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Some Results on Zero-Sum Games with Incomplete Information: The Dependent Case

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Abstract: In games with incomplete information, the players' states of information may be determined either through independent chance moves or through a unique one. Generally, a unique chance move generates some dependance in the players' state of information thus giving rise to significant complications in the analysis. However, it turns out that many results obtained in the simpler independent case have their counterpart in the dependent one. This is proved in this paper for several previous results of the authors.

1. Introduction

The class of games under consideration is the following.

(i) Let G be a finite two person game tree with its rules (sequence of moves and information sets).

(ii) Let M_h be the zero sum payoff associated with a play h of G, M_h is a discrete random variable defined by:

Prob $(M_h = m_h^k) = p^k$ where $p \in P$ (the simplex of $\mathbf{R}^K$) is a common knowledge probability. Moreover the private information structure is the following [*Mertens/Zamir*].

There are two partitions of $K = \{1, \ldots, k, \ldots, L\}$ denoted by:

$$K^{\mathrm{I}} = \{K_1^{\mathrm{I}}, \ldots, K_A^{\mathrm{I}}\}$$

$$K^{\mathrm{II}} = \{K_1^{\mathrm{II}}, \ldots, K_B^{\mathrm{II}}\}$$

such that if chance chooses k according to p, player 1 – the maximizer – is informed of a and player 2 – the minimizer – is informed of b where k belongs to $K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}$.

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All the above description is common knowledge and we shall denote by $V(p)$ the value of this game.

The independent case is obtained when $K = \{(a, b): a = 1, \ldots, A, b = 1, \ldots, B\}$, $K_a^{\mathrm{I}} = \{(a, b): b = 1, \ldots, B\}$, $K_b^{\mathrm{II}} = \{(a, b): a = 1, \ldots, A\}$ and $p^k = r^a q^b$ where $r = (r^1, \ldots, r^A)$, $q = (q^1, \ldots, q^B)$ are two probability vectors.

We shall extend to the dependent case results proved for the simpler independent case.

In section 2, we shall work with games with almost perfect information [*Ponssard*] and generalize a recursive formula for the value of such games. Section 3 will be devoted to the value of repeated sequential games with incomplete information [*Sorin*].

In the last section, we shall prove properties of $V(p)$ using linear programming [*Ponssard/Sorin*].

2. Value of Games with Almost Perfect Information

We shall assume in this section that the sequences of moves and the information sets are as follows.

After chance's move k player 1 chooses some $i_1 \in I_1$ which is told to player 2. Then player 2 chooses some $j_1 \in J_1$ which is told to player 1 and so on. Finally player 1 receives

$$m^k (i_1, j_1, \ldots, i_T, j_T)$$

from player 2.

We assume that T and all the sets I_t, J_t, $t \in T$ are finite (so that we are in the framework of section 1).

Let us introduce the following nonempty convex compact subsets of P. [*Mertens/Zamir*].

$\Pi_{\mathrm{I}}(p) = \{(\alpha^1 p^1, \ldots, \alpha^k p^k, \ldots, \alpha^L p^L) \mid \alpha^k \geqslant 0, k = 1, \ldots, L, \sum_k \alpha^k p^k = 1$ and $\alpha^k = \alpha^{k'}$ if k and k' belong to the same $K_a^{\mathrm{I}}\}$.

$\Pi_{\mathrm{II}}(p) = \{(\beta^1 p^1, \ldots, \beta^k p^k, \ldots, \beta^L p^L) \mid \beta^k \geqslant 0, k = 1, \ldots, L, \sum_k \beta^k p^k = 1$ and $\beta^k = \beta^{k'}$ if k and k' belong to the same $K_b^{\mathrm{II}}\}$.

For every real function defined on P, we shall denote by $\operatorname{Cav}_{\mathrm{I}} f$ the smallest real function u such that $\forall p \in P$, u restricted to $\Pi_{\mathrm{I}}(p)$ is concave (such a function will be said to be I-*concave*) and $u(p) \geqslant f(p)$ on P; and similarly, $\operatorname{Vex}_{\mathrm{II}} f$ is the greatest II-convex function smaller than f [see *Mertens/Zamir*].

