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"Big Match" with Lack of Information on one Side (Part I)

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Abstract: For a special class of two-person zero-sum infinitely repeated stochastic games with incomplete information, we prove the existence of the maxmin, minmax and lim v_n. However the value may not exist and moreover maxmin and lim v_n may be transcendental functions.

1. Introduction

Blackwell/Ferguson have introduced a specific stochastic game called "the Big Match".

Here we are concerned with a similar class of repeated zero-sum stochastic games, but with lack of information on one side. Recall that stochastic games are games in which the payoff function at each stage depends on the state reached at this time. The transition probabilities on the state space are functions of the moves of the players and the current state as well as these transition probabilities are common knowledge.

Games with incomplete information are also games in which the payoff function depends on some state. However, in this case the state is not known by all the players but is constant along the play. More precisely the state is chosen at random once for ever, according to some probability which is itself common knowledge and the players have some information about this choice.

The study of infinitely repeated games can be done along two ways. The first one is to consider the game as the limit of the finitely repeated games and to study the existence of the limit of their values (called the asymptotic value). On the other hand, one can directly define a concept of value (called the infinite value) for the infinitely repeated game and look for the existence of this value.

It was proved by *Aumann/Maschler* [1966] that games with lack of information on one side do have an infinite value (hence the asymptotic value also exists). *Blackwell/Ferguson* proved the existence of the infinite value for the Big Match and *Kohlberg* extended this result to games with absorbing states. Later *Bewley/Kohlberg* showed the existence of the asymptotic value for stochastic games and recently *Mertens/Neyman* succeed in proving the existence of the infinite value for such games.

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Here we show that these results do not extend to stochastic games with incomplete information on one side. More precisely the infinite value may not exist.

However we prove for the class of games under consideration the existence of the asymptotic value and the existence of maxmin and minmax for the infinite game.

2. The Model

Let

$$A = \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21} & b_{22} \end{bmatrix}$$

be two payoff matrices of a zero-sum game, where the star (*) denotes an absorbing payoff [*Blackwell/Ferguson; Kohlberg*]. We denote by $I = \{\text{Top, Bottom}\} = \{T, \text{Bo}\}$ (resp. $J = \{\text{Left, Right}\} = \{L, R\}$) the set of moves of player I (resp. player II). Player I is the row player and the maximizer.

For $p \in [0, 1]$, $n = 1, 2, \ldots$, we consider the n-stage repeated game $G_n(p)$, defined as follows:

- at stage 0, C which is one of the two payoff matrices is chosen once and for all by the referee (with probability p for A), and this is told to player I;
- at stage 1, player I (PI) chooses $i_1 \in I$, player II (PII) chooses $j_1 \in J$ and the couple (i_1, j_1) is told to both players;
- at stage m, $m = 2, \ldots$, PI (resp. PII) knowing the previous history up to this stage namely the sequence of moves $h_{m-1} = (i_1, j_1, \ldots, i_{m-1}, j_{m-1})$, chooses $i_m \in I$ (resp. $j_m \in J$) and (i_m, j_m) is announced to both players;
- after stage n, PI receives from PII the following amount:

$$\bar{q}_n(h_n) = \frac{1}{n} \sum_{m=1}^{n} c_{h_m}$$

where

$$\begin{aligned} c_{h_m} &= c_{h_t} && \text{if } t < m \text{ and } c_{i_t j_t} \text{ is absorbing} \\ &= c_{i_m j_m} && \text{otherwise.} \end{aligned}$$

The whole description of the game including this sentence is common knowledge [*Aumann*, 1974].

We denote by $v_n(p)$ the value of $G_n(p)$. $G_\infty(p)$ is the infinitely repeated game.

Since the game under consideration is a game with perfect recall we can assume [*Kuhn; Aumann*, 1964] that both players actually use behavioral strategies. Hence a strategy for PI in $G_\infty(p)$ is defined by a couple $\sigma = (\sigma^A, \sigma^B)$: σ^A is a sequence $(s_1^A, \ldots, s_m^A, \ldots,)$ where for each $m = 1, 2, \ldots, s_m^A$ is a function from the set of previous histories, $H_{m-1} = (I \times J)^{m-1}$ into the set of "one stage mixed strategies", namely the set of probabilities over I (denoted by I*), and similarly for σ^B. For PII a strategy in $G_\infty(p)$ is given by $\tau = (t_1, \ldots, t_m, \ldots)$ where t_m is a function from J^{m-1} into the set of probabilities over J (denoted by J^*), since it is enough to define t_m on $i_k = Bo$, $k = 1, \ldots, m-1$.

Obviously the strategies of both players in $G_n(p)$ are the restriction to the n first stage of the strategies in $G_\infty(p)$ and are denoted by σ_n, τ_n. For each $m = 1, 2, \ldots, \sigma$, τ and p induce a probability on $H_m \times \{A, B\}$ and we define $\bar{\gamma}_n(\sigma, \tau)$ to be the expected payoff in $G_n(p)$ and $\gamma_m(\sigma, \tau)$ to be the expected payoff at stage m. Hence:

$$\gamma_m(\sigma, \tau) = E_{p,\sigma,\tau}\, c_{h_m}$$

$$\bar{\gamma}_n(\sigma, \tau) = E_{p,\sigma,\tau}\, \bar{q}_n(h_n)$$

and

$$\bar{\gamma}_n = \frac{1}{n} \sum_{m=1}^{n} \gamma_m.$$

In order to study $G_\infty(p)$ we use the following definitions [*Mertens/Zamir*, 1980]:

$\underline{v}(p)$ is the *maxmin* of $G_\infty(p)$ if

(i) $\forall \epsilon > 0, \exists\, \sigma$ and $\exists N \in \mathrm{N}$ such that, for all τ and all $n \geqslant N$

$$\bar{\gamma}_n(\sigma, \tau) > \underline{v}(p) - \epsilon;$$

(ii) $\forall \epsilon > 0, \forall \sigma, \exists\, \tau$ and $\exists N \in \mathrm{N}$ such that $n \geqslant N$ implies

$$\bar{\gamma}_n(\sigma, \tau) < \underline{v}(p) + \epsilon.$$

We shall refer to these conditions by saying that PI can guarantee $\underline{v}(p)$, (*i*); and that he cannot guarantee more, (ii). The *minmax* $\bar{v}(p)$ is defined in a dual way. $G_\infty(p)$ has a *value* iff $\bar{v}(p) = \underline{v}(p)$.

The paper is organized as follows:

In Part III we prove the existence of the minmax. Part IV and V are independent of III. In Part IV we construct an auxiliary game to prove the existence of lim v_n. Part

V is devoted to the study of the maxmin: $\underline{v}$ exists and equals $\lim v_n$. In Part VI several examples are given. After some remarks in Part VII we show in Part VIII that some games with signalling matrices can be reduced to the class of games studied here.

In the proofs the following notation will be used: Let $\mathcal{H}_\infty$ be the product σ-field on H_∞ and $\mathcal{H}_n$ be the σ-field induced on H_∞ by the sets $h_n \times H_\infty$, $h_n \in H_n$. We define the $(\mathcal{H}_n)_1^\infty$ stopping time m as:

$$\mathrm{m}(h_\infty) = \min [\{m; i_m = T\} \cup \{+\infty\}].$$

$L = \max \{c_{ij} \mid i \in I, j \in J, C \in \{A, B\}\}$.

If $x \in [0, 1]$ then $\hat{x}$ denotes $1 - x$.

Given $x \in I^*$ and $y \in J^*$ then xCy stands for $\sum_{I \times J} c_{ij} x_i y_j$.

3. Minmax

Proposition 1: *$\bar{v}(p)$ exists and $\bar{v}(p) = v_1(p)$ on* $[0, 1]$.
(Recall that v_1 is the value of the one-stage game.) The proposition will follow from the next two lemmas.

