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REPEATED GAMES AND QUALITATIVE DIFFERENTIAL GAMES: APPROACHABILITY AND COMPARISON OF STRATEGIES*

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Abstract. We study the notion of approachability in a repeated game with vector payoffs from a new point of view using techniques recently developed for qualitative differential games. Namely, we relate the sufficient condition for approachability (**B**-set) to the notion of discriminating domain for a suitably chosen differential game. The other goal of the present article is to obtain a new precise link between the strategies in the differential game and in the repeated game preserving approachability properties.

Key words. differential games, repeated games, viability

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1. Introduction.

Blackwell [7] considered a finite repeated game G with vector payoffs, where two players play in discrete time and generate at each stage of the game an outcome in $\mathbb{R}^k$. He introduced and studied the notion of approachability, which is a generalization of max-min level in a one-shot game with real payoff. Namely, a closed subset C of $\mathbb{R}^k$ is approachable for Player 1 if he can guarantee that asymptotically the average outcome reaches C, regardless of the strategy employed by Player 2.

It is difficult to check directly whether a given set is approachable or not; however, Blackwell [7] also introduced a sufficient condition upon the target set—called a **B**-set—and described explicitly an approachability strategy in this case. The **B**-set condition is purely geometric in nature: roughly speaking, it says that for any point z outside the set, Player 1 can force an expected one-stage outcome to lie on the other side of the tangent hyperplane to the set at the projection of z onto the set (cf. Definition 2 below for a more precise statement). He also proved that this condition was necessary in the convex case. In [29], [30], Spinat proved that every approachable set contains a **B**-set; see also [20].

Since the pioneering work of Blackwell [7], several developments of the notion of approachability have been studied: e.g., determinacy for weak approachability, extension to the infinite-dimensional case, and approachability with automata (cf., for instance, [22], [28], [29], [30], [32], [23], [24]). In the last few years approachability tools have been extensively applied in calibration and adaptive learning procedures; see, e.g., [18], [19], [16].

A powerful method for analyzing a repeated game may be the introduction of a continuous time analogue and the use of differential games techniques, especially for approachability.

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In the general theory of repeated games two main approaches have been studied:
- asymptotic analysis, such as limit of values of finitely repeated games or discounted games: this amounts to considering finer and finer time discretizations of a continuous time game played on $[0, 1]$;
- uniform analysis, through robustness properties of strategies: they should be approximately optimal in any sufficiently long game.

Our paper deals with uniform analysis for approachability. Recall that the asymptotic analysis for approachability corresponds to "weak approachability" and was studied in Vieille [32], first by introducing a deterministic repeated game $\bar{G}$, then an adapted differential game with fixed duration $\bar{\Gamma}$. Results on differential games with fixed duration were then used to obtain properties on strategies in $\bar{G}$. The next step was to translate these properties into strategies in G and to eventually deduce that every set in $\mathbb{R}^k$ is either weakly approachable or weakly excludable in G. In Spinat [29], the helpfulness of the differential games techniques in approachability theory has also been noticed. Finally in [5], [6], general properties of stochastic approximation for differential inclusions are used to study approachability procedures.

In our context of approachability, the differential game Γ that we use is of a completely different nature than the game used in [32]: it has different dynamics and is not of the same type in terms of payoffs. We are interested in a qualitative differential game where there are no payoffs for the players, but Player 1's goal is to force the trajectory to remain in a certain set, while Player 2 wants the trajectory to leave this set.

One of the purposes of the paper is thus to show that the correspondence between repeated and differential games extends from the framework "asymptotic analysis versus games of fixed duration" to the framework "uniform analysis versus qualitative games." This may be of interest for topics other than approachability.

We proceed by introducing an alternative repeated game G^*, played in a deterministic way and corresponding to the stage-by-stage "expected version" of G. We define then $*$approachability and note that the notion of **B**-set is the same in both games. G^* is associated in a natural way to a qualitative differential game Γ.

Then our first main result is the following: A set C is a **B**-set for the repeated game G if and only if it is a discriminating domain of the associated differential game Γ. (The discriminating property was first defined by Aubin [1], [2] and extensively studied in [11] and [12].) Mathematically this amounts to considering the **B**-set condition as a local geometrical property from inside rather than a global property from outside.

This result provides a useful bridge between the theory of differential games and dynamic games. Indeed, numerical methods are developed in [14] to compute the largest discriminating domain contained in a given closed set (also called the discriminating kernel). Hence, to show if a given set C is approachable is equivalent to checking if the largest discriminating domain contained in C is nonempty. This furnishes an algorithmic criterion of approachability.

Our second main result shows that a necessary and sufficient condition for $*$approachability of a given set is the existence of a nonempty **B**-set included in it, and the proof relies on differential games tools. In fact we analyze the set of limit points of trajectories in Γ compatible with strategies induced by $*$approachability strategies in G^*.

Our third main result deals with the correspondence between strategies in the different models. Actually we prove that a strategy in the game G^* can be interpolated by a nonanticipative strategy with delay in Γ. We also show that the range of nonanticipative strategies in the differential game can be approached by nonanticipative strategies with delay. By preserving nonanticipative strategies with delay in Γ, we thus construct $*$approachability strategies in the repeated game G^*. A last step is to use these strategies to construct approachability strategies in the repeated game G.

The paper is organized as follows: In section 2 we introduce the repeated games G and G^* and the notions of approachability and of **B**-set. Then we define differential games and discriminating domain and finally specify the differential game Γ. In section 3, we exhibit the relationship between the notions of **B**-set for G and G^* and discriminating domain for Γ. Then we prove that a necessary and sufficient condition for a closed set to be $*$approachable is to contain a **B**-set. In section 4 we discuss the relationship between strategies in the differential and the repeated games. In the appendix, we recall basic formulations and results for qualitative differential games.

2. Preliminaries. This section defines the main objects of our analysis: repeated games and associated differential game, **B**-sets, and discriminating domains.

2.1. The repeated games G and G^*. We introduce here two related repeated two-person games played in discrete time. The first one corresponds exactly to Blackwell's framework [7].

2.1.1. The "random" repeated game G. Given an $I \times J$ matrix A with coefficients in $\mathbb{R}^k$, a two-person infinitely repeated game G is defined as follows. At each stage $n = 1, 2, \ldots$, each player chooses an element in his set of actions:[1] $i_n \in I$ for Player 1 (resp., $j_n \in J$ for Player 2). The corresponding outcome is $g_n = A_{i_n j_n} \in \mathbb{R}^k$ and the pair of actions (i_n, j_n) is announced to both players. Denote by $\overline{g}_n$ the average outcome up to stage n, thus $\overline{g}_n = \frac{1}{n}\sum_{m=1}^{n} g_m$. Let $H_n = (I \times J)^n$ be the set of possible histories at stage $n+1$ and let $H_\infty = (I \times J)^\infty$ be the set of plays. $\mathcal{H}_\infty$ stands for the σ-field spanned by the cylinders $B^h = \{(i_m, j_m)_{m\in\mathbb{N}} \in H_\infty;\ (i_m, j_m)_{1\le m\le n} = h\}$ $(h \in H_n,\ n \in \mathbb{N})$.

Denote by Σ (resp., $\mathcal{T}$) the set of strategies of Player 1 (resp., Player 2): mappings from $H = \cup_{n\ge 0} H_n$ to the sets of mixed actions $U = \Delta(I)$ (probabilities on I) (resp., $V = \Delta(J)$). At stage n, given the history $h_{n-1} \in H_{n-1}$, Player 1 chooses an action in I according to the probability distribution $\sigma(h_{n-1}) \in U$ (and similarly for Player 2). Notice that this distribution is unknown to the opponent: only its realization $i_n \in I$ is announced. A pair of strategies (σ, τ) induces a probability on $(H_\infty, \mathcal{H}_\infty)$, and $\mathbb{E}_{\sigma,\tau}$ denotes the corresponding expectation.

