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ON REPEATED GAMES WITH COMPLETE INFORMATION*

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We consider N person repeated games with complete information and standard signalling. We first prove several properties of the sets of feasible payoffs and Nash equilibrium payoffs for the n-stage game and for the λ-discounted game. In the second part we determine the set of equilibrium payoffs for the Prisoner's Dilemma corresponding to the critical value of the discount factor.

0. Introduction. We consider N-person repeated games with complete information and standard signalling. We introduce the n-stage game, the λ-discounted game and the infinitely repeated game; then we prove several properties concerning the sets of feasible payoffs and of Nash equilibrium payoffs.

The properties studied are mainly the relation between convexity and stationarity and the simply-connectedness of the set of feasible payoffs.

The second part of the paper is devoted to the study of the λ-discounted Prisoner's Dilemma. If λ is greater than a critical value $\bar{\lambda}$ the only Nash equilibrium payoff is the usual one (like in any finite repetition). Then we determine exactly the set of Nash equilibrium in the game with this discount factor $\bar{\lambda}$, and this is a connected set of dimension 2 which differs from the set of individually rational feasible payoffs.

1. Notations and preliminaries. Let G_1 be an N-person game in normal form with finite pure strategy sets T_i, $i \in N$ and payoff function X from $T = \prod_{i=1}^{N} T_i$ into R^N. We denote by $\mathscr{C}_i$ the set of mixed strategies of player i. We associate to G_1 a repeated game with perfect recall played as follows: at each stage m, knowing the previous history h_m (i.e. the sequence of moves of all players up to stage $m-1$), each player i chooses a move t_i in T_i and this choice is told to all players.

We denote by S_i (resp. Σ_i) the set of pure (resp. mixed) strategies of player i in this repeated game and $S = \prod_{i=1}^{N} S_i$, $\Sigma = \prod_{i=1}^{N} \Sigma_i$. We now define 3 games according to the following payoffs:

$(1/n) \cdot \sum_{m=1}^{n} x_m$, $n \in N$ for G_n (n-stage repeated game),
$\lambda \cdot \sum_{m=1}^{\infty} (1-\lambda)^{m-1} x_m$, $\lambda \in (0,1]$ for G_λ (λ-discounted game),
$L((1/n) \cdot \sum_{m=1}^{n} x_m)$ for G_∞ (L-infinitely repeated game),
where x_m is the payoff at stage m and L a Banach limit.[1]

Let us now define D_n (resp. D_λ, D_∞) to be the set of feasible payoffs using mixed strategies and E_n (resp. E_λ, E_∞) to be the set of Nash equilibrium payoffs in G_n (resp. G_λ, G_∞).

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[1] Remark that the payoff in the L-infinitely repeated game is defined as the L-limit of the expectation. Nevertheless in our set-up the results would be the same by taking the expectation of the L-limit (this is no longer true for games with incomplete information).

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Note that G_n and G_λ are special cases of games $\tilde{G}:(\tilde{S}_i,\tilde{\Sigma}_i,f_i,\ i\in N)$ where $\tilde{S}_i$ are compact strategy spaces, $\tilde{\Sigma}_i$ regular probabilities on S_i and f_i continuous (real) functions on $\tilde{S}=\prod_{i=1}^{N}\tilde{S}_i$. The (vector) payoff function is defined on $\tilde{\Sigma}=\prod_{i=1}^{N}\tilde{\Sigma}_i$ by

$$F(\sigma)=\int_{\tilde{S}}f(s)\prod_{i=1}^{N}\sigma_i(ds_i).$$

It follows that D_n and D_λ will share all the properties of $\tilde{D}$ (set of feasible payoffs in $\tilde{G}$) and similarly for E_n and E_λ with respect to $\tilde{E}$ (set of equilibrium payoffs in $\tilde{G}$).

In particular we have:

(1) $\tilde{D}$ is a nonempty, path-connected, compact set,

(2) $\tilde{E}$ is a nonempty compact set (Nash theorem).

Recall that $\tilde{D}$ is usually not convex and $\tilde{E}$ not connected.

Let F be the finite set of feasible payoffs in pure strategies in G_1 and let $C=\operatorname{co}F$ denote the convex hull of F. Hence C is the set of payoffs achievable by using correlated strategies in G_1.

Finally define a_i to be the individually rational level of player i and Δ to be the set of individually rational payoffs in C, namely:

$$\Delta=\left\{y\,|\,y\in C,y_i\geqslant a_i=\min_{\mathscr{C}^i}\max_{T_i}X_i(\tau^i,t_i)\ \forall i,\text{ where }\mathscr{C}^i=\prod_{j\neq i}\mathscr{C}_j\right\}.$$

Then the following asymptotic properties hold:

(3) D_n (resp. D_λ) converges in the Hausdorff topology as n goes to ∞ (resp. as λ goes to 0) to C and D_∞ equals C (see [2], [6] and Proposition 4 below).

(4) E_λ converges in the Hausdorff topology, as λ goes to 0, to Δ (see [2] or Lemma 2 below)[2] and E_∞ equals Δ (Folk theorem see [1] or [6]). It is well known that E_n does not necessarily converge to Δ, see e.g. example in §3.

Thus Property (4) shows an important difference with zero-sum two-person repeated games; in this framework the asymptotic behaviour of v_n (value of G_n) and v_λ (value of G_λ) is the same, even for stochastic games (where it converges to v_∞ (value of G_∞), see [4] and [8]) or for a large class of games with incomplete information (where v_∞ may not exist, see [9]).

2. Study of G_n and G_λ.

We first recall and prove briefly easy results.

LEMMA 1. (5) $F\subset D_1\subset D$,

(6) $D\subset C$,

(7) D *convex* $\Leftrightarrow D=C$,

where D *stands for* D_n *or* D_λ.

(8) $E_1\subset E\subset\Delta$ *where* E *stands for* E_n *or* E_λ.

PROOF. If an N-tuple τ of strategies in $\prod_{i=1}^{N}\mathscr{C}_i$ generates the payoff x in D_1, then $\sigma(\tau)$ defined in Σ by playing τ i.i.d. at each stage gives the same payoff in D, hence (5).

Now each payoff in D is the expectation of barycenters of (random) points in F, hence lies in C (6).

Finally since the extreme points of C lie in F, (5) and (6) imply (7). The first inclusion in (8) is proved like in (5). The second follows from the fact that at each stage m, conditionally to the history h_m, each player can obtain an individually rational payoff. ∎

LEMMA 2. E_λ *converges in the Hausdorff topology to* Δ, *as* λ *goes to* 0.[3]

[2] A condition is needed, see added in proof.

