The LP Formulation of Finite Zero-Sum Games with Incomplete Information[1])

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Abstract: This paper gives the LP formulation for finite zero sum games with incomplete information using Bayesian mixed strategies. This formulation is then used to derive general properties for the value of such games, the well known concave-convex property but also the "piecewise bilinearity". These properties may offer considerable help for computational purposes but also provide structural guidelines for the analysis of special classes of games with incomplete information.

1. Introduction

The class of zero sum games under consideration is the following:

Let R, S, I, J be finite sets and P, Q be the symplexes of $\mathbf{R}^R, \mathbf{R}^S$. For each $r \in R$, $s \in S$, let $A^{rs} = ((a^{rs}_{ij}))$ be the $I \times J$ payoff matrix of a zero sum game in normal form, where I is the set of strategies of player 1 (the maximizer) and J for player 2.

Now chance chooses $r \in R$ (resp. $s \in S$) according to $p \in P$ (resp. $q \in Q$) and r is announced to player 1 (resp. s to player 2). Then the game A^{rs} is played. We also assume that all of the above description is common knowledge.

This game will be denoted by $G(p, q)$ and its value by $V(p, q)$. We are interested in the behavior of $V(p, q)$ as p and q vary in P and Q. It is clear that this class of games includes the finite repeated games with incomplete information and information matrices [*Aumann/Maschler*, 1967, 1968] as well as the games with "almost" perfect information [*Ponssard*, 1975a]. However, it should be noted that our attention is restricted here to the "independent" case.

The purpose of this paper is to give the LP (linear programming) formulation of these games using Bayesian mixed strategies, that is, strategies which are behavioral with respect to the initial chance move and normalized with respect to what is taking place after the chance move in whatever form it may appear: sequential, repeated, or other, provided it remains finite. A special case of this formulation was given in

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Ponssard [1975b]. This formulation is then used to derive general properties for the value of such games; this is done in section 3. Several applications are presented in section 4, in particular it is thought that the LP formulation may provide structural guidelines, for the analysis of optimal strategies as well as for the value. However the development of these ideas is reserved for a subsequent paper since more specific assumptions need to be made on the class of games under consideration. Finally, section 5 illustrates by an example the considerable aid for computational purposes which result from the properties proved in this paper.

2. The LP Formulation

Let us introduce the Bayesian mixed strategies of player 1, that is,

$x = (x_i^r, r \in R)$ where $x_i^r = $ Prob (move i/r is announced)

and denote by X the set of all these strategies. Similarly define, Y for player 2. Given x and y, the associated payoff in $G(p, q)$ is

$$M(x, y) = \sum_{r,s,i,j} p^r q^s a_{ij}^{rs} x_i^r y_j^s.$$

Thus we obtain the following.

Lemma 1: $V(p, q)$ is the value of the linear program

$$\text{Max} \sum_s q^s v^s$$

subject to

$$\sum_{i,r} a_{ij}^{rs} p^r x_i^r \geq v^s \quad \forall j, \forall s \tag{0}$$

$$\sum_i x_i^r = 1 \quad \forall r$$

$$x_i^r \geq 0 \quad \forall i, \forall r$$

Proof: Since

$$V(p, q) = \text{Max}_x \text{Min}_y M(x, y)$$

$$= \text{Max}_x \sum_s q^s \text{Min}_j \sum_{i,r} p^r a_{ij}^{rs} x_i^r$$

the result follows.

Let us now make a change of variable and define

$$\forall i, \ r \ \ \alpha_i^r = p^r \times x_i^r = \text{Prob}(i \text{ and } r)$$

$$\forall j, \ s \ \ \beta_j^s = q^s \times y_j^s = \text{Prob}(j \text{ and } s).$$

Now we have

Lemma 2: $V(p, q)$ is the value of the following dual programs

$$\text{Max } \sum_s q^s v^s \qquad\qquad\qquad \text{Min } \sum_r p^r w^r$$

s.t.   s.t.

$$\forall j, \forall s \ \sum_{i,r} a_{ij}^{rs} \alpha_i^r - v^s \geqslant 0 \qquad\qquad \forall i, \forall r \ \sum_{j,s} a_{ij}^{rs} \beta_j^s - w^r \leqslant 0$$

$$\forall r, \sum_i \alpha_i^r = p^r \qquad (1) \qquad \forall s, \sum_j \beta_j^s = q^s \qquad (2)$$

$$\forall i, \forall r, \alpha_i^r \geqslant 0 \qquad\qquad\qquad \forall j, \forall s, \beta_j^s \geqslant 0$$

(1) is the program for player 1 derived from program (0), but observe that p is now on the right side of the constraints so that the matrix of the LP is independent of p and q.