Notation. Given u in U, define $uAV = \{uAv = \sum_{ij} u_i A_{ij} v_j;\ v \in V\} \subset \mathbb{R}^k$, similarly UAv for $v \in V$, and let $M = \sup_{u\in U,\, v\in V} \|uAv\| = \max_{i,j} \|A_{i,j}\|$. For every nonempty closed subset C of $\mathbb{R}^k$ let $d_C(\cdot)$ be the distance to it: $d_C(x) = \min_{z\in C} \|x - z\|$, where $\|.\|$ is the Euclidean norm on $\mathbb{R}^k$. For every $x \in \mathbb{R}^k$, the set of projections of x on C is defined by

$$\pi_C(x) = \{z \in C;\quad d_C(x) = \|x - z\|\}$$

and the proximal normal set to C at x (cf., e.g., [9]) is given by

$$NP_C(x) = \{p \in \mathbb{R}^k;\ d_C(x + p) = \|p\|\}.$$

Finally $B(x, r)$ denotes the open ball centered in x with radius r, and $\overline{B}$ the closed unit ball centered at 0.

[1]We obviously identify I with the set $\{1, 2, \ldots, I\}$.

DEFINITION 1 (see [7]). *A nonempty closed set C in $\mathbb{R}^k$ is approachable by Player 1 in G if, for every $\varepsilon > 0$, there exists a strategy σ of Player 1 and $N \in \mathbb{N}$ such that, for any strategy τ of Player 2 and any $n \geq N$,*

$$\mathbb{E}_{\sigma,\tau}(d_C(\overline{g}_n)) \leq \varepsilon.$$

In other words, given a closed subset $C \subset \mathbb{R}^k$, the goal of Player 1 is to guarantee that asymptotically the average outcome gets close in expectation to the target C, uniformly with respect to his opponent's behavior.

2.1.2. The "expected deterministic" repeated game G^*. Define now another two-person infinitely repeated game G^* associated, as the previous one, to the matrix A. At each stage $n = 1, 2, \dots$, Player 1 (resp., Player 2) chooses $u_n \in U = \Delta(I)$ (resp., $v_n \in V = \Delta(J)$), the outcome is $g^*_n = u_n A v_n$, and (u_n, v_n) is announced. Accordingly, a strategy σ^* for Player 1 in G^* is a map from $H^* = \cup_{n\geq 0} H^*_n$ to U, where $H^*_n = (U \times V)^n$. A strategy τ^* for Player 2 is defined similarly. A couple of strategies induces a play $\{(u_n, v_n)\}$ and a sequence of outcomes $\{g^*_n\}$, and $\overline{g}^*_n = \frac{1}{n}\sum_{m=1}^{n} g^*_m$ denotes the average outcome up to stage n. G^* is the game played in "mixed moves" or in distribution.

DEFINITION 2. *A nonempty closed set C in $\mathbb{R}^k$ is $*$approachable by Player 1 in G^* if, for every $\varepsilon > 0$, there exists a strategy σ^* of Player 1, and $N \in \mathbb{N}$ such that, for any strategy τ^* of Player 2 and any $n \geq N$,*

$$d_C(\overline{g}^*_n) \leq \varepsilon.$$

2.2. B-set.

DEFINITION 3 (see [7]). *A closed set C in $\mathbb{R}^k$ is a* **B***-set for Player 1 (given A) if, for any $z \notin C$, there exists $y \in \pi_C(z)$ and a mixed action $u = \hat{u}(z)$ in $U = \Delta(I)$ such that the hyperplane through y orthogonal to the segment $[yz]$ separates z from uAV:*

$$\langle uAv - y, z - y\rangle \leq 0 \qquad \forall v \in V.$$

This notion was introduced by Blackwell [7] and is crucial in the analysis of approachability. The basic result is as follows.

PROPOSITION 4. *Let C be a* **B***-set for Player 1. Then it is approachable in G and $*$approachable in G^* by that player. An approachability strategy is given by $\sigma(h_n) = \hat{u}(\bar{g}_n)$ (resp., $\sigma^*(h^*_n) = \hat{u}(\bar{g}^*_n)$).*

The proof for approachability is Proposition 8 in [7]. The other one is a simple adaptation where the outcome $\bar{g}_n$ is replaced by $\bar{g}^*_n$; see [29].

Remark 1. The previous proposition implies that a **B**-set remains approachable (resp., $*$approachable) in the game where the only information of Player 1 after stage n is the average outcome $\bar{g}_n$ (resp., $\bar{g}^*_n$) rather than the complete previous history h_n (resp., h^*_n).

2.3. Differential games. We briefly recall the basic notion of differential games and discriminating domains that will be needed in the analysis. Assume the following hypotheses:

$$(1) \qquad \begin{cases} \text{(i)} & U \text{ and } V \text{ are compact subsets of } \mathbb{R}^k. \\ \text{(ii)} & f : \mathbb{R}^k \times U \times V \mapsto \mathbb{R}^k \text{ is continuous.} \\ \text{(iii)} & f(., u, v) \text{ is an } l\text{-Lipschitz map for all } (u, v) \in U \times V. \\ \text{(iv)} & U \text{ is convex, and } f \text{ is affine in } u. \end{cases}$$

Introduce the sets of controls defined by

$$\mathcal{U}(t_0) = \{\mathbf{u} : [t_0, +\infty) \mapsto U;\ \mathbf{u} \text{ is measurable}\},$$

$$\mathcal{V}(t_0) = \{\mathbf{v} : [t_0, +\infty) \mapsto V;\ \mathbf{v} \text{ is measurable}\},$$

and let us use the shorter notation $\mathcal{U}$ and $\mathcal{V}$ when $t_0 = 0$.

Consider a differential game whose evolution is governed by the two-controlled dynamical system (2), where $x_0 \in \mathbb{R}^k$ and $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$:

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)) & \text{for almost every } t \geq 0, \\ \mathbf{x}(0) = x_0. \end{cases} \tag{2}$$

DEFINITION 5. *A map $\alpha : \mathcal{V} \to \mathcal{U}$ is a* nonanticipative strategy *if, for any $t \geq 0$ and for any $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathcal{V}$, which coincide almost everywhere on $[0, t]$ of $[0, +\infty)$, the images $\alpha(\mathbf{v}_1)$ and $\alpha(\mathbf{v}_2)$ coincide also almost everywhere on $[0, t]$.*

Nonanticipative strategies (introduced in [17], [27], [31]) are very classical in the differential games literature.

Denote by $\mathcal{M}(\mathcal{U}, \mathcal{V})$ the set of nonanticipative strategies from $\mathcal{U}$ to $\mathcal{V}$.

For all $\mathbf{u} \in \mathcal{U}$, $\mathbf{v} \in \mathcal{V}$, and $t \geq 0$, $\mathbf{x}[x_0, \mathbf{u}, \mathbf{v}](t)$ stands for the value at time t of the unique solution of (2) (and $\mathbf{x}[t_0, x_0, \mathbf{u}, \mathbf{v}](t)$ stands for the solution associated to initial condition $x(t_0) = x_0$).

DEFINITION 6. *A nonempty closed set C in $\mathbb{R}^k$ is a* discriminating domain *for Player 1, given f, if*

$$\forall x \in C, \quad \forall p \in NP_C(x) \qquad \sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq 0. \tag{3}$$

The interpretation is that, at any boundary point $x \in C$, Player 1 can react to any control of Player 2 in order to keep the trajectory in the half space facing a proximal normal p.

The following theorem, due to Cardaliaguet [10], [11], states that Player 1 can ensure his remaining in a discriminating domain as soon as he knows, at each time t, Player 2's control up to time t.

THEOREM 7 (interpretation theorem [11]). *Assume that f satisfies conditions (1), and that C is a closed subset of $\mathbb{R}^k$. Then C is a discriminating domain if and only if, for every x_0 belonging to C, there exists a nonanticipative strategy $\alpha \in \mathcal{M}(\mathcal{V}, \mathcal{U})$, such that for any $\mathbf{v} \in \mathcal{V}$, the solution $\mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t)$ remains in C for every $t \geq 0$.*

We shall say that such a strategy α *preserves* the set C. We refer the reader to the appendix for another formulation of the interpretation theorem and for basic results on qualitative differential games (cf. also [15]).

