JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **98**, 296–303 (1984)

On a Pair of Simultaneous Functional Equations

S. SORIN

Laboratoire d'Économétrie, Université Paris VI,
4 Place Jussieu, 75230 Paris Cedex 05, France

Submitted by R. P. Boas

For each p in the simplex P of $\mathbb{R}^k$ we introduce convex subsets of P, $\Pi_{\mathrm{I}}(p)$ and $\Pi_{\mathrm{II}}(p)$. For f a real function on P we define $\mathrm{Cav}_{\mathrm{I}}\, f$ to be the smallest function greater than f on P and concave on $\Pi_{\mathrm{I}}(p)$ for each p in P (and similarly $\mathrm{Vex}_{\mathrm{II}}\, f$). Given u a continuous real function on P we prove that the following problems:

Minimize f; $f: P \to \mathbb{R}, f \geqslant \mathrm{Cav}_{\mathrm{I}}\, \mathrm{Vex}_{\mathrm{II}} \max\{u, f\}$
Maximize f; $f: P \to \mathbb{R}, f \leqslant \mathrm{Vex}_{\mathrm{II}}\, \mathrm{Cav}_{\mathrm{I}} \min\{u, f\}$

have the same solution which is also the only solution of $f = \mathrm{Vex}_{\mathrm{II}} \max\{u, f\} = \mathrm{Cav}_{\mathrm{I}} \min\{u, f\}$. This is an extension of a former proof by Mertens and Zamir for the case where P is a product of convex R and S with $\Pi_{\mathrm{I}}(p) = r \times S$ and $\Pi_{\mathrm{II}}(p) = R \times s$.

1. INTRODUCTION

A certain problem in game theory gives rise to a pair of simultaneous functional equations involving the operations of concavification and convexification of a function. Using game theoretical arguments and techniques it was proved in [1] that this set of equations has a unique solution. This result was proved in the independent case in [2] by purely analytic means. The purpose of this paper is to extend this demonstration to the dependent case. The tools used here were introduced in [4]. We shall follow the plan and the numbering of [2] and just state without proof the propositions, corollaries or lemmas the extensions of which are straightforward.

2. NOTATIONS AND STATEMENTS OF THE THEOREMS

Let P be the simplex of the k-dimensional euclidean space $\mathbb{R}^k$. Let u be a continuous real-valued function on P. We denote by F the set of all real-valued function on P. Let $K = \{1,..., r,..., k\}$ and $K^{\mathrm{I}} = \{K_1^{\mathrm{I}},..., K_l^{\mathrm{I}},..., K_L^{\mathrm{I}}\}$, $K^{\mathrm{II}} = \{K_1^{\mathrm{II}},..., K_m^{\mathrm{II}},..., K_M^{\mathrm{II}}\}$ be two partitions of the set K. We shall say that

0022-247X/84 $3.00

$g: K \to \mathbb{R}$ is I-measurable if g is measurable with respect to the σ-field generated by K^{I}, and similarly for II-measurable. Given c and p in $\mathbb{R}^k$ we define $c * p$ in $\mathbb{R}^k$ by $(c * p)_r = c_r p_r$, $\forall r \in K$. Let us now introduce, for any $p \in P$, the following subsets of P (see [1]):

$$\Pi_{\mathrm{I}}(p) = \{q = a * p \mid q \in P; a: r \to a_r \text{ is I-measurable}\},$$

$$\Pi_{\mathrm{II}}(p) = \{q = b * p \mid q \in P; b: r \to b_r \text{ is II-measurable}\}.$$

A function $f \in F$ will be called I-concave if for any $p_0 \in P$, f restricted to $\Pi_{\mathrm{I}}(p_0)$ is concave, and similarly for II-convex.

DEFINITION. Let $f \in F$. The I-concavification of f is denoted by $\mathrm{Cav}_{\mathrm{I}} f$ and is defined by $\mathrm{Cav}_{\mathrm{I}} f = \min\{g \in F \mid g \text{ is I-concave and } g(p) \geqslant f(p) \text{ for all } p \in P\}$. The II-convexification of f is denoted by $\mathrm{Vex}_{\mathrm{II}} f$ and is defined by $\mathrm{Vex}_{\mathrm{II}} f = \max\{g \in F \mid g \text{ is II-convex and } g(p) \leqslant f(p) \text{ for all } p \in P\}$. Here min and max always mean a pointwise minimization and maximization, respectively, of the functions under consideration.

