Int J Game Theory (1998) 27:315–330

International Journal of Game Theory

On the impact of an event

Sylvain Sorin

Laboratoire d'Econométrie, Ecole Polytechnique, 1 rue Descartes, 75005 Paris and MODAL'X, U.F.R. SEGMI, Université Paris X Nanterre, 200 Avenue de la République, 92001 Nanterre Cedex, France (e-mail: sorin @ poly.poytechnique.fr)

Received November 1994/Revised version September 1996

Abstract. Given an information structure a function that measures how the inferences made by the agents spread among the states is defined; it specifies for each event its impact at each state, for each agent. Several properties are established.

Key words: Information structure, transmission of knowledge

1. Presentation

The present study has two main sources: on the one hand results on propagation effects and diffusion of knowledge (Neyman (1996), Rubinstein (1989)), on the other hand research on perturbations of games (Kreps, Milgrom, Roberts and Wilson, (1982)) and measures of irrationality (Aumann (1992)).

More precisely, the first pair of papers deal with the comparison between common knowledge and mutual knowledge of high order of an event. They exhibit a "lack of continuity": the results obtained are qualitatively different under the alternative hypotheses on knowledge.

The second topic shows how introducing a small perturbation in the behavior of one player or allowing for a small amount of irrationality, may drastically modify the overall strategic results.

In both cases, there is lack of common knowledge of some event at some stage and attention is concentrated on the analysis of how this uncertainty propagates and affects the behavior of the players in the game. Note that, given the game parameters, some "critical amount" of uncertainty is needed for the phenomenon to occur. One would like to have a measure of the propagation effect. Thus it seems useful to have an analysis that treats only the informational aspects and eliminates any strategic consideration.

This paper builds on two approaches: one quantitative, in the framework of one-person Bayesian problems, the second qualitative, for multiperson information structures.

Consider first a one person decision problem under uncertainty and take the simple case of a finite state space Ω with an initial probability Q and a partition X corresponding to the agent's information. $\omega \in \Omega$ is chosen at random according to Q and the element $x = x(\omega)$ of X containing it is known by the agent. He must then choose an action s in some set S. Given a payoff function f defined on $S \times \Omega$, at each ω the agent will maximize on S his expected conditional payoff given $x(\omega)$. Assume that the function f changes on a subset A of Ω and consider the new problem faced by the agent. At each state ω the new behavior will depend on the change in f, but it is clear that for specific classes, such as $f + cI_A$ (where c is some constant and I_A the indicator function of the subset A), the change in the choice of action at ω will depend only on the conditional expectation of A given $x(\omega)$. If $A \cap x(\omega) = \emptyset$, there is no change in the agent's behavior at that state ω: he knows, given $x(\omega)$, that A does not occur. Otherwise, the information is more precise and has quantitative content: the way the change in f will influence the agent's behavior depends upon the conditional expectation of A at ω, i.e., the conditional probability of A given $x(\omega)$: we call this quantity the *impact* of A at ω. It is a measure of how a change in A may affect the behavior at ω.

If we consider now the n person case, the natural framework is a Bayesian game G defined by an information structure and a state dependent strategic game.

Formally, an *information structure* $\mathscr{I}$ for n players is a probability state space $(\Omega, \mathscr{A}, Q)$ endowed with a sub σ-field $\mathscr{A}^i$ for each player i, corresponding to his private information. (An equivalent representation is a correlated device with an $\mathscr{A}$-measurable private signaling map θ^i for each player i, $\mathscr{A}^i$ being the sub σ-field generated by θ^i on Ω.) For simplicity, we will consider the case where each private information σ-field $\mathscr{A}^i$ is generated by a finite partition X_i, thus $x_i(\omega)$, the element of X_i that contains ω, is player i's knowledge at ω.

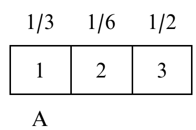
In addition one has for each state ω a strategic game $G(\omega)$ where each player i's action set $S^i(\omega)$ depends only on his signal $\theta^i(\omega)$.

Consider a profile of strategies σ corresponding to some equilibrium in the game G. One could modify the data of $G(.)$ on some subset A of Ω and ask whether σ is still an equilibrium of the new game. In the same vein one could change the strategies of some players at some states (such as imposing a perturbed behavior) and study the consequences on the others. Remark that the perturbation concerns only the "game part" and not the information structure since its role is to reveal the characteristics of the latter. A "distance" on the space of games to compare the size of the perturbations seems called for, and similarly one on the payoff or strategy spaces to evaluate the consequences of the perturbations.

The paper concentrates on the specific role played by the information structure. Qualitative properties related to the knowledge operators K_i – for each player i and event A, $K_i(A)$ is defined as the set $\{\omega : x_i(\omega) \subset A\}$ – are known. If the complement of A is common knowledge at some state ω, no perturbation of the game on A will change the equilibrium conditions at ω. However, a purely qualitative analysis based on the order of mutual knowledge of the complement of A at ω will not capture the notion of impact of A

(for example if the probability of A is very small). This corresponds to the difference between a quantitative approach in terms of beliefs with an initial probability and σ-fields (see e.g., Monderer and Samet (1989)), compared to a qualitative approach in terms of knowledge through information functions or u-fields (see Aumann (1995)). Similarly, a study in terms of beliefs with probability 1 will miss the point. It is necessary to consider beliefs of all "sizes" (p-belief operators for all $0 \leq p \leq 1$, Monderer and Samet (1989)). Moreover there is a complex interplay between the size and the order of the beliefs, both playing a role in such events as "I believe with probability p_1, that he believes with probability p_2, that I believe ... that the event A occurs".