It is worth pointing out that the strategy α does not contain any randomness: it is a deterministic nonanticipative strategy. Later on we will apply this strategy for a differential game where $V = \Delta(J)$, which is the simplex and which can be viewed as a set of probability measures on J. Nevertheless the corresponding strategy α is still deterministic. Contrary to the repeated game case, one cannot easily construct a random process in continuous time with value in J and which "represents" the strategy α.

2.4. The associated differential game Γ. Let us define now a differential game Γ associated to the $I \times J$ matrix A and the infinitely repeated game G^*. Recall that $U = \Delta(I)$ and $V = \Delta(J)$. We mimic the average payoff $\overline{g}_n^*$ of the repeated game

G^* by a continuous time average payoff, denoted by γ, and defined on $\mathcal{U} \times \mathcal{V} \times \mathbb{R}^+$ as follows: $\gamma[\mathbf{u}, \mathbf{v}](0) = 0$ and for $t > 0$

$$\gamma[\mathbf{u}, \mathbf{v}](t) = \frac{1}{t} \int_0^t \mathbf{u}(s) A \mathbf{v}(s) ds. \tag{4}$$

Differentiating the above expression, we obtain

$$\frac{d}{dt}\gamma[\mathbf{u}, \mathbf{v}](t) = \frac{-\gamma(t) + \mathbf{u}(t) A \mathbf{v}(t)}{t}. \tag{5}$$

Set

$$t = e^s, \tag{6}$$

and introduce $\mathbf{x}(s) = \gamma(e^s)$; hence (5) becomes

$$\dot{\mathbf{x}}(s) = -\mathbf{x}(s) + \mathbf{u}(s) A \mathbf{v}(s). \tag{7}$$

This is the dynamics of a differential game Γ of the form (2) if we state

$$f(x, u, v) = -x + uAv. \tag{8}$$

Notice that $\mathbf{x}(0) = \gamma[\mathbf{u}, \mathbf{v}](1)$.

Remark 2. Note that assumptions (1) hold true for this game.

In addition Isaacs' condition, namely, for any $\zeta \in \mathbb{R}^k$

$$\sup_{v \in V} \inf_{u \in U} \langle \zeta, f(x, u, v) \rangle = \inf_{u \in U} \sup_{v \in V} \langle \zeta, f(x, u, v) \rangle, \tag{9}$$

which will play a central role in the proofs, is also satisfied by f.

Remark 3. The dynamics (7) appeared in [29].

On the other hand, to study the limit of "compact" repeated games and weak approachability, [32] used the same deterministic repeated game G^* but an alternative differential game $\bar{\Gamma}$ with dynamics given by $\dot{\mathbf{x}}(s) = \mathbf{u}(s) A \mathbf{v}(s)$ and with finite duration.

3. Discriminating domain and approachability.

This section is mainly devoted to the relations between **B**-sets, discriminating domains, and $*$approachability. The first result is that the first two objects coincide. Then we prove that a necessary and sufficient condition for $*$approachability of C is the existence of a nonempty discriminating domain included in C.

3.1. B-set and discriminating domain.

The following theorem states that the notions of **B**-set (for A) and discriminating domain (for f) coincide.

THEOREM 8. *Consider f satisfying* (8). *A closed set $C \subset \mathbb{R}^k$ is a discriminating domain for Player* 1 *if and only if C is a* **B***-set for Player* 1.

Proof. (i) *Necessary condition.* Suppose that C is a **B**-set for Player 1. Let $x \in C$ and $p \in NP_C(x)$. If $p = 0$, condition (3) obviously holds true. Otherwise let $z = x + p/2$ and observe that $\pi_C(z)$ is reduced to the singleton $\{x\}$. Hence there exists a mixed move $u \in U$ such that, for every $v \in V$,

$$\langle uAv - x, z - x \rangle \leq 0.$$

This yields, a fortiori,

$$\sup_{v \in V} \inf_{u \in U} \langle f(x, u, v), p \rangle \leq \inf_{u \in U} \sup_{v \in V} \langle f(x, u, v), p \rangle \leq 0.$$

Thus, C is a discriminating domain for Player 1, given f.

(ii) *Sufficient condition.* Suppose that C is a discriminating domain for Player 1. For every $z \notin C$ and every $y \in \pi_C(z)$, we have $(z - y) \in NP_C(y)$, and hence

$$\sup_{v \in V} \inf_{u \in U} \langle f(y, u, v), z - y \rangle = \inf_{u \in U} \sup_{v \in V} \langle f(y, u, v), z - y \rangle \leq 0.$$

Thus by continuity and compactness, there exists u in U such that for every v in V, $\langle uAv - y, z - y \rangle \leq 0$. Hence C is a **B**-set for Player 1, given A. □

Remark 4. We used in the proof the fact that Isaacs' condition was satisfied by f. In a more general framework one could define a **B**-set for a function T from $U \times V$ to $\mathbb{R}^k$ and then let $f(x, u, v) = T(u, v) - x$. The previous proof shows that a set C is a **B**-set given T if and only if it satisfies for all $x \in C$ and all $p \in NP_C(x)$,

$$\inf_U \sup_V \langle p, f(x, u, v) \rangle \leq 0.$$

(C is a "leadership domain" for f; see the appendix.)

Theorem 7 states that in Γ, if the initial state belongs to a discriminating domain C for f, Player 1 can force the trajectory to remain in C as soon as he knows the actions of his opponent. The question that naturally arises is what would happen if the initial state is outside of C.

THEOREM 9. *If a closed set $C \subset \mathbb{R}^k$ is a **B**-set for Player 1, there exists a nonanticipative strategy of Player 1 in Γ, $\alpha \in \mathcal{M}(\mathcal{V}, \mathcal{U})$, such that for every $\mathbf{v} \in \mathcal{V}$*

$$\forall t \geq 1 \quad d_C(\gamma[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{2M}{t}. \tag{10}$$

Proof. Let $y_0 \in C$, which, being a **B**-set, is a discriminating domain. By Theorem 7, there exists $\alpha \in \mathcal{M}(\mathcal{V}, \mathcal{U})$ such that for any $\mathbf{v} \in \mathcal{V}$, $\mathbf{x}[y_0, \alpha(\mathbf{v}), \mathbf{v}](t)$ remains in C on $[0, +\infty)$. Given x_0, let Player 1 use the previous α. Then, denoting $y_t = \mathbf{x}[y_0, \alpha(\mathbf{v}), \mathbf{v}](t)$ and $x_t = \mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t)$, one has $\dot{x}_t = u_t A v_t - x_t$, and $\dot{y}_t = u_t A v_t - y_t$ with $u_t = \alpha(\mathbf{v})(t)$ and $v_t = \mathbf{v}(t)$. Hence

$$\frac{d}{dt}(x_t - y_t) = \dot{x}_t - \dot{y}_t = -(x_t - y_t)$$

so that

$$\|x_t - y_t\| = \|x_0 - y_0\| e^{-t},$$

and since $y_t \in C$, for all $t \geq 0$

$$d_C(x_t) \leq \|x_0 - y_0\| e^{-t}.$$

Changing the time scale as in (6) we obtain that

$$\forall t \geq 1 \quad d_C(\gamma[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq \frac{2M}{t}. \qquad \square \tag{11}$$

Remark 5. The previous proof, showing that if α preserves C, then α approaches C, extends to any dynamics of the form $f(x, u, v) = T(u, v) - x$ or, more generally, $f(x, u, v) = T(x, u, v) - x$ with T strictly contracting in the following sense:

$$\|T(x, u, v) - T(y, u, v)\| \leq k\|x - y\| \qquad \forall u, v$$

with $k < 1$.

3.2. B-set and ∗approachability.

THEOREM 10. *A closed set C is ∗approachable for Player 1 in G^* if and only if it contains a* **B***-set for Player 1 (given A).*

Proof. If C contains a **B**-set for Player 1, Proposition 4 implies that it is ∗approachable.