3. A Theorem

We first need the following definition.

Definition: A real function defined on $C \times D$ where C (resp. D) is a convex polyedron in $\mathbf{R}^m$ (resp. $\mathbf{R}^n$) is "piecewise bilinear" 'if there exists a finite partition of C (resp. D) into convex polyedra C_k, $k \in K$, (resp. D_l, $l \in L$) such that the restriction of f to each product $C^k \times D^l$ is bilinear.

Remark: If f is piecewise bilinear on $C \times D$ it follows that $f(\cdot, y)$ is piecewise linear on C for all $y \in D$ and similarly for $f(x, \cdot)$; but the converse is false.

Theorem: $V(p, q)$ is concave w.r.t. p, convex w.r.t. q and piecewise bilinear on $P \times Q$.

Proof: The concavity is straightforward since if (α', v') (resp. α'', v'') is feasible in (1) for (p', q) (resp. (p'', q)) then for all $\lambda \in [0, 1]$, $(\lambda\alpha' + (1 - \lambda)\alpha'', \lambda v' + (1 - \lambda)v'')$ is feasible for $p = \lambda p' + (1 - \lambda)p''$.

The convexity follows from duality.

Now we can write (1) as follows

$$\text{Max } \widetilde{q} \cdot z$$

subject to

$$B z \leqslant \widetilde{p} \tag{3}$$

where z is the vector (v, α), $\widetilde{q}$ is the $(S + I \times R)$ vector $(q, 0)$ and $\widetilde{p}$ is the $(2R + J \times S + I \times R)$ vector $(p, -p, 0)$, and B is independent of p and q. Note that $\varphi : p \rightarrow \widetilde{p}$ (resp. $\psi : q \rightarrow \widetilde{q}$) is a one to one linear mapping from P to $\widetilde{P} = \varphi(P)$ (resp. from Q to $\widetilde{Q} = \psi(Q)$). Hence, letting $\widetilde{V}(\widetilde{p}, \widetilde{q}) = V(p, q)$, it is enough to prove that $\widetilde{V}(\widetilde{p}, \widetilde{q})$ is piecewise bilinear.

For each $\widetilde{p}$ in $\widetilde{P}$, $\widetilde{V}(\widetilde{p}, \cdot)$ is piecewise linear on $\widetilde{Q}$: as $\widetilde{q}$ moves linearly, $\widetilde{V}(\widetilde{p}, \cdot)$ is linear until some face of the set of the feasible points in (3) is weakly parallel to $\widetilde{q}$. Hence the faces of this set define a finite subset of $\widetilde{Q}$, $\widetilde{Q}^*(\widetilde{p})$, which induces a finite partition of $\widetilde{Q}$ into convex polyedra by joining all the points of $\widetilde{Q}^*(\widetilde{p})$ to each other and to the extreme points of $\widetilde{Q}$. The restriction of $\widetilde{V}(\widetilde{p}, \cdot)$ to each polyedra is now linear.

It remains to show that, for all $\widetilde{p}$ in $\widetilde{P}$, $\widetilde{Q}^*(\widetilde{p})$ is included in the finite set $\widetilde{Q}^*$ defined as follows. Let us denote by B_n, $n \in N = \{1, \ldots, 2R + J \times S + I \times R\}$, the lines of the matrix B; and for all $T \subset N$ let

$$B_T = \{z \in \mathbf{R}^{S+IR} \mid B_n \cdot z = 0 \quad \forall n \in T\}$$

Let $H(\widetilde{q}) = \{z \in \mathbf{R}^{S+IR} ; \widetilde{q} \cdot z = 0\}$. Then

$$\widetilde{Q}^* = \{\widetilde{q} \in \widetilde{Q} \mid \exists T \subset N \text{ such that } B_T \neq \{0\} \text{ and } B_T \subset H(\widetilde{q})\}.$$

Since, as T varies, B_T describes all the directions of the faces of the feasible points, for all $\widetilde{p}$, this implies $\widetilde{Q}^*(\widetilde{p}) \subset \widetilde{Q}^*$. Hence the finite partition of $\widetilde{Q}$ induced by $\widetilde{Q}^*$ is independent of $\widetilde{p}$ and is a refinement of the partition induced by $\widetilde{Q}^*(\widetilde{p})$, for all $\widetilde{p}$. Thus $\widetilde{V}(\widetilde{p}, \cdot)$ is piecewise linear with respect to this partition for all $\widetilde{p}$.

Now by duality we have a similar result for $\widetilde{V}(\cdot, \widetilde{q})$ and this proves that $\widetilde{V}$ is piecewise bilinear.

Remark: The fact that $V(p, q)$ is concave convex, is a general property of zero-sum games with incomplete information [see *Aumann/Maschler*, 1967].

