JOURNAL OF ECONOMIC THEORY **69**, 431–446 (1996)
ARTICLE NO. 0062

Strategic Market Games with Exchange Rates

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Received September 20, 1994; revised July 16, 1995

Strategic market games describe mechanisms of formation of prices and of redistribution of goods. We introduce here a model without commodity money but where the use of fiat money does not imply the possibility of debt, hence the necessity for penalty. This is done through a collection of exchange rates that allows for a coherent price system and a redistribution that clears the market for any initial bids of the agents. *Journal of Economic Literature* Classification Numbers: C72, D51, E31.

1. PRESENTATION

We are dealing with the basic problem of redistribution of goods in the framework of an exchange economy. Following the literature on strategic market games (initiated by Shapley [12], Shubik [14], and Shapley and Shubik [13]), we model this procedure by describing explicitly the behavior of the agents and the corresponding process of formation of prices and exchange of goods.

There is first a mechanism (game form), where the agents are the players and their strategies are signals (in term of money and/or commodities to buy or sell on each trading post), which specifies as outcome a new allocation of the quantities announced. Price appear at this stage as an interim technical device.

Once endowments and utilities are added to the model, one can describe the set of feasible strategies for each player and evaluate the outcome in terms of utilities: one faces a strategic game.

* This work was initiated during a course given at the Université Paris X Nanterre and greatly benefited from the comments of the participants, especially Carlo Benetti and Jean Cartelier. I also thank Pradeep Dubey for fertile discussions and promising ideas on this topic and an associate editor for very helpful comments and references.

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0022-0531/96 $18.00

One could very schematically classify the existing models of strategic market games according to the following three categories:

(i) Models with commodity money (Shapley [12], Shapley and Shubik [13], Dubey and Shubik [5], Dubey and Shapley [4]). Assume that there are $m+1$ commodities; one good (say good $m+1$) is exchangeable for any of the others, it may have utility of its own and it is the sole medium of exchange. Hence there are m trading posts, corresponding to all exchange markets between good i, $i=1, ..., m$, and good $m+1$, where each agent can bid a quantity of good i for sale or of commodity money $m+1$ for buying.

All bids are real (cash in advance) so that there is no possibility for default and no need for penalty rules. Moreover all markets operate independently in term of prices and final utilities.

(ii) Models with fiat money (Shubik and Wilson [15], Postlewaite and Schmeidler [10], Mas Colell [6], Peck and Shell [7, 8], Peck *et al.* [9], Dubey and Shapley [4]). One "paper good" (bank money) is exchangeable for any of the m goods, it has no utility of its own and it is the sole medium of exchange. Bids are real or paper (goods or fiat money), so there is a possibility for default, hence a need for penalty rules. Here also all the m trading posts for goods operate independently to determine the prices but the final utility of each agent depends upon a budget constraint that links the operations on the different markets. In case of bankruptcy there might be a disutility (Dubey and Shapley [4]), a confiscation of the purchase (Postlewaite and Schmeidler [10]) or of all goods (Peck and Shell [8], Peck *et al.* [9]), or a global status quo (Peck and Shell [7]).

(iii) Models where all goods can be used for trade (Shapley windows, see Sahi and Yao [11]). Again all bids are real and there are no defaults hence no penalty rule. Each player bids a part of his initial endowment in exchange for good i, for all i. However, here, a coordination device is needed to obtain consistent prices and the markets do not operate independently even at this level. (If one wants to preserve the independence of the markets in this model one obtains exchanges without consistent prices, see Amir *et al.* [1].)

Note that a generalisation of both (i) and (iii) is possible, in terms of a graph describing the markets open for trade between goods i and j, see Dubey and Sahi [3].

Our approach here will be a combination of the above: the number of markets is equal to the number of goods. Signals in quantities are real (no promises to deliver, hence no "real" defaults—compare with Peck and Shell [7] where the fact that short sales are permitted allows asymptotic optimality) and signals in "money" are unconstrained. Then for each profile of signals, prices are determined and markets clear. Transactions occur through a paper money that has no intrinsic value. However to avoid default, this money is private, namely, the quantity depends only upon the agent that creates it, and its relative value, hence the exchange rate between these monies, will be determined endogeneously, at the same time as the prices, to clear the market and to respect budget constraints.