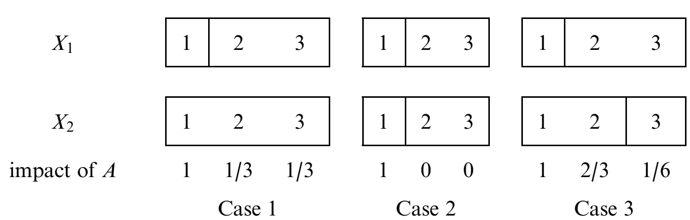
To clarify the ideas consider a very simple example with $\Omega = \{1, 2, 3\}$ and two agents. The initial probability is (q_1, q_2, q_3) and the event A is $\{1\}$. The figure below ilustrates the case when $(q_1, q_2, q_3) = (1/3, 1/6, 1/2)$.



Assume first that agent 1's partition X_1 is $(\{1\}, \{2, 3\})$ and that the other agent has no information. If the state is 1, agent 1 knows that A occurs and the impact of A at that state will be 1 for him. As for player 2, it is as if he were in the one person case, and the impact of A for him will be $Q(A) = q_1$ at each state. However this fact is known by player 1, even at states 2 or 3 when he knows that A does not occur. The overall profile describing the impact of A will thus be $(1, q_1, q_1)$. (Note that the number of states is irrelevant: the impact is 1 on A and $Q(A)$ elsewhere).

Suppose now that agent 2's information is increased so that $X_2 = X_1$. It is clear that, in so far as the information is concerned, this is a one person decision situation and the impact of A becomes $(1, 0, 0)$. The impact has decreased.

Finally consider the case where $X_2 = (\{1, 2\}, \{3\})$. As before the impact is 1 at state 1 (for player 1) and $q_1/(q_1 + q_2)$ at states 1 and 2 (for player 2). Now at states 2 or 3, player 1 faces with probability $q_2/(q_2 + q_3)$ a player for whom the impact is $q_1/(q_1 + q_2)$. The impact is defined there as $(q_1/(q_1 + q_2)) \times (q_2/(q_2 + q_3))$. This quantity is also the impact of A on player 2 at state 3. Note that at this state, both players know that A does not occur, however A's impact will be positive, expressing the fact that the complement of A is not common knowledge at that state.



Returning to the alternate definition of the information with signals one could imagine an approach via the entropy of the signals. However, the above examples show that the issue is not the complexity of each partition, or of their intersection or union, but instead the complexity of their interaction. The impact of an event is small (relative to its initial probability) if the knowledge of it is essentially the same among the agents. On the other hand, a large impact will reflect a long sequence of new deductions on the influence of A.

In a related context (the measure of proximity of information structures) Monderer and Samet (1996) emphasize the fact that concepts adapted to the one person case miss the point when interaction occur. In parallel with the usual conception of game theory as interactive decision theory, one could consider this work as part of the field of interactive information theory.

The paper is constructed as follows. The definition of the impact (local and global) of a function is presented in §2 and the main properties of local impact are studied in §3. In §4 we develop them in the case of an event while §5 is devoted to global impact. Examples are discussed in §6 and extensions proposed in §7.

2. The construction

Assume that there are two players (the extensions to n players will be described in section 7) and consider the simple case where the joint information of the agents defines the discrete partition of the state space. The relevant space is then $X = X_1 \times X_2$ (see also generalization in section 7), corresponding to the pairs of signals for the two players. Denote by P the probability induced on X by the initial probability Q on $(\Omega, \mathscr{A})$, i.e., defined by the formula $P(x_1, x_2) = Q(x_1 \cap x_2)$, for $x_1 \in X_1, x_2 \in X_2$. An event A is simply a subset of X.

For example, starting with $\Omega = \{1, 2, 3\}$ endowed with a uniform probability, and $a = \{1, 2\}$, $b = \{3\}$ (resp. $\alpha = \{1\}, \beta = \{2, 3\}$) being the partition of player 1 (resp. player 2), one obtains the following matrix description:

	α	β
a	1/3	1/3
b	0	1/3

$X_1 = \{a, b\}$, $X_2 = \{\alpha, \beta\}$, player 1 knows the line, player 2 the column and the numbers define the initial probability distribution P on the signal space $X_1 \times X_2$.

Denote by $X^P = \{x \in X; P(x) > 0\}$ the support of P and assume that the marginals of P on X_1 and X_2 have full support (hence that any signal of a player has a positive probability of occurence).

Given x_1 in X_1, let $\pi_1(x_1) = \{y \in X^P; y_1 = x_1\}$ describe the subset of states where player 1's signal is x_1; π_2 is defined similarly and for an arbitrary element σ in $\Sigma = X_1 \cup X_2$, $\pi(\sigma)$ will denote $\pi_i(\sigma)$, if $\sigma \in X_i$. (If X is viewed as a matrix, σ is the name of a line or of a column and $\pi(\sigma)$ the intersection with the support of P.)