We can now state the following proposition (extension of theorem 1 of *Ponssard* [1975]).

Proposition 1:

$$V(p) = \operatorname*{Cav}_{\mathrm{I}} \ \max_{i_1 \in I_1} \ \operatorname*{Vex}_{\mathrm{II}} \ \min_{j_1 \in J_1} \ \ldots \ \operatorname*{Cav}_{\mathrm{I}} \ \max_{i_T \in I_T} \ \operatorname*{Vex}_{\mathrm{II}} \ \min_{j_T \in J_T}$$

$$\{\sum_{k=1}^{L} p^k m^k (i_1, j_1, \ldots, i_T, j_T)\}.$$

The proof of this proposition can be obtain as in *Ponssard* [1975] by using a compounded game.

Let us denote by $V^{i_1}(p)$ the value of G^{i_1}, i.e. the game G where player's 1 first set of strategies I_1 is reduced to $\{i_1\}$.

Proposition 2:

$$V(p) = \underset{\mathrm{I}}{\mathrm{Cav}} \underset{i_1 \in I_1}{\mathrm{Max}} V^{i_1}(p)$$

Proof: We obviously have $V(p) \geqslant V^{i_1}(p) \ \forall i_1 \in I_1, \ \forall p \in P$.

Since $V(p)$ is I-concave ([see *Mertens/Zamir*, p. 49] an alternative proof is given in section 4) it follows that:

$$V(p) \geqslant \underset{\mathrm{I}}{\mathrm{Cav}} \underset{i_1 \in I_1}{\mathrm{Max}} V^{i_1}(p).$$

To prove the opposite inequality let us normalize the strategies in the game where player 1's first move is i_1. Since I_t and J_t are finite for all t, there exist two finite sets $\mathcal{I}$ and $\mathcal{J}$ such that the strategies in G^{i_1} are given by:

$y(i_1, a)$, a probability distribution over $\mathcal{I}$ for all $i_1 \in I_1, a \in A$

$z(i_1, b)$, a probability distribution over $\mathcal{J}$ for all $i_1 \in I_1, b \in B$

and let $M(i_1, k)$ be the $|\mathcal{I}| \times |\mathcal{J}|$ payoff matrix of the compounded game if chance choses k.

Let $x(a)$ be a probability distribution over I_1, for all $a \in A$. Then the payoff associated with the strategies $((x, y), z)$ in G is

$$H(p, x, y, z) = \sum_{a,b} \sum_{\substack{i_1, i, j \\ k \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}}} p^k x_{i_1}(a) y_i(i_1, a) m_{ij}^k(i_1) z_j(i_1, b).$$

Let us now introduce

$$\bar{x}_{i_1} = \sum_a \sum_{k \in K_a^{\mathrm{I}}} p^k x_{i_1}(a)$$

and define for all i_1, a, k with $k \in K_a^{\mathrm{I}}$,

$$p_{i_1}^k = \frac{p^k \cdot x_{i_1}(a)}{\bar{x}_{i_1}} \qquad \text{if } \bar{x}_{i_1} \neq 0$$

$$= \text{arbitrary} \qquad \text{if } \bar{x}_{i_1} = 0.$$

Since $x_{i_1}(a)$ depends on k only through a it follows that p_{i_1} belongs to $\Pi_\text{I}(p)$ for all i_1 whenever $\bar{x}_{i_1} \neq 0$.