2. The Model

We will mainly follow the notations of Sahi and Yao [11] and most of the proofs are simple adaptations of theirs.

There is a set A of n agents and a set I of m goods.

There are m trading posts, one for each good $i \in I$, where each agent $\alpha \in A$ puts up a (real) quantity q_i^α of good i and an amount b_i^α of money for its purchase of good i.

In a model with "universal" money, each market (for good) i operates independently, defining a price p_i through the balancing equation

$$p_i\left(\sum_\alpha q_i^\alpha\right)=\sum_\alpha b_i^\alpha.$$

Then each agent α obtains back from this trading post i, $q_i^\alpha p_i$ units of money and b_i^α / p_i units of good i.

Obviously this is a redistribution of the initial offers in goods, hence all markets clear, but the budget constraint ($\sum_i q_i^\alpha p_i \geqslant \sum_i b_i^\alpha$) may be violated. Hence in a game theoretical framework penalities are necessary (like in Dubey and Shapley [4]) or one has to work with discontinuous violation rules or constrained strategies—in the spirit of a generalized game—like in the general equilibrium approach; see Postlewaite and Schmeidler [10], Peck and Shell [7], and Peck *et al.* [9].

To avoid this phenomenon, we will assume that each agent creates his own money (with no constraints) and we will show how an endogeneous value of each private money emerges in order to equilibrate each budget. We thus define for each α, an exchange rate t^α (which is a nonnegative number) to be understood as the value, in some unit of account, of one unit of the money used by agent α.

The equations defining the prices (in terms of unit of account) are thus

$$p_i\left(\sum_\alpha q_i^\alpha\right)=\sum_\alpha t^\alpha b_i^\alpha \tag{1}$$

and the budget rule gives

$$t^{\alpha}\left(\sum_i b_i^{\alpha}\right) = \sum_i q_i^{\alpha} p_i. \tag{2}$$

Each agent α will get back from trading post i the quantities

$$q_i^{\alpha} p_i \text{ units of account} \tag{3}$$

and

$$t^{\alpha} b_i^{\alpha}/p_i \text{ units of good } i. \tag{4}$$

Intuitively, if agent α spends too much, the value of his money, and hence of his bids, will decrease. Since his sales (in quantity) do not change but his purchases (in quantity) decrease, his deficit will decrease too. (Note that this "intuition" avoids taking into consideration the impact of the new value of α's money on prices).

It may appear strange that each agent receives back some "universal money" (3), since it is a useless good, and there is no debt to refund, like in the model with fiat money where one can consider that there is a credit system. In fact, as we will see later, the different monies are basically in this model part of a unit of account and the only use of Eqs. (1) and (2) is to define (4) or the following amount, which is the net trade in good i for agent α, namely,

$$z_i^{\alpha} = -q_i^{\alpha} + t^{\alpha} b_i^{\alpha}/p_i. \tag{5}$$

Remarks

(1) The mechanism is global (like in case (iii), described in Section 1 above): the prices and transactions cannot be defined independently on each trading post.

Note that prices on one hand and exchange rates on the other, are determined by a very similar procedure, namely, Eqs. (1) and (2). They do no emerge, like in Walrasian setting, from an outside announcement, meaningful only at equilibrium. For each profile of bids of the agents, only an accountant (or a computer) is needed to solve (1) and (2)—which translates the fact that the markets are not only centralized but also connected. Comparing with case (ii), Eq. (1) is the price formation, market by market (once the exchange rates are determined) and Eq. (2) is the budget constraint, linking the markets (once the prices are known). Here both operations are done at once to avoid defaults.

(2) Obviously the above system (1), (2) is homogeneous of degree zero: if (p, t) is a solution, the same is true for $(\lambda p, \lambda t)$, for all positive λ.

(3) The exchange rates are consistent: $t^\alpha/t^\gamma = (t^\alpha/t^\beta) \times (t^\beta/t^\gamma)$.

(4) Even if all exchange rates are the same, given some profile of bids the model differs from the one with bank money, since the consequences of a change in the bids are not the same.

EXAMPLE. Consider an economy with two agents and two goods. Assume that agent α sells good 1 and buys good 2, and assume a dual situation for agent β.