To obtain the converse implication, consider C a closed ∗approachable set. The proof follows several steps: First, we construct a map Ψ from strategies of Player 1 in G^* to nonanticipative strategies in Γ. Then given $\varepsilon > 0$ and a strategy σ_ε that ε-approaches C in G^*, we define its image $\alpha_\varepsilon = \Psi(\sigma_\varepsilon)$. The next step consists in proving that the trajectories in the differential game Γ compatible with α_ε approach $C + \varepsilon\bar{B}$ asymptotically. Then we prove that the ω-limit set of any trajectory compatible with some $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$, denoted by D^α, is a nonempty compact discriminating domain for f. By introducing a limit set of D^{α_ε} as ε tends to 0, we conclude that there exists a nonempty closed discriminating domain contained in C.

(1) We first define a map Ψ from strategies of Player 1 in G^* to nonanticipative strategies of Player 1 in Γ.

Given σ, we proceed by induction to construct $\alpha = \Psi(\sigma)$. Fix $\mathbf{v} \in \mathcal{V}$ and let u_1 be the first move of the strategy σ. Let $\alpha(\mathbf{v}) = u_1$ on $[0, 1)$. Suppose that $\alpha(\mathbf{v})$ is specified on $[0, i)$ for some $i \in \mathbb{N}$, $i \geq 1$, and let us define it on $[i, i + 1)$. Introduce $\bar{v}_i = \int_{i-1}^{i} \mathbf{v}(s)ds$, which belongs to V since V is closed and convex. This variable will play the role of the move of Player 2 at stage i in G^*.

Explicitly, denote by $\bar{h}_j(\sigma, \mathbf{v})$ the jth component of the history of the moves of both players in G^*. Namely,

$$\bar{h}_1(\sigma, \mathbf{v}) = [u_1, \bar{v}_1],$$

and inductively for every integer $j > 1$

$$\bar{h}_j(\sigma, \mathbf{v}) = [\sigma(\bar{h}_{j-1}), \bar{v}_j].$$

Now, we define the map $\Psi[\sigma]$ on $[i, i + 1)$ as follows:

$$\forall t \in [i, i + 1) \quad \Psi[\sigma](\mathbf{v})(t) = \sigma(\bar{h}_1(\sigma, \mathbf{v}), \ldots, \bar{h}_i(\sigma, \mathbf{v})).$$

(2) C being ∗approachable, given $\varepsilon > 0$, there exists a strategy σ^*_ε in G^* and $N \in \mathbb{N}$ such that, for any strategy τ^* of Player 2 in G^* and any $n \geq N$,

$$d_C(\overline{g}^*_n) \leq \varepsilon. \tag{12}$$

Introduce $\alpha_\varepsilon = \Psi[\sigma^*_\varepsilon]$ as defined above and let us show that the trajectories induced by this strategy in the differential game Γ approach $C + \varepsilon B$ asymptotically, in the sense that

$$\exists T > 0, \quad \forall t > T, \quad \forall \mathbf{v} \in \mathcal{V} \quad d_C(\mathbf{x}[x_0, \alpha_\varepsilon(\mathbf{v}), \mathbf{v}](t)) \leq 2Me^{-t} + \varepsilon. \tag{13}$$

We first prove the inequality for the trajectories expressed with γ, and then we will deduce (13) from changing time scale.

Since

$$\begin{aligned} d_C(\gamma[\alpha_\varepsilon(\mathbf{v}), \mathbf{v}](t)) \leq{} & \|\gamma[\alpha_\varepsilon(\mathbf{v}), \mathbf{v}](t) - \gamma[\alpha_\varepsilon(\mathbf{v}), \mathbf{v}](E(t))\| \\ & + d_C(\gamma[\alpha_\varepsilon(\mathbf{v}), \mathbf{v}](E(t))), \end{aligned} \tag{14}$$

where $E(t)$ defines the larger integer smaller than or equal to t, we will treat the two terms separately.

Note that

$$\gamma[\alpha_\varepsilon(\mathbf{v}),\mathbf{v}](E(t)) = \frac{1}{E(t)}\int_0^{E(t)} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds = \frac{1}{E(t)}\sum_{i=0}^{E(t)-1}\int_i^{i+1} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds.$$

Since for every $s \in [i, i+1)$, $\alpha_\varepsilon(\mathbf{v})(s)$ depends only on the values of $\mathbf{v}$ on $[0, i)$ and hence is constant, one obtains

$$\int_i^{i+1} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds = \alpha_\varepsilon(\mathbf{v})A\int_i^{i+1} \mathbf{v}(s)ds = \alpha_\varepsilon(\mathbf{v})A\bar{v}_{i+1}.$$

Hence

$$\frac{1}{E(t)}\int_0^{E(t)} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds = \frac{1}{E(t)}\sum_{i=0}^{E(t)-1} \alpha_\varepsilon(\mathbf{v})A\bar{v}_{i+1}$$
$$= \frac{1}{E(t)}\sum_{i=0}^{E(t)-1} \sigma^*_\varepsilon(\bar{h}_1(\sigma_\varepsilon,\mathbf{v}),\bar{h}_2(\sigma_\varepsilon,\mathbf{v}),\ldots,\bar{h}_i(\sigma_\varepsilon,\mathbf{v}))A\bar{v}_{i+1},$$

which is an average outcome in G^* compatible with σ^*_ε. Thus $t+1 > N$ and inequality (12) yields

$$d_C\left(\frac{1}{E(t)}\int_0^{E(t)} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds\right) \leq \varepsilon.$$

On the other hand, we have

$$\begin{aligned}&\|\gamma[\alpha_\varepsilon(\mathbf{v}),\mathbf{v}](t) - \gamma[\alpha_\varepsilon(\mathbf{v}),\mathbf{v}](E(t))\| \\ &= \left\|\frac{1}{t}\int_0^t \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds - \frac{1}{E(t)}\int_0^{E(t)} \alpha_\varepsilon(\mathbf{v})(s)A\mathbf{v}(s)ds\right\| \leq \frac{2M}{t}.\end{aligned} \tag{15}$$

Hence, changing time scale in the last inequality yields (13).

(3) We now introduce the set of limit points compatible with a nonanticipative strategy in Γ.

Let $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$. Given $x_0 \in \mathbb{R}^k$, $\mathbf{v} \in \mathcal{V}$, and $\theta > 0$, define

$$D(\alpha,\mathbf{v},\theta) = \text{cl}\{\mathbf{x}[x_0,\alpha(\mathbf{w}),\mathbf{w}](t);\quad t \geq \theta,\quad \mathbf{w} \in \mathcal{V},\quad \mathbf{w}(s) = \mathbf{v}(s)\quad \forall s \in [0,\theta)\}$$

(where *cl* is the closure operator) and $D(\alpha,\mathbf{v}) = \bigcap_{\theta\geq 0} D(\alpha,\mathbf{v},\theta)$.

We need the following lemma, which will be proven later on.

LEMMA 11. *$D(\alpha,\mathbf{v})$ is a nonempty compact discriminating domain for Player 1 given f.*

(4) Let $\mathbf{v} \in \mathcal{V}$. Lemma 11 implies that $D(\alpha_\varepsilon,\mathbf{v})$ is a nonempty discriminating domain. Then let $D^n := D(\alpha_{\frac{1}{n}},\mathbf{v})$ and define $D = \limsup_{n\to\infty} D^n$. Note that D is nonempty because it is the upper limit of a sequence of nonempty compact sets contained in the compact ball $\overline{B(0,4M)}$.

(5) Note that $\langle p, f(x,u,v)\rangle$ is uniformly Lipschitz in (p,x); hence the function $\sup_v \inf_u \langle p, f(x,u,v)\rangle$ is lower semicontinuous and, by Proposition 1.2 in [12], D is a

discriminating domain for Player 1; thus it is a **B**-set. We finally deduce from (13) that $\sup_{x\in D} d_C(x) = 0$, and thus $D \subset C$.

This ends the proof of Theorem 10. ∎

We now provide the last missing element.

Proof of Lemma 11. Let $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$ and $\mathbf{v} \in \mathcal{V}$. First note that since $\|\gamma\|_\infty \le M$, for all $\theta > 0$ the set $D(\alpha, \mathbf{v}, \theta)$ is bounded and closed. Hence $D(\alpha, \mathbf{v})$ is nonempty and compact being an intersection of a decreasing sequence of nonempty compact sets.