Let us now consider the following pair of dual problems:

Problem I: Minimize f subject to

$$f \geqslant \underset{\mathrm{I}}{\mathrm{Cav}}\ \underset{\mathrm{II}}{\mathrm{Vex}} \max(u, f). \tag{2.1}$$

Problem II: Maximize g subject to

$$g \leqslant \underset{\mathrm{II}}{\mathrm{Vex}}\ \underset{\mathrm{I}}{\mathrm{Cav}} \min(u, g). \tag{2.2}$$

The independent case considered in [2] is obtained when

$$K = \{(l, m) \mid l = 1,..., k_1, m = 1,..., k_2\}, \quad p^{l,m} = s^l t^m, \quad \sum_1^{k_1} s^l = \sum_1^{k_2} t^m = 1$$

and

$$K_l^{\mathrm{I}} = \{(l, m) \mid m = 1,..., k_2\}, \qquad K_m^{\mathrm{II}} = \{(l, m) \mid l = 1,..., k_1\}.$$

THEOREM 2.1. *Both Problems* I *and* II *have solutions and the two solutions are equal.*

THEOREM 2.2. *The common solution of Problems* I *and* II *is also a simultaneous solution, and the only simultaneous solution, of the following two functional equations*:

$$f = \operatorname*{Vex}_{\text{II}} \max(u, f), \tag{2.3}$$

$$f = \operatorname*{Cav}_{\text{I}} \min(u, f). \tag{2.4}$$

3. Proofs

Denote by F_1 the set of functions satisfying (2.1) and F_2 the set of functions satisfying (2.2).

PROPOSITION 3.1. *$F_1 \neq \varnothing$ and $F_2 \neq \varnothing$.*

Let $\underline{v} = \inf\{f \mid f \in F_1\}$ and $\bar{v} = \sup\{g \mid g \in F_2\}$.

PROPOSITION 3.2. *$\underline{v} \in F_1$ and $\bar{v} \in F_2$.*

COROLLARY 3.3. *$\underline{v} = \min\{f \mid f \in F_1\}$ and is the solution of Problem* I. *$\bar{v} = \max\{f \mid f \in F_2\}$ and is the solution of Problem* II.

PROPOSITION 3.4. $\underline{v} = \operatorname{Cav}_{\text{I}} \operatorname{Vex}_{\text{II}} \max(u, \underline{v})$, $\bar{v} = \operatorname{Vex}_{\text{II}} \operatorname{Cav}_{\text{I}} \min(u, \bar{v})$.

LEMMA 3.5. *For any $f \in F$, each of* $\operatorname{Cav}_{\text{I}} \operatorname{Vex}_{\text{II}} f$ *and* $\operatorname{Vex}_{\text{II}} \operatorname{Cav}_{\text{I}} f$ *is both* I-*concave and* II-*convex.*

Proof. It is enough to prove that if g is II-convex, then $\operatorname{Cav}_{\text{I}} g$ is II-convex. So we want to show that for each $p \in P$, $b_1 * p$ and $b_2 * p \in \Pi_{\text{II}}(p)$ such that $\lambda b_1^r p^r + (1-\lambda) b_2^r p^r = p^r$, $\forall r \in K$, where $\lambda \in \,]0, 1[$ we have

$$\operatorname*{Cav}_{\text{I}} g(p) \leqslant \lambda \operatorname*{Cav}_{\text{I}} g(b_1 * p) + (1-\lambda) \operatorname*{Cav}_{\text{I}} g(b_2 * p) \tag{3.1}$$

We shall use the fact that, for $n \geqslant k$,

$$T^n g = \operatorname*{Cav}_{\text{I}} g,$$

where T is defined by

$$Tg(p) = \sup_{\mu, a_1, a_2} \{\mu g(a_1 * p) + (1-\mu) g(a_2 * p) \mid a_1 * p \text{ and } a_2 * p \in \Pi_{\text{I}}(p),$$
$$\mu \in [0, 1], \mu a_1^r + (1-\mu) a_2^r = 1, \forall r \in K\}. \tag{3.2}$$