Without an exchange rate the deficit of agent α is $b_2^\alpha - b_1^\beta$. If there are no debts then α exchanges q_1^α for q_2^β.

Adding an exchange rate allows one to obtain this result for any positive bid in private money.

3. Basic Properties

Let us first introduce the following notations: $q_i = \sum_\alpha q_i^\alpha$ is the total amount of good i for sale and $b^\alpha = \sum_i b_i^\alpha$ is the total amount of money created by agent α. If q_i is 0, good i is eliminated, and similarly, if b^α is 0, agent α disappears from the economy.

A. Determination of the Prices

From (1) and (2), the exchange rates will be such that

$$t^\alpha = \sum_\beta t^\beta \left(\sum_i (q_i^\alpha / q_i)(b_i^\beta / b^\alpha) \right) \tag{6}$$

and the prices will satisfy

$$p_i q_i = \sum_\beta \left(\sum_j q_j^\beta p_j \right) b_i^\beta / b^\beta. \tag{7}$$

The above equation expresses the fact that the value of the quantity of good i on the trading post i is equal to the sum over all agents of their revenue multiplied by the fraction of their total bid that they offer for good i. An equivalent relation is

$$p_i = \sum_j p_j \left(\sum_\beta (q_j^\beta / q_i)(b_i^\beta / b^\beta) \right). \tag{8}$$

Defining $M_{ji} = \sum_\beta q_j^\beta (b_i^\beta / b^\beta)$ we will write (7) under the form

$$p_i q_i = \sum_j p_j M_{ji}. \tag{9}$$

Since we have $\sum_i M_{ji} = q_j$, we will normalize these quantities and introduce $\bar{M}$ defined by $\bar{M}_{ji} = M_{ji}/q_j$. So that if a vector $\bar{p}$ satisfies

$$\bar{p} = \bar{p}\bar{M}, \tag{10}$$

then p defined by $p_i = \bar{p}_i/q_i$ will be a solution of (9).

Since $\bar{M}$ is a stochastic matrix there always exists a nonzero solution of (10) with $\bar{p} \geqslant 0$: $\bar{p}$ corresponds to an invariant measure of the Markov chain defined by $\bar{M}$ on I. Moreover, there is unicity (modulo multiplication by a positive constant) on each irreducible component of $\bar{M}$. (Recall that a subset I' of I is irreducible if for each couple i, j in I', i leads to j in the sense that there is a chain $i_1, ..., i_n$ in I', starting at i and ending at j with $\bar{M}_{i_k i_{k+1}} > 0$, for all adjacent states. In the current framework it means that any good is directly or indirectly used to buy any other good).

Since M and $\bar{M}$ have the same set of zeros we have the same ergodic decomposition for both.

For a finite state Markov chain, any invariant distribution has full support (or is 0) on each irreducible component. Hence to avoid zero prices we have only to check that there are no transient states, i.e., a collection of goods that need the complementary goods to be exchangeable while the opposite is not true. (For example, if the demand for good i is zero, $b_i^\alpha = 0$, $\forall \alpha$).

Otherwise two solutions of (10) will induce the same net trade. In fact let I' be an irreducible component of I. Then if b_i^α or q_i^α is positive for some $i \in I'$, one has $b_j^\alpha = 0$ and $q_j^\alpha = 0$ for all $j \notin I'$. It follows that the two exchange rates for α will be proportional to the prices, hence by (5) the assertion holds. So we have shown:

LEMMA 1. *If $\bar{M}$ defines a recurrent chain, the market mechanism determines the net trade.*

In particular if M itself is irreducible, (relative) prices are uniquely determined.

B. Bids on Both Sides

We consider here what occurs if an agent bids on both sides of the market. Starting from a configuration of bids of the form (q, b) inducing prices p and exchange rates t we assume that agent α modifies his sale on good i to $q_i^\alpha + \theta$ ($\theta \geqslant 0$) and simultaneously increased his bid on trading post i up to $b_i^\alpha + \theta p_i/t^\alpha$. Then it is easy to see (using (6) and (7)) that: (a) the previous (p, t) are still solutions and (b) the new net trade is identical to the initial one.

As a consequence, as soon as all prices are defined, any feasible final allocation can also be achieved through a bid where all agents are requested to put up for sale all their initial endowments.