Now we have (where $\sum\limits_k$ means $\sum\limits_{a,b} \sum\limits_{k \in K_a^\text{I} \cap K_b^\text{II}}$)

$$H(p, x, y, z) = \sum_{k,i_1,i,j} p_{i_1}^k \bar{x}_{i_1} y_i(i_1, a)\, m_{ij}^k(i_1)\, z_j(i_1, b)$$

$$= \sum_{i_1} \bar{x}_{i_1} \sum_{k,i,j} p_{i_1}^k y_i(i_1, a)\, m_{ij}^k(i_1)\, z_j(i_1, b).$$

Using the minimax theorem we obtain

$$V(p) = \max_{x,y} \min_z H(p, x, y, z)$$

$$= \max_x \sum_{i_1} \bar{x}_{i_1} \max_{y(i_1,.)} \min_{z(i_1,.)} \sum_{k,i,j} p_{i_1}^k y_i(i_1, a)\, m_{ij}^k(i_1)\, z_j(i_1, b)$$

$$= \max_x \sum_{i_1} \bar{x}_{i_1} V^{i_1}(p_{i_1}) \text{ by definition of } y, z \text{ and } m.$$

Thus

$$V(p) \leqslant \max_x \sum_{i_1} \bar{x}_{i_1} \operatorname*{Max}_{l \in I_1} V^l(p_{i_1})$$

Using the fact that $\sum\limits_{i_1} \bar{x}_{i_1} p_{i_1} = p$ and $\sum\limits_{i_1} \bar{x}_{i_1} = 1$ it follows that

$$\sum_{i_1} \bar{x}_{i_1} \operatorname*{Max}_l V^l(p_{i_1}) \leqslant \operatorname*{Cav}_\text{I} \operatorname*{Max}_l V^l(p).$$

Hence $V(p) \leqslant \operatorname*{Cav}\limits_\text{I} \operatorname*{Max}\limits_l V^l(p)$.

(This proof is a generalization of the proof of the main theorem of *Ponssard/Zamir* [1973]).

Proof of Prop. 1:

This follows from recursive application of Prop.2 to each move of the game since the value of the $i_1, j_1, \ldots, i_T, j_T$ restricted game is precisely

$$V^{i_1,j_1,\ldots,i_T,j_T} = \sum_{k=1}^{L} p^k m^k(i_1, j_1, \ldots, i_T, j_T).$$

3. Convergence of the Value for Sequential Games

We now consider a specification of the game described in section 2: in fact we assume that

$$I_t = I, J_t = J \text{ for all } t \text{ and } m^k(i_1, j_1, \ldots, i_n, j_n) = \frac{1}{n} \sum_{t=1}^{n} m^k(i_t, j_t)$$
for all k, n.

G is now a sequential repeated game. We shall denote by $v_n(p)$ the value of the n-th repeated game and allow n to vary in order to study the asymptotic behavior of the value.

Let us introduce some notation.
$u(p)$ is the value of the average game with payoff matrix given by:

$$\sum_{k=1}^{L} p^k M^k = M(p).$$

Consider the following system:

$$\begin{aligned} f(p) &= \underset{\mathrm{I}}{\mathrm{Cav}} \min \{f(p), u(p)\} \\ f(p) &= \underset{\mathrm{II}}{\mathrm{Vex}} \max \{f(p), u(p)\}. \end{aligned} \tag{1}$$

In *Mertens/Zamir* [1971] it is shown that this system has only one solution which we shall denote by $v(p)$. Moreover they prove that $\lim_{n\to\infty} v_n$ exists and equals $v(p)$.

Proposition 3:

(i) The sequence $v_n(p)$ is increasing in n for all p.

(ii) $v(p) - v_n(p)$ is bounded by C/n for some $C \in \mathbf{R}^+$ and this is the best bound.

Proof: The above proposition is proved in the independent case by using only formula (1) and the recursive formula of *Ponssard* [1975] [see *Sorin*]. Since this one extends to the dependent case (section 2) the proposition follows.