Now, for each μ, a_1, a_2, satisfying the constraints in (3.2) we shall construct p_{ij}, $i = 1, 2$, $j = 1, 2$, λ_j, $j = 1, 2$, and μ_i, $i = 1, 2$ such that

$$\begin{aligned} &p_{ij} \in \Pi_{\text{II}}(a_i * p), && j = 1, 2, i = 1, 2, \\ &\lambda_i p_{i1} + (1-\lambda_i) p_{i2} = a_i * p, && \lambda_i \in [0, 1], i = 1, 2. \end{aligned} \tag{3.3}$$

$$p_{ij} \in \Pi_1(b_j * p), \qquad i = 1, 2, j = 1, 2,$$
$$\mu_j p_{1j} + (1-\mu_j) p_{2j} = b_j * p, \qquad \mu_j \in [0, 1], j = 1, 2. \tag{3.4}$$

$$\mu\lambda_1 = \lambda\mu_1, (1-\mu)\lambda_2 = \lambda(1-\mu_1), \mu(1-\lambda_1) = (1-\lambda)\mu_2, (1-\mu)(1-\lambda_2) = (1-\lambda)(1-\mu_2). \tag{3.5}$$

Assuming that (3.3)–(3.5) hold true we get

$$g(a_i * p) \leqslant \lambda_i g(p_{i1}) + (1-\lambda_i) g(p_{i2})$$

since g is II-convex. So we have

$$\begin{aligned} &\mu g(a_1 * p) + (1-\mu) g(a_2 * p) \\ &\quad \leqslant \lambda_1 \mu g(p_{11}) + (1-\mu)\lambda_2 g(p_{21}) \\ &\qquad + (1-\lambda_1)\mu g(p_{12}) + (1-\mu)(1-\lambda_2) g(p_{22}). \end{aligned}$$

Using (3.5) the majorant is

$$\lambda(\mu_1 g(p_{11}) + (1-\mu_1) g(p_{21})) + (1-\lambda)(\mu_2 g(p_{12}) + (1-\mu_2) g(p_{22}))$$

which is smaller than

$$\lambda Tg(b_1 * p) + (1-\lambda) Tg(b_2 * p). \tag{3.6}$$

Since this inequality holds true for all μ, a_1, a_2 we use (3.2) and obtain the following: g is II-convex implies Tg is II-convex, hence by induction $T^k g = \mathrm{Cav}_{\mathrm{I}}\, g$ is II-convex.

Let us now construct the auxiliary variables. If $\mu = 0$ or 1, the majorization (3.6) is obvious. Now let $\mu \in \,]0, 1[$. From (3.1) and (3.2) it follows that we can assume that $a_1 \cdot (b_1 * p) \neq 0$ and $a_2 \cdot (b_2 * p) \neq 0$. Now if $a_1 \cdot (b_2 * p) \neq 0$ and $a_2 \cdot (b_1 * p) \neq 0$, we take, with $\delta = 1/\sum_{r=1}^{k} a_1^r b_1^r p^r$,

$$p_{11} = \delta a_1 * (b_1 * p), \qquad p_{12} = \frac{\delta(1-\lambda)}{(\delta-\lambda)} a_1 * (b_2 * p),$$

$$p_{21} = \frac{\delta(1-\mu)}{(\delta-\mu)} \cdot a_2 * (b_1 * p), \qquad p_{22} = \frac{(1-\lambda)(1-\mu)}{\left(1-\lambda-\mu+\dfrac{\lambda\mu}{\delta}\right)} (a_2 * (b_2 * p)),$$

$$\lambda_1 = \frac{\lambda}{\delta} \lambda_2 = \frac{\lambda}{\delta}\left(\frac{\delta-\mu}{1-\mu}\right), \qquad \mu_1 = \frac{\mu}{\delta}, \qquad \mu_2 = \frac{\mu}{\delta}\left(\frac{\delta-\mu}{1-\lambda}\right). \tag{3.7}$$