C. Restrictions and Normalizations

(a) Let us consider the restricted game where the players are forced to put their entire initial allocations on the market, hence $q_i^\alpha = \omega_i^\alpha$ for all α and all i. Following the previous discussion, if the trading posts are active at some equilibrium of this restricted game, this one induces an equilibrium in the original game. We will in fact construct equilibria sharing this property when dealing with the restricted game. It will be pleasant to normalize the bids in private money, using b_i^α / b^α as a parameter, so that a strategy of agent α is described by a point x^α in the simplex on I, $\Delta(I) = \{x \in \mathbb{R}_+^n ; \sum_i x_i = 1\}$.

(b) Other normalizations are possible, since if given $(b^{-\alpha}, q^{-\alpha})$, (b^α, q^α) induces $((t^\alpha, t^{-\alpha}), p)$, then $(\lambda b^\alpha, q^\alpha)$ gives rise to $((\lambda t^\alpha, t^{-\alpha}), p)$, for all positive λ, hence the same net trade occurs. One could then induce it with all exchange rates equal to 1 (but recall the previous Remark 4).

D. Manipulability

We show in this section that an agent α cannot benefit by pretending being two different agents, say types α' and α''.

Fix the $(n-1)$ profile of bids, $(\bar{q}, \bar{b})$, of the other $n-1$ agents. Consider the economy with $n+1$ agents where the two types of agent α are present and denote by (q', b') and (q'', b'') their bids. The induced price is p and $\bar{t}$ is the profile of exchange rates of the regular agents. t' and t'' are the exchange rates for the two types; z' and z'' are their corresponding vectors of net trade.

Then there exists, in the economy with n agents, a bid (q, b) of agent α inducing the same total net trade $z = z' + z''$ for him, when facing the profile $(\bar{q}, \bar{b})$. In fact let $q = q' + q''$ and $b = t'b' + t''b''$. It follows that the same price p and the n exchange rate profile $(1, \bar{t})$ will satisfy Eq. (1) and (2), hence the result by (5).

(Note that this property is an axiom in Dubey and Sahi [3].)

E. Discussion

If one agent α wants to transfer some fixed quantity (in units of account) of his bids from one trading post to another, several phenomena occur:

- first the relative prices will vary,
- then his own exchange rate will change,
- and finally the exchange rate of the other agents also will change.

If only the first two aspects were present, one would get the same equilibria as in the fiat money model (Peck *et al.* [9]), since the rule of price formation is identical. More precisely, starting from a profile of bids (b, q) satisfying the budget constraints (with exchange rate one for all agents), call an alternative bid $(\bar{b}^\alpha, \bar{q}^\alpha)$ of agent α nice if it equalizes α's budget constraint in the PSS game. If the other agents had fixed exchange rates, the use of a flexible exchange rate for an agent will simply prevent him from default, thus transforming any bid to a nice one as follows. An alternative bid $(\bar{b}^\alpha, \bar{q}^\alpha)$ gives rise to new prices $\bar{p}$ and a new exchange rate $\bar{t}^\alpha$ for agent α according to the equations

$$\bar{p}_i\left(\sum_{\beta \neq \alpha} q_i^\beta + \bar{q}_i^\alpha\right) = \sum_{\beta \neq \alpha} b_i^\beta + \bar{t}^\alpha \bar{b}^\alpha, \qquad \forall i$$

$$\bar{t}^\alpha \sum_i \bar{b}_i^\alpha = \sum_i \bar{q}_i^\alpha \bar{p}_i$$

so that $(\bar{t}^\alpha \bar{b}^\alpha, \bar{q}^\alpha)$ would be a nice alternative bid in the PSS game.

However the change in prices usually affects the budget constraints, for agent α as well as for the others, and this is reflected in all exchange rates and hence, again, in prices.

4. Existence of Equilibria

We consider now an economy with n agents, where each agent α has a nonnegative initial endowment $\omega^\alpha = (\omega_i^\alpha)$ and a utility function u^α defined on $\mathbb{R}_+^m$.

We assume that all goods are present in the economy and we normalize the quantities, hence $\omega_i = \sum_\alpha \omega_i^\alpha = 1$.