[3] A condition is needed, see added in proof.

PROOF. By (8) it is enough to prove that in any neighbourhood of a point from Δ lies a point from E_λ, for λ small enough.

Let x in Δ and assume first $x_i - a_i \geqslant \epsilon > 0$, $\forall i = 1, \ldots, N$. Then we can write $x = \sum_{k=1}^{N+1} \alpha_k x^k$ with x^k in F, α_k in $[0, 1]$ and $\sum_k \alpha_k = 1$. Hence there exist n_k in $\mathbf{N}$ such that, if $\sum_k n_k = R$ and $\sum_k (n_k/R)x^k = y$, we have: $y_i \geqslant a_i + \epsilon/2$ and $|y_i - x_k| \leqslant \epsilon/2$, $\forall$i.

Choose now $\bar{\lambda}$ such that $(1-\bar{\lambda})^{R-1} \geqslant 1 - \epsilon/4$. It follows then that by playing n_1 times a move inducing $x_1, \ldots, n_k$ times a move inducing x^k and so on and starting again at stage $R+1$, the payoff in G_λ will be some z with: $|x_i - z_i| \leqslant \epsilon$ and $z_i \geqslant a_i + \epsilon/4$, for $\lambda \leqslant \bar{\lambda}$.

We now claim that this payoff can be obtained by equilibrium strategies for λ small enough. In fact since the strategies described above are pure any deviation can be observed and the deviator's payoff reduced to a_i.

Defining by L the greatest absolute value of the payoffs it follows that the gain by deviating is at most: $2L(1-(1-\lambda)^{R+1}) - (\epsilon/4)(1-\lambda)^{R+1}$ which is negative for λ small enough. This ends the proof if Δ is full dimensional.

If now, for some i, $x_i = a_i$, for all x in Δ, player i will always play a best reply and no profitable deviation for him is profitable. It is then enough to specify the strategies of the other players and the proof goes by induction. ∎

Note that contrary to the "Perfect Folk Theorem" (see [2]) the previous result does not extend to perfect equilibria, for a counterexample see [5].

For any set X and any t in N we define:

$tX = \{tx;\ x \in X\}$,

$t * X = \{y;\ y = \sum_{m=1}^{t} x_m,\ x_m \in X\}$.

LEMMA 3. *Let $n = mp + r$ in N, then*

(9) $nD_n \supset m * (pD_p) + rD_r$,

(10) $nE_n \supset m * (pE_p) + rE_r$.

PROOF. Let a_0 in D_r and a_j in D_p, $j = 1, \ldots, m$, be obtained by the N-tuple of strategies $\sigma(j)$, $j = 0, \ldots, m$. Then the strategy σ in Σ, defined by: play $\sigma(0)$ up to stage r, $\sigma(j)$ from stage $r + (j-1)p$ up to stage $r + jp - 1$ (independently from the history at stage $r + (j-1)p$, induces a payoff in G_n equal to $n^{-1}(ra_0 + \sum_{j=1}^{m} pa_j)$ hence (9).

Now if $\sigma(0)$ is an equilibrium strategy in G_r and similarly for $\sigma(j)$ in G_p, $j = 1, \ldots, m$, then the strategy σ defined above is still an equilibrium in G_n hence (10). ∎

In particular this gives $D_n \subset D_{kn}$ $\forall k \geqslant 1$, $k \in N$ hence $D_{kn} \subset D_n$ for some $k > 1$ implies D_n convex and similarly for E_n.

Nevertheless there are games for which:

(11) the sequences D_n and E_n are not monotonic.

EXAMPLE 1. G_1 is a 2-person game defined by the following payoff matrix:

(1, 0)	(0, 0)
(0, 0)	(0, 1)

Note that $(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1, 0) + \frac{1}{2}(0, 1)$ belongs to E_2 hence to D_2. Obviously $(\frac{1}{2}, \frac{1}{2})$ is not in D_1.

Now since this payoff is Pareto Optimal, the only way to achieve it in G_3 is to play a pure strategy at each stage. This gives the payoffs $(n/3, 1 - n/3)$, $n = 0, 1, 2, 3$ and $(\frac{1}{2}, \frac{1}{2}) \notin D_3$. Since $E_n \subset D_n$ (11) follows.

Note in this example that $D_n \neq C$ for all n. Remark also that by duplicating one strategy of one of the players, D_1 and E_1 will not change, but D_2 will increase and $(\frac{1}{2},\frac{1}{2})$ will belong to D_3.

Moreover the variations of D_n and E_n are not related:

(12) $D_n = D_{n+1}$ does not imply $E_n = E_{n+1}$.

EXAMPLE 2.

(1,0)	(2,2)
(0,0)	(0,1)

In this game $D_1 = C$ hence $D_1 = D_n$ for all n. E_1 is reduced to (2,2) since each player has a strictly dominating strategy. Now we claim that (1,1) belongs to E_2.

In fact this payoff is achievable through the following equilibrium strategies: (Bottom, Left) at the first stage, and at the second stage:

—for player I: Bottom if player II played Right at the first stage.
Top otherwise.

—for player II: Left if player I played Top at the first stage.
Right otherwise.

Similarly we have:

(13) $E_n = E_{n+1}$ does not imply $D_n = D_{n+1}$.

EXAMPLE 3.

(1,0)	(1,1)
(0,0)	(1,0)

$E_1 = \{(1,x);\ x \in [0,1]\} = E_n$ for all n and $(\frac{1}{2},\frac{1}{2}) \in D_2 \backslash D_1$. Note that Example 2 shows also:

(14) E_n is not contained in the convex hull of E_1.

Moreover:

(15) $E_{n+1} \subset E_n$ does not imply $E_{n+2} \subset E_n$.

EXAMPLE 4.

$(m,0)$	$(m+1,m+1)$
$(0,0)$	$(0,m)$

Since by playing first Bottom player I can achieve at most $(n-1)(m+1)/n$ in G_n, the fact that he can guarantee m by playing always top implies by induction that E_n is reduced to $(m+1,m+1)$ for all $n < m$.

Now it is easy to see that (m,m) belongs to E_{m+1} (play (0,0) once then $(m+1,m+1)$, see Example 2). As for the game G_λ we have, as in (11):

(16) the nets D_λ and E_λ are not monotonic.

EXAMPLE 1 (revisited). By playing once (1,0) and then always (0,1), the players achieve $(7/8, 1/8)$ in $E_{7/8}$.