Suppose by contradiction that $D(\alpha, \mathbf{v})$ is not a discriminating domain. Then there exists $\bar{z} \in D(\alpha, \mathbf{v})$, $p \in NP_{D(\alpha,\mathbf{v})}(\bar{z})$, and $a > 0$ such that

$$\exists\ \bar{v} \in V, \quad \forall\ u \in U \quad \langle f(\bar{z}, u, \bar{v}), p\rangle > a. \tag{16}$$

Let $\bar{\mathbf{v}}$ be the constant control equal to $\bar{v}$ and, given $\mathbf{u} \in \mathcal{U}$, let $u_s = \mathbf{u}(s)$. Define $\mathbf{z}$ to be the solution of the differential equation

$$\begin{cases} \dot{\mathbf{z}}(s) = f(\mathbf{z}(s), u_s, \bar{v}) = -\mathbf{z}(s) + u_s A\bar{v}, \quad s \ge 0, \\ \mathbf{z}(0) = \bar{z}. \end{cases} \tag{17}$$

We claim that for $s > 0$ small enough, there exists a map ρ from $(0, s)$ to $(0, +\infty)$ and independent of $\mathbf{u}$, such that

$$B(\mathbf{z}(t), \rho(t)) \cap D(\alpha, \mathbf{v}) = \emptyset \quad \forall t \in [0, s). \tag{18}$$

To prove our claim let us compute the derivative of $\|\mathbf{z}(s) - (\bar{z} + p)\|^2$. For almost every t,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{z}(t) - (\bar{z} + p)\|^2 = \langle f(\mathbf{z}(t), u_t, \bar{v}), \mathbf{z}(t) - (\bar{z} + p)\rangle$$

$$= \langle f(\mathbf{z}(t), u_t, \bar{v}), \mathbf{z}(t) - \bar{z}\rangle - \langle f(\mathbf{z}(t), u_t, \bar{v}), p\rangle + \langle f(\bar{z}, u_t, \bar{v}), p\rangle - \langle f(\bar{z}, u_t, \bar{v}), p\rangle$$

$$\le \|\mathbf{z}(t) - \bar{z}\|(\|f(\mathbf{z}(t), u_t, \bar{v})\| + l\|p\|) - \langle f(\bar{z}, u_t, \bar{v}), p\rangle,$$

taking into consideration that f is l-Lipschitz with respect to x. Choose s small enough such that for every $t \in [0, s]$ and every $\mathbf{u}$ (f is uniformly bounded in a neighborhood of $\bar{z}$),

$$\|\mathbf{z}(t) - \bar{z}\|(\|f(\mathbf{z}(t), u_t, \bar{v})\| + l\|p\|) \le \frac{a}{2}.$$

Hence, using (16) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{z}(t) - (\bar{z} + p)\|^2 < -\frac{a}{2} \qquad \text{for a.e. } t \in [0, s],$$

and thus, integrating on $[0, t]$,

$$\|\mathbf{z}(t) - \bar{z} - p\|^2 < \|p\|^2 - \frac{at}{2},$$

which means that $\mathbf{z}(t) \in B(\bar{z} + p, \sqrt{\|p\|^2 - \frac{at}{2}})$ for every $t \in (0, s]$; hence, setting $\rho(t) = \|p\| - \sqrt{\|p\|^2 - \frac{at}{2}} > 0$, the claim is proved.

Since $\bar{z} \in D(\alpha, \mathbf{v})$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = +\infty$, and a sequence $\mathbf{v}_n \in \mathcal{V}$ such that, letting $z_n = \mathbf{x}[x_0, \alpha(\mathbf{v}_n), \mathbf{v}_n](t_n)$, one has $\lim_{n\to+\infty} z_n = \bar{z}$. For every $n \in \mathbb{N}$ we define

$$\tilde{\mathbf{v}}_n(s) = \begin{cases} \mathbf{v}_n(s) & \forall s \in [0, t_n), \\ \bar{v} & \forall s \in [t_n, +\infty) \end{cases} \tag{19}$$

and

$$\mathbf{z}_n(s) = \mathbf{x}[x_0, \alpha(\tilde{\mathbf{v}}_n), \tilde{\mathbf{v}}_n](t_n + s),$$
$$\mathbf{z}_n(0) = z_n.$$

Let $u_s^n = \alpha(\tilde{\mathbf{v}}_n)(t_n + s)$. Then $\mathbf{z}_n(\cdot)$ is also a solution of (17) with $\bar{z} = z_n$ and $u_s = u_s^n$; hence we have

$$\frac{d}{dt}(\mathbf{z}(t) - \mathbf{z}_n(t)) = -(\mathbf{z}(t) - \mathbf{z}_n(t))$$

so that

$$\|\mathbf{z}(t) - \mathbf{z}_n(t)\| \leq e^{-t}\|\bar{z} - z_n\|. \tag{20}$$

Fix $t \in (0, s)$ and choose N large enough to have for every $n > N$

$$\|\bar{z} - z_n\| \leq \frac{\rho(t)}{4} \quad \text{and} \quad D(\alpha, \mathbf{v}, t_n) \subset D(\alpha, \mathbf{v}) + \frac{\rho(t)}{4}\bar{B}. \tag{21}$$

In view of (20) and (21) we obtain

$$\|\mathbf{z}(t) - \mathbf{z}_n(t)\| \leq \frac{\rho(t)}{4} \quad \text{and} \quad \mathbf{z}_n(t) \in D(\alpha, \mathbf{v}, t_n) \subset D(\alpha, \mathbf{v}) + \frac{\rho(t)}{4}\bar{B}.$$

Thus $\mathbf{z}(t) \in D(\alpha, \mathbf{v}) + \frac{1}{2}\rho(t)\bar{B}$, which yields a contradiction with the claim (18).

This ends the proof that $D(\alpha, \mathbf{v})$ is a discriminating domain for f. ∎

Remark 6. The same proof shows that, given $\alpha \in \mathcal{M}(\mathcal{V}, \mathcal{U})$, the set

$$D(\alpha) = \cap_{\theta \geq 0}\, cl\{\mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t);\ t \geq \theta,\ \mathbf{v} \in \mathcal{V}\}$$

is a discriminating domain for f.

Remark 7. Recall that Theorem 2.5 in Spinat [30] states that a closed set C is approachable in G if and only if it contains a **B**-set; hence we deduce that approachability and $*$approachability do coincide.

3.3. Minimal approachable sets. Since every set containing an approachable (or $*$approachable since both notions coincide) set is also approachable, it is natural to study approachable sets which are minimal for inclusion. This notion was first studied in [29] and [30].

DEFINITION 12. *A closed subset C is minimal approachable if it is approachable, and every approachable closed subset D of C satisfies $D = C$.*

Spinat [30] proved that a compact approachable set contains a minimal approachable subset and that any minimal approachable subset is connected.

Remark. Note that a minimal approachable set, if it exists, is not necessarily unique. Note also that a minimal approachable set must be a **B**-set.

Let C be a closed subset of $\mathbb{R}^k$. The discriminating kernel, denoted $Disc_f(C)$, is the largest closed set—possibly empty—contained in C and satisfying (3).

PROPOSITION 13. *Every minimal approachable set B included in C is included in $Disc_f(C)$.*

Proof. Assume that B is approachable minimal. Then B is a **B**-set and thus it is discriminating. Since $Disc_f(C)$ is the largest discriminating set it is clear that B is included in $Disc_f(C)$. ∎

Remark 8. In [14] algorithms finding the discriminating kernel are provided (cf. also [8] for the one-player case). It is worth pointing out that these algorithms do not require explicit computation of trajectories.

4. On strategies in the differential game and the repeated games. To the best of our knowledge, the links between nonanticipative strategies in differential games and strategies in repeated games have never been developed. Here, we introduce a kind of interpolation from strategies in the repeated game G^* to nonanticipative strategies with delay in Γ. Reciprocally we prove that existence of preserving nonanticipative strategies in Γ implies existence of $*$approachability strategies in the repeated game G^* and finally of approachability strategies in the repeated game G.