Lemma 2: PII *can guarantee $v_1(p)$.*

Proof: Let $t_1 = t \in J^*$ be an optimal strategy for PII in $G_1(p)$ and define τ by $t_m \equiv t$ for all $m \geqslant 1$. (Hence PII is using a sequence of independent identically distributed (i.i.d.) one-stage mixed strategies.)

Now for each σ we introduce the distribution of the stopping time m:

$$z_C(m) = \mathrm{Prob}_{\sigma^C, \tau} \{\mathrm{m} \leqslant m\} \quad C = A, B$$

and if $x_C(m)$ denotes the vector $(z_C(m), \hat{z}_C(m))$ it will follow that

$$\gamma_m(\sigma, \tau) = p\, x_A(m) A t + \hat{p}' x_B(m) B t. \tag{1}$$

Actually we have:

$$\gamma_m(\sigma, \tau) = E_{p,\sigma,\tau}\, c_{h_m}$$

$$= p E_{\sigma^A, \tau}\, a_{h_m} + \hat{p}' E_{\sigma^B, \tau}\, b_{h_m}.$$

Now

$$E_{\sigma^A,\tau}(a_{h_m}) = \sum_{l=1}^{m} \operatorname{Prob}_{\sigma^A,\tau}\{\mathrm{m}=l\}\,[a_{11}\,E_{\sigma^A,\tau}(t_l(L)\mid \mathrm{m}=l)$$

$$+\,a_{12}\,E_{\sigma^A,\tau}(t_l(R)\mid \mathrm{m}=l)]$$

$$+\operatorname{Prob}_{\sigma^A,\tau}\{\mathrm{m}>m\}\,[a_{21}\,E_{\sigma^A,\tau}(t_m(L)\mid \mathrm{m}>m)$$

$$+\,a_{22}\,E_{\sigma^A,\tau}(t_m(R)\mid \mathrm{m}>m)].$$

Since τ is i.i.d. we get

$$E_{\sigma^A,\tau}(a_{h_m}) = [\sum_{l=1}^{m} \operatorname{Prob}_{\sigma^A,\tau}\{\mathrm{m}=l\}]\,(a_{11}\,t(L)+a_{12}\,t(R))$$

$$+\operatorname{Prob}_{\sigma^A,\tau}\{\mathrm{m}>l\}\,(a_{21}\,t(L)+a_{22}\,t(R))$$

and similarly for $E_{\sigma^B,\tau}(b_{h_m})$, hence (1).

The choice of t then implies

$$\gamma_m(\sigma,\tau)\leqslant v_1(p)$$

hence we have

$$\bar{\gamma}_n(\sigma,\tau)\leqslant v_1(p) \text{ for all } n\geqslant 1 \text{ and all } \sigma. \qquad \text{Q.E.D.}$$

Remarks:

- The above proof also implies that $v_n(p)\leqslant v_1(p)$ for all $n\geqslant 1$.
- The result holds as soon as there exists an optimal strategy t for PII in $G_1(p)$ such that if $J(t)$ denotes the support of t then for all $i\in I$ and all C the payoffs $(c_{ij}; j\in J(t))$ are either all absorbing or all non absorbing.

Lemma 3: PII *cannot guarantee less than* $v_1(p)$.

Proof: Let τ be a strategy of PII in G_∞, and let us define, for all $m\geqslant 1$, $\bar{t}_m = E_\tau(t_m)$, hence $\bar{t}_m\in J^*$. Now given an optimal strategy $s_1 = s$ for PI in $G_1(p)$, let us introduce, for each $t\in J^*$:

$$f(s, t) = p\, s^A\,(T)\,(a_{11}\, t\,(L) + a_{12}\, t\,(R)) + \hat{p}'\, s^B(T)\,(b_{11}\, t\,(L) + b_{12}\, t\,(R)),$$

$$g\,(s, t) = p\, \hat{s}^A\,(T)\,(a_{21}\, t\,(L) + a_{22}\, t\,(R)) + \hat{p}'\, \hat{s}^B\,(T)\,(b_{21}\, t\,(L) + b_{22}\, t\,(R)).$$

f is the absorbing part of the payoff in $G_1\ (p)$ given s and t, and g is the non absorbing part.

Given $\epsilon > 0$ let $n \geq 1$ be such that

$$f(s, \bar{t}_n) \geq \sup_m f(s, \bar{t}_m) - \frac{\epsilon}{3}. \tag{2}$$

We can now define a strategy σ for PI as follows:

– Play (Bo, Bo) up to stage $n-1$ included

$$(\text{i.e., } s_m^C\,(h_{m\text{-}1})\,(T) = 0\ \forall\, m, 1 \leq m \leq n, \forall\, h_{m\text{-}1} \in H_{m\text{-}1}, \forall\, C \in \{A, B\}).$$

– Play s at stage n

$$(\text{i.e., } s_n^C\,(h_{n\text{-}1}) = s^C, \forall\, h_{n\text{-}1} \in H_{n\text{-}1}).$$

– At each following stage play Bo

$$(\text{i.e., } s_m^C\,(h_{m\text{-}1}) = \text{Bo}, \forall\, m > n, \forall\, h_{m\text{-}1}, \forall\, C).$$

Given σ and τ the expected payoff at stage m, $m \geq n$, will be

$$\gamma_m\,(\sigma, \tau) = f(s, \bar{t}_n) + g\,(s, \bar{t}_m).$$

It follows by (2) that

$$\gamma_m\,(\sigma, \tau) \geq f(s, \bar{t}_m) + g\,(s, \bar{t}_m) - \frac{\epsilon}{3}$$

hence by the choice of s

$$\gamma_m\,(\sigma, \tau) \geq v_1\ (p) - \frac{\epsilon}{3} \text{ for all } m \geq n.$$

Thus we get, for $m \geq n$

$$\bar{\gamma}_m\,(\sigma, \tau) \geq -\left(\frac{n-1}{m}\right)L + \left(\frac{m-n+1}{m}\right)\left(v_1\ (p) - \frac{\epsilon}{3}\right)$$

so that $m \geq N = (3L\,(n-1))/\epsilon$ implies

$$\bar{\gamma}_m\,(\sigma, \tau) \geq v_1\ (p) - \epsilon.$$

Q.E.D.

Remark: We use explicitly in the definition of τ the fact that there is only one and the same line non absorbing in both payoff matrices.

It follows then from the previous remark that for all games satisfying this condition we do have

$$\bar{v}(p) = v_1(p).$$

4. Lim v_n

In this part we prove that lim v_n exists and give an explicit formula for it.

Lemma 4 (Recursive formula.): *For each $n \geqslant 0$ and each $p \in [0, 1]$ the following holds*:

$$(n+1)\, v_{n+1}(p) = \max_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1}} \; \min_{0 \leqslant t \leqslant 1} D(x, y, t) \tag{3}$$

with $D(x, y, t) = (n+1)\,(px\,(t\,a_{11} + \hat{t}\,a_{12}) + \hat{p}\,y\,(t\,b_{11} + \hat{t}\,b_{12}))$

$$+ p\,\hat{x}\,(t\,a_{21} + \hat{t}\,a_{22}) + \hat{p}\,\hat{y}\,(t\,b_{21} + \hat{t}\,b_{22})$$

$$+ n\,(p\,\hat{x} + \hat{p}\,\hat{y})\, v_n\left(\frac{p\,\hat{x}}{p\,\hat{x} + \hat{p}\,\hat{y}}\right)$$

and

$$v_0 \equiv 0 \text{ on } [0, 1].$$

Proof: Given σ and τ strategies of both players in $G_{n+1}(p)$, we introduce the following simpler notations which correspond to the moves of the players at the first stage:

$$x = s_1^A(T), y = s_1^B(T), t = t_1(L).$$

We first prove

$$\forall\, \sigma, \exists\, \tau \text{ such that } (n+1)\,\bar{\gamma}_{n+1}(\sigma, \tau) \leqslant \min_{0 \leqslant t \leqslant 1} D(x, y, t). \tag{4}$$