We will at some point use the following assumptions:

A.1. All u^α are continuous, concave and increasing.

A.2. There is a set A^* containing at least two agents, having positive initial endowments of any good and continuously differentiable utilities satisfying $u^\alpha(w) \geqslant u^\alpha(\omega^\alpha)$ implies $w_i > 0$, $\forall i$. (This amounts to ask for some (weak) complementarity in goods and will prevent one agent from manipulating a trading post by himself.)

The game Γ induced by the previous mechanism has for strategy the bids (q, b) satisfying $q_i^\alpha \leqslant \omega_i^\alpha$, for all i, α and for payoff the utility of the reallocation $\omega + z$.

Obviously Γ has equilibria like $q^\alpha = 0$, $\forall \alpha$. However, we will also exhibit *active* ones, i.e., all trading posts are active (p_i and q_i are positive, for all i).

This will follow from the usual procedure using an ε-perturbation of the game (see Shapley [12]). Then a unique price will be defined for any bids and we will deduce from A.1 the existence of an active equilibrium of the perturbed game. Assuming A.2 implies that all corresponding (relative) prices p_ε will remain bounded uniformly (in ε). Then a limit of equilibrium strategies of the perturbed games will be an equilibrium in the initial one.

The natural way of perturbing the game is to add a small amount of goods for sale, on each market, and to do similarly for the bids. Then all trading posts are active and one can use the normalization defined in Section 3.C.a. The corresponding strategies are called *saturated.*

To define, for any positive ε, the perturbed game Γ_ε, it remains to specify the payoff associated with a profile $(x^\alpha, \alpha \in A)$ of strategies.

Add a fictitious agent (called 0) with an initial endowment $\omega_i^0 = \varepsilon$, $\forall i$ and a uniform monetary bid $x_i^0 = 1/n$, $\forall i$. Let also $A^0 = A \cup \{0\}$. Note that the exchange matrix of the economy with $n+1$ agents, corresponding to any of these bids $((\omega^\beta, x^\beta), \beta \in A^0)$, has all positive coefficients, and hence is irreducible. Hence any *saturated* equilibrium, i.e., where the players are restricted to saturated strategies, is an equilibrium in the original game. The corresponding net trade (given (5)) is $z(\varepsilon)$ and agent α's payoff is $u^\alpha(\omega^\alpha + z^\alpha(\varepsilon))$.

Recall that the prices are given by (9), hence in our new notations by

$$p_i(1+\varepsilon) = \sum_j p_j \left(\sum_{\beta \in A^0} \omega_j^\beta x_i^\beta \right). \tag{11}$$

Similarly (1) and (2) are now

$$p_i(1+\varepsilon) = \sum_{\alpha \in A^0} t^\alpha x_i^\alpha \tag{12}$$

and

$$t^\alpha = \sum_i \omega_i^\alpha p_i, \qquad \forall \alpha \in A^0. \tag{13}$$

Hence by (5)

$$z_i^\alpha(\varepsilon) = -q_i^\alpha + (1+\varepsilon) - \sum_{\beta \neq \alpha,\, \beta \in A^0} \left(\sum_j (p_j/p_i)\, \omega_j^\beta \right) x_i^\beta \tag{14}$$

and the final holding in good i is

$$f_i^\alpha = 1 + \varepsilon - \sum_{\beta \neq \alpha,\, \beta \in A^0} \left(\sum_j (p_j/p_i)\, \omega_j^\beta \right) x_i^\beta. \tag{15}$$

So that, as remarked by Shapley [12] and by Sahi and Yao [11], the outcome for agent α depends upon his bid only through its effect on the prices.

Let us now study properties of the best reply correspondence; we fix the strategies x^β for $\beta \neq \alpha$, $\beta \in A^0$.

We are first led to consider the set of feasible prices that agent α can generate as his own strategy x^α varies in $\Delta(I)$, and the analysis will be very similar to the one used by Sahi and Yao (1989).

Define two matrices C and D by $C_{ji} = \sum_{\beta \neq \alpha} \omega_j^\beta x_i^\beta$ and $D_{ji} = \omega_j^\alpha x_i^\alpha$. Then (11) is now

$$p(1+\varepsilon) = pC + pD \tag{16}$$

Lemma 2. *p is feasible iff all its components are positive and it satisfies*

$$p(1+\varepsilon) \geqslant pC.$$

Proof. The condition is clearly necessary by (16).