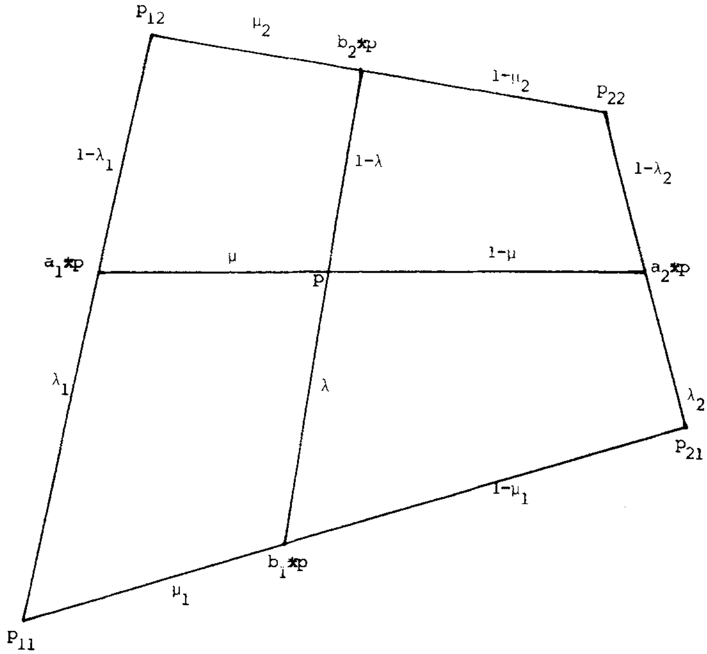

FIGURE 1

If $a_1 \cdot (b_2 * p) = 0$ and $a_2 \cdot (b_1 * p) \neq 0$, we take

$$p_{11} = \lambda a_1 * (b_1 * p) = p_{12}$$

and the other variables as above with $\delta = \lambda$, similarly if $a_1 \cdot (b_2 * p) \neq 0$ and $a_2 \cdot (b_1 * p) = 0$. Finally if $a_1 \cdot (b_2 * p) = a_2 \cdot (b_1 * p) = 0$, we have $\lambda = \mu$ and we choose

$$p_{11} = \lambda a_1 * (b_1 * p), \qquad \lambda_1 = \mu_1 = 1,$$
$$p_{22} = \lambda a_2 * (b_2 * p), \qquad \lambda_2 = \mu_2 = 0.$$

This completes the proof of the lemma.[1]

COROLLARY 3.6. *Each of* $\underline{v}$ *and* $\bar{v}$ *is both* I-*concave and* II-*convex.*

[1] I am indebted to the referee for calling my attention to an inaccuracy in the first version of this lemma.

LEMMA 3.7.

$$\underline{v} = \underset{\text{II}}{\text{Vex}} \max(u, \underline{v}),$$

$$\bar{v} = \underset{\text{I}}{\text{Cav}} \min(u, \bar{v}).$$

Define now two sequences of functions $\{\underline{u}_n\}$ and $\{\bar{u}_n\}$ by $\underline{u}_0 \equiv -\infty$ and $\bar{u}_0 \equiv +\infty$ and

$$\underline{u}_{n+1} = \underset{\text{I}}{\text{Cav}}\, \underset{\text{II}}{\text{Vex}} \max(u, \underline{u}_n), \qquad n \geqslant 1, \tag{3.8}$$

$$\bar{u}_{n+1} = \underset{\text{II}}{\text{Vex}}\, \underset{\text{I}}{\text{Cav}} \min(u, \bar{u}_n), \qquad n \geqslant 1. \tag{3.9}$$

PROPOSITION 3.8. *$\{\underline{u}_n\}$ is an increasing sequence, uniformly converging to a finite continuous function $\underline{u}$. $\{\bar{u}_n\}$ is a decreasing sequence uniformly converging to a finite continuous function $\bar{u}$.*

PROPOSITION 3.9.

$$\underline{u} \geqslant \underline{v},$$
$$\bar{u} \leqslant \bar{v}.$$

PROPOSITION 3.10.

$$\underline{u} = \underline{v},$$
$$\bar{u} = \bar{v}.$$

Let $\mathfrak{U} = \{u \in F \mid u(p) = \max_{i \in I} \min_{j \in J} \sum_r a^r_{ij} p^r$, *where* $a^r_{ij} \in \mathbb{R}$ *for all* i, j, r, I *and* J *are finite sets*$\}$.