Let us compute the expected payoff in $G_{n+1}(p)$ given σ and τ:

$$\bar{\gamma}_{n+1}(\sigma, \tau) = E_{p,\sigma,\tau}(\bar{q}_{n+1} \mid \mathrm{m} = 1) \cdot \mathrm{Prob}_{p,\sigma,\tau}(\mathrm{m} = 1) \tag{5}$$

$$+ E_{p,\sigma,\tau}(\bar{q}_{n+1} \mid \mathrm{m} > 1) \cdot \mathrm{Prob}_{p,\sigma,\tau}(\mathrm{m} > 1).$$

But we have

$$E_{p,\sigma,\tau}(\bar{q}_{n+1} \mid \mathrm{m} = 1) = E_{p,\sigma,\tau}(q_1 \mid \mathrm{m} = 1)$$

and

$$E_{p,\sigma,\tau}(q_1 \mid \mathrm{m} = 1) \cdot \mathrm{Prob}_{p,\sigma,\tau}(\mathrm{m} = 1) = px\,(t\,a_{11} + \hat{t}\,a_{12}) + \hat{p}y\,(t\,b_{11} + \hat{t}\,b_{12}).$$

Write $(n + 1)\,\bar{q}_{n+1} = q_1 + n\,\bar{q}'_n$ where $\bar{q}'_n$ is the average

$$\bar{q}'_n = \frac{1}{n} \cdot \sum_{m=2}^{n+1} c_{h_m}.$$

Now

$$E_{p,\sigma,\tau}(q_1 \mid \mathrm{m} > 1) \cdot \mathrm{Prob}_{p,\sigma,\tau}(\mathrm{m} > 1) = p\hat{x}\,(t\,a_{21} + \hat{t}\,a_{22}) + \hat{p}\hat{y}\,(tb_{21} + \hat{t}b_{22})$$

and

$$\mathrm{Prob}_{p,\sigma,\tau}(\mathrm{m} > 1) = p\,\hat{x} + \hat{p}\hat{y}.$$

Hence it remains to majorize $E_{p,\sigma,\tau}(\bar{q}'_n \mid \mathrm{m} > 1)$. But knowing σ, PII can compute the new posterior on $\{A, B\}$ given $\{\mathrm{m} > 1\}$ which is equal to

$$\mathrm{Prob}\,(C = A \mid \mathrm{m} > 1) = \frac{p\,\hat{x}}{p\,\hat{x} + \hat{p}\,\hat{y}} = p_1$$

and then by playing optimally in $G_n(p_1)$, PII can guarantee a payoff less than $v_n(p_1)$ for the last n stages, hence

$$E_{p,\sigma,\tau}(\bar{q}'_n \mid \mathrm{m} > 1) \leqslant v_n(p_1).$$

Now, replacing in (5) gives (4). Using the minmax theorem (4) implies

$$\exists\,\tau \text{ such that } \forall\,\sigma \qquad (n + 1)\,\bar{\gamma}_{n+1}(\sigma, \tau) \leqslant \max_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1}} \; \min_{0 \leqslant t \leqslant 1} D(x, y, t).$$

On the other hand, by playing according to x_0, y_0 at the first stage, where $D(x_0, y_0, t) \geqslant \max\limits_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1}} \min\limits_{0 \leqslant t \leqslant 1} D(x, y, t)$ for all t, and then playing optimally in $G_n\left(\dfrac{p\,\hat{x}_0}{p\,\hat{x}_0 + \hat{p}\,\hat{y}_0}\right)$ PI obtain at least $D(x_0, y_0, t)$, hence the result follows.

Q.E.D.

Corollary 5: PI *has an optimal strategy in* $G_n(p)$ *which is independent of the histories.*

Proof: Using (3) recursively it is easy to see that PI can compute an optimal strategy which can be chosen independently of the history since the posterior probability does not depend of the moves of PII and appears only if PI plays Bo before.

Q.E.D.

Corollary 6: The value of G_n remains the same if both players, strategy sets are restricted to be independent of the histories.

Proof: Assume σ_n to be such that

$$s_m^C(h_{m-1}) = \bar{s}_m^C,\ \forall\, m,\ 1 \leqslant m \leqslant n,\ \forall\, h_{m-1},\ \forall\, C.$$

It is now enough to prove that for any τ_n there exists τ_n^* independent of the histories such that

$$\bar{\gamma}_n(\sigma_n, \tau_n) = \bar{\gamma}_n(\sigma_n, \tau_n^*).$$

It is clear that τ_n^* defined by $t_m^* = E_{\tau_n}(t_m)$, $1 \leqslant m \leqslant n$ satisfies this condition.

Q.E.D.

Hence we shall now consider strategies $\sigma_n = (\sigma_n^A, \sigma_n^B)$ and τ_n where σ_m^C is a sequence s_m^C, $m = 1, \ldots, n$ in I^* and τ_n is a sequence t_m, $m = 1, \ldots, n$ in J^*. (Note that σ_n defines the law of the stopping time m.)

In order to study the asymptotic behaviour let us think of the game $G_n(p)$ as being played between time 0 and 1, the m-th stage being at time $(m-1)/n$. As $n \to \infty$ we have a continuous time of play and a couple of strategies defined by

$$\rho_C(x) = \text{Prob (PI plays } T \text{ at some time } y,\ y \leqslant x \mid C) \quad C = A, B$$

$$f(x) = \text{Prob (PII plays } L \text{ at time } x).$$

Given these strategies let us compute the expected payoff, if $C = A$ for example. With probability $d\,\rho_A(x)$ the payoff will be absorbing for the first time at time x and its total contribution with the remaining time will be $(1-x)(a_{11} f(x) + a_{12} \hat{f}(x))$. On the other hand, the payoff will be non absorbing at time x, with probability $(1 - \rho_1(x))$ and equal to $a_{21} f(x) + a_{22} \hat{f}(x)$. Integrating on $[0, 1]$ and adding the similar part for B gives the expected payoff.

We are thus led to introduce the following sets:

$$F = \{f; f\colon [0,1] \to [0,1], \text{ Borel measurable}\}$$

$$Q = \{\rho; \rho \text{ positive Borel measure on } [0,1] \text{ with } \rho([0,1]) \leqslant 1\}$$

and given $\rho \in Q$ we shall write $\rho(t)$ for $\rho([0, t])$.

Definition 7 (The auxiliary game):

For $f \in F$, ρ_A and ρ_B in Q we define

$$\varphi_A(f, \rho) = \int_0^1 (1-x)(a_{11} f(x) + a_{12} \hat{f}(x))\, d\rho(x) + \int_0^1 (a_{21} f(x) + a_{22} \hat{f}(x))(1-\rho(x))\, dx$$

similarly for $\varphi_B(f, \rho)$, and finally

$$\varphi_p(f, \rho_A, \rho_B) = p\varphi_A(f, \rho_A) + \hat{p}\varphi_B(f, \rho_B).$$

Now let $\Gamma(p)$ be the two-person zero-sum game where the strategy sets are $Q \times Q$ for PI and F for PII and the payoff function for PI (the maximizer) is φ_p.

Then we have:

Theorem 8: *$\Gamma(p)$ has a value $v(p)$, $\lim_{n\to\infty} v_n(p)$ exists, and* $\lim v_n(p) = v(p)$ *on* $[0, 1]$.