For each $\sigma \in \Sigma$ we introduce a deduction operator T_σ, defined on the set $\mathscr{F}$ of positive functions f on X by:

$$T_\sigma f(x) = \begin{cases} f(x) & \text{if } x \notin \pi(\sigma) \\ \max\{(\sum_{x \in \pi(\sigma)} P(x) f(x) / P(\pi(\sigma)), f(x)\} & \text{if } x \in \pi(\sigma). \end{cases}$$

Hence on all states x compatible with σ, $T_\sigma f$ is the supremum of f and of its conditional expectation given σ; otherwise it remains equal to f. $T_\sigma f$ has to be thought as the new amount of information starting from level f and making the inference according to σ.

Assume for example that σ is a signal to agent 1. His new belief, as in the one person case, will be the conditional expectation of f given σ. However the fact that there is another player induces him to keep at least the previous level of information, hence the "max" that appears in the formula.

For each ordered subset of Σ, say $\Lambda = (\sigma_1, \ldots, \sigma_m)$, with $\sigma_i \neq \sigma_j$, for $i \neq j$, let:

$$T_\Lambda = T_{\sigma_m} \circ \cdots \circ T_{\sigma_1}.$$

Finally introduce the following:

Definition 1. *The* **(local) impact** *of f on the information structure $\mathscr{I}$ at x is:*

$$\varphi(x, f, \mathscr{I}) = \begin{cases} \max\{T_\Lambda f(x) : \Lambda \text{ ordered subset of } \Sigma^P\} & \text{for } x \in X^P \\ 0 & \text{for } x \notin X^P. \end{cases}$$

The intuition behind the above definition is the following: one considers all possible sequences of deductions compatible with the signals of the players. Each such chain of deduction operators applied to f defines a function and one takes the maximum.

Remarks

a) Since $T_\sigma f \geq f$ for all σ an equivalent expression is:

$$\varphi(x, f, \mathscr{I}) = \max\{T_\Lambda f(x); \Lambda \text{ permutation of } \Sigma\}.$$

b) Moreover for any $\Lambda = (\sigma_1, \ldots, \sigma_m)$:

$$T_\Lambda f(x) = T_{\sigma_{j(x)}} \circ \cdots \circ T_{\sigma_1} f(x)$$

where $j(x)$ is the smallest index k such that both $\pi_i(x_i)$, $i = 1, 2$ are included in $\bigcup_{\ell=1}^{k} \pi(\sigma_\ell)$.

Define the range of the chain Λ as $\pi(\Lambda) = \bigcup_{\sigma \in \Lambda} \pi(\sigma)$. Then:

c) $T_\Lambda f(x) = f(x)$ if $x \notin \pi(\Lambda)$

d) $T_\Lambda f(x)$ depends only upon the values of f on $\pi(\Lambda) \cup \{x\}$ and is less than the maximum of f on this subset.

The impact of f for player 1 at $x_1 \in X_1$ is:

$$\varphi_1(x_1, f, \mathscr{I}) = \min_{y \in \pi_1(x_1)} \varphi(y, f, \mathscr{I})$$

and similarly for player 2. Note that this function is measurable with respect to the player's information: this corresponds to the fact that after the last deduction of some player, one does not have to keep the max operator.

In the introductory example the impact for player 1 is respectively $(1, 1/3, 1/3)$; $(1, 0, 0)$; $(1, 1/6, 1/6)$, that for player 2 $(1/3, 1/3, 1/3)$; $(1, 0, 0)$; $(2/3,\ 2/3,\ 1/6)$.

Definition 2. *The* **global impact** *of f on the information system $\mathscr{I}$, is the expectation of the local impact:*

$$\Phi(f, \mathscr{I}) = E_P(\varphi(x, f, \mathscr{I})).$$

3. General properties of the local impact φ

Consider now the properties of φ as a function of the state x, of the function f and of the probability distribution P. Since the signalling structure $X = X_1 \times X_2$ is fixed, φ may be viewed as a function of P rather than of $\mathscr{I}$.

3.1 Normalization

f may be replaced by $f^P = f \cdot I_{X^P}$ in all computations of the impact.

The domain of φ extends to the set $\mathscr{M}^+(X)$ of non zero finite positive measures on X by homogeneity of degree 0. For all $t > 0$, $\varphi(x, f, tP) = \varphi(x, f, P)$ and the formula of definition 1 still holds.

3.2 Monotonicity

φ satisfies the following properties:

Proposition 1. *(P1) If $f \geq g$, then $\varphi(x, f, P) \geq \varphi(x, g, P)$ for all x, P;*

(P2) Let $k \geq 1$ and μ, μ' in $\mathscr{M}^+(X)$ with $\mu = \mu'$ on $\{f < t\}$ and $\mu = k\mu'$ on $\{f \geq t\}$ for some function f and some real number t, then:

$$\varphi(x, f, \mu) \geq \varphi(x, f, \mu') \textit{ for all } x;$$

(P3) $\varphi(x, f + g, P) \leq \varphi(x, f, P) + \varphi(x, g, P)$ for all x, f, g, P.