4. The L.P. Formulation

Since any finite zero-sum game in normal form is equivalent to a linear program we shall use the normalized strategies of the game G defined in section 1. Let us denote by i, $i \in I = \{1, \ldots, m\}$ and j, $j \in J = \{1, \ldots, n\}$ the moves of player 1 and player 2. A Bayesian mixed strategy for player 1 is now given by

$$x = (x_i(a); i \in I, a \in A) \text{ such that } x_i(a) = \text{Prob (move } i \mid k \in K_a^{\mathrm{I}})$$

and similarly for player 2

$$y = (y_j(b); j \in J, b \in B) \text{ such that } y_j(b) = \text{Prob (move } j \mid k \in K_b^{\mathrm{II}}).$$

Finally M^k is the $m \times n$ payoff matrix if chance chooses k. Now we have:

Lemma 1: $V(p)$ is the value of the following program.

$$\begin{aligned} &\operatorname{Max} \sum_b v^b \\ \text{s.t.}\quad & \forall b \forall j \sum_a \sum_{\substack{k \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}} \\ i}} p^k x_i(a) m_{ij}^k \geqslant v^b \\ & \forall a \ \sum_i x_i(a) = 1 \\ & \forall a \forall i \ \ x_i(a) \geqslant 0. \end{aligned} \tag{2}$$

Since the payoff associated with the strategies x and y is

$$H(x, y) = \sum_{k,i,j} p^k x_i(a) y_j(b) m_{ij}^k$$

we have by the min-max theorem

$$\begin{aligned} V(p) &= \operatorname*{Max}_x \operatorname*{Min}_y H(x, y) \\ &= \operatorname*{Max}_x \sum_b \operatorname*{Min}_j \sum_a \sum_{k_i \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}} p^k x_i(a) m_{ij}^k \end{aligned}$$

so that the result follows.

Now, for all $p \in P$ let us introduce the following set:

$$Q(p) = \{q \in P; q^k = \delta \alpha^k \beta^k p^k \text{ where } p_\alpha = (\ldots, \alpha^k p^k, \ldots) \text{ belongs to } \Pi_{\mathrm{I}}(p) \text{ and } p_\beta = (\ldots, \beta^k p^k, \ldots) \text{ belongs to } \Pi_{\mathrm{II}}(p)\}.$$

Remark: Note that we always have $\Pi_{\mathrm{I}}(p) \subset Q(p)$ and $\Pi_{\mathrm{II}}(p) \subset Q(p)$. Moreover if $q \in Q(p)$ with $q^k = \delta \alpha^k \beta^k p^k$ then $q \in \Pi_{\mathrm{I}}(p_\beta) \cap \Pi_{\mathrm{II}}(p_\alpha)$.

Definition: A function on $D \subset P$ is said to be I-*linear* if its restriction to each $D \cap \Pi_{\mathrm{I}}(p)$ is linear. Similarly, for II-*linear*, and I-II-*bilinear* if it is both I and II linear.

Theorem I: $V(p)$ is I-concave and II-convex on P. For all $p \in P$ there is a finite partition of $Q(p)$ into convex polyhedra C_t such that $V(p)$ is I-II-*bilinear* on each C_t.

From lemma 1 it follows that if $q \in Q(p_0)$, $V(q)$ is the value of the following program.

$$\begin{aligned} &\text{Max } \Sigma\, \beta^b v^b \\ &\text{s.t.} \\ &\forall b\, \forall j\ \sum_{\substack{a \\ i}} \sum_{k \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}} \delta p_0^k\, x_i(a)\, \alpha^a\, m_{ij}^k \geqslant v^b \\ &\forall a\ \sum_i x_i(a) = 1. \\ &\forall i\ \forall a\ x_i(a) \geqslant 0. \end{aligned} \tag{3}$$

Let us denote $\delta x_i(a)\alpha^a$ by $y_i(a)$ for all i, a, so that the constraints are now

$$\begin{aligned} &\forall j \forall b\ \sum_{\substack{a \\ i}} \sum_{k \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}} p_0^k\, y_i(a)\, m_{ij}^k \geqslant v^b \\ &\forall a\ \sum_i y_i(a) = \delta \alpha^a \\ &\forall i\, \forall a\ y_i(a) \geqslant 0. \end{aligned} \tag{4}$$

Now let $q' = (\delta' \alpha'^k \beta^k p^k)$ and $q'' = (\delta'' \alpha''^k \beta^k p^k)$ belong to $\Pi_{\mathrm{I}}(p_\beta)$ with $\lambda q' + (1-\lambda) q'' = q$.