On the other hand, assume p normalized ($\sum_i p_i = 1$) and recall that $\omega^\alpha = (1+\varepsilon)\, u - Cu$, where u has all its components equal to 1. Define $a = p(1+\varepsilon) - pC$. If ω is not the null vector, the scalar product $(p \cdot \omega)$ is a positive number r (and equal to $(a \cdot u)$). Then $x_i^\alpha = a_i / r$ will induce a matrix D such that (16) holds. ▮

This characterization allows us to deduce the following:

Corollary 3. *Given two strategies $x^\alpha[1]$ and $x^\alpha[1]$ of agent α inducing prices $p[1]$ and $p[2]$ and final holdings $f^\alpha[1]$ and $f^\alpha[2]$, there exists a strategy $\bar{x}^\alpha$ inducing a price $\bar{p}$ with $\bar{p}_i = \sqrt{p_i[1]\, p_i[2]}$ and a final holding $\bar{f}^\alpha \geqslant (f^\alpha[1] + f^\alpha[2])/2$, with strict inequality if $p[1] \neq p[2]$.*

Proof. By the previous Lemma 2, $\bar{p}$ is feasible, using Cauchy Schwartz inequality. Using (15) and the comparison of arithmetic and geometric means gives the second assertion. ▮

From now on we assume A.1 and then we can state an existence theorem.

Theorem 4. *For each positive ε, the perturbed game Γ_ε has a saturated equilibrium.*

Proof. The strategy sets are convex and compact for each agent. By Corollary 4 above and assumption A.1, there is only one price associated to a best reply in term of final allocation. Finally by (11) the set of bids x^α compatible with a specific price is closed and convex. It is easy to see that the payoff function is jointly continuous, hence the best reply correspondence satisfies the hypotheses of Kakutani's fixed point theorem. ∎

It remains to show the existence of equilibria in Γ.

From now on we assume A.2.

We consider a sequence of equilibria in perturbed games Γ_ε as ε goes to zero. We will obtain a uniform property for the prices that will thus hold at the limit and will imply that for at least two agents (namely in A^*) the limit of equilibrium bids are in the interior of the simplex. This will in turn imply that the limits of equilibrium strategies do form equilibrium strategies. Again the proof is deeply related to the one in Sahi and Yao [11].

Lemma 5. *There exist two positive constants η and δ such that, for all $\varepsilon \geqslant 0$, all saturated equilibria of Γ_ε have bids satisfying $x_i^\alpha \leqslant \delta$, $\forall i$, $\forall \alpha \in A^*$ and induce a normalized price with $p_i \geqslant \eta$, $\forall i$.*

Proof. The proof follows that of Lemma 9 in Sahi and Yao [11] and we will sketch the main lines.

Consider a saturated equilibria with a profile of bids x and prices p. We can suppose that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$. By feasibility, f^α remains in a compact set K^α and by assumption A.2, $f_i^\alpha \geqslant \bar{f}_i^\alpha > 0$, $\forall i$, $\forall \alpha \in A^*$. Using (15) and Lemma 2, this strict inequality implies that any of these rich agents α can generate any price in a small neighborhood of p, in particular $q = (p_1, ..., p_k, (1+\lambda)\, p_{k+1}, ..., (1+\lambda)\, p_m)$, for λ positive and small enough. We use again (15) (and the notations in (16)) to compute the impact on the new reallocation:

$$\Delta f_i^\alpha = \begin{cases} -\sum_{j \geqslant k+1} \lambda C_{ji} p_j / p_i & \text{for} \quad i \leqslant k \\ \sum_{j \leqslant k} (\lambda/(1+\lambda))\, C_{ji} p_j / p_i & \text{for} \quad i \geqslant k+1. \end{cases}$$

Hence we obtain for the utility

$$\Delta u^\alpha = (\lambda/(1+\lambda)) \sum_{i \geqslant k+1,\, j \leqslant k} C_{ij} (p_j/p_i)\, \partial_i u^\alpha(f^\alpha) - \lambda \sum_{i \leqslant k,\, j \geqslant k+1} C_{ij}(p_j/p_i)\, \partial_i u^\alpha(f^\alpha) + o(\lambda). \tag{17}$$