Proof: We first denote by X (resp. Y) the infsup (resp. supinf) of Γ, namely

$$X(p) = \inf_{f\in F} \sup_{\substack{\rho_A\in Q\\ \rho_B\in Q}} \varphi_p(f, \rho_A, \rho_B)$$

$$Y(p) = \sup_{\substack{\rho_A\in Q\\ \rho_B\in Q}} \inf_{f\in F} \varphi_p(f, \rho_A, \rho_B).$$

The proof will be divided into three steps. (p is any fixed point in $[0, 1]$ and we write φ for φ_p.)

Step 1:

Lemma 9: $\Gamma(p)$ has a value.

Proof: We denote by F_0 the set of continuous functions in F and by Y_0 the corresponding supinf, namely

$$Y_0(p) = \sup_{\substack{\rho_A\in Q\\ \rho_B\in Q}} \inf_{f\in F_0} \varphi(f, \rho_A, \rho_B).$$

Given $\epsilon > 0$, choose ρ_A and ρ_B such that

$$\varphi(f_0, \rho_A, \rho_B) \geqslant Y_0(p) - \epsilon \quad \text{for all } f_0 \in F_0.$$

But for each $f \in F$ there exists, by Lusin's theorem, a sequence (f_n) in F_0 which converges to f a.e. with respect to ρ_A, ρ_B and l (Lebesgue measure). By Lebesgue's dominated convergence theorem it follows that

$$\varphi(f, \rho_A, \rho_B) = \lim_{n\to\infty} \varphi(f_n, \rho_A, \rho_B) \geqslant Y_0(p) - \epsilon.$$

Hence $Y_0(p) = Y(p)$.

Now for the weak topology, Q is compact and the mapping $(\rho_A, \rho_B) \to \varphi(f, \rho_A, \rho_B)$ is continuous for each $f \in F_0$. Since moreover $\varphi(f, ., .)$ and $\varphi(., \rho_A, \rho_B)$ are affine for all $f \in F_0$, $\rho_A \in Q$, $\rho_B \in Q$, the minmax theorem 3.5 in *Sion* implies that

$$\sup_{Q\times Q} \inf_{F_0} \varphi(f, \rho_A, \rho_B) = \inf_{F_0} \sup_{Q\times Q} \varphi(f, \rho_A, \rho_B)$$

so that we have

$$Y(p) = \sup_{Q\times Q} \inf_{F} \varphi = \sup_{Q\times Q} \inf_{F_0} \varphi = \inf_{F_0} \sup_{Q\times Q} \varphi \geqslant \inf_{F} \sup_{Q\times Q} \varphi = X(p).$$

Since the reverse inequality always holds we get

$$Y(p) = X(p).$$

Q.E.D.

Step 2: $\overline{\lim_{n\to\infty}}\, v_n(p) \leqslant X(p)$.

The idea of the proof is the following. Given f "optimal" for $X(p)$ we shall define a strategy τ_n of PII in G_n; on the other hand from any σ_n of PI we shall construct a strategy (ρ_A, ρ_B) in Γ. These choices will be done in such a way that the payoff in G_n induced by σ_n, τ_n will be approximated by the payoff corresponding to f and (ρ_A, ρ_B) in Γ. By the property of f this will prove the claim.

Let $\epsilon > 0$ be given. By the proof of Lemma 9 we can choose f in F_0, $\epsilon/2$ optimal for $X(p)$, i.e. such that

$$\varphi(f, \rho_A, \rho_B) \leqslant X(p) + \frac{\epsilon}{2} \quad \text{for all } \rho_A, \rho_B \text{ in } Q.$$

Let now τ_n be such that:

$$t_m(L) = f\left(\frac{m-1}{n}\right) \quad m = 1, \ldots, n.$$

Given σ_n, the strategy of PI, we introduce (using Corollary 6)

$$z_C(m) = \operatorname{Prob}_{\sigma_n}{}_{C}(\mathrm{m} \leqslant m) \quad m = 1, \ldots, n, \quad C = A, B$$

and we define ρ_C in Q by

$$\rho_C(x) = z_C(m) \text{ on } \left[\frac{m-1}{n}, \frac{m}{n}\right[, \quad m = 1, \ldots, n$$

and $\rho_C(1) = z_C(n)$.

Let us now compute $\bar{G}_A$, the expected payoff in $G_n(p)$ given σ_n, τ_n and $C = A$. In order to simplify the notation we put

$$g_1(x) = a_{11} f(x) + a_{12} \hat{f}(x).$$

$$g_2(x) = a_{21} f(x) + a_{22} \hat{f}(x).$$

Thus

$$\begin{aligned} n\bar{G}_A = {} & n g_1(0) z_A(1) + \ldots + (n-m) g_1\left(\frac{m}{n}\right)(z_A(m+1) - z_A(m)) + \ldots \\ & + g_1\left(1 - \frac{1}{n}\right)(z_A(n) - z_A(n-1)) \\ & + g_2(0)(1 - z_A(1)) + \ldots + g_2\left(\frac{m}{n}\right)(1 - z_A(m+1)) + \ldots \\ & + g_2\left(1 - \frac{1}{n}\right)(1 - z_A(n)). \end{aligned}$$

Recall that

$$\varphi(f, \rho_A, \rho_B) = p\,\varphi_A(f, \rho_A) + p'\,\varphi_B(f, \rho_B)$$

hence it is enough to compare $\bar{G}_A$ and $\varphi_A(f, \rho_A)$, but we have

$$\begin{aligned} \varphi_A(f, \rho_A) = {} & \sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) g_1\left(\frac{m}{n}\right)(z_A(m+1) - z_A(m)) \\ & + \sum_{m=0}^{n-1} (1 - z_A(m+1)) \int_{m/n}^{(m+1)/n} g_2(x)\,dx \quad \text{with } z_A(0) = 0 \end{aligned}$$

since g_2 is continuous.

Now $f \in F_0$ implies that there exists N such that $n \geqslant N$ gives

$$\left| f\left(\frac{m}{n}\right) - n \int_{m/n}^{(m+1)/n} f(x)\,dx \right| \leqslant \frac{\epsilon}{4L} \quad \text{for } m = 0, \ldots, n-1.$$

It follows that for $n \geqslant N$ we have

$$|\bar{G}_A - \varphi_A(f, \rho_A)| \leqslant \frac{\epsilon}{2}$$

and by defining similary $\bar{G}_B$

$$|\bar{G}_B - \varphi_B(f, \rho_B)| \leqslant \frac{\epsilon}{2}.$$

Hence for $n \geqslant N$

$$\bar{\gamma}_n(\sigma_n, \tau_n) = p\,\bar{G}_A + \hat{p}\,\bar{G}_B \leqslant p\,\varphi_A(f, \rho_A) + \hat{p}\,\varphi_B(f, \rho_B) + \frac{\epsilon}{2}$$

$$\leqslant \varphi(f, \rho_A, \rho_B) + \frac{\epsilon}{2} \leqslant X(p) + \epsilon.$$

So we get the result: $\forall\, \epsilon$, $\exists\, N$ such that for $n \geqslant N$ there exists τ_n strategy of PII satisfying $\bar{\gamma}_n(\sigma_n, \tau_n) \leqslant X(p) + \epsilon$ for all σ_n.

Step 3: $\underline{\lim}_{n\to\infty}\, v_n(p) \geqslant Y(p)$.

Let us choose ρ_A and ρ_B in Q optimal (by compactness) for $Y(p)$. Like in step 2 we shall use ρ to define a strategy σ_n of PI and construct from any τ_n of PII a function f in F such that $\gamma_n(\sigma_n, \tau_n)$ will be near $\varphi(f, \rho_A, \rho_B)$.