It is clear that this payoff is not in D_1. To prove that it does not belong to $D_{3/4}$ note that since it is Pareto optimal it can only be achieved by using pure strategies. The payoff for player I in $G_{3/4}$ is at most $\frac{1}{4}$ if $X_1 = (0,1)$ hence X_1 has to be $(1,0)$. Now if $X_2 = (1,0)$ player I get at least $\frac{15}{16}$ and at most $\frac{13}{16}$ if $X_2 = (0,1)$.

We shall now focus on the sets of feasible payoffs and study properties of convexity and stationarity.

For small values of λ the description of D_λ is easy since we have the following (compare with (3) and example 1 where $D_n \neq C\ \forall n$):

PROPOSITION 4.
(17) $D_\lambda = C$ *for all* $\lambda \leqslant 1/N$.

PROOF. By (5) and (6) C is the convex hull of D_1 and D_1 is connected (1). A theorem of Fenchel (see e.g. [10, p. 169, Proposition 3.3]) now implies that each point of C is a convex combination of at most N points of D_1. Thus given x in C, there exist x_i in D_1 and λ_i in $[0, 1]$, $i = 1, \ldots, N$, with $x = \sum_{i=1}^N \lambda_i x_i$.

Now we can assume $\lambda_1 \geqslant 1/N$ and we can introduce x' in C defined by:

$$x' = \frac{1}{1-\lambda}\Big((\lambda_1 - \lambda)x_1 + \sum_{i>1} \lambda_i x_i\Big),$$

such that $x = \lambda x_1 + (1-\lambda)x'$.

Doing the same decomposition for x' we obtain inductively:

$$x = \lambda \sum_{m=0}^{\infty} (1-\lambda)^m x_1^{(m)} \qquad \text{with } x_1^{(m)} \text{ in } D_1 \quad \text{for all} \quad m = 0, 1, \ldots$$

This implies that x is in D_λ, by playing at stage $m+1$ a strategy in $\prod_{i=1}^N \mathscr{C}_i$ achieving $x_1^{(m)}$. ∎

Note that this bound is the best one:

EXAMPLE 5. $T_i = \{1, \ldots, N\}$ for all $i = 1, \ldots, N$. The payoff function X from T to R^N is defined by:

$$\begin{aligned} X(t_1, \ldots, t_N) &= e_j \qquad (j\text{-unit vector in } R^N) \quad \text{if} \quad t_i = j \quad \text{for all } i, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then $(1/N, \ldots, 1/N)$ does not belong to D_λ for $\lambda > 1/N$.

PROPOSITION 5.
(18) *If* D_n *is convex then* $D_{n+1} = D_n$, *hence* $D_m = C$ *for all* $m \geqslant n$.

PROOF. Let x in D_n be induced by an N-tuple of strategies σ and let x_m, $m = 1, \ldots, n$ be the corresponding expected payoff at stage m. It follows that $nx = \sum_{m=1}^n x_m$ with x_1 in D_1 and x_m in C for all m.

Now $y = (\sum_{m>1} x_m)/(n-1)$ still belongs to the convex set C which equals D_n by (7). By (5) this implies that the line segment $[x_1, y]$ lies in D_n hence: $z = x_1/n^2 + (1 - 1/n^2)y$ belongs to D_n and is induced by some τ.

Since we have $x = (x_1 + nz)/(n+1)$ it follows that x is achievable in G_{n+1} by playing σ at the first stage and then τ. ∎

Reciprocally the following obviously holds:
(19) $D_m = D_n$ for all $m \geqslant n$ implies $D_n = C$ (by (3) or (9)).
Nevertheless we have:
(20) D_n convex does not imply D_{n-1} convex.
EXAMPLE 6. Let G_1 be the following two-person game:

(0, 1)	(1, 1)	(2, 0)	(3, 0)
(0, 0)	(1, 0)	(2, 1)	(3, 1)

$(3/2, 1)$ does not belong to D_1 (a payoff 1 to player II implies that player I is using a pure strategy) but D_1 contains the two squares $C' = \mathrm{co}\{(0,0), (0,1), (1,1), (1,0)\}$ and $C'' = \mathrm{co}\{(2,0), (3,0), (3,1), (2,1)\}$. Thus we have

$$C = \tfrac{1}{2}(C' + C'') \subset \tfrac{1}{2}(D_1 + D_1) \subset D_2.$$

Remark that for a two-person game where each player has only two pure strategies, either $D_1 = C$ (see Example 2) or $D_n \neq C$ for all n (see Example 1).

In a similar way one can prove:

PROPOSITION 6.
(21) *If D_λ is convex then $D_\delta = C$ for all $0 < \delta \leqslant \lambda$.*

PROOF. Let x in D_λ be induced by some σ and denote by x_m the expected payoff at stage m. Here also x_1 is in D_1 and x_m is in C with $x = \lambda\sum_{m=1}^{\infty}(1-\lambda)^{m-1}x_m$. Define y to be $\lambda\sum_{m\geqslant 2}(1-\lambda)^{m-2}x_m$, then y belongs to $C = D_\lambda$ and $x = \lambda x_1 + (1-\lambda)y$.

D_1 being included in the convex set D_λ it follows that x' defined to be $((\lambda-\delta)/(1-\delta))x_1 + ((1-\lambda)/(1-\delta))y$ belongs to D_λ and $x = \delta x_1 + (1-\delta)x'$. Doing the same decomposition for x' we obtain by induction $x = \delta\sum_{m=0}^{\infty}(1-\delta)^m x_1^{(m)}$ with $x_1^{(m)}$ in D_1 for all $m = 0, 1, \ldots$. By playing σ_m at stage $m+1$, where σ_m achieves $x_1^{(m)}$ in G_1, the players can obtain x in G_δ hence x belongs to D_δ. ∎

Reciprocally we have:

(22) $D_\lambda = D_\delta$ for all $0 < \delta \leqslant \lambda$ implies $D_\lambda = C$ (by (17)).

Recall that $C = \mathrm{co}\, F$ is a convex polyhedron. Denote by L a one-dimensional face of C. Then by (5), $L \cap D_\lambda$ and $L \cap D_n$ are nonempty for all λ in $(0,1]$ and all $n \geqslant 1$.

We now consider the feasible payoffs lying on L and prove that if this set is decreasing then it contains all L. For $N = 2$, this property has interesting consequences (see Corollary 12).