Proof: (P1) $f \geq g$ implies $T_\sigma f \geq T_\sigma g$ for any σ. It follows that if $\varphi(x, g, P)$ can be written as $T_A g(x)$ one has $\varphi(x, f, P) = \max_{A'} T_{A'} f(x) \geq T_A f(x) \geq \varphi(x, g, P)$;

(P2) If T is associated with μ and T' with μ', one has: $T_\sigma f \geq T'_\sigma f$ for any σ since the conditional expectation of f under μ is everywhere greater than under μ'. The proof is then as above.

(P3) Again it is enough to check that : $T_\sigma(f + g) \leq T_\sigma f + T_\sigma g$. For then by induction, using the monotonicity of T_σ one has: $T_A(f + g) \leq T_A f + T_A g$.

Hence if $\varphi(x, f + g, P) = T_A(f + g)(x)$, since both $\varphi(x, f, P) \geq T_A f(x)$ and $\varphi(x, g, P) \geq T_A g(x)$, the result follows. ■

3.3 Other properties

The following properties are straightforward but useful:

Proposition 2. *(P4) If $f' = f + t$ for some real number t, then:*

$\varphi(x, f', P) = \varphi(x, f, P) + t$;

(P5) For any $t \geq 0$, $\varphi(x, tf, P) = t\varphi(x, f, P)$

Corollary 1. *φ as a mapping from $\mathscr{F}$ to $\mathscr{F}$ (with the uniform norm) is non-expansive.*

Proof. Use *(P1)* and *(P4)*. ■

4. The local impact of an event

The *impact* of an event A is the impact of its indicator function I_A and will also be denoted by $\varphi(x, A, \mathscr{I})$ (meaning local at x) or $\Phi(A, \mathscr{I})$ (meaning global).

We first translate the general properties to obtain:

(P6) $A \subset B \Rightarrow \varphi(x, A, P) \leq \varphi(x, B, P)$,
which means that if the event A is included in the event B, its impact is less at any state.

(P7) $\varphi(x, A \cup B, P) \leq \varphi(x, A, P) + \varphi(x, B, P)$,
namely the impact of the union of two events is less than the sum of the impacts.

(P8) $\varphi(x, A, P) = 0$ for $x \notin X^P$ or if $A^P = A \cap X^P = \emptyset$, since then either x or A cannot occur under P.
Note that property (P2) can be strengthened to:

(P9) Let μ, μ' in $\mathscr{M}^+(X)$ with $\mu = \mu'$ on the complement of A and $\mu \geq \mu'$ on A, then:

$\varphi(x, A, \mu) \geq \varphi(x, A, \mu')$ for all x.

This means that if the probability of any element in A increases relatively to any element of its complement, then the impact of A increases everywhere. It is also easy to see that:

(P10) $\varphi(x, A, P) = 1$ iff $x \in A^P$.

To state the next properties several more definitions and notations are needed:

x leads to y if there exists a path from x to y, namely, a sequence $\{x_0, \ldots, x_k, \ldots, x_m\}$ in X^P with $x_0 = x$ and $x_m = y$ and for all $k = 0, \ldots, m-1$, an index $i = 1$ or 2 such that $x^i_{k+1} = x^i_k$ (we denote by α_{k+1} the corresponding point in Σ).

A path is *minimal* (or an m-path) if $x^i_{k+j} = x^i_k$ for some $i = 1, 2$, $k = 0, \ldots, m-2$ and $j \geq 2$ is impossible. Hence an m-path can also be defined by a sequence $\{x, \alpha_1, \ldots, \alpha_m, y\}$ with all $\alpha_k \in \Sigma$ distinct (but remark that not all such sequences are possible, because the nodes have to be in the support of P).

Given an m-path from x to y, let $C_{xy} = \bigcup_1^m \pi(\alpha_k)$ be its graph.

A path from a state x to some event A is a path from x to some $y \in A$ with $x_k \notin A$ for $k < m$. If such a path exists, we say that x is *connected to A*.

C_x is taken to be the union of the graphs C_{xy}, over all m-paths from x to A. Finally define A_x as $A \cap C_x$, the subset of A connected to x.

Remark that: (P11) $\varphi(x, A, P) > 0$ iff x is connected to A, i.e. iff $A_x \neq \emptyset$.

Another way to look at this property is to define $E(A)$ as the smallest evident event containing A (or equivalently $(CK(A^c))^c$, where A^c is the complement of A and CK is the common knowledge operator associated to the K_i's). Then $x \in E(A)$ if and only if $A_x \neq \emptyset$.

In particular, if $x \notin E(A)$, A^c is common knowledge at x and the impact of A at x is 0.

The next proposition states that if any path from A reaches y (strictly) before x, then the influence of A at y is (strictly) greater than at x.