Let us define the strategy σ_n of PI such that

$$\rho_C\left(\frac{m}{n}\right) = \text{Prob}_{\sigma_n^C}\{\mathrm{m} \leqslant m\} \qquad m = 1, \ldots, n$$

i.e., $s_m^C(T) = \dfrac{\rho_C(m/n) - \rho_C((m-1)/n)}{1 - \rho_C((m-1)/n)}$ as long as $\rho_C\left(\dfrac{m-1}{n}\right) < 1$.

For each τ_n, we introduce (using Corollary 6) f in F defined by

$$f(x) = t_1(L) \text{ on } \left[0, \frac{1}{n}\right] \text{ and } f(x) = t_m(L) \text{ on } \left]\frac{m-1}{n}, \frac{m}{n}\right] \qquad m = 2, \ldots, n$$

and let

$$u_1(m) = a_{11}\, t_m(L) + a_{12}\, t_m(R),\; u_2(m) = a_{21}\, t_m(L) + a_{22}\, t_m(R),$$
$$m = 1, \ldots, n.$$

The expected payoff for $G_n(p)$, given σ_n, τ_n and $C=A$, is now $\underline{G}_A$ with

$$n\,\underline{G}_A = n\,u_1(1)\,\rho_A\left(\frac{1}{n}\right)+\ldots+(n-m+1)u_1(m)\left(\rho_A\left(\frac{m}{n}\right)-\rho_A\left(\frac{m-1}{n}\right)\right)+\ldots$$

$$+u_1(n)\left(\rho_A(1)-\rho_A\left(\frac{n-1}{n}\right)\right)$$

$$+u_1(1)\left(1-\rho_A\left(\frac{1}{n}\right)\right)+\ldots+u_2(\mathrm{m})\left(1-\rho_A\left(\frac{m}{n}\right)\right)+\ldots+u_2(n)(1-\rho_A(1))$$

On the other hand we have:

$$\varphi_A(f,\rho_A)=u_1(1)\int_0^{1/n}(1-x)\,d\rho_A(x)+\ldots+u_1(m)\int_{((m-1)/n)^+}^{m/n}(1-x)\,d\rho_A(x)+\ldots$$

$$+u_1(n)\int_{(1-(1/n))^+}^{1}(1-x)\,d\rho_A(x)$$

$$+u_2(1)\int_0^{1/n}(1-\rho_A(x))\,dx+\ldots+u_2(m)\int_{(m-1)/n}^{m/n}(1-\rho_A(x))\,dx+\ldots$$

$$+u_2(n)\int_{(1-(1/n))}^{n}(1-\rho_A(x))\,dx.$$

Now it is clear that

$$\left(1-\frac{1}{n}\right)\rho_A\left(\frac{1}{n}\right)\leqslant\int_0^{1/n}(1-x)\,d\rho_A(x)\leqslant\rho_A\left(\frac{1}{n}\right)$$

$$\left(1-\frac{m}{n}\right)\left(\rho_A\left(\frac{m}{n}\right)-\rho_A\left(\frac{m-1}{n}\right)\right)\leqslant\int_{((m-1)/n)^+}^{m/n}(1-x)\,d\rho_A(x)\leqslant$$

$$\leqslant\left(1-\frac{m-1}{n}\right)\left[\rho_A\left(\frac{m}{n}\right)-\rho_A\left(\frac{m-1}{n}\right)\right]$$

$$m=2,\ldots,n$$

$$\left(1-\rho_A\left(\frac{m}{n}\right)\right)\frac{1}{n}\leqslant\int_{(m-1)/n}^{m/n}(1-\rho_A(x))\,dx\leqslant\left(1-\rho_A\left(\frac{m-1}{n}\right)\right)\frac{1}{n}.$$

If follows that

$$|\underline{G}_A - \varphi_A(f, \rho_A)| \leqslant \frac{|u_1(1)|}{n}\rho_A\left(\frac{1}{n}\right) + \sum_{m=2}^{n}\frac{|u_1(m)|}{n}\left(\rho_A\left(\frac{m}{n}\right) - \rho_A\left(\frac{m-1}{n}\right)\right)$$

$$+\frac{1}{n}\sum_{m=1}^{n}|u_2(m)|\left(\rho_A\left(\frac{m}{n}\right) - \rho_A\left(\frac{m-1}{n}\right)\right)$$

$$\leqslant \frac{2L}{n}.$$

Obviously we have a similar result for $\underline{G}_B$ and $\varphi_B(f, \rho_B)$, hence $n \geqslant N = \frac{2L}{\epsilon}$ implies

$$p\,\underline{G}_A + \hat{p}\,\underline{G}_B \geqslant p\,\varphi_A(f, \rho_A) + \hat{p}\,\varphi_B(f, \rho_B) - \epsilon = \varphi(f, \rho_A, \rho_B) - \epsilon \geqslant Y(p) - \epsilon.$$

This proves that

$\forall\, \epsilon > 0\ \exists\, N$ such that for all $n \geqslant N$ there exists σ_n satisfying

$\tilde{\gamma}_n(\sigma_n, \tau_n) \geqslant Y(p) - \epsilon$ for all τ_n.

This achieves Step 3 and the proof of the theorem. Q.E.D.

We conclude this Part IV by showing that a result similar to Theorem 9 holds for the discounted game [see also *Mertens/Zamir*, 1971/1972; *Bewley/Kohlberg*].

More precisely, given $0 < \lambda \leqslant 1$, let $G_\lambda(p)$ be the infinitely repeated game with payoff function $\tilde{\gamma}_\lambda(\sigma, \tau) = \sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1}\gamma_m(\sigma, \tau)$, and denote its value by $\tilde{v}_\lambda(p)$.

Then we have:

Proposition 10: $\lim_{\lambda\to 0} \tilde{v}_\lambda(p)$ *exists and equals* $v(p)$ *on* $[0, 1]$.

Proof: The idea is exactly the same as in Theorem 9, Steps 2 and 3. The modifications are as follows: Define $x_m = 1 - (1-\lambda)^m$. To prove that $\overline{\lim}\ \tilde{v}_\lambda \leqslant X$ we introduce:

$t_m = f(x_{m-1})$ and $\rho_C(x) = z_C(m)$ on $[x_{m-1}, x_m[$

and to prove that $\underline{\lim}\ \tilde{v}_\lambda \geqslant Y$ we define σ such that

$\rho_C(x_m) = \mathrm{Prob}_{\sigma}{}_C(\mathrm{m} \leqslant m)$ and $f(x) = t_m$ on $]x_{m-1}, x_m]$.

The corresponding computations are similar to the previous one and hence omitted. Q.E.D.

5. Maxmin

The main purpose of this part is to prove the following

Theorem 11: $\underline{v}(p)$ *exists and* $\underline{v}(p) = v(p)$ *on* $[0, 1]$.

This part is divided into 5 sections. In the first one we exhibit properties of optimal strategies for PII in $\Gamma(p)$ (Proposition 13). Using this result one can prove in section 2 that PI cannot guarantee more than $v(p)$ (Proposition 20), and also derive in section 3 some characterizations of PI's optimal strategies in $\Gamma(p)$. This will enable us to construct in section 4 a strategy for PI in $G_\infty(p)$ which guarantees $v(p)$ (Proposition 26). In the last section a new formula for $v(p)$ will be obtained.

Let us first recall that, if $u_n(p)$ denotes the value of the n-stage "non-revealing" game (i.e. with complete information and payoff matrix $pA + \hat{p}B$), *Kohlberg* proved that

$$u_\infty(p) \text{ exists and } u_\infty(p) = u_n(p)\ (= u_1(p))\ \forall n \geq 1, \quad \forall p \in [0, 1] \tag{6}$$

and we shall denote this common value by $u(p)$.

Now we may and shall assume without loss of generality that $u(0) = u(1) = 0$ by subtracting from all the payoffs the constant $p\,u(1) + \hat{p}\,u(0)$.