PROPOSITION 7.
(23) *If for some δ, $0 < \delta < \lambda$, $D_\delta \cap L$ is included in $D_\lambda \cap L$ then L is included in D_δ.*

PROOF. Let us suppose that there exists a point in L which is not in D_δ. Without loss of generality we can assume that L is the line segment $[X_0, Y_0]$ with $X_0 = (0, \ldots, 0)$, $Y_0 = (1, 0, \ldots, 0)$ in R^N, and X_0, Y_0 belonging to $F \subset D_\delta$.

For each point Z in L, let $d(Z)$ denotes its distance to the compact set $D_\delta . \cap L$. The maximum of $d(Z)$ on L, denoted by $\bar{d}$, is taken at some point $\bar{Z} = (\bar{z}, 0, \ldots, 0)$ and is strictly positive by hypothesis. Let us introduce: $X = (x, 0, \ldots, 0)$ and $Y = (y, 0, \ldots, 0)$ with $x = \bar{z} - \bar{d}$ and $y = \bar{z} + \bar{d}$. Then we have:

(*) X and Y belong to $D_\delta \cap L$ and $(X, Y) \cap D_\delta$ is empty.

(**) No other couple of points X', Y' with $\|X' - Y'\| > 2\bar{d}$ satisfy (*).

Let X be induced by σ. Since X lies on a face of C, at each stage the random payoff induced by σ will belong to this face. Hence it is enough to consider the first component of the payoff.

Let H be the set of histories at stage 2, having positive probability $p(h)$, under σ. For each h in H, let $\sigma(h)$ be the strategy from stage 2 on defined by σ conditionally on h.

Denote by x_1 the expected payoff at stage 1 and by $x_2(h)$ the payoff induced in G_δ by $\sigma(h)$, for each h in H. Thus:

$$x = \delta x_1 + (1-\delta)\sum_{h\in H} p(h)x_2(h).$$

(a) If for some h_0 in H, $x_2(h_0)$ is strictly less than 1, then by (**) there exists $Z = (z, 0, \ldots, 0)$ in D_δ with: $x_2(h_0) < z \leqslant x_2(h_0) + 2\bar{d}$.

If Z is achievable by τ in G_δ, then the following strategy: play σ, unless the history at stage 2 is h_0 and from this stage on use τ, gives a payoff w with:

$$w = \delta x_1 + (1-\delta)\left(p(h_0)z + \sum_{\substack{h \neq h_0 \\ h\in H}} p(h)x_2(h)\right).$$

Note that $0 < w - x \leqslant (1-\delta)2\bar{d}$; thus $W = (w, 0, \ldots, 0)$ belongs to $D_\delta \cap (X, Y)$ contradicting $(*)$.

(b) Since we can do the same construction starting from Y it remains to consider the case where:

$$x = \delta x_1 + (1-\delta), \qquad y = \delta y_1 .$$

We now use the fact that $D_\delta \cap L$ is included in $D_\lambda \cap L$, hence x can be written as $\lambda u_1 + (1-\lambda)u_2$ with $U_1 = (u_1, 0, \ldots, 0)$ in D_1. Since u_2 is less than one and $\delta < \lambda$ we have $u_1 > x_1$. Hence:

(§) $\delta u_1 < \lambda u_1 \leqslant x$,

(§§) $\delta u_1 + (1-\delta) > x$.

Let us consider the following set: $A = \{\delta u_1 + (1-\delta)t;\ T = (t, 0, \ldots, 0) \text{ is in } D_\delta \cap L\}$. By $(**)$, (§) and (§§) it follows that there exists z in A satisfying: $0 < z - x \leqslant (1-\delta)2\bar{d}$. Now if z is $\delta u_1 + (1-\delta)t$, let U_1 be induced by σ (in G_1) and T be induced by τ (in G_δ).

The strategy defined by playing σ at stage 1 and τ from stage 2 on gives as a payoff in G_δ, $Z = (z, 0, \ldots, 0)$ contradicting $(*)$. ∎

As for the feasible payoffs in the finitely repeated game G_n we have:

PROPOSITION 8.

(25) *Let $n \geqslant Nm$, then $D_{n+m} \subset D_n$ implies $D_{n+m} = C$.*

PROOF. The proof goes by induction on the dimension of the faces of C and follows obviously from the following:

PROPOSITION 9.

(26) *Let P be a face of C of dimension p ($p \leqslant N$). If $n \geqslant pm$ and $D_{n+m} \cap P \subset D_n \cap P$ then $P \subset D_{n+m}$.*

PROOF. By induction (the proof follows from (5) if $p = 0$) we assume that each face of P of dimension at most $p - 1$ is in D_{n+m} and we write D'_m for $D_m \cap P$, for all m.

Note that by (9) we can and shall assume $m < n$. Suppose that P is not included in D'_{n+m}. For each point Z in P, $d(Z)$ denotes its distance to the compact D'_{n+m} and the maximum, $\bar{d} > 0$, is taken at some $\bar{Z}$.

Let $B = B(\bar{Z}, \bar{d}) \cap P$ where $B(\bar{Z}, \bar{d})$ is the closed ball in R^N with center $\bar{Z}$ and radius $\bar{d}$.

We first need the following:

LEMMA 10. *$\bar{Z}$ belongs to the convex hull of $B \cap D'_{n+m}$.*

PROOF. By definition of $\bar{d}$, $B \cap D'_{n+m}$ is not empty. Define H to be the convex hull of $B \cap D'_{n+m}$, H is a compact convex set. If $\bar{Z}$ is not in H, let Y be a closest point to $\bar{Z}$ in H. Thus:

$$\langle \bar{Z} - Y, \bar{Z} \rangle > \langle \bar{Z} - Y, T \rangle \qquad \text{for all } T \text{ in } H. \tag{$*$}$$

For every $\epsilon > 0$ let $B_\epsilon = B(\bar{Z}_\epsilon, \bar{d}) \cap P$ where $\bar{Z}_\epsilon = \bar{Z} + \epsilon(\bar{Z} - Y)$. Since by induction the frontier of P is in D'_{n+m}, $\bar{Z}_\epsilon$ is in P for ϵ small enough, hence $B_\epsilon \cap D'_{n+m}$ is not empty. Note now that if T belongs to B_ϵ and

$$\langle \bar{Z} - Y, T \rangle < \langle \bar{Z} - Y, \bar{Z} + \tfrac{\epsilon}{2}(\bar{Z} - Y) \rangle$$

then T belongs to the interior $\mathring{B}$ of B. By the choice of $\bar{Z}$, $\mathring{B} \cap D'_{n+m}$ is empty hence there exists T in $B_\epsilon \cap D'_{n+m}$ with $\langle \bar{Z} - Y, T \rangle > \langle \bar{Z} - Y, \bar{Z} \rangle$. By compacity we thus obtain a point $\bar{T}$ in $B \cap D'_{n+m}$ satisfying $\langle \bar{Z} - Y, \bar{T} \rangle \geqslant \langle \bar{Z} - Y, \bar{Z} \rangle$ contradicting $(*)$.