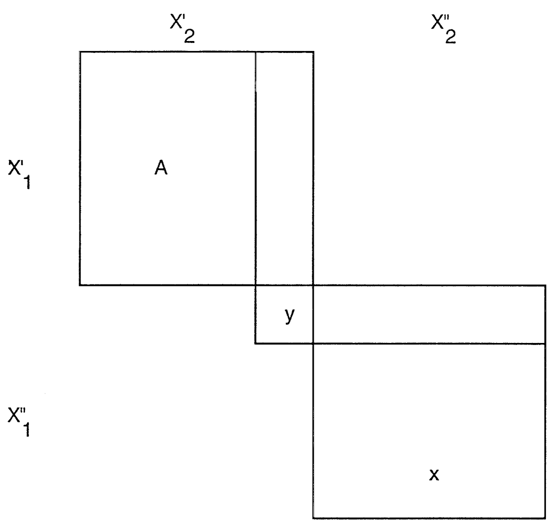
Proposition 3. *(P12) If x is connected to A and any m-path from x to A contains y ≠ x as a node, then:*

$$\varphi(x, A, P) < \varphi(y, A, P).$$

(P13) If y belongs to any graph of an m-path from x to A then:

$$\varphi(x, A, P) \leq \varphi(y, A, P).$$

Proof: (P12) One may assume $A = A^P$. The hypothesis implies that there exist partitions X_1', X_1'' of X_1 and X_2', X_2'' of X_2 such that: $A \subset X_1' \times X_2'$, $x \in X_1'' \times X_2''$ and $y_1 \in X_1''$, $y_2 \in X_2'$ (or a dual property).

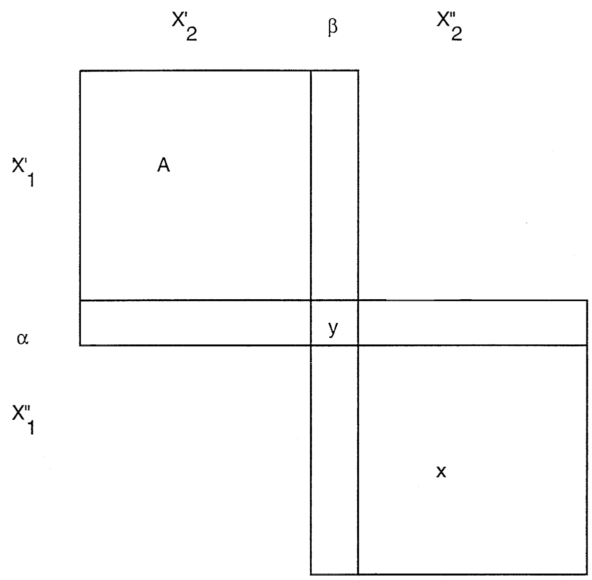


Let then $\alpha = y_1$ and note that if $\alpha \notin \Lambda$, $T_\Lambda f(x) = T_{\Lambda'} f(x)$ for $\Lambda' = \Lambda \cap \{X_1'' \times X_2''\}$; in particular $T_\Lambda I_A(x) = 0$.

So that if $\varphi(x, A, P) = T_\Lambda I_A(x)$ with $\Lambda = (\sigma_1, \dots, \sigma_k, \alpha, \sigma_{k+1}, \dots, \sigma_m)$, write $\Lambda' = (\sigma_1, \dots, \sigma_k)$ and $\Lambda'' = (\sigma_{k+1}, \dots, \sigma_m) \cap \{X_1'' \times X_2''\}$. We have $\varphi(x, A, P) = T_{\Lambda''} \circ T_\alpha \circ T_{\Lambda'} I_A(x)$. Clearly $T_{\Lambda'} I_A(y) \leq \varphi(y, A, P)$ and $T_{\Lambda'} I_A(z) = 0$ on $X_1'' \times X_2''$. This in turn implies that $T_\alpha \circ T_{\Lambda'} I_A(z) < \varphi(y, A, P)$ on $X_1'' \times X_2''$ (recall that $\varphi(y, A, P) > 0$). The result obtains by remark d).

P13) The hypothesis in this case implies the existence of partitions X_1', α, X_1'' of X_1 and X_2', β, X_2'' of X_2 such that:

$$A \subset X_1' \cup \{\alpha\} \times X_2' \cup \{\beta\}, x \in X_1'' \cup \{\alpha\} \times X_2'' \cup \{\beta\} \text{ and } y_1 = \alpha, y_2 = \beta.$$



Obviously if $\{\alpha, \beta\} \cap A = \emptyset$, then $T_A I_A(x) = 0$. Hence if $\varphi(x, A, P) = T_A I_A(x)$, one can assume that the elements of A before the first of α or β are in $X_1' \cup X_2'$ while the ones after the last of α and β are in $X_1'' = \cup X_2''$. Write for example, T_A as $T_{A_3} \circ T_\beta \circ T_{A_2} \circ T_\alpha \circ T_{A_1}$.

Now we have $T_\alpha \circ T_{A_1} I_A(z) \leq T_\alpha \circ T_{A_1} I_A(y) \leq \varphi(y, A, P)$ for $z \in X_1'' \times X_2''$. Again on $X_1'' \times X_2''$, $T_{A_2} \circ T_\alpha \circ T_{A_1} I_A(z)$ will remain less than $\varphi(y, A, P)$ so the same inequality holds for $T_\beta \circ T_{A_2} \circ T_{A_1} I_A(z)$, since $T_\beta \circ T_{A_2} \circ T_\alpha \circ T_{A_1} I_A(y) \leq \varphi(y, A, P)$. The result follows again from remark d). ■

The proof shows:

Corollary 2. *Under the hypotheses of Proposition 3 i), $\varphi(x, A, P)$ depends only on the value of $\varphi(y, A, P)$ and on the restriction of P to $X_1'' \times X_2''$.*

The next property means that the impact of the event on x depends only on the "exposed elements" of A at x (i.e. the ones in A_x) and that the states not "between" x and A do not influence the impact.