We first introduce a new notion of strategies which is crucial for time discretization.

DEFINITION 14. *A map $\delta : \mathcal{V} \mapsto \mathcal{U}$ is a* nonanticipative strategy with delay (NAD) *if there exits a sequence of times $0 < t_1 < t_2 < \cdots < t_n < \cdots$ going to ∞ with the following property:*

For every control $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{U}$ such that

$$\mathbf{v}_1(s) = \mathbf{v}_2(s) \textit{ for almost every } s \in [0, t_i],$$
$$\textit{then } \delta(\mathbf{v}_1)(s) = \delta(\mathbf{v}_2)(s) \textit{ for almost every } s \in [0, t_{i+1}].$$

(In particular this definition means that $\delta(\mathbf{v})$ does not depend on $\mathbf{v}$ on the interval $[0, t_1)$.)

Denote by $\mathcal{M}_d(\mathcal{V},\mathcal{U})$ the set of such NAD strategies with delay from $\mathcal{V}$ to $\mathcal{U}$. Observe that clearly $\mathcal{M}_d(\mathcal{V},\mathcal{U}) \subset \mathcal{M}(\mathcal{V},\mathcal{U})$.

4.1. From strategies in G^* to NAD strategies. We first verify that the construction using Ψ in section 3.2 leads to elements in $\mathcal{M}_d(\mathcal{V},\mathcal{U})$.

PROPOSITION 15. *Let σ be a strategy in G^* and let $\Psi[\sigma]$ be the converted strategy as introduced in Theorem* 10. *Then $\Psi[\sigma]$ is a NAD strategy.*

Proof. Let $0 < 1 < 2 < \cdots < n < \cdots$ be the subdivision of time. Given $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{V}$ with $\mathbf{v}(s) = \mathbf{w}(s)$ on $[0, i]$, one has, for every $j < i$, $\bar{v}_j = \bar{w}_j$. From the definition of $\bar{h}_i(\sigma, \mathbf{v})$, this implies for every $j < i$ that $\bar{h}_j(\sigma, \mathbf{v}) = \bar{h}_j(\sigma, \mathbf{w})$. Hence for every $s \in [i, i+1)$, $\Psi[\sigma](\mathbf{v}(s)) = \sigma(\bar{h}_0(\sigma, \mathbf{v}), \bar{h}_1(\sigma, \mathbf{v}), \ldots, \bar{h}_i(\sigma, \mathbf{v})) = \sigma(\bar{h}_0(\sigma, \mathbf{w}), \bar{h}_1(\sigma, \mathbf{w}), \ldots, \bar{h}_i(\sigma, \mathbf{w})) = \Psi[\sigma](\mathbf{w}(s))$. Thus $\Psi[\sigma]$ is a NAD strategy. ∎

Reciprocally we exhibit now a link between preserving nonanticipative strategies in the differential game Γ and approachability strategies in the repeated game G.

The idea of the construction is the following:

(a) Given a nonanticipative strategy α, we will prove that it can be approximated in term of range by a piecewise constant NAD strategy $\bar{\alpha}$.

(b) When applied to α preserving C (hence approaching C, by Theorem 9), we will obtain a NAD strategy $\bar{\alpha}$ approaching C.

(c) Considering the repeated game G^*, this NAD strategy $\bar{\alpha}$ will produce an $*$approachability strategy.

(d) Finally, $*$approachability strategies in G^* will induce approachability strategies in G.

4.2. From nonanticipative strategies to NAD strategies. In this part we compare nonanticipative strategies to NAD strategies. This is based on a result of Cardaliaguet [13] dealing with approximation on compact time intervals (Proposition 16). We give the proof for sake of completeness of the paper, because this result was never published. The second result is new and extends the approximation of nonanticipative strategies by NAD strategies to any time $t > 1$ (Proposition 17).

We first introduce the range associated to a nonanticipative strategy $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$ as

$$R(\alpha, t) = cl\{y \in \mathbb{R}^k \mid \exists \mathbf{v} \in \mathcal{V} \text{ with } y = \mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t)\}.$$

PROPOSITION 16. *Consider the differential game* (2) *with assumptions* (1) *and* (9). *Fix* $x_0 \in \mathbb{R}^k$. *For any* $\varepsilon > 0$, $T > 0$ *and any nonanticipative strategy* $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$, *there exists some nonanticipative strategy with delay* $\overline{\alpha} \in \mathcal{M}_d(\mathcal{V},\mathcal{U})$ *such that, for all* $t \in [0, T]$ *and all* $\mathbf{v} \in \mathcal{V}$,

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon.$$

Proof. The partition for the NAD strategy is $0 = t_0 < t_1 < \cdots < t_n = T$ with $t_k = kT/n$, $k = 0, \ldots, n$, for some large n to be specified later. The construction of the NAD strategy $\overline{\alpha}$ is by induction on k.

Observe first that any trajectory of (2) is bounded on $[0, T]$: there exists some constant R such that, for any control $\mathbf{u}$ and $\mathbf{v}$,

$$\forall t \in [0, T] \quad \|\mathbf{x}[x_0, \mathbf{u}, \mathbf{v}](t)\| \leq R.$$

Let us set

$$M' = \sup_{\|x\| \leq R,\, u \in U,\, v \in V} \|f(x, u, v)\|. \tag{22}$$

For $k = 0$, choose any $u_0 \in U$ and set $\overline{\alpha}(\bar{\mathbf{v}}) = u_0$ on $[t_0, t_1)$ for any $\bar{\mathbf{v}} \in \mathcal{V}(t_0)$.

Assume $\overline{\alpha}$ defined on $[t_0, t_k)$. Given $\mathbf{v} \in \mathcal{V}(0)$, write $x_k = \mathbf{x}[t_0, x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t_k)$. If x_k belongs to $R(\alpha, t_k)$, then choose some $u_k \in U$ and set $\overline{\alpha}(\mathbf{v}) = u_k$ on $[t_k, t_{k+1})$.

Otherwise, and from now on, assume that x_k does not belong to $R(\alpha, t_k)$. Then there exists some control $\mathbf{v}_k \in \mathcal{V}$ such that $y_k := \mathbf{x}[t_0, x_0, \alpha(\mathbf{v}_k), \mathbf{v}_k](t_k)$ is an approximate closest point to x_k in $R(\alpha, t_k)$ in the sense that

$$d^2_{R(\alpha,t_k)}(x_k) \geq \|y_k - x_k\|^2 - \frac{1}{n^2}. \tag{23}$$

Note that $p_k := x_k - y_k$ and take $u_k \in U$ such that

$$\sup_{v \in V} \langle f(x_k, u_k, v), p_k \rangle = \inf_{u \in U} \sup_{v \in V} \langle f(x_k, u, v), p_k \rangle = A_k. \tag{24}$$

In other words, u_k is optimal in the local game at x_k in direction p_k. Then set $\overline{\alpha}(\mathbf{v})(t) = u_k$ for all $t \in [t_k, t_{k+1})$, which clearly depends only on the restriction of $\mathbf{v}$ to $[0, t_k)$.