∎

Using Caratheodory's theorem we can now introduce X^k in $B \cap D'_{n+m}$, $k = 1, \ldots, q$, $q \leq p + 1$, such that $\bar{Z}$ lies in the convex hull of the X^k, and this family is minimal with respect to this property. If X^k is generated by σ^k in G_{n+m}, let us denote by S^k the average expected payoff up to stage m and for each history in H_k: set of histories at stage $m+1$ having positive probability $p(h)$ under σ^k, let $U^k(h)$ be the average expected payoff for the remaining n stages in G_{n+m}, conditionally on h. Thus:

$$(n+m)X^k = mS^k + n\sum_{H_k} p(h)U^k(h). \qquad (**)$$

Since X^k belongs to the face P, S^k and $U^k(h)$ have the same property.

(a) Assume that there exists h_0 in H_k such that:

$$\langle U^k(h_0), X^k - \bar{Z}\rangle > \min_{T \in P} \langle T, X^k - \bar{Z}\rangle = a^k. \qquad (***)$$

Since the frontier of P is in D'_{n+m}, the intersection of D'_{n+m} with the closed ball $B^k(h_0)$ centered at $U^k(h_0) - X^k + \bar{Z}$ and of radius $\bar{d}$ is not empty.

Using (***) there exists a point $Z^k(h_0)$ in this intersection and different from $U^k(h_0)$.

Since D'_{n+m} is included in D'_n, $Z^k(h_0)$ is in D'_n hence $\bar{X}^k$ defined by

$$\bar{X}^k = \frac{1}{n+m}\left(mS^k + n\left[p(h_0)Z^k(h_0) + \sum_{h \neq h_0} p(h)U^k(h)\right]\right)$$

belongs to D'_{n+m} (see the proof of Proposition 7).

It remains to compute the distance from this new point to $\bar{Z}$. But

$$\|\bar{X}^k - \bar{Z}\|^2 = \|\bar{X}^k - X^k\|^2 + \|X^k - Z\|^2 + 2\langle \bar{X}^k - X^k, X^k - \bar{Z}\rangle.$$

Note that

$$\langle \bar{X}^k - X^k, X^k - \bar{Z}\rangle = p(h_0)\frac{n}{(n+m)}\langle Z^k(h_0) - U^k(h_0), X^k - \bar{Z}\rangle$$

hence this quantity is negative.

Moreover, since $Z^k(h_0)$ is in $B^k(h_0)$:

$$|\langle Z^k(h_0) - U^k(h_0), X^k - \bar{Z}\rangle| \geq \tfrac{1}{2}\|Z^k(h_0) - U^k(h_0)\|^2.$$

Thus:

$$\|\bar{X}^k - \bar{Z}\|^2 \leq \bar{d}^2 + \left(p^2(h_0)\frac{n^2}{(n+m)^2} - p(h_0)\frac{n}{n+m}\right)\|Z^k(h_0) - U^k(h_0)\|^2 < \bar{d}^2$$

which contradicts the definition of $\bar{Z}$ and $\bar{d}$.

(b) We are now left the case where for each k and each h in H_k

$$\langle U^k(h), X^k - \bar{Z}\rangle = a^k. \qquad (***)$$

Let L be the linear space generated by the X^k and denote by Q the projection on L of the points T in $\mathbb{R}^n$ satisfying: $\langle T, X^k - \bar{Z}\rangle \geq a^k$ for all k. Note that Q contains the projection of P on L and that Q is homeomorphic to a simplex of dimension $q - 1 \leq p$.

We shall write $\tilde{T}$ for the projection of T on L and introduce barycentric coordinates $(\alpha^1, \ldots, \alpha^q)$ for the points in Q such that the set of α's with $\alpha^k = 0$ corresponds to the set of $\tilde{T}$ in Q with $\langle \tilde{T}, X^k - Z\rangle = a^k$. Let $(\bar{\alpha}^1, \ldots, \bar{\alpha}^q)$ corresponding to $\bar{\tilde{Z}}$. It follows from (**) and (***) that $\bar{\alpha}^k < m/(m+n)$ for all $k = 1, \ldots, q$. Since $\sum_1^q \bar{\alpha}^k = 1$, this inequality implies $pm > n$ contradicting the assumption. ∎

In order to obtain more precise results for $N = 2$ we shall prove and use the following property (recall that $\tilde{D}$ is the set of feasible payoffs in a game $\tilde{G}$):

PROPOSITION 11.
(27) *If $N = 2$ then $\tilde{D}$ is simply connected.*

PROOF. Let γ be a closed continuous path in $\tilde{D}$ (i.e. γ is a continuous map from $[0, 1]$ to $\tilde{D}$ with $\gamma(0) = \gamma(1)$) and assume that there exists y in $\mathbb{R}^2 \setminus \tilde{D}$ such that:

$$\operatorname{Ind}(y, \gamma) \neq 0. \tag{*}$$

For each t in $[0, 1]$ and each σ (resp. τ) strategy of player I (resp. player II) in $\tilde{G}$ such that $X(\sigma, \tau) = \gamma(t)$ we define a closed continuous path $\Gamma[t; \sigma, \tau]$ as follows:

Fix σ_0, τ_0, such that $X(\sigma_0, \tau_0) = \gamma(0)$. Now $\Gamma[t; \sigma, \tau]$ coincides with γ on $\{\gamma(0), \gamma(t)\}$. Starting from $\gamma(t)$ it follows the two line segments:

first $X(\sigma, s\tau_0 + (1-s)\tau)$ where s goes from 0 to 1,

then $X(u\sigma_0 + (1-u)\sigma, \tau_0)$ where u goes from 0 to 1.

By construction we have $\operatorname{Ind}(y, \Gamma[0, \sigma_0, \tau_0]) = \operatorname{Ind}(y, \gamma(0)) = 0$ and, since $\gamma(0) = \gamma(1)$, $\operatorname{Ind}(y, \Gamma[1, \sigma_0, \tau_0]) = \operatorname{Ind}(y, \gamma) \neq 0$.