It is easy to see that if (y', v') is admissible in (4) for q' (resp. (y'', v'') for q'') then $(\lambda y' + (1-\lambda) y'', \lambda v' + (1-\lambda) v'')$ is admissible for q so that $V(q)$ is I-concave. By duality $V(q)$ is also II-convex.

But $V(q)$ is also the solution of the following program.

$$\begin{aligned} &\text{Max } \sum_b \delta \beta^b v^b \\ &\text{s.t.} \\ &\forall b\, \forall j\ \sum_{\substack{a \\ i}} \sum_{k \in K_a^{\mathrm{I}} \cap K_b^{\mathrm{II}}} p_0^k\, z_i(a)\, m_{ij}^k \geqslant v^b \\ &\forall a\ \sum_i z_i(a) = \alpha^a \\ &\forall i\ \forall a\ z_i(a) \geqslant 0 \end{aligned}$$

where $z_i(a) = \alpha^a\, x_i(a)$.

It follows that $V(q)$ is II-piecewise linear on $Q(p_0)$. Moreover there exists a finite number of vectors β_s, $s \in S$ such that for every p_α, $V(q)$ is II-linear on each convex polyhedron whose vertices are some of the $p_s(\alpha)$'s where $p_s(\alpha) = (\ldots, \delta(\alpha, \beta_s) \beta_s^k \alpha^k p_0^k, \ldots)$ and whose interior contains none of these points . [See *Ponssard/Sorin*]. It remains to note that $p_s(\alpha)$ belongs to $\Pi_{\mathrm{I}}(p_s)$, for all α, where $p_s = = (\ldots, \beta_s^k p_0^k, \ldots)$. By duality $V(q)$ is also I-piecewise linear on $Q(p_0)$ and there exists similarly a finite number of vectors α_r, $r \in R$.

Now if we define $p_{rs} \in P$ by $p_{rs}^k = \delta\, \alpha_r^k \beta_s^k p_0^k$ whenever this is possible it follows that $V(q)$ is I-II-linear on each of the convex polyedras constructed over the p_r, p_s and p_{rs}, and that these are a finite number.

Following theorem 1 we shall now say that $V(p)$ is I-II-piecewise bilinear on each $Q(p), \forall p \in P$.

5. Example

Let P be the simplex in $\mathbf{R}^3$; $K = \{1, 2, 3\}$, $K^{\mathrm{I}} = (\{1\}, \{2, 3\})$, and $K^{\mathrm{II}} = (\{1, 2\}, \{3\})$ (so that $K^{\mathrm{I}}, K^{\mathrm{II}}$ is proper and complete). (See the Appendix.)

From the definition of K^{I} and K^{II}, it follows that for all $p \in P$, $\Pi_{\mathrm{I}}(p)$ is the intersection of the line joining $(1, 0, 0)$ to p with P, and $\Pi_{\mathrm{II}}(p)$ the intersection with P of the line joining $(0, 0, 1)$ to p. Let

$$A^1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad A^3 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

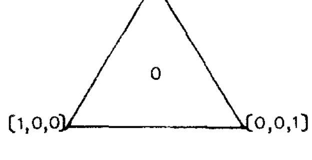
Since the first row is constant we can consider this game as a sequential game where player 1 plays first.

We shall use Proposition 1 in order to compute V_1. It will be observed that V_1 is I-II-piecewise bilinear in accordance with theorem 1. We have

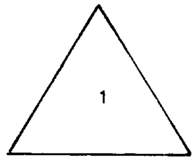
$$V_1(p) = \underset{\mathrm{I}}{\mathrm{Cav}}\ \underset{i}{\mathrm{Max}}\ \underset{\mathrm{II}}{\mathrm{Vex}}\ \underset{j}{\mathrm{Min}} \left(\sum_k p^k a_{ij}^k\right).$$