Let us introduce $m_1 = \max\{\partial_i u^\alpha(y);\, i \in I,\, y \in K^\alpha,\, \alpha \in A^*\}$ and similarly m_2 for the min. From (17) we obtain

$$\Delta u^{\alpha} \geqslant (\lambda/(1+\lambda))\, m_2/p_{k+1} \sum_{i \geqslant k+1, j \leqslant k} C_{ij} p_j$$
$$-\lambda m_1/p_k \sum_{i \leqslant k, j \geqslant k+1} C_{ij} p_j + o(\lambda). \tag{18}$$

Recall that $M = D + C$ (the decomposition depending on α) and that (9) says

$$p_i \sum_i M_{ij} = \sum_j p_j M_{ji}.$$

Hence,

$$\sum_{i \geqslant k+1, j \leqslant k} M_{ij} p_j = \sum_{i \leqslant k, j \geqslant k+1} M_{ij} p_j (= v).$$

In particular, for one agent α one has $\sum_{i \geqslant k+1, j \leqslant k} C_{ij} p_j \geqslant v/2$ and since $\sum_{i \leqslant k, j \geqslant k+1} C_{ij} p_j \leqslant \sum_{i \leqslant k, j \geqslant k+1} M_{ij} p_j$, one obtains from (18)

$$\Delta u^{\alpha} \geqslant \lambda v \left(\frac{m_2}{p_{k+1}}\right)\left(\frac{1}{2(1+\lambda)} - \left(\frac{m_1}{m_2}\right)\left(\frac{p_{k+1}}{p_k}\right)\right) + o(\lambda)$$

The equilibrium condition thus implies

$$\frac{p_k}{p_{k+1}} \leqslant 2(1+\lambda)\left(\frac{m_1}{m_2}\right)$$

so that

$$p_m \left(2(1+\lambda)\left(\frac{m_1}{m_2}\right)\right)^{m-1} \geqslant p_1,$$

and we can choose $\eta = (1/m)(3(m_1/m_2))^{1-m}$.

The bound on this bids follows since the prices are bounded and the rich agents will end with at least $\bar{f}^{\alpha}$. ∎

THEOREM 6. *Γ has a saturated active equilibrium.*

Proof. Consider a sequence x_ε of saturated equilibria and corresponding prices p_ε in Γ_ε converging as ε goes to 0, to some (x, p) that will satisfy the conditions of Lemma 5. It is clear that p is a price induced by the bids (ω, x) in Γ.

It remains to be shown that these bids form an equilibrium in Γ. Assume not: then one agent α obtains a payoff, greater by some positive amount ρ, by using an alternative strategy (q^α, y^α). Since there exists at least one other rich agent $\beta \neq \alpha$ with positive bids, the exchange matrix is irreducible and we can assume that $q^\alpha = \omega^\alpha$. It follows then by continuity that the same saturated deviation would be $\rho/2$ profitable against the profile x_ε^β, $\beta \neq \alpha$, in Γ_ε for ε positive and small enough. A contradiction. ∎

5. Convergence to Walrasian Equilibrium

Denote by ${}^k\Gamma$ the k-fold replication of the game Γ where each agent α is replaced by k copies of himself, i.e., with the same endowment and utility. We define a symmetric equilibrium of ${}^k\Gamma$ as a kn equilibrium profile x where each of the k α-like agents has the same strategy x^α. $\tilde{x}$ will be the reduced profile of size n with components x^α.

We first have:

Lemma 7. *For any k, ${}^k\Gamma$ has a saturated active symmetric equilibria. Moreover any of these satisfies the properties of Lemma* 5.

Proof. To get the existence of a symmetric equilibrium, one considers the restriction of the best reply correspondence to symmetric strategies. It has nonempty values since all α-like agents are facing the same situation. Then the Kakutani theorem applies as above.