Using the continuity of $\Gamma[\cdot; \cdot, \cdot]$ and the compactness of the strategies' sets we obtain the existence of two couples of strategies (σ, τ) and (σ', τ') and of a point t in $[0, 1]$ such that: $\gamma(t) = X(\sigma, \tau) = X(\sigma', \tau')$ and $\operatorname{Ind}(y, \Gamma[t; \sigma, \tau]) \neq \operatorname{Ind}(y, \Gamma[t; \sigma', \tau'])$.

Defining $\bar{\gamma}$ by $\Gamma[t; \sigma, \tau] - \Gamma[t; \sigma', \tau']$ we obviously have: $\operatorname{Ind}(y, \bar{\gamma}) \neq 0$. The idea of the proof now is to introduce a new path γ^*, such that $\operatorname{Ind}(y, \bar{\gamma}) = \operatorname{Ind}(y, \gamma^*)$, with the additional property that γ^* will be the image under X of a path in the strategy's space. The latter being simply connected (in fact contractile) this will imply $\operatorname{Ind}(y, \gamma^*) = 0$, hence the contradiction.

Recall that $\bar{\gamma}$ is defined by:

$$\begin{aligned}\gamma(0) = X(\sigma_0, \tau_0) &\to X(\sigma', \tau_0) \to X(\sigma', \tau') \\ &= \gamma(t) = X(\sigma, \tau) \to X(\sigma, \tau_0) \to X(\sigma_0, \tau_0).\end{aligned}$$

We define γ^* by adding to $\bar{\gamma}$ from the point $\gamma(t)$ the closed path ρ

$$\gamma(t) = X(\sigma', \tau') \to X(\sigma, \tau') \to X(\sigma, \tau) = \gamma(t).$$

Note that, since X is linear in each variable, ρ consists of a line segment in D in both directions hence $\operatorname{Ind}(y, \gamma^*) = \operatorname{Ind}(y, \bar{\gamma})$. Obviously γ^* is now the image under X of the following closed continuous path in the strategy's space:

$$(\sigma_0, \tau_0) \to (\sigma', \tau_0) \to (\sigma', \tau') \to (\sigma, \tau') \to (\sigma, \tau) \to (\sigma, \tau_0) \to (\sigma_0, \tau_0),$$

hence the result. ∎

COROLLARY 12.

$$\begin{aligned}&\textit{If} \quad N = 2, \quad 0 < \delta < \lambda, \qquad D_\delta \subset D_\lambda \quad \textit{implies} \quad D_\delta = C, \\ &\qquad\qquad\qquad m > 0, \qquad\quad D_{n+m} \subset D_n \quad \textit{implies} \quad D_{n+m} = C.\end{aligned} \tag{28}$$

PROOF. Using (23) D_δ contains the frontier of C hence is equal to C by (27). The proof is similar for D_{n+m} by using (26) with $p = 1$, then (9) to reduce to the case $m < n$, and finally (27). ∎

Open problem: is D simply connected or even contractile for $N > 2$?

3. Study of the prisoner's dilemma. In this part we shall study the following two-person game:

	L	R
T	(4, 4)	(0, 5)
B	(5, 0)	(1, 1)

We first remark that $D_1 = C$ hence $D_n = C$ for all n and that $\Delta = \{x = (x_1, x_2) \mid x \in C, x_i \geqslant 1, i = 1,2\}$. Moreover $E_1 = \{(1,1)\}$ since B and R are strictly dominating strategies in G_1.

This game has been widely analyzed and it is well known that $E_n = \{(1,1)\}$, see e.g. [7, pp. 95–102]. Nevertheless this property is not a consequence of the existence of strictly dominating strategies (see Example 4) and backwards induction arguments lead only to perfect Nash equilibrium payoffs.

A more general class of games for which an analog property holds is described by the following result:(recall that $a_i = \min_{\mathscr{E}^i} \max_{T^i} X_i(\tau^i, t_i)$.)

PROPOSITION 13. *Let G_1 be an N-person game such that $E_1 = \{a\}$ then $E_n = \{a\}$ for all n.*

PROOF. Let σ be a Nash equilibrium N-tuple of strategies in G_n corresponding to a payoff different from a. Denote by $H_m(\sigma)$ the set of histories up to stage m having a positive probability under σ.

Obviously, since a is the only one-stage Nash equilibrium payoff, the payoff induced by σ at stage n conditionally to any history in $H_n(\sigma)$ is a. Hence there exists a stage m and an history h_m in $H_m(\sigma)$ such that:

—the payoff induced by σ at stage m conditionally to h_m is different from a,

—the payoff at any further stage $k \geqslant m + 1$ conditionally to any h_k that follows h_m and belongs to $H_k(\sigma)$ is a.

In particular this implies that σ is not in equilibria at stage m, conditionally to h_m; hence we can assume that player 1 can strictly increase his payoff at that stage by using some τ_1.

Now, by definition of a_1, whatever being σ^1, player 1 can obtain at least a_1 for the remaining stages, which was his payoff under σ.

It follows that by deviating at stage m if h_m, player can strictly increase his average payoff; since h_m belongs to $H_m(\sigma)$ we obtain a contradiction. ∎

Note that this condition is also necessary since a recent result states that for $N = 2$, $E_1 \neq \{a\}$ implies that E_n converges to Δ (see [3]).

We now turn to the study of the discounted game.

The following result was already announced in [2].

PROPOSITION 14. *E_λ is reduced to $\{(1,1)\}$ for all λ in $(\frac{3}{4}, 1]$.*

PROOF. Let (σ, τ) be an equilibrium pair in G_λ. H_n will denote the set of histories up to stage n and H_n^* those histories in H_n having positive probability under (σ, τ). We write a_n for the random payoff of player I at stage n, and $s_n(h_n)$ (resp. $t_n(h_n)$) for the probability of playing T (resp. L) at stage n, conditionally on h_n, σ and τ.

The equilibrium condition can be written as follows:

For each h_n in H_n^* and each σ' which coincides with σ up to stage $n - 1$:

$$E_{\sigma,\tau}\left(\lambda \sum_0^\infty (1-\lambda)^m a_{n+m} \mid h_n\right) \geqslant E_{\sigma',\tau}\left(\lambda \sum_0^\infty (1-\lambda)^m a_{n+m} \mid h_n\right) \qquad (*)$$

(and similarly for player II).