5.1 Study of the Optimal Strategies of PII in $\Gamma(p)$

Recall that from Theorem 9 we have

$$v(p) = \inf_{F_0} \sup_{Q \times Q} \varphi(f, \rho_A, \rho_B).$$

Hence by taking the extreme points in Q, namely the Dirac mass at point x, denoted by δ_x, $x \in [0, 1]$, and the measure 0, we obtain:

$$v(p) = \inf_{F_0} \{p \max_{0 \leq x \leq 1} M_A(f, x) + \hat{p} \max_{0 \leq x \leq 1} M_B(f, x)\}$$

with

$$M_C(f, x) = \varphi_C(f, \delta_x) = (1 - x)\, c_1(f(x)) + \int_0^x c_2(f(y))\, dy$$

where for any y in $\mathbf{R}$ we define $c_i(y) = c_{i1}\, y + c_{i2}\, \hat{y}$, and write c_i for $c_{i1} - c_{i2}$, $i = 1, 2$, $C = A, B$.

This led us to introduce the differential equation

$$(\mathrm{E}_C) \quad (1 - x)(c_{11} - c_{12}) f'(x) - \Delta(C) f(x) + c_{22} - c_{21} = 0$$

where $\Delta(C) = c_{11} + c_{22} - c_{12} - c_{21}$, $C = A, B$. Hence it is clear that if f satisfies (E_C) at x_0 then

$$\frac{d}{dx} M_C(f, x_0) = 0.$$

We denote by $T(C)$ the set of PII's optimal strategies in the game C, $C = A, B$. We shall often write t for the strategy $(t, \hat{t})$, no confusion will result.

Finally we define F_1 to be the set of functions f in F satisfying one of the 3 following properties:

(i) $f(x) = t \quad \forall\, x \in [0, 1]$ *with* $t \in T(A) \cup T(B)$

(ii) $f(x) = f_A(x)$ *on* $[0, x_B]$
$\qquad = f_A(x_B)$ *on* $[x_B, 1]$

with f_A *a solution of* (E_A) *on* $[0, x_B]$ *and* $f_A'(x) \cdot (a_{11} - a_{12}) \geqslant 0$ *on* $[0, x_B]$

(iii) *the dual statement with* f_B *and* E_B.

Proposition 13: $\forall\, p \in [0, 1]$, PII *has an optimal strategy for* $\Gamma(p)$ *in* F_1.

Note that this proposition implies the existence of optimal strategies for PII.

The proof goes from lemma 14 to lemma 19. We will first eliminate trivial cases.

Lemma 14: If $T(A) \cap T(B) \neq \emptyset$, *then* $\forall\, t \in T(A) \cap T(B)$, $f(x) = t$ *on* $[0, 1]$ *is optimal (and* $v_1(p) = v(p) = 0\ \forall\, p \in [0, 1]$).

Proof: PII can guarantee the vector payoff $(0, 0)$ by playing t i.i.d. with $t \in T(A) \cap T(B)$ (i.e., his payoff in both games will be less than 0, for all σ). On the other hand, by (6), PI can obviously obtain 0 for all p, hence the result.

Q.E.D.

We shall now assume $T(A) \cap T(B) = \emptyset$. Let t_A be a closest point to $T(B)$ in $T(A)$ and symmetrically for t_B. We denote by (t_A, t_B) the closed interval between t_A and t_B.

Now we introduce the following sub-intervals of (t_A, t_B);

$$TT = \{t \in (t_A, t_B); a_1(t) \geqslant a_2(t), b_1(t) \geqslant b_2(t)\}$$

$$\mathring{TT} = \{t \in (t_A, t_B); a_1(t) > a_2(t), b_1(t) > b_2(t)\}.$$

Note that if PII plays $t \in TT$ (resp. $\mathring{TT}$) a best reply (resp.; the only best reply) of PI is to play T in both games. TB, $\mathring{TB}$, BT, $\mathring{BT}$, BB, $\mathring{BB}$ are define similarly.

Obviously the extreme points of these intervals belong to $\{t_A, t_B, y_A, y_B\}$ where y_C satisfies $c_1(y_C) = c_2(y_C)$, $C = A, B$. Finally let $L_C(f) = \max_{0 \leqslant x \leqslant 1} M_C(f, x)$ be the maximum payoff PI can obtain against f in $\Gamma(p)$, if C is the true game; we shall write $L_C(t)$ for $L_C(f)$ where $f(x) = t$ on $[0, 1]$. To prove Proposition 13 we shall show that for any $g \in F_0$ and any $p \in [0, 1]$ there exists f in F_1 such that:

$$p L_A(f) + \hat{p} L_B(f) \leqslant p L_A(g) + \hat{p} L_B(g).$$

In order to do this we shall first compare $L_C(g)$ to $L_C(t)$ with t in (t_A, t_B).

Lemma 15: If $L_{C'}(g) \geqslant L_{C'}(t_C)$ for some $C = A$ or B and $\{C, C'\} = \{A, B\}$ then

$f(x) = t_C$ on $[0, 1]$, *dominates* g.

Proof: By definition of t_C we have $L_C(t_C) = 0$. Moreover PI can always guarantee 0 in both games (by (6)), hence $L_C(g) \geqslant L_C(t_C)$. Q.E.D.

Since $L_C(\cdot)$ is continuous on (t_A, t_B), we can now assume that $L_C(g) \in \{L_C(t);\ t \in (t_A, t_B)\}$.

Lemma 16: If $L_C(g) \in L_C(\overset{\circ}{TT} \cup \overset{\circ}{BB})$ for some $C = A$ or B, then $\forall\ p, \exists\ t \in \in \{t_A, t_B, y_A, y_B\}$ such that $f(x) = t$ on $[0, 1]$ dominates g.

Proof: Assume for example that $L_A(g) \in L_A(\overset{\circ}{TT})$ and let $TT = [\underline{t}, \overline{t}]$ with $\underline{t} < \overline{t}$.
Remark first that by definition of t_A and t_B, $\overset{\circ}{TT} \neq \phi$ implies $a_1 \cdot b_1 < 0$.
Hence there exists $p_T \in]0, 1[$ such that

$$p_T a_{11} + \hat{p}_T b_{11} = p_T a_{12} + \hat{p}_T b_{12}.$$

It follows that PI can guarantee this quantitiy at p_T and that he cannot get more if PII plays t in TT, thus

$$v_1(p_T) = u(p_T)$$

which gives (Lemma 2)

$$v(p_T) = u(p_T). \tag{7}$$

Suppose now that $L_A(\underline{t}) \leqslant L_A(g) \leqslant L_A(\overline{t})$. Then $\underline{t}$ and $\overline{t}$ being optimal at p_T and L_C being monotonic implies by (7) that $L_B(\overline{t}) \leqslant L_B(g)$. Hence again by (7) we obtain that g is dominated by $\overline{t}$ on $[0, p_T]$ and by $\underline{t}$ on $[p_T, 1]$. Q.E.D.

Lemma 17: There is no optimal g such that

$$L_A(g) \in L_A(\overset{\circ}{BT}) \text{ and } L_B(g) \in L_B(\overset{\circ}{TB}).$$

Proof: Assume that

$$L_A(g) = L_A(t) \text{ and } L_B(g) = L_B(t') \text{ with } t \in \overset{\circ}{BT} \text{ and } t' \in \overset{\circ}{TB}. \tag{8}$$

Suppose that $t' > t$, hence there exists $t_0 \in TT \cup BB$ such that $t' > t_0 > t$.

Remark that since $\overset{\circ}{BT}$ and $\overset{\circ}{TB}$ are not empty this implies that $a_1 \cdot b_1$ and $a_2 \cdot b_2$ are strictly negative, hence, as in Lemma 16, there exists $p_0 \in]0, 1[$ such that

$$v(p_0) = u(p_0)$$

and t_0 is optimal at p_0.

