Sorin S (2012) A continuous time approach for the asymptotic value in two-person zero-sum repeated games. *SIAM J. Control Optim.* 50(3):1573–1596.

[9] Cardaliaguet P, Rainer C, Rosenberg D, Vieille N (2014) Markov games with frequent actions and incomplete information: The limit case. *Math. Oper. Res.* 41(1):9–71.

[10] Crandall MG, Lions P-L (1981) Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre. *C.R. Acad. Sci. Paris* 292:183–186.

[11] Crandall MG, H Ishii H, Lions P-L (1992) User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27(1):1–67.

[12] Elliot RJ, Kalton NJ (1972) *The Existence of Value in Differential Games*. Memoirs of the American Mathematical Society, No. 126 (AMS, Providence, RI).

[13] Evans LC, Souganidis PE (1984) Differential games and representation formulas for solutions of Hamilton-Jacobi equations. *Indiana Univ. Math. J.* 33(5):773–797.

[14] Fleming WH (1957) A note on differential games of prescribed duration. Dresher M, Tucker AW, Wolfe P, eds. *Contributions to the Theory of Games, III*, Annals of Mathematical Studies, Vol. 39 (Princeton University Press, Princeton, NJ), 407–412.

[15] Fleming WH (1961) The convergence problem for differential games. *J. Math. Anal. Appl.* 3(1):102–116.

[16] Fleming WH (1964) The convergence problem for differential games II. Dresher M, Shapley LS, Tucker AW, eds. *Advances in Game Theory*, Annals of Mathematical Studies, Vol. 52 (Princeton University Press, Princeton, NJ), 195–210.

[17] Friedman A (1971) *Differential Games* (John Wiley & Sons, New York).

[18] Friedman A (1974) *Differential Games*, CBMS Regional Conference Series in Mathematics, Vol. 18 (AMS, Providence, RI).

[19] Guo X, Hernandez-Lerma O (2003) Zero-sum games for continuous-time Markov chains with unbounded transition and average payoff rates. *J. Appl. Probab.* 40(2):327–345.

[20] Guo X, Hernandez-Lerma O (2005) Zero-sum continuous-time Markov games with unbounded transition and discounted payoff rates. *Bernoulli* 11(6):1009–1029.

[21] Isaacs R (1965) *Differential Games* (John Wiley & Sons, New York).

[22] Jimenez C, Quincampoix M, Xu Y (2016) Differential games with incomplete information on a continuum of initial positions and without Isaacs condition. *Dynam. Games Appl.* 6(1):82–96.

[23] Laraki R (2002) Repeated games with lack of information on one side: The dual differential approach. *Math. Oper. Res.* 27(2):419–440.

[24] Mertens J-F, Sorin S, Zamir S (2015) *Repeated Games* (Cambridge University Press, Cambridge, UK).

[25] Neyman A (2017) Continuous-time stochastic games. *Games Econom. Behavior* 104:92–130.

[26] Neyman A (2013) Stochastic games with short-stage duration. *Dynam. Games Appl.* 3(2):236–278.

[27] Neyman A, Sorin S, eds. (2003) *Stochastic Games and Applications*, NATO Science Series, Vol. C 570 (Kluwer Academic Publishers, Dordrecht, Netherlands).

[28] Prieto-Rumeau T, Hernandez-Lerma O (2012) *Selected Topics on Continuous-Time Controlled Markov Chains and Markov Games* (Imperial College Press, London).

[29] Rosenberg D, Sorin S (2001) An operator approach to zero-sum repeated games. *Israël J. Math.* 121(1):221–246.

[30] Scarf HE (1957) On differential games with survival payoff. Dresher M, Tucker AW, Wolfe P, eds. *Contributions to the Theory of Games, III*, Annals of Mathematical Studies, Vol. 39 (Princeton University Press, Princeton, NJ), 393–405.

[31] Shapley LS (1953) Stochastic games. *Proc. Natl. Acad. Sci. USA* 39(10):1095–1100.

[32] Sorin S (1984) "Big Match" with lack of information on one side (Part I). *Internat. J. Game Theory* 13(4):201–255.

[33] Sorin S (2002) *A First Course on Zero-Sum Repeated Games* (Springer, Berlin).

[34] Sorin S (2011) Zero-sum repeated games: Recent advances and new links with differential games. *Dynam. Games Appl.* 1(1):172–207.

[35] Sorin S (2017) Limit value of dynamic games zero-sum games with vanishing stage duration: The symmetric case. In preparation.

[36] Sorin S, Vigeral G (2015) Reversibility and oscillations in zero-sum discounted stochastic games. *J. Dynam. Games* 2(1):103–115.

[37] Sorin S, Vigeral G (2016) Operator approach to values of stochastic games with varying stage duration. *Internat. J. Game Theory* 45(1):389–410.

[38] Souganidis PE (1985) Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations* 59(1):1–43.

[39] Tanaka K, Wakuta K (1977) On continuous Markov games with the expected average reward criterion. *Sci. Rep. Niigata Univ. Ser. A* 14:15–24.

[40] Vieille N (1992) Weak approachability. *Math. Oper. Res.* 17(4):781–791.

[41] Vigeral G (2013) A zero-sum stochastic game with compact action sets and no asymptotic value. *Dynam. Games Appl.* 3(2):172–186.

[42] Zachrisson LE (1964) Markov games. Dresher M, Shapley LS, Tucker AW, eds. *Advances in Game Theory*, Annals of Mathematical Studies, Vol. 52 (Princeton University Press, Princeton, NJ), 211–253.

[43] Ziliotto B (2016) Zero-sum repeated games: Counterexamples to the existence of the asymptotic value and the conjecture $\max\min = \lim v_n$. *Ann. Probab.* 44(2):1107–1133.