4. Application

4.1 Optimal Strategies

If we return now to the linear program (1) and denote by $\bar{v}(p, q)$ and $\bar{\alpha}(p, q)$ an optimal solution, it follows that they both can be choosen constant w.r.t. q and linear

w.r.t. p on each product $C_k \times D_l$.

Hence, the optimal Bayesian mixed strategies $\bar{x}(p, q)$ are constant w.r.t. q and differentiable w.r.t. p on each $C_k \times D_l$. In particular, player 1 randomizes over the same set of pure strategies in each $C_k \times D_l$. Observe also that if one knows the optimal $\bar{v}(p, q)$ then the optimal strategy for player 1 may be derived without further reference to q. It turns out that this property of the LP formulation will play a significant role in the derivation of optimal behavioral strategies in extensive games, p and v playing the role of state variables for player 1. These ideas will be explored further in a subsequent paper.

Finally, it is clear that this LP formulation may be extended to non zero sum-games and also offer in that case interesting insights into the structure of equilibrium strategies in the sense of Nash.

4.2 Computation of $V(p, q)$

From our theorem it follows that, in order to define V on $P \times Q$ it is enough to compute a finite number of values $V(p_m, q_n)$, (p_m, q_n) defined by the finite partitions of P and Q.

4.3 Repeated Games with Incomplete Information

The above description applies to finitely repeated games with incomplete information so that for these games $v_n(p, q)$ is "piecewise bilinear" for all n.

Nevertheless, this property does not extend to the limiting value for infinitely repeated games. In fact examples of such values which do no satisfy it can be founded in *Mertens/Zamir* [1971–1972].

It is worth noting that the value of the "average" game which plays a significant role in the value for infinitly repeated games does not possess the "piecewise bilinear" property. In the average game, the players do not use their information; its value may be obtained from the following program

$$\text{Max } v$$

subject to

$$\forall j \quad \sum_{i,r,s} a_{ij}^{rs} p^r q^s x_i \geqslant v$$

$$\sum_{i=1}^{m} x_i = 1$$

$$\forall i \quad x_i \geqslant 0$$

where p and q cannot be isolated as in program (1). Use of information is the key to the elimination of p and q from the LP matrix.

5. An Example

Consider the following

$$|I| = |J| = |R| = |S| = 2$$

$$A^{11} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \qquad A^{12} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$A^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \qquad A^{22} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

We shall denote (p^1, p^2) by $(p, p' = 1 - p)$ and similarly for q. $V(p, q)$ shall first be computed on the boundaries of the simplexes P and Q. Clearly

$$V(p, 0) = 0 \quad \forall p, \ V(1, q) = 0 \quad \forall q.$$

To compute $V(0, q)$, observe that player 2 has a dominating strategy, hence
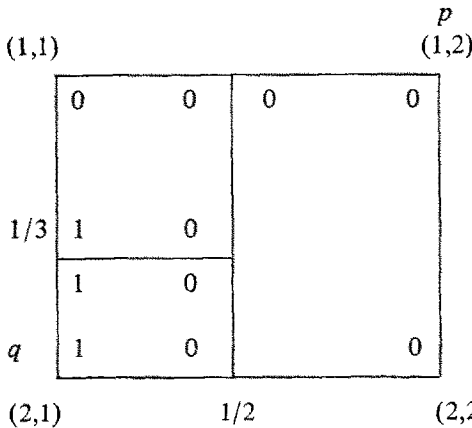
$$V(0, q) = 2q - 1 \qquad \text{for } q \geqslant 1/2.$$

$$V(0, q) = 0 \qquad \text{for } q \leqslant 1/2.$$

The derivation of $V(p, 1)$ requires some computation to obtain

$$V(p, 1) = 1 \qquad \text{for } p \leqslant 1/3$$

$$V(p, 1) = 3(1 - p)/2 \qquad \text{for } p \geqslant 1/3.$$

At this point, the simplest candidate for the value is the following function:

p

(1,1)   (1,2)

0	0	0	0	
1/3	1	0		
	1	0		
q	1	0		0

(2,1)   1/2   (2,2)

(In this figure, $V(p, q)$ is defined by its value at the extremal points of the polyedra $C_k \times D_1$.)

That this function is indeed the value may be proved by first checking that the

value of the game at $p = 1/3, q = 1/2$ is indeed 0, concavity of $V(p, 1/2)$ is then used to obtain $V(p, 1/2) = 0$, and by again checking the linearity of $V(1/3, q)$ for $q \geqslant 1/2$ in a similar fashion. Of course, in more complicated cases the first candidate may not be a good one; nevertheless, the piecewise bilinearity may be of considerable help.

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