Proposition 4. *(P14) For any state x and any probability P' that coincides with P on C_x:*

$$\varphi(x, A, P) = \varphi(x, A_x, P')$$

Proof: Let $\varphi(x, A, P) = T_\Lambda(x)$ with $\Lambda = (\sigma_1, \ldots, \sigma_m)$. It is enough to check the two following facts:

$$T_{\sigma_{k+1}} \circ T_{\sigma_k} \circ \cdots \circ T_{\sigma_1} I_A = T_{\sigma_k} \circ \cdots \circ T_{\sigma_1} I_A,$$

if $\pi(\sigma_{k+1})$ is disjoint from $A \cup \{\bigcup_{j=1}^{k} \pi(\sigma_j)\}$, and

$$T_{\sigma_m} \circ \cdots \circ T_{\sigma_k} f(x) = T_{\sigma_m} \circ \cdots \circ T_{\sigma_{k+1}} f(x),$$

if $\pi(\sigma_k)$ is disjoint from $\{x\} \cup \{\bigcup_{j=k+1}^{m} \pi(\sigma_j)\}$.

Applying these properties to Λ, one obtains a reduced chain with graph in C_x, hence the result. ∎

This property also explains that the notion of impact is not well covered by the order of mutual knowledge. There might exist states where A is mutual knowledge of order k which can be deleted when computing the impact at a state x with mutual knowledge order k' lower than k. On the other hand, one may have to consider states with mutual knowledge order less than k' to compute the impact at x. (One may also have dual relations when considering the mutual knowledge of the complement of A).

5. Properties of the global impact Φ

We now consider the global impact of a function f or of an event A on the information structure, or the expectation of the local impact.

The first properties follow easily from the corresponding local properties of φ.

(P15) $f \geq g \Rightarrow \Phi(f, P) \geq \Phi(g, P)$
(P16) $\Phi(f + g) \leq \Phi(f) + \Phi(g)$
(P17) $\Phi(f + t, P) = \Phi(f, P) + t$
(P18) $\Phi(tf, P) = t\Phi(f, P)$ for $t \geq 0$.

The property analogous to *(P9)* also holds:

Proposition 5. *(P19) Let P and Q be probabilities on X with $P \geq Q$ on A and $P = tQ$ on A^c. Then:*

$$\Phi(A, P) \geq \Phi(A, Q).$$

Proof: One has $\varphi(x, A, P) \geq \varphi(x, A, Q)$ for all x and $\varphi(x, A, P) = 1$ on A^P. So:

$$\begin{aligned}
\Phi(A, P) &= P(A) + \sum_{x \notin A} P(x)\varphi(x, A, P) \\
&\geq P(A) + \sum_{x \notin A} tQ(x)\varphi(x, A, Q) \\
&\geq Q(A) + \sum_{x \notin A} Q(x)\varphi(x, A, Q) \\
&= \Phi(A, Q)
\end{aligned}$$

since $P(A) = 1 - t(1 - Q(A))$. ∎

Finally:

Proposition 6. *Φ is jointly continuous on $\mathscr{F} \times P$ (each with the uniform norm).*

Proof: Due to Corollary 1 it is enough to prove the continuity with respect to P. Assume that $\|P - P'\| < \varepsilon^2$ with $P(x) \geq \varepsilon$ for $x \in X^P \subset X^{P'}$. Write T (resp. T') for the operator associated with P (resp. P'). Then, for ε small enough, $\|T'_\sigma f - T_\sigma f\| \leq 3\varepsilon\|f\|$ for any $\sigma \in \Sigma^P = X_1^P \cup X_2^P$. It follows that $\varphi(x, f, P') \geq \varphi(x, f, P) - 3k\|f\|\varepsilon$, where k is the cardinality of Σ.

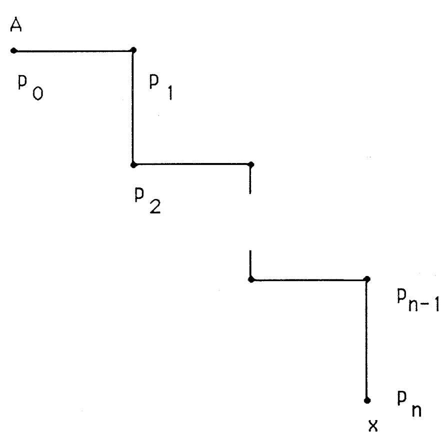
On the other hand, $|T'_\sigma f(x) - T_\sigma f(x)| \leq 3\varepsilon\|f\|$ for any $\sigma \in \Sigma'^P$ and any $x \in X^P$. Hence $\varphi(x, f, P) \geq \varphi(x, f, P') - 3k\|f\|\varepsilon$ on X^P. Since φ is bounded by $\|f\|$, the result follows. ■

6. Examples

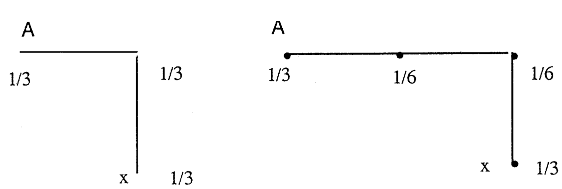
6.1 Single path

The basic example corresponds to the following stair-like information structure: the nodes are the states in the support of P and the numbers the probabilities; as usual, player 1 knows the horizontal sections and player 2 the vertical ones; finally there is one and only one path between any two nodes.