Therefore, since $y_k \in R(\alpha, t_k)$, we obtain for any $t \in [t_k, t_{k+1})$

$$d^2_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \inf_{\mathbf{v}' \in \mathcal{V}(t_k)} \|\mathbf{x}[t_0, x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t) - \mathbf{x}[t_k, y_k, \alpha(\mathbf{v}'), \mathbf{v}'](t)\|^2. \tag{25}$$

The main part of the proof consists in estimating the right-hand side of the previous inequality (25). Fix $\mathbf{v}' \in \mathcal{V}(t_k)$ and denote $x(t) = \mathbf{x}[t_0, x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t) = \mathbf{x}[t_k, x_k, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)$ and $y(t) = \mathbf{x}[t_k, y_k, \alpha(\mathbf{v}'), \mathbf{v}'](t)$. We evaluate the difference:

$$\begin{aligned} \|x(t) - y(t)\|^2 \leq{}& \|x(t) - x_k\|^2 + \|y_k - y(t)\|^2 + \|p_k\|^2 \\ &+ 2\langle x(t) - x_k, p_k\rangle - 2\langle y(t) - y_k, p_k\rangle \\ &+ 2\langle x(t) - x_k, y_k - y(t)\rangle . \end{aligned} \tag{26}$$

For any $t \in [t_k, t_{k+1})$, we have

$$\|x(t) - x_k\|^2 = \left\| \int_{t_k}^{t} f(x(s), u_k, \mathbf{v}(s))ds \right\|^2 \leq M'^2(t - t_k)^2 \leq M'^2T^2/n^2$$

because f is bounded by M', and, in the same way,

$$\|y(t) - y_k\|^2 \leq M'^2(t - t_k)^2 \leq M'^2T^2/n^2.$$

We now estimate the term $\langle x(t) - x_k, p_k\rangle$. For any $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} \langle x(t) - x_k, p_k\rangle &= \int_{t_k}^{t} \langle f(x(s), u_k, \mathbf{v}(s)), p_k\rangle\, ds \\ &\leq \int_{t_k}^{t} \langle f(x_k, u_k, \mathbf{v}(s)), p_k\rangle\, ds + M'L\|p_k\|T^2/n^2 \\ &\leq (t - t_k)A_k + M'L\|p_k\|T^2/n^2, \end{aligned}$$

denoting by L the Lipschitz constant of f.

We now deal with the term $\langle y(t) - y_k, p_k\rangle$. We have

$$\begin{aligned} \langle y(t) - y_k, p_k\rangle &= \int_{t_k}^{t} \langle f(y(s), \alpha(\mathbf{v}')(s), \mathbf{v}'(s)), p_k\rangle\, ds \\ &\geq \int_{t_k}^{t} \langle f(y_k, \alpha(\mathbf{v}')(s), \mathbf{v}'(s)), p_k\rangle\, ds - M'L\|p_k\|T^2/n^2 \\ &\geq \int_{t_k}^{t} \inf_{u\in U} \langle f(x_k, u, \mathbf{v}'(s)), p_k\rangle\, ds - L\|p_k\|^2T/n - M'L\|p_k\|T^2/n^2. \end{aligned}$$

Therefore

$$\begin{aligned} &\sup_{\mathbf{v}'\in\mathcal{V}(t_k)} \langle \mathbf{x}[t_k, y_k, \alpha(\mathbf{v}'), \mathbf{v}'](t) - y_k, p_k\rangle \\ &\quad\geq (t - t_k) \sup_{v\in V} \inf_{u\in U} \langle f(x_k, u, v), p_k\rangle - L\|p_k\|^2T/n - M'L\|p_k\|T^2/n^2. \end{aligned}$$

Using Isaacs' assumption (9), which gives $A_k = \sup_{v\in V} \inf_{u\in U} \langle f(x_k, u, v), p_k\rangle$, the previous inequalities provide an estimation of (26), which together with (25) give, for any $t \in [t_k, t_{k+1})$,

$$d^2_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \|p_k\|^2(1 + 2LT/n) + +4(M'L\|p_k\| + M'^2)T^2/n^2.$$

Since $\|p_k\| \leq 2R$ and in view of (23), we obtain

$$d^2_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq d^2_{R(\alpha,t_k)}(x_k)(1 + 2LT/n) + CT^2/n^2$$

for some constant C depending only on R, L, and M.

By induction we have thus constructed a NAD strategy such that, for any $\mathbf{v} \in \mathcal{V}(0)$ and any $t \in [0, T]$,

$$d^2_{R(\alpha,t)} \leq \frac{CT^2}{n^2}(1 + 2LT/n)\frac{(1 + 2LT/n)^n - 1}{2LT/n}. \tag{27}$$

The right-hand side goes to 0 when $n \to +\infty$. Thus for n large enough

$$\forall t \in [0, T],\ \forall v \in \mathcal{V} \quad d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(v), v](t)) \leq \varepsilon. \tag{28}$$ ◻

Remark 9. The above proof used only the fact that f satisfied (1) (i), (ii), (iii), and (9).

It is inspired by the "extremal aiming" method of Krasovskii and Subbotin [21] and is very much in the spirit of proximal normals and approachability.

The next result relies explicitly on the specific form (7) of the dynamics f in Γ and extends the approximation from a compact interval to $\mathbb{R}^+$.

PROPOSITION 17. *Fix $x_0 \in \mathbb{R}^k$. For any $\varepsilon > 0$ and any nonanticipative strategy $\alpha \in \mathcal{M}(\mathcal{V}, \mathcal{U})$ in the game Γ, there is some NAD strategy $\overline{\alpha} \in \mathcal{M}_d(\mathcal{V}, \mathcal{U})$ such that, for all $t \geq 0$ and all $\mathbf{v} \in \mathcal{V}$,*

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon.$$

Proof. Choose some $T > 0$. We will construct the NAD strategy $\overline{\alpha}$ in two steps: first on the interval $[0, T]$ and then on $[lT, (l + 1)T]$ for any integer l.

Let $\varepsilon'(T) > 0$ satisfy

$$\varepsilon'(T)\left(1 + \frac{1}{1 - e^{-T}}\right) < \varepsilon.$$

Observe now that if $x(\cdot)$ is a trajectory in the differential game Γ, $x'(t) = -x(t) + u(t)Av(t)$, $x(0) = x_0$, then

$$\langle x(t), x'(t)\rangle \leq -\|x(t)\|^2 + M\|x(t)\| \leq \frac{1}{2}(-\|x(t)\|^2 + M^2).$$

Thus for any $t \geq 0$

$$\|x(t)\|^2 \leq \|x_0\|^2 e^{-t} + M^2(1 - e^{-t}).$$

So there exists some R such that $\|\mathbf{x}[x_0, u, v](t)\| \leq R$ for all $t \in [0, +\infty)$. Define $\overline{\alpha}$ adapted to $[0, T]$ and $\epsilon'(T)$, as in Proposition 16. We now construct it on $[T, 2T]$ for $\mathbf{v} \in \mathcal{V}(0)$. By the property of $\overline{\alpha}$ there exists y_T achievable at time T under α such that

$$\|\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](T) - y_T\| \leq \varepsilon'(T).$$

Repeating the arguments of Proposition 16, starting from (T, y_T), we construct $\overline{\alpha}(\mathbf{v})$ on $[T, 2T)$ and obtain

$$\forall t \in [T, 2T] \qquad \|\mathbf{x}[T, y_T, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t) - y'_t\| \leq \varepsilon'(T), \tag{29}$$

where y'_t is achievable a time $t - T$ under α when starting from y_T at time 0 (and in particular belongs to $R(\alpha, t)$).

By the same estimation as that used in the proof of Theorem 9, we now have the following upper bound of the deviation:

$$\|\mathbf{x}[T, x(T), \overline{\alpha}(\mathbf{v}), \mathbf{v}](t) - \mathbf{x}[T, y_T, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)\| \leq e^{-(t-T)}\|x(T) - y_T\|.$$

In view of (29), this yields for any $t \in [T, 2T]$

$$\|\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t) - y_t\| \leq \varepsilon'(T)e^{-(t-T)} + \varepsilon'(T).$$

Thus

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon'(T)e^{-(t-T)} + \varepsilon'(T) \quad \forall t \in [0, 2T].$$

Finally, by an analogous iteration, we construct $\overline{\alpha}$ on any $[lT, (l+1)T]$ (for $l \geq 1$) with

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon'(T)\left[1 + \sum_{l=0}^{\infty} e^{-lT}\right] < \varepsilon \quad \forall t \in [0, +\infty)$$

and the proof is complete. ∎

In particular if we start with a nonanticipative strategy that preserves a set C, we obtain the next result.