Concerning the properties of saturated active equilibria, the proof relies only on assumptions A.1 and A.2 and the bounds depend only on endowments and utilities—and not on the number of agents. ∎

Given an economy composed of agents with endowments ω^α and utility u^α we define as usual a price π and an allocation y^α as a competitive equilibrium if $\sum_\alpha y^\alpha = \sum_\alpha w^\alpha$, and for each α, y^α maximizes $u^\alpha(\cdot)$ under the constraint $\sum_i y_i^\alpha \pi_i = \sum_i \omega_i^\alpha \pi_i$.

We consider now a sequence of replica ${}^k\Gamma$ as k goes to infinity and a corresponding sequence of saturated active symmetric equilibria with prices $p(k)$, (reduced) profile of strategies $\tilde{x}(k)$, and (reduced) final holding $\tilde{f}(k)$ converging to some (p^*, x^*, f^*). Then we have the following convergence result:

Theorem 8. *(f^*, p^*) is a competitive equilibrium of the economy associated with Γ.*

Proof. Once more the proof follows Sahi and Yao [11] and we will sketch it.

Note first that f^* is a reallocation and budget balancing given p^*, by continuity, using (5) or equivalently

$$f_i^{*\alpha} = x_i^{*\alpha} \left(\sum_j \omega_j^\alpha p_j^* \right) \Big/ p_i^*. \tag{19}$$

Now if f^* is not a competitive allocation, let α and φ be an agent and an allocation with $(\varphi \cdot p^*) = (\omega^\alpha \cdot p^*)$ and $u^\alpha(\varphi) > u^\alpha(f^{*\alpha})$. Assuming first the prices independent of the bids there exists a strategy χ that induces by (19) the allocation φ.

If one agent of type α uses this strategy rather than $x^\alpha(k)$ in the game ${}^k\Gamma$, the price will be some $p'(k)$ and the new allocation of this agent $\varphi(k)$. Now it is clear that, as k goes to infinity, due to the uniform bound on prices the influence of one agent on the prices goes to 0, hence the difference $p'(k) - p(k)$ converges to 0, so that by (19) $\varphi(k)$ converges to φ. This implies that for k large enough the strategy χ would induce a profitable deviation in ${}^k\Gamma$. ▮

6. Comments

(1) The convergence result is slightly stronger than that in Sahi and Yao [11] since all saturated equilibria converge (we do not need to ask for δ positiveness since with our strategy sets this is implied by A.2 and the bound on prices). As remarked in Sahi and Yao [11] the hypotheses needed for the convergence to competitive equilibrium are weaker than in related models with commodity money (Dubey and Shubik [5]), where in addition to desirability hypotheses, conditions on the amount and the distribution of commodity money are necessary.

(2) In the game Γ there might exist active equilibria with $q_i^\alpha \neq \omega_i^\alpha$. Moreover they might not be equivalent, in terms of final trade, to sell-all equilibria, since the best reply correspondence will differ.

(3) Similarly ${}^k\Gamma$ may have nonsymmetric equilibria but then one should compare the limit to an atomless economy with types.

The present model has some features in common with both model (ii) and model (iii) introduced in Section 1. However, there are conceptual differences:

(4) Comparison concerning the use of money: If the means of payment is bank money or fiat money, there is a need for penalty. Moreover, with bank money the amount available is bounded and one has to introduce quotas or allocation rules. In the case of private money, there

is no penalty but a social evaluation through exchange rates which avoids a domino effect.

(5) Comparison with Shapley windows: The clearing system (iii) allows for more exchange than the first one since one can at the same time buy with the money coming from the sales (while in (i) one has to wait for the next round to spend the new income) without taking the risk of spending too much (which is the case in (ii)).

Recall that in this framework one sets on trading post i an amount of goods that one wants to exchange for good i (offers for buying). Then all these goods corresponding to all trading posts are registered and classified according to their nature. This defines the quantity for sale on each trading post.

In the present model, each agent puts on each trading post i the quantity of good i he wants to sell, then he bids for each good using his private money. We already remarked that an equivalent formulation is that the agent bids a fraction of his (unknown) income.

In fact, in the Shapley windows framework, if all the bids are proportional to some fixed vector z, say $z(i)$ is the bid on trading post i with $z(i) = x_i z$, $x \in \Delta(I)$, then they correspond to the strategies (z, x) in the present model. Hence the present game can be viewed as a restriction in terms of strategies of the previous one. Obviously this does imply in general a relation in terms of equilibrium sets.

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