In the example below $\varphi(x, A, P) = \prod_0^{n-1} p_i/(p_{i+1} + p_i)$. In particular, the impact of A strictly decreases along the path towards x.



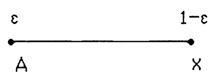
Note that the impact reflects the uncertainty;

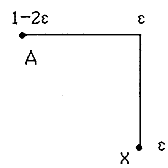


The impact of A at x is $1/4$ on the left and $1/6$ on the right where the node through which the uncertainty propagates has smaller probability.

6.2 *Impact and knowledge*

The impact of A is larger if the degree of mutual knowledge of A^c is smaller but the initial probabilities that induce the beliefs have to be taken into account:

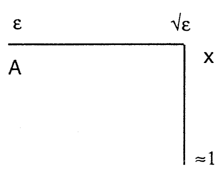


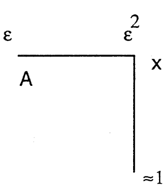


The impact of A at x is ε on the left and almost $1/2$ on the right. The corresponding global impact is of the order of 2ε on the left and $1-\varepsilon/2$ on the right.

6.3 *Local versus global*

In the following examples the local impact of A at x is very different (almost one on the left and zero on the right) while the global impact is almost the same.





6.4 *Impact and uncertainty*

The next example shows how the impact is related to uncertainty:

	α	β
a	A	x
b	A	y

Case 1

	α	β	γ
a	$*$	A	x
b	A	$*$	y

Case 2

	α	β	γ	δ
a	$*$	A	$*$	x
b	A	$*$	y	$*$

Case 3

The $*$ corresponds to states with probability 0. In the first case player 2 does not know the beliefs of 1 on A because at α or β he does not know the signal of player 1; while in the second case he may deduce it from his information, but only when he knows A: for example at α, he knows that the signal of player 1 is b and that his belief on A is $P(\alpha,b)/(P(\alpha,b)+P(\gamma,b))$, but the impact of A at (α,b) is already 1. Finally in the third case player 2 can compute player 1's beliefs at any state. Thus the impact of A will be greater in the first two cases, because the uncertainty is higher.

6.5 *Impact and order of knowledge*

The example below shows how the way A affects a state depends upon the state itself and not only the "proximity" of A.

	α	β
a	A	x
b	$*$	y
c	A	z

Note that $T_b = id$, $T_\alpha I_A = id$. The impact at x is obtained through $T_a \circ T_\beta \circ T_c$. In fact, in addition to the direct propagation effect due to the signal a at x one has to take into account that at that stage, player's 2 signal is β; thus player 2 considers as possible state z, where player 1's signal is c and the impact of A is positive. For y the composition $T_\beta \circ T_a \circ T_c$ is used, and for z, $T_c \circ T_\beta \circ T_a$.

6.6 *Impact and private impact*

For the following and final example

A	x
y	$*$

states

1/2	1/3
1/6	0

probabilities

the following values are obtained:

1	3/5
3/4	0

φ

3/5	3/5
3/4	0

φ_1

3/4	3/5
3/4	0

φ_2

The function φ is known by both players but the impact as a function of the signal is given by φ_i for player i.

7. Concluding remarks

7.1 Extensions

The definition of impact has been given under several simplifying assumptions. We extend it here to more general cases.

a) The situation with n agents is straightforward using the product signal space $X = \prod_i X_i$ and deduction operators T_{x_i} for all x_i and all i.

b) When the σ-algebra $\mathscr{A}$ of events is finer that the one generated by all the signals of all the players (denoted by $\mathscr{B}$), the definition of the impact is extended as follows: Given A in $\mathscr{A}$, let $E(A\,|\,\mathscr{B})$ be its conditional expectation, which is an element of $\mathscr{F}$. The previous construction thus applies and one defines the impact of A at x as $\varphi(x, E(A\,|\,\mathscr{B}), \mathscr{P})$.

In particular if A and B in $\mathscr{A}$ are such that $E(B|\mathscr{B}) = tE(A\,|\,\mathscr{B})$ then $\varphi(\cdot, B, P) = t\varphi(\cdot, A, P)$ on X.

c) Finally when dealing with countable partitions, to compute the impact of A at x, choose a finite collection of elements of the partitions, containing x such that its union B has a probability larger than $1 - \varepsilon P(x)$. By considering the case where the complement of A is common knowledge at any point of the complement of B one the one hand and the case where the complement of B belongs to A on the other hand, one obtains by monotonicity lower and upper bounds on the impact of A at x, that differ by at most 2ε.

7.2 Related approaches

a) Aumann (1992) considers a situation consisting of an information structure, a game G and for each player a map from his signal space to his strategy space in the game G. This corresponds to the profile of strategies in the extended game. A player is rational given his signal if he plays a best reply to the distribution of the other players' moves in G, given the initial distribution of signals and the profile of strategies. If all players are rational at each state, hence given any profile of signals, the distribution induced on the moves in G is a correlated equilibrium distribution (but not reciprocally). By allowing irrationality at some states one obtains induced distributions that differ drastically from the equilibrium distributions.