From the following figures

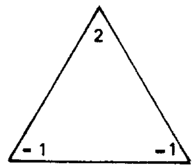
$V^{11} = V^{12}$



V^{21}



V^{22}

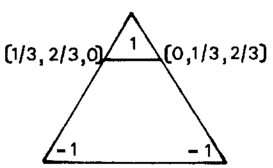


we obtain

$$V^1 = \underset{\text{II}}{\text{Vex}} \min(V^{11}, V^{12}) = V^{11} = V^{12}.$$

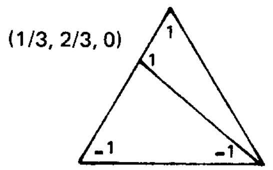
Thus we have

$\min(V^{21}, V^{22})$



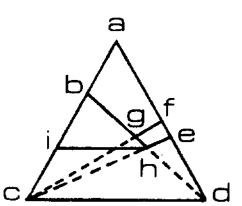
So that

$$V^2 = \underset{\text{II}}{\text{Vex}} \min(V^{21}, V^{22})$$



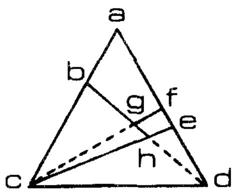
In order to obtain V_1, we first compute

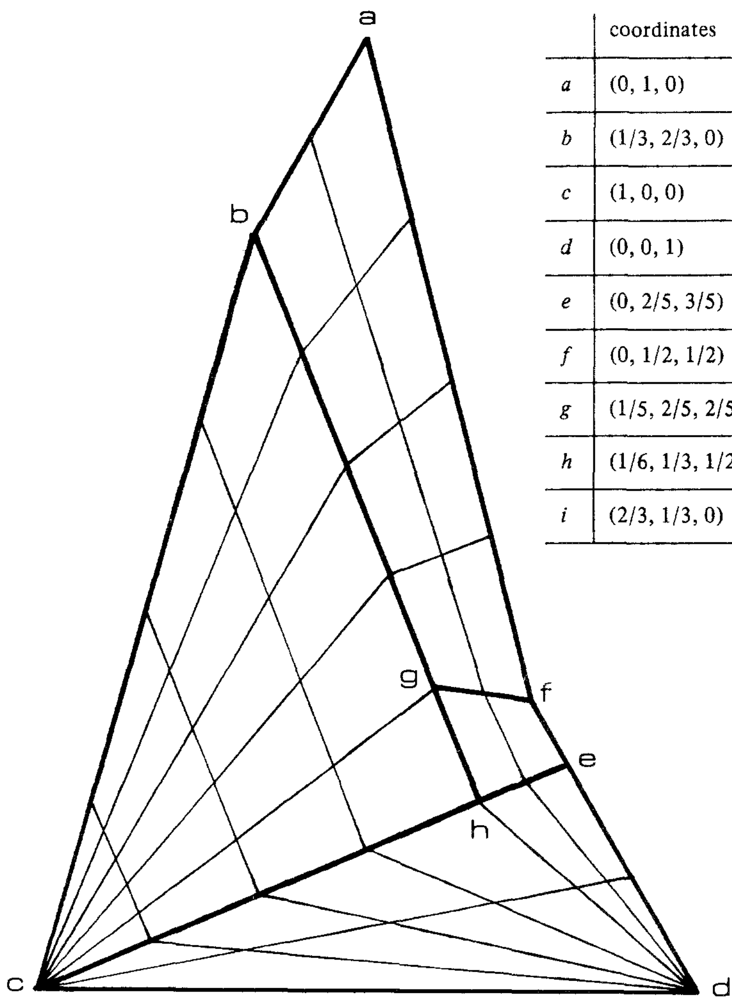
$\max(V^1, V^2)$



and then

$$V_1 = \underset{I}{\mathrm{Cav}} \max (V^1, V^2)$$





	coordinates	values
a	(0, 1, 0)	1
b	(1/3, 2/3, 0)	1
c	(1, 0, 0)	0
d	(0, 0, 1)	0
e	(0, 2/5, 3/5)	0
f	(0, 1/2, 1/2)	0
g	(1/5, 2/5, 2/5)	1/5
h	(1/6, 1/3, 1/2)	0
i	(2/3, 1/3, 0)	0