LEMMA 3.11. *For all* $u \in \mathfrak{U}$, $\bar{v} \leqslant \underline{v}$.

Proof. Let us introduce the following sequence:

$$v_1(p) = \underset{\text{I}}{\text{Cav}}\, \max_i\, \underset{\text{II}}{\text{Vex}}\, \min_j \sum_r a^r_{ij} p^r,$$

$$n v_n(p) = \underset{\text{I}}{\text{Cav}}\, \max_i\, \underset{\text{II}}{\text{Vex}}\, \min_j \left\{ \sum_r a^r_{ij} p^r + (n-1)\, v_{n-1}(p) \right\}.$$

We shall first prove that $v_n(p) \leqslant \underline{u}_n(p)$ for all $p \in P$. In fact, we have

$v_1(p) \leqslant \mathrm{Cav}_{\mathrm{I}}\, \mathrm{Vex}_{\mathrm{II}} \max_i \min_j \sum_r a_{ij}^r p^r = \mathrm{Cav}_{\mathrm{I}}\, \mathrm{Vex}_{\mathrm{II}}\, u(p) = \underline{u}_1(p)$. Assume now that $v_{n-1}(p) \leqslant \underline{u}_{n-1}(p)$, then

$$\begin{aligned} nv_n(p) &\leqslant \underset{\mathrm{I}}{\mathrm{Cav}}\ \underset{\mathrm{II}}{\mathrm{Vex}} \left\{ \max_i \min_j \sum_i a_{ij}^r p^r + (n-1)\, v_{n-1}(p) \right\} \\ &\leqslant \underset{\mathrm{I}}{\mathrm{Cav}}\ \underset{\mathrm{II}}{\mathrm{Vex}} \{u(p) + (n-1)\, \underline{u}_{n-1}(p)\} \\ &\leqslant \underset{\mathrm{I}}{\mathrm{Cav}}\ \underset{\mathrm{II}}{\mathrm{Vex}} \{n \max(u, \underline{u}_{n-1})\}(p) = n\underline{u}_n(p). \end{aligned}$$

Now we shall show that $v_n(p) \geqslant \bar{u}_n(p) + K/n$ for some $K \in \mathbb{R}$. Let us define, for all $i \in I$, $f_i(p) = -\min_j \sum_r a_{ij}^r p^r$ and $f(p) = \sum_i f_i(p) - L$ where L is chosen such that $v_1(p) \geqslant \bar{u}_1(p) + f(p)$ (v_1 and $\bar{u}_1$ are bounded on P). Assume now that $v_n(p) \geqslant \bar{u}_n(p) + f(p)$. Then we get

$$\begin{aligned} (n+1)\, v_{n+1}(p) &\geqslant \underset{\mathrm{I}}{\mathrm{Cav}} \max_i \underset{\mathrm{II}}{\mathrm{Vex}} \min_j \left\{ \sum_r a_{ij}^r p^r + f(p) + n\bar{u}_n(p) \right) \\ &\geqslant \underset{\mathrm{I}}{\mathrm{Cav}} \max_i \left(\underset{\mathrm{II}}{\mathrm{Vex}} \left(\min_j \sum_r a_{ij}^r p^r + f(p) \right) + n\bar{u}_n(p) \right) \end{aligned}$$

since $\bar{u}_n$ is II-convex by Lemma 3.5. But by construction $\min_j \sum_r a_{ij}^r + f(p)$ is convex, thus it is II-convex:

$$\begin{aligned} (n+1)\, v_{n+1} &\geqslant \underset{\mathrm{I}}{\mathrm{Cav}}\, (u + f + n\bar{u}_n) \\ &\geqslant \underset{\mathrm{I}}{\mathrm{Cav}}\, (u + n\bar{u}_n) - \underset{\mathrm{I}}{\mathrm{Cav}}(-f) \\ &\geqslant \underset{\mathrm{II}}{\mathrm{Vex}}\ \underset{\mathrm{I}}{\mathrm{Cav}} \{(n+1) \min(u, \bar{u}_n)\} + f = (n+1)\, \bar{u}_n + f \end{aligned}$$

and since f is bounded, the result follows. Hence letting $n \to \infty$ we obtain $\underline{v} \geqslant \bar{v}$.