PROPOSITION 18. *Let C be a $\mathbf{B}$-set. For any $\varepsilon > 0$ there is some NAD strategy $\overline{\alpha} \in \mathcal{M}_d(\mathcal{V},\mathcal{U})$ in the game Γ and some T such that for any $\mathbf{v}$ in $\mathcal{V}$*

$$d_C(\gamma[\overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon \qquad \forall t \geq T.$$

Proof. Apply the previous Proposition 17 to a strategy α satisfying (10) to obtain a strategy $\overline{\alpha}$ satisfying the requirement. ∎

4.3. From preserving strategies to ∗approachability strategies.

PROPOSITION 19. *For any $\varepsilon > 0$ and any nonanticipative strategy $\alpha \in \mathcal{M}(\mathcal{V},\mathcal{U})$ preserving C in the game Γ, there is some nonanticipative strategy with delay $\overline{\alpha} \in \mathcal{M}_d(\mathcal{V},\mathcal{U})$ that induces an ε-approachability strategy σ^* for C in G^*.*

Proof. Let α be a nonanticipative strategy preserving C. Given ε choose N large enough so that the majorant in (27) satisfies

$$\frac{CT^2}{n^2}(1 + 2LT/n)\frac{(1 + 2LT/n)^n - 1}{2LT/n} \leq \varepsilon/3$$

for $\log(1+1/N)^{-1} = n$ and $T = \log 2$. The stages k in G^* will define times in Γ using the map $t_k = \log(1 + k/N)$. The state $x_k = x(t_k)$ will correspond to $\gamma(e^{t_k})$, hence to the average payoff $\bar{g}_k^*$ in G^*. Assume the trajectory defined up to time t_k in Γ (i.e., up to stage k in G^*). Given the current range $R(\alpha, t_k)$ and the current position x_k, the NAD strategy $\overline{\alpha}$ specifies, as in Proposition 17, a constant control u_{k+1} on the time interval $[t_k, t_{k+1}]$. The strategy σ^* in G^* is now defined by the move u_{k+1} at stage $k + 1$. Given the move v_{k+1} of Player 2 at stage $k + 1$ in G^* the trajectory is now defined up to time t_{k+1} in Γ. Note that the step size $t_{k+1} - t_k$ is decreasing, and hence the construction is consistent with the majorization in Proposition 17. (Recall that $\overline{\alpha}$ is not "readjusted" every N stages (in G^*) but every time mT in Γ—or at times $t_{k(m)}$ for the smallest k such that $\log(1 + k/N)$ exceeds mT.) ∎

So the result of approximation of nonanticipative strategies by NAD strategies *on compact intervals* is valid for quite general differential games. This fact was observed in [14].

4.4. From $*$approachability strategies to approachability strategies.

PROPOSITION 20. *Given a strategy σ^* that $*$approaches C up to $\varepsilon > 0$ in the game G^*, there exists a strategy σ that approaches C up to 2ε in the game G.*

Proof. Let σ^* be a $*$approachability strategy in G^* (with parameters ε and N) and let us construct a σ in G.

First $\sigma(\emptyset) = \sigma^*(\emptyset)$. Given j_1, let $x_1 = g_{i_1 j_1}$ be the outcome in G and denote $y_1 = \sigma(\emptyset)A_{j_1}$. Given $h_1 = (i_1, j_1)$, let $h_1^* = (s_1^* = \sigma(\emptyset), j_1)$ and define the strategy by $\sigma(h_1) = \sigma^*(h_1^*)$. Inductively given $h_n = (h_{n-1}, i_n, j_n)$, let $h_n^* = (h_{n-1}^*, \sigma(h_{n-1}^*), j_n)$ and define $\sigma(h_n) = \sigma^*(h_n^*)$. Again the outcome is $x_n = A_{i_n j_n}$ and denote $y_n = \sigma(h_{n-1}^*)A_{j_n}$. The sequence of outcomes induced by σ and a strategy τ of Player 2 is $(x_1, \ldots, x_n, \ldots)$ and the associated sequence $(y_1, \ldots, y_n, \ldots)$ is compatible with σ^* in G^*. Hence $d_C(\bar{y}_n) \le \varepsilon$ for $n \ge N$.

Let $z_n = x_n - y_n$, $Z_n = \sum_{m=1}^n z_m$. Since $E(z_n|h_{n-1}) = 0$, observe that Z_n is a martingale for the filtration generated by $\{h_n\}_n$

$$E(Z_n|h_{n-1}) = \sum_{m=1}^{n-1} z_m + E(z_n|h_{n-1}) = Z_{n-1}$$

with increments bounded by $K = 2\|A\|$. It follows from the Hoeffding–Azuma inequality (see, e.g., [16, p. 363]) that for each component $l \in \{1, 2, \ldots, k\}$ and each positive number a

$$P(Z_n^l > a) \le \exp\left(-\frac{2a^2}{nK^2}\right),$$

and hence

$$P(\bar{Z}_n^l > a) \le \exp\left(-\frac{2a^2 n}{K^2}\right) \to 0.$$

In particular since $\bar{Z}_n$ is bounded this implies that $E(\|\bar{x}_n - \bar{y}_n\|)$ goes uniformly to 0, and hence there exists N' such that $n \ge N'$ implies

$$E(d(\bar{x}_n, C)) \le 2\varepsilon. \qquad \square$$

Remark 10. The result of Spinat [29] shows that approachability implies $*$approachability. We have obtained, as a byproduct of Proposition 20, another proof of the equivalence between $*$approachability and approachability (cf. Remark 7).

Appendix. In the differential games literature, the geometrical condition (3) is often written in different forms. We provide here relations between several of its formulations.

Introduce the tangent (Bouligand) cone

$$T_C(x) = \left\{ v \in \mathbb{R}^n, \quad \liminf_{h \to 0^+} \frac{d_C(x + hv)}{h} = 0 \right\}.$$

PROPOSITION 21 (see [2, 10]). *Let C be a nonempty closed subset of $\mathbb{R}^k$ and suppose that assumptions* (1) *hold true. Then the following assertions are equivalent:*

(i) *C is a discriminating domain for f;*

(ii) *for all $x \in C$ and all $v \in V$, there exists $u \in U$ such that $f(x, u, v) \in T_C(x)$;*

(iii) *for all* $x \in C$ *and all* $v \in V$*, there exists* $u \in U$ *such that* $f(x,u,v) \in co(T_C(x))$*, where* $co(A)$ *denotes the closed convex hull of* A*.*

The relations between (Bouligand) tangent cones and proximal normals are given by the following.

LEMMA 22 (cf. [26, Prop. 6.5, p. 200; Ex. 6.16, p. 212; Ex. 6.18, p. 214]). *Let* C *be a nonempty closed subset of* $\mathbb{R}^k$*; then we have that*

(i) $T_C(x) \subset NP_C(x)^-$*;*

(ii) $\liminf_{y\in C,\, y\to x} T_C(y) = (\limsup_{y\in C,\, y\to x} NP_C(y))^-$*;*

(iii) $\liminf_{y\in C,\, y\to x} T_C(y) \subset T_C(x)$*.*

We give another form of the interpretation theorem (Theorem 7) when Isaacs' condition (9) is satisfied.

THEOREM 23 (see [11]). *Let* C *be a nonempty closed subset of* $\mathbb{R}^k$ *and let* f *satisfy* (1) *and also Isaacs' condition* (9)*. Then* C *is a discriminating domain for* f *if and only if*

$$\text{(i) } \forall x_0 \in C, \quad \exists \alpha \in \mathcal{M}(\mathcal{V},\mathcal{U}), \quad \forall \mathbf{v} \in \mathcal{V}, \quad \forall t \geq 0 \quad \mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t) \in C,$$

or equivalently

$$\text{(ii) } \forall x_0 \in C, \quad \forall \varepsilon > 0, \quad \forall T > 0, \quad \forall \beta \in \mathcal{M}(\mathcal{U},\mathcal{V}), \quad \exists u \in \mathcal{U}, \quad \forall t \in [0,T] \\ \mathbf{x}[x_0, u, \beta(u)](t) \in C + \varepsilon B.$$

Note that the equivalence of (i) and (ii) is valid when Isaacs condition holds true. In fact, property (ii)—called leadership domain property—is equivalent to

$$\forall x \in C, \quad \forall p \in NP_C(x) \qquad \inf_{u\in U} \sup_{v\in V} \langle f(x,u,v), p\rangle \leq 0,$$

which reduces to (3) when Isaacs' condition (9) is fulfilled (cf. [10], [11]).

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