Now if $L_A(t') > L_A(t_0) > L_A(t) = L_A(g)$ then we have

$$L_B(g) > L_B(t_0) > L_B(t')$$

which contradicts (8), and otherwise

$$L_A(t) > L_A(t_0) > L_A(t')$$

implies $L_B(t') > L_B(t_0) > L_B(t)$, hence t_0 dominates g. Q.E.D.

Thus we are left with the case where

$$L_A(g) \in L_A(TB) \text{ (or symmetrically).} \tag{9}$$

We shall now introduce a class of functions (f_K) in F_1 such that, for each g satisfying (9), there will exist some K with

$$L_A(f_K) = L_A(g) \text{ and } L_B(f_K) \leqslant L_B(g).$$

First we may and shall assume that $a_1 \geqslant 0$. Let $N_C(f) = \{0 \leqslant x \leqslant 1; M_C(f, x) = L_C(f)\}$ and write $TB = [\alpha, \beta]$. Then we have:

Lemma 18: $\forall K \in [\alpha, \beta], \exists f_K \in F_1$ *such that*

(i) $f_K(0) = K$ *and* f_K *is non decreasing*

(ii) $N_A(f_K) = [0, x_B]$

(iii) $f_K(x) \equiv \beta$ *on* $]x_B, 1]$.

Proof:

1) If $a_1 = 0$, then $TB \subset (t_A, t_B)$ implies $TB = \{y_A\}$. Thus let $f(x) \equiv y_A$.

2) If $a_1 > 0$, we introduce the following:

– if $\Delta(A) = a_{11} + a_{22} - a_{12} - a_{21} \neq 0$

$$\tilde{f}_K(x) = y_A + (K - y_A)(1 - x)^{-(\Delta(A)/(a_{11} - a_{12})}$$

– if $\Delta(A) = 0$

$$\tilde{f}_K(x) = \frac{a_{22} - a_{21}}{a_{11} - a_{12}} \operatorname{Log}(1 - x) + K$$

and $f_K(x) = \min(\tilde{f}_K(x), \beta)$.

Now it is easy to see that $\tilde{f}_K(x)$ is increasing and satisfies (E_A). Letting $x_B = \min(1, \{x; \tilde{f}_K(x) \geqslant \beta\})$ it follows that if $x_B < 1$, then $a_1(f(x)) > a_2(f(x))$ on $]x_B, 1]$ which gives (ii). Q.E.D.

Thus given g satisfying (9), let K be such that

$$L_A(g) = L_A(K)$$

and choose f_K according to the previous lemma. It follows from (i) and (ii) that

$$L_A(f_K) = M_A(f_K, x) \quad \forall x \in N_A(f)$$

hence

$$L_A(f_K) = M_A(f_K, 0) = M_A(K, 0) = L_A(K) = L_A(g).$$

Thus it remains to prove the following:

Lemma 19: $L_B(f_K) \leqslant L_B(g)$.

Proof: (During this proof we shall write f for f_K.)

1) If $a_1 = 0$ (recall that in this case $TB = \{y_A\}$), then we can assume $a_2 > 0$, hence $b_2 < 0$.
Now $L_A(g) \leqslant L_A(f)$ implies $\int_0^1 g(x)\,dx \leqslant y_A$, but then

$$L_B(f) = M_B(f, 1) \leqslant M_B(g, 1) \leqslant L_B(g).$$

2) Assume now $a_1 > 0$. If $b_2 \geqslant 0$, then again $TB = \{y_B\}$ and $b_1 < 0$.
Now $L_A(g) \leqslant L_A(f)$ implies $g(0) \leqslant y_B$, but then

$$L_B(f) = M_B(f, 0) \leqslant M_B(g, 0) \leqslant L_B(g).$$

3) There remains the case where $a_1 > 0$ and $b_2 < 0$. Let us first prove the following.

Claim: If $a_2 \leqslant 0$, then $g(x) \leqslant f(x)$ on $[0, x_B]$.

In fact let $x_0 = \inf\{x; g(x) > f(x)\}$ and assume $x_0 < x_B$. On $[x_0, x_0 + \epsilon] \subset [0, x_B]$, $g(x) - f(x)$ takes its maximum at some $\tilde{x}$. Recall that

$$M_A(g, \tilde{x}) \leqslant L_A(g) = L_A(f) = M_A(f, \tilde{x})$$

which gives

$$(1 - \tilde{x})(a_{11} - a_{12})(g(\tilde{x}) - f(\tilde{x})) \leqslant \int_0^{\tilde{x}} (a_{22} - a_{21})(g(x) - f(x))\, dx.$$

Thus, a fortiori, by the choice of x_0 and $\tilde{x}$

$$(1 - \tilde{x})(a_{11} - a_{12})(g(\tilde{x}) - f(\tilde{x})) \leqslant \epsilon\,(a_{22} - a_{21})(g(\tilde{x}) - f(\tilde{x}))$$

hence a contradiction as ϵ goes to zero. Q.E.D.

We now split the study into four subcases:

3.1) $a_2 \leqslant 0, b_1 \geqslant 0$.

Obviously we have $N_B(f) \subset [x_B, 1]$. (Recall that the range of f is in TB.) Moreover, since $a_2 \leqslant 0$, $\beta = y_B$ or $\beta = 1$.

- if $\beta = 1$, it follows from 3) that $g \leqslant f$ on $[0, 1]$. Hence $M_B(g, 1) \geqslant M_B(f, 1) = L_B(f)$.
- if $\beta = y_B$, then $N_B(f) = [x_B, 1]$.

Now either $g \leqslant f$ on $[0, 1]$ and we conclude as above, or there exists some $\bar{x} \in [x_B, 1]$ such that $f(x) \geqslant g(x)$ on $[0, \bar{x}]$ and $f(\bar{x}) = g(\bar{x})$. But this implies

$$M_B(g, \bar{x}) \leqslant M_B(f, \bar{x}) = L_B(f).$$

3.2) $a_2 \leqslant 0, b_1 < 0$.

If $N_B(f) \cap [0, x_B] \neq \emptyset$, choose some $\bar{x}$ inside. Now by 3) we get

$$M_B(g, \bar{x}) \leqslant M_B(f, \bar{x}) = L_B(f).$$

Otherwise we conclude as in 3.1).

3.3) $a_2 > 0, b_1 > 0$.

As in 3.1) we have $N_B(f) \subset [x_B, 1]$. Now either $f(x_B) < g(x_B)$ but

$$M_A(g, x_B) \leqslant L_A(g) = L_A(f) = M_A(f, x_B)$$

implies $\int_0^{x_B} f(x)\,dx > \int_0^{x_B} g(x)\,dx$, hence

$$M_B(g, x_B) \geqslant M_B(f, x_B) = L_B(f).$$

Or let $[\underline{x}, \bar{x}]$ be a maximal interval such that $\underline{x} \leqslant x_B \leqslant \bar{x}$ and $f(x) \geqslant g(x)$ on $[\underline{x}, \bar{x}]$. It follows that $\int_0^{\bar{x}} f(x)\,dx \geqslant \int_0^{\bar{x}} g(x)\,dx$ and

$$M_B(g, \bar{x}) \geqslant M_B(f, \bar{x}) = L_B(f).$$

3.4) $a_2 > 0, b_1 < 0$.

- if $0 \in N_B(f), M_A(g, 0) \leqslant L_A(f)$ implies $g(0) \leqslant f(0)$ hence $L_B(g) \geqslant L_B(f)$.
- if $N_B(f) \cap [x_B, 1] \neq \emptyset$.

First of all let us prove that $x_B < 1$ implies $\beta \neq y_B$.
Otherwise $x_B \in N_B(f)$, but $a_1(f(x_B)) - a_2(f(x_B)) > 0$ and $b_1(f(x_B)) =$
$= b_2(f(x_B))$.