In particular if σ' is after h_n, play always bottom:

$$E_{\sigma,\tau}(a_n \mid h_n) - E_{\sigma' t}(a_n \mid h_n) = -s_n(h_n) \quad \text{and} \quad E_{\sigma',\tau}(a_{n+m} \mid h_n) \geq 1 \qquad \forall m.$$

(*) now implies

$$E_{\sigma,\tau}\left(\sum_1^\infty (1-\lambda)^m (a_{n+m} - 1) \mid h_n\right) \geq s_n(h_n) \qquad \text{for all } h_n \text{ in } H_n^*.$$

Define: $\gamma_{n+m}(h_n) = E_{\sigma,\tau}(a_{n+m} + b_{n+m} \mid h_n) - 2$ and note that

$$\begin{aligned} \gamma_n(h_n) &= (4+4)s_n(h_n)t_n(h_n) + (5+0)(1 - s_n(h_n))t_n(h_n) \\ &\quad + (0+5)s_n(h_n)(1 - t_n(h_n)) + (1+1)(1 - s_n(h_n))(1 - t_n(h_n)) - 2 \\ &= 3(s_n(h_n) + t_n(h_n)). \end{aligned}$$

From (*) and the similar inequality for player II we obtain:

$$\sum_1^\infty (1-\lambda)^m \gamma_{n+m}(h_n) \geq \tfrac{1}{3}\gamma_n(h_n) \qquad \text{for all } h_n \text{ in } H_n^*. \tag{**}$$

Let us introduce, for all m, $\bar\gamma_m = E_{\sigma,\tau}(a_m + b_m - 2)$. Since $P_{\sigma,\tau}(H_n^*) = 1$, integrating (**) gives

$$\sum_1^\infty (1-\lambda)^m \bar\gamma_{n+m} \geq \tfrac{1}{3}\bar\gamma_n \qquad \text{for all} \quad n > 1.$$

Letting γ be the supremum of the $\bar\gamma_n$ it follows that $\bar\gamma\sum_1^\infty(1-\lambda)^m \geq \frac{1}{3}\bar\gamma_n$. Thus

$$\bar\gamma\left(\frac{1-\lambda}{\lambda}\right) \geq \frac{1}{3}\bar\gamma.$$

Hence either $\gamma = 0$, i.e. $\bar\gamma_m = 0$ for all m, and the payoff is always $(1,1)$ or $\lambda \leq \frac{3}{4}$. ∎

In the last proposition we shall describe explicitly the set of equilibrium payoffs in $G_{3/4}$.

We first define S to be the square with extreme points $(1,1)$, $(1,4)$, $(4,4)$, $(4,1)$ and A to be the union of S with the two line segments: $[(4,1),(19/4,1)]$, $[(1,4),(1,19/4)]$:

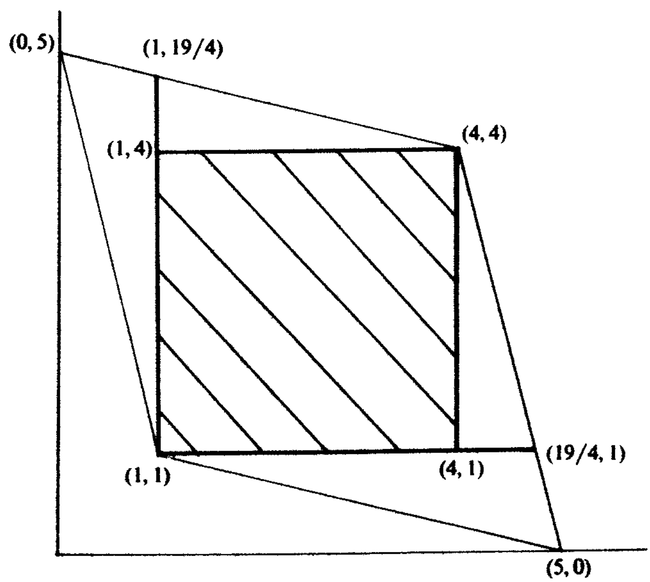

FIGURE 1. Equilibrium payoffs in the prisoner's dilemma with discount factor 3/4.

We can now state the result:

PROPOSITION 15. $E_{3/4} = A$.

PROOF.
First part: $A \subset E_{3/4}$. Straightforward computation shows that the extreme points of S are equilibrium payoffs (we describe the pure equilibrium strategies by their sequence of payoffs on H^*, both players using their dominating strategies outside H^*):
for $(4,4)$ always $(4,4)$,
for $(1,1)$ always $(1,1)$,
for $(4,1)$ alternating sequence of $(5,0),(0,5)\ldots$ starting from $(5,0)$ and symmetrically for $(1,4)$.
Now given (a,b) in S, write a as $4t+1-t$ and b as $4s+1-s$, t and s being in $[0,1]$.
The strategies are now:
At the first stage, for player I plays T with probability s, for player II plays L with probability t.
From stage 2 on:

if	$(4,4)$ is the payoff at stage 1 play always	$(4,4)$
	$(5,0)$	$(0,5)\,(5,0)\ldots$
	$(0,5)$	$(5,0)\,(0,5)\ldots$
	$(1,1)$	$(1,1)\ldots$

It is easy to see that the payoffs of both players are independent of their first moves. Since from stage 2 on no deviation is profitable the above description gives an equilibrium and it is easy to see that the corresponding payoff is (a,b).

Finally, in order to obtain a payoff equal to $(4+\frac{3}{4}\alpha,1)$, $\alpha\in[0,1]$ in $E_{3/4}$ the strategies are:
play $(5,0)$ at the first stage, then achieve the equilibrium payoff $(1+3\alpha,4)$, which belongs to S, in $G_{3/4}$ starting from stage 2. Here also none of the players has incentive to deviate at stage 1, hence the equlibrium with the right payoff: $\frac{3}{4}(5,0)+\frac{1}{4}(1+3\alpha,4)$.

Second part: $E_{3/4}\subset A$. Obviously $E_{3/4}$ is included in Δ, hence it remains to prove, by symmetry that there exists no (a,b) in $E_{3/4}$ with $a \geqslant 4$, $b \geqslant 1$ and $(a-4)(b-1) = \lambda(a,b) > 0$.

We shall prove that if there is such a payoff this implies the existence of another payoff (a',b') in $E_{3/4}$ with $a' \geqslant 4$, $b' \geqslant 1$ and $\lambda(a',b') \geqslant 4\lambda(a,b)$, hence the contradiction.

(a) Let σ,τ be the equilibrium strategies corresponding to (a,b). We define a_1 to be the maximal payoff that player I can achieve in $G_{3/4}$ if player II is using $\tau(h_2)$ with $h_2=(T,L)$.