Now it is easy to see, applying the last part of [2] that Theorems 2.1 and 2.2 hold true for all $u \in \mathfrak{U}$, the solution being $\underline{v} = \bar{v}$. Denote by $C(P)$ the space of all continuous functions on P.

PROPOSITION 3.12. (a) *$\mathfrak{U}$ is a vector lattice which contains the affine functions*

(b) *Hence $\mathfrak{U}$ is dense in $C(P)$.*

Proof. (a)(1) $\mathfrak{U}$ obviously contains the affine functions.

(2) $u \in \mathfrak{U} \Rightarrow -u \in \mathfrak{U}$.

Let $u(p) = \max_{i \in I} \min_{j \in J} \sum_r a_{ij}^r p^r$. Define $J' = J^I$ for all $i \in I$, $j' = (j'(1), ..., j'(i), ..., j'(I)) \in J'$, $b_{ij'}^r = a_{ij'(i)}$. Then we have

$$u(p) = \max_I \min_{J'} \sum_r b_{ij'}^r p^r = \min_{J'} \max_I \sum_r b_{ij'}^r p^r.$$

Letting $c_{j'i}^r = -b_{ij'}^r$ it follows that

$$\operatorname*{Max}_{J'} \operatorname*{Min}_I \sum_r c_{j'i}^r p^r = -\operatorname*{Min}_{J'} \operatorname*{Max}_I \sum_r b_{ij'}^r p^r = -u(p).$$

(3) $u \in \mathfrak{U}$, $\lambda \geqslant 0 \Rightarrow \lambda u \in \mathfrak{U}$. λu comes from λa_{ij}^r

(4) $u_1 \in \mathfrak{U}$, $u_2 \in \mathfrak{U} \Rightarrow u_1 + u_2 \in \mathfrak{U}$.

Let $I = I_1 \times I_2$ and $J = J_1 \times J_2$ and define, where $i = (i_1, i_2)$, $j = (j_1, j_2)$,

$$a_{ij}^r = a_{i_1 j_1}^r + a_{i_2 j_2}^r,$$

then

$$\max_I \min_J \sum_r a_{ij}^r p^r = u_1(p) + u_2(p).$$

(5) $u \in \mathfrak{U} \Rightarrow \max(u, 0) \in \mathfrak{U}$. Add to I some letter α and define $a_{\alpha j} = 0$ for all j, then $\max(u, 0) = \max_{i \in I \cup \{\alpha\}} \min_J \sum_r a_{ij}^r p^r$.

(b) This follows from (a) and the Stone–Weierstrass theorem for lattices, see, e.g., [3, p. 243].

Now define $\varphi: \mathfrak{U} \to C(P)$ where $\varphi(u)$ is the solution of Problems I and Problem II. It was proved in [1] (Proposition 5.2) that φ has a unique continuous extension $\tilde{\varphi}: C(P) \to C(P)$ whenever $\mathfrak{U}$ is dense in $C(P)$. Moreover Theorem 5.3 in [1] implies that Theorems 2.1 and 2.2 remain true for all u in $C(P)$ and that the solution is $\tilde{\varphi}(u)$.

References

1. J. F. Mertens and S. Zamir, The value of two-person zero-sum repeated games with lack of information on both sides, *Inter. J. Game Theory* **1** (1971), 39–64.
2. J. F. Mertens and S. Zamir. A duality theorem on a pair of simultaneous functional equations, *J. Math. Anal. Appl.* **60** (1977), 550–558.
3. H. H. Schaefer, "Topological Vector Spaces," Macmillan Company, New York, 1967.
4. S. Sorin, A note on the value of zero-sum sequential repeated games with incomplete information, *Inter. J. Game Theory* **8** (1979), 217–223.

Printed in Belgium