In our framework let A be the set of states where irrationality occurs. The states where the impact of A is 0 correspond to a correlated equilibrium distribution. On the other hand, the larger is the impact (local or global) the "easier" it is to perturb the game. In other words given same initial bound on the amount of irrationality (like $P(A)$), irrationality at event A will have more important consequences if the corresponding impact is higher.

b) Monderer and Samet (1996) compare information structures (on the same probability space) by considering by how much the equilibria they induce in a game will differ. The properties of the impact (which is clearly continuous for their topology) could be used to refine their notion of distance: in words, two information structures could be very different far from A but the impact of A could be essentially the same for both. In particular, for some games the behavior of the players will be similar under both information structures. A dual approach would focus on "test" events that will reveal the distance between two information structures (even allowing for different initial probability distributions) by inducing different impacts.

c) Morris, Rob and Shin (1995) define the belief potential of an event A as the smallest p such that the event will be iteratively p-believed everywhere and they deduce a notion of belief potential of an information system.

They consider a related two person game with the following property (q-dominance): there exist two strategies s_1 and s_2 of the two players such that at any state, s_1 is the unique best reply to any mixed strategy of player 2 giving a weight at least q to s_2, and the dual property for s_2. Let A be a set of states where player 1 is playing s_1; if A has a belief potential larger than q, (s_1, s_2) will be played everywhere. This describes precisely the propagation effect in Rubinstein's (1989) example.

It is also possible to define the belief potential of an event A at a state x as the smallest p such that the iterative p-beliefs of A will eventually cover x, and then study propagation effects locally. Our approach allows also the "level" of beliefs to vary (this could correspond to different underlying q-dominance relations) while their procedure amounts to applying uni!formly a kind of 0–1 test to the belief operators: A extends to $A \cup K_i^p(A)$ if p is larger than q and remains A otherwise.

7.3 Comments

The way impact is defined implies that in the one person case the global impact of an event A is $P(A)$. It is thus explicitly independent of the entropy of the information structure. It would nevertheless be interesting to know whether an axiomatic approach could be obtained to define impact. One difficulty is the number of variables when dealing with local impact.

A second difficulty lies in the monotonicity properties. The local and global impacts, φ and Φ, considered as functions of the σ – fields $\mathscr{A}_i$'s, describe the way the private informations of both players are interconnected. When the σ-fields are identical for the players one obviously obtains the same result as in the one person case and the impact $\Phi(A)$ is minimal. More precisely this minimum is attained when A is evident (this property is independent of P) but otherwise the value of the impact (which thus somehow measures the lack of common knowledge) depends crucially on P.

The fact that impact is related to the coordination of the signals implies that when one σ-algebra is kept fixed the impact is not a monotonic function of the other (for the inclusion), as in the following example:

$$\Omega = \{1,2,3\}, P = (p_1, p_2, p_3), X_1 = \{\{1,2\},3\} \text{ and } A = \{1\}.$$

The values of $\Phi(A)$ corresponding to the different partitions of player 2 are given by:

(a) $X_2 = \{\Omega\}$, $\Phi(A) = p_1 + p_1 p_3$
(b) $X_2 = \{\{1\},\{2,3\}\}$,
$\Phi(A) = p_1 + p_1 p_2/(p_1 + p_2) + p_1 p_2 p_3/(p_1 + p_2)(p_2 + p_3)$
(c) $X_2 = \{\{1,3\},\{2\}\}$, $\Phi(A) = p_1 + p_1 p_2/(p_1 + p_2) + p_1 p_3/(p_1 + p_3)$
(d) $X_2 = \{\{1,2\},\{3\}\}$, $\Phi(A) = p_1$
(e) $X_2 = \{\{1\},\{2\},\{3\}\}$, $\Phi(A) = p_1 + p_1 p_2/(p_1 + p_2)$

Acknowledgments. Thanks are due to a referee, to an associate editor and to the editor for clear and deep comments. The help of Michel Balinski in writing this new version is appreciated.

References

Aumann RJ (1992) Irrationality in game theory. In: Dasgupta P, Gale D, Hart O, Maskin E (eds.) Economic analysis of markets and games (Essays in honor of F. Hahn), M.I.T. Press pp. 214–227

Aumann RJ (1995) Interactive epistemology. CRIDT DP 67

Kreps D, Milgrom P, Roberts J, Wilson R (1982) Rational cooperation in the finitely repeated Prisoner's Dilemma. Journal of Economic Theory 27:245–252

Morris S, Rob R, Shin HS (1995) *p*-Dominance and stochastic potential. Econometrica 63:145–157

Monderer D, Samet D (1989) Approximating common knowledge with common beliefs. Games and Economic Behavior 1:170–190

Monderer D, Samet D (1996) Proximity of information in games with incomplete information. Mathematics of Operations Research 21:707–725

Neyman A (1996) Cooperation in repeated games when the number of stages is not commonly known. Econometrica, forthcoming

Rubinstein A (1989) The electronic mail game: strategic behavior under "almost common knowledge". American Economic Review 79:385–391