Figure : $V_1(p)$

6. Appendix

Definition: The information structure K^{I}, K^{II} is *complete* if for all $K' \subset K$ there exists $k \in K'$, and some $a \in A$ or some $b \in B$ such that $\{k\} = K^{\mathrm{I}}_{a|K'}$ or $\{k\} = K^{\mathrm{II}}_{b|K'}$ where $K^t\big|_{K'}$ denote the restriction of the partition K^t to K'.

Lemma 2: If K^{I}, K^{II} is complete then for all $p \in \mathring{P}$, we have $Q(p) = P$. $\mathring{P}$ is the interior of P.

Indeed, the result is true for $L = 1$ and we shall assume for $L - 1$. Since K^{I}, K^{II} is complete, we can suppose that $\{L\}$ belongs to K^{I}.

Now let $p \in \mathring{P}$ and $q \in P$. We can change the letters such that if $q^k \underset{\circ}{=} 0$ then $q^{k'} = 0\ \forall k' \geqslant k$. We introduce q' in the simplex P' of $\mathbf{R}^{L\text{-}1}$ and $p' \in P'$ defined by

$$q'^k = \frac{q^k}{\bar{q}} \quad p'^k = \frac{p^k}{\bar{p}} \quad \text{for all } k = 1, \ldots, L-1$$

where

$$\bar{q} = \sum_{k=1}^{L-1} q^k \quad \bar{p} = \sum_{k=1}^{L-1} p^k.$$

Now let K'^{I}, K'^{II} be the projection of K^{I}, K^{II} on $K' = \{1, \ldots, L-1\}$ and note that K'^{I}, K'^{II} is complete. So q' belongs to $Q'(p')$. Then there exist δ', α'^k, β'^k, $k = 1, \ldots, L-1$ such that

$$q'^k = \delta' \alpha'^k \beta'^k p'^k \quad \forall k: 1, \ldots, L-1$$

and

$$p'_\alpha = (\ldots, \alpha'^k p'^k, \ldots) \qquad \text{belongs to } \Pi'_{\mathrm{I}}(p')$$

$$p'_\beta = (\ldots, \beta'^k p'^k, \ldots) \qquad \text{belongs to } \Pi'_{\mathrm{II}}(p').$$

Thus we have

$$q^k = \delta' \frac{\bar{q}}{\bar{p}} \frac{1}{\lambda\mu} \lambda\alpha'^k \mu\beta'^k p^k \quad \forall k = 1, \ldots, L-1$$

where λ and μ are strictly positive.

Now choose $\beta'^L = \beta'^{L\text{-}1}$ and μ such that $p_\beta = (\ldots, \mu\beta'^k p^k, \ldots)$ belongs to $\Pi_{\mathrm{II}}(p)$. Then we take α'^L such that

$$q^L = \delta' \frac{\bar{q}}{\bar{p}} \beta'^L \alpha'^L p^L.$$

(If $\beta'^L = 0$ then $\beta'^{L\text{-}1} = 0$ so that, $q^{L\text{-}1} = 0$. But in this case $q = (q', 0)$ and there exists $p' \in \mathring{P}'$ such that $q' \in Q'(p')$; let $\tilde{p} = (p', 0)$ and then $q \in Q(\tilde{p})$ so that we can assume that $q' \in \mathring{P}'$.)

Finally, the coefficient λ is taken such that $p = (\ldots, \lambda\alpha'^k p^k, \ldots)$ belongs to $\Pi_{\mathrm{I}}(p)$ and the result follows.

Corollary 1: If $K^{\mathrm{I}}, K^{\mathrm{II}}$ is complete, $V(q)$ is I-II-piecewise bilinear on P.