In words a_1 corresponds to the normalized payoff from stage 2 on if player I uses a best response to τ, conditionally on (T,L) at stage 1.

b_1 is defined in the same way for player II and similarly (a_2,b_2) correspond to (T,R), (a_3,b_3) to (B,L) and (a_4,b_4) to (B,R). (Note that $a_i \geqslant 1, b_i \geqslant 1$.) Hence the players face a matrix with current and future payoffs

$(4,4)$ (a_1,b_1)	$(0,5)$ (a_2,b_2)
$(5,0)$ (a_3,b_3)	$(1,1)$ (a_4,b_4)

Let s and t be the strategies induced by σ and τ at the first stage. If we define $f(\bar{s},\bar{t})$ for every $\bar{s}$, $\bar{t}$ by:

$$f(\bar{s},\bar{t}) = \tfrac{3}{4}\big(4\bar{s}\bar{t} + 5(1-\bar{s})\bar{t} + (1-\bar{s})(1-\bar{t})\big)$$

$$+ \tfrac{1}{4}\big(\bar{s}\bar{t}a_1 + \bar{s}(1-\bar{t})a_2 + (1-\bar{s})\bar{t}a_3 + (1-\bar{s})(1-\bar{t})a_4\big),$$

then the equilibrium condition implies:

$$f(s,t) \geqslant f(\bar{s},t) \qquad \text{for all } \bar{s} \tag{$*$}$$

(and similarly for a function g corresponding to player II's payoff) and

$$\begin{aligned} &s,t > 0 \Rightarrow (a_1,b_1) \in E_{3/4}, \qquad s(1-t) > 0 \Rightarrow (a_2,b_2) \in E_{3/4},\\ &(1-s)t > 0 \Rightarrow (a_3,b_3) \in E_{3/4}, \qquad (1-s)(1-t) > 0 \Rightarrow (a_4,b_4) \in E_{3/4}. \end{aligned} \tag{$**$}$$

(b) We can assume $t > 0$ otherwise a is less than 4; and $s < 1$ otherwise by playing $\bar{t} = 0$ player II obtains $\frac{3}{4}5 + \frac{1}{4}b_2 \geqslant \frac{3}{4}5 + \frac{1}{4} = 4$ hence a is again less than 4.

Using ($*$) we now obtain, with $a = 4 + x$, $b = 1 + y$,

(1) $4 + x = \frac{3}{4}[5t + (1-t)] + \frac{1}{4}(ta_3 + (1-t)a_4)$,

(2) $1 + y = \frac{3}{4}(4s) + \frac{1}{4}[sb_1 + (1-s)b_3]$

As $a_4 \leqslant 5$ we obtain from (1) that:

(3) $a_3 \geqslant 1 + 4x$.

(c) If $s = 0$. By (2) we have $b_3 = 4 + 4y$. By ($**$), this implies that (a_3,b_3) is in $E_{3/4}$, hence by symmetry (b_3,a_3) also. Now $b_3 \geqslant 4$, $a_3 \geqslant 1$ and $\lambda(b_3,a_3) \geqslant 16xy = 16\lambda(a,b)$.

(d) Assume now $s > 0$. ($*$) implies:

(4) $4 + x = \frac{3}{4}(4t) + \frac{1}{4}(ta_1 + (1-t)a_2)$,

(5) $1 + y \geqslant \frac{3}{4}(5s + (1-s)) + \frac{1}{4}(sb_2 + (1-s)b_4)$.

From (2) and (5) it follows that

(6) $sb_1 + (1-s)b_3 \geqslant 3 + sb_2 + (1-s)b_4 \geqslant 4$;

hence using again (2)

(7) $s \leqslant y/3$.

Finally from (4) we get:

(8) $a_1 \geqslant 4 + 4x$.

We now consider two cases:

—either $b_1 \geqslant 1 + y$

Then by (8) we obtain $\lambda(a_1,b_1) \geqslant 4xy = 4\lambda(a,b)$ hence the result.

—or $b_1 < 1 + y$.

Using again (2) we have:

$$4 + 4y \leqslant s(12 + 1 + y - b_3) + b_3$$

$$\leqslant \frac{y}{3}(12 + 1 + y - b_3) + b_3 \qquad \text{by (7).}$$

This inequality gives:

(9) $(4 + y)(3 - y) \leqslant b_3(3 - y)$.

Since $x > 0$ implies $y < 3$ it follows that

(10) $b_3 \geqslant 4 + y$.

By (3) and (10) we obtain $\lambda(b_3,a_3) \geqslant 4\lambda(a,b)$ and this achieves the proof. ∎

CONCLUDING REMARKS. (1) The computations made in the proof of Proposition 14 show also that for the following values of the parameters

$\beta - x, \beta - x$	$\alpha - x, \beta$
$\beta, \alpha - x$	α, α

with $\beta - x > \alpha$

$x > 0$

similar results hold with a critical value $\bar{\lambda} = (\beta - \alpha - x)/(\beta - \alpha)$.

(2) One can also prove that the analog of Proposition 15 holds, at least for $\beta \geqslant \max\{1 + \alpha, 1 + 2x\}$.

(3) To compute E_λ for other values of the discount factor seems quite difficult. It is nevertheless easy to see that E_λ is not monotonic: there are denumbrably many points on the Pareto boundary for $1/2 < \lambda < 3/4$.

(4) We use deeply the fact that the "gain of deviating" was uniform, namely x. In the more general case:

$\beta - y, \beta - y$	$\alpha - x, \beta$
$\beta, \alpha - x$	α, α

with $\beta - y > \alpha$

$x > 0, \quad y > 0$

the critical value is $\bar{\lambda} = \max\{\beta - \alpha - x, \beta - \alpha - y\}/(\beta - \alpha)$.

In fact if $y \geqslant x$ an alternate sequence $(\beta, \alpha - x), (\alpha - x, \beta), \ldots$ gives an equilibrium at $\bar{\lambda}$; and similarly if $x \geqslant y$ a stationary sequence of $(\beta - y, \beta - y)$ is an equilibrium. Now if for some $\lambda > \bar{\lambda}$, (σ, τ) is an equlibrium, it keeps this property as x or y decrease, in particular for $x' = y' = \min(x, y)$ contradicting Remark 1. The explicit computation of $E_{\bar{\lambda}}$ seems more delicate.

Added in proof. Lemma 2 holds under the following additional assumption: Δ is full dimensional or $N = 2$, as used in the proof. Forges, Mertens and Neyman have a counterexample where $N = 3$ and Δ is 2-dimensional.

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