Remark: Let $\bar{H}$ be the coarsest common refinement of K^{I} and K^{II} and say that $K^{\mathrm{I}}, K^{\mathrm{II}}$ is ***regular*** if $\bar{H} = [\{1\}, \ldots \{k\}, \ldots \{L\}]$.

Then it is easy to see that $K^{\mathrm{I}}, K^{\mathrm{II}}$ regular is a necessary condition in Lemma 2 since if $\{1, 2\}$ belongs to $K_1^{\mathrm{I}} \cap K_1^{\mathrm{II}}$, then for each p, q in $\mathring{P}$ such that $p_1 q_2 \neq p_2 q_1$ we have $p \notin Q(q), q \notin Q(p)$ so that there is an infinity of distincts sets $Q(p)\ p \in P$. Nevertheless, this condition is not sufficient since if $K = \{1,2,3,4\}$ with $K^{\mathrm{I}} = [\{1,2\}, \{3,4\}]$ and $K^{\mathrm{II}} = [\{1,3\}, \{2,4\}]$ then $q_1 q_4 / q_2 q_3$ is constant on $Q(p)$.

Definition: Let $\underline{H}$ be the finest common coarsening of K^{I} and K^{II} and say that $K^{\mathrm{I}}, K^{\mathrm{II}}$ is ***proper*** if $\underline{H} = [\{1, \ldots, L\}]$.

Note that in this case $Q(p)$ is the set of posterior probabilities which can arise from p and the partitions K^{I} and K^{II}. Note also that $Q(p)$ may still be strictly smaller than P (see previous example).

It is easy to see that $\Pi_{\mathrm{II}}(p_\alpha) \cap \Pi_{\mathrm{I}}(p_\beta)$ is reduced to at most one point for all $p \in P$, $p_\alpha \in \Pi_{\mathrm{I}}(p)$, $p_\beta \in \Pi_{\mathrm{II}}(p)$ if and only if $K^{\mathrm{I}}, K^{\mathrm{II}}$ is proper.

It follows that if $K^{\mathrm{I}}, K^{\mathrm{II}}$ is complete and proper we get a coordinate system, since we have the following.

Lemma 3: If $K^{\mathrm{I}}, K^{\mathrm{II}}$ is proper and complete, then for all $p \in \mathring{P}$, $q \in \mathring{P}$ there is a unique couple (p_α, p_β) such that:

$$P_\alpha \in \Pi_{\mathrm{I}}(p), p_\beta \in \Pi_{\mathrm{II}}(p) \text{ and } q \in \Pi_{\mathrm{II}}(p_\alpha) \cap \Pi_{\mathrm{I}}(p_\beta). \qquad (*)$$

Since $K^{\mathrm{I}}, K^{\mathrm{II}}$ is complete the existence follows. For all p_α, p_β satisfying (*)

$\Pi_{\mathrm{II}}(p_\alpha) \cap \Pi_{\mathrm{I}}(p_\beta)$ is reduced to one point ($K^{\mathrm{I}}, K^{\mathrm{II}}$ proper).

Suppose now that $q \in \Pi_{\mathrm{II}}(p_\alpha) \cap \Pi_{\mathrm{I}}(p_\beta) \cap \Pi_{\mathrm{II}}(p_{\alpha'}) \cap \Pi_{\mathrm{I}}(p_{\beta'})$. It follows that $q \in \Pi_{\mathrm{II}}(p_\alpha) \cap \Pi_{\mathrm{I}}(p_{\beta'})$, so we have

$$q^k = \delta(\alpha, \beta)\, \alpha^k \beta^k p^k = \delta(\alpha, \beta')\, \alpha^k \beta'^k p^k$$

and thus $\delta(\alpha, \beta)\, \beta^k p^k = \delta(\alpha, \beta')\, \beta'^k p^k$ since $p \in \mathring{P}$, $q \in \mathring{P}$. Now p_β and $p_{\beta'} \in P$ which implies $\delta(\alpha, \beta) = \delta(\alpha, \beta')$, so that $\beta = \beta'$; and similarly for α, α'.

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