Now since f satisfies (E_A) on $[0, x_B]$ we obtain

$$\frac{d}{dx} M_B(f, x) = \left(\frac{b_{11} - b_{12}}{a_{11} - a_{12}} \right) (a_1(f(x)) - a_2(f(x))) + b_2(f(x)) - b_1(f(x)) \tag{10}$$

thus $(d/dx) M_B(f, x_B) < 0$, a contradiction.

Hence we can assume that $1 \in N_B(f)$ and it remains to show that

$$\int_0^1 f(x)\,dx \geqslant \int_0^1 g(x)\,dx.$$

If $g \leqslant f$ on $[0, x_B]$ the inequality follows.

Otherwise let $\bar{x} = \sup\{0 \leqslant x \leqslant x_B; g(x) > f(x)\}$, and $M_A(g, \bar{x}) \leqslant L_A(g) =$
$= L_A(f) = M_A(f, \bar{x})$ implies $\int_0^{\bar{x}} f(x)\,dx \geqslant \int_0^{\bar{x}} g(x)\,dx$ hence the required inequality.

- Finally there remains the case where $N_B(f) \subset]0, x_B[$.

Obviously this implies the function (10) to be increasing, hence

$$(b_{11} - b_{12})(a_{21} - a_{22}) < (a_{11} - a_{12})(b_{22} - b_{21}). \tag{11}$$

Let now $\bar{x} \in N_B(f)$. Since $\bar{x} \in N_A(f)$ we have

$$(a_{21} - a_{22}) \int_0^{\bar{x}} (f(x) - g(x))\, dx \geqslant (1 - \bar{x})(a_{11} - a_{12})(g(\bar{x}) - f(\bar{x}))$$

hence if $g(\bar{x}) \geqslant f(\bar{x})$ (11) implies:

$$M_B(g, \bar{x}) \geqslant M_B(f, \bar{x}). \tag{12}$$

On the other hand if $g(\bar{x}) < f(\bar{x})$, let $\underline{x}$ be such that $[\underline{x}, \bar{x}]$ is a maximal interval on which $g \leqslant f$. Since $\underline{x} \in N_A(f)$ this implies $\int_0^{\bar{x}} f(y)\, dy \geqslant \int_0^{\bar{x}} g(y)\, dy$, hence (12) by the choise of $\underline{x}$.

Q.E.D.

This completes the proof of Proposition 13.

5.2 PI Cannot Guarantee more than $v(p)$

From Proposition 13 we can already deduce

Proposition 20: PI *cannot guarantee more than* $v(p)$

The proof will follow from Corollary 21 and Lemma 23.

Let us first introduce on $Q \times \{(t_1, t_2); 1 \geqslant t_2 \geqslant t_1 \geqslant 0\}$

$$\psi_C^+(\mu, t_1, t_2) = c_1(t_1)\mu(t_1) + \int_{t_1^+}^{t_2} c_1(y)\, d\mu(y) + c_2(t_2)(1 - \mu(t_2)), \ C = A, B \tag{13}$$

Now let

$$\psi^+(p, \mu_A, \mu_B, t_1, t_2) = p\, \psi_A^+(\mu_A, t_1, t_2) + \hat{p}\, \psi_B^+(\mu_B, t_1, t_2) \tag{14}$$

$$Z^+(p, \mu_A, \mu_B, t_1) = \inf_{1 \geqslant t_2 \geqslant t_1} \psi^+(p, \mu_A, \mu_B, t_1, t_2). \tag{15}$$

$$Z^+(p) = \inf_{0 \leqslant t \leqslant 1} \sup_{\substack{\mu_A \in Q \\ \mu_B \in Q}} Z^+(p, \mu_A, \mu_B, t). \tag{16}$$

Lemma 21: PI *cannot guarantee more than* $Z^+(p)$.

Proof: The idea of the proof is as follows. Assume that PII plays t_1 i.i.d. Knowing σ he can compute the probability $\mu_C(t_1)$ that PI will eventually play Top in game C. He then uses t_1 a long enough time to reach this probability and then increases his frequency by now playing $t_1 + \epsilon$ i.i.d. Doing the same computation at this new level he obtains a new absorbing payoff; then he increases again and so on up to some t_2 where he plays i.i.d. from this time on. He then gets at t_2 a non absorbing payoff, with probability $1 - \mu_C(t_2)$ in game C hence ψ^+ by computing the total payoff.

So first choose R large in N and for any $t_1 \in [0, 1]$ let $r_0 \in \mathrm{N}, 0 \leqslant r_0 \leqslant R$ such that

$$\frac{r_0}{R} \leqslant t_1 < \frac{r_0 + 1}{R}. \tag{17}$$

Given $\epsilon > 0$ and a strategy σ of PI, we define inductively three sequences of strategies for PII, numbers, and probabilities as follows:

$$\tau_0 : \text{play} \left(\frac{r_0}{R}, \left(\frac{\hat{r}_0}{R}\right)\right) \text{ i.i.d.}$$

$$\tilde{P}_C(0) = \mathrm{Prob}_{\sigma^C, \tau_0} (\mathrm{m} < \infty) \quad C = A, B.$$

Now n_0 and $P_C(0), C = A, B$, are such that

$$P_C(0) = \mathrm{Prob}_{\sigma^C, \tau_0} (\mathrm{m} \leqslant n_0) > \tilde{P}_C(0) - \epsilon.$$

Given $\tau_{r-1}, n_{r-1}, 0 \leqslant r - 1 < R - r_0$, let

$$\tau_r\text{: play } \tau_{r-1} \text{ up to stage } n_{r-1} \text{ and then} \left(\frac{r_0 + r}{R}, \left(\widehat{\frac{r_0 + r}{R}}\right)\right) \tag{18}$$

$$\tilde{P}_C(r) = \mathrm{Prob}_{\sigma^C, \tau_r} (\mathrm{m} < \infty) \quad C = A, B \tag{19}$$

and $n_r, P_C(r)$ such that

$$n_r \geqslant n_{r-1} \text{ and } P_C(r) = \mathrm{Prob}_{\sigma^C, \tau_r} (\mathrm{m} \leqslant n_r) > \tilde{P}_C(r) - \epsilon \quad C = A, B. \tag{20}$$

Let us majorize $\bar{\gamma}_n^A(\sigma, \tau_r)$, the average expected payoff for the first n stages given σ, τ_r and A, with $n > n_r$. Note that the payoff is absorbing with probability $P_A(0)$ before n_0, with value $a_1(r_0/R)$, then absorbing with probability $P_A(1) - P_A(0)$ between n_0 and n_1 with value $a_1(r_0 + 1/R)$ and so on.

Hence we have

$$
\begin{aligned}
\tilde{\gamma}_n^A(\sigma,\tau_r) \leqslant\ & P_A(0)\cdot a_1\left(\frac{r_0}{R}\right) + 2L\frac{n_0}{n}P_A(0) \\
& + (P_A(1)-P_A(0))\,a_1\left(\frac{r_0+1}{R}\right) + 2L\frac{n_1}{n}(P_A(1)-P_A(0)) + \dots \\
& + (P_A(r)-P_A(r-1))\,a_1\left(\frac{r_0+r}{R}\right) + 2L\frac{n_r}{n}(P_A(r)-P_A(r-1)) \\
& + (\tilde{P}_A(r)-P_A(r))\cdot L \\
& + (1-P_A(r))\,a_2\left(\frac{r_0+r}{R}\right) + 2L\frac{n_r}{n}(1-P_A(r)) \\
& + (\tilde{P}_A(r)-P_A(r))\cdot L.
\end{aligned}
$$

Thus for $n \geqslant N_r = (n_r/\epsilon)$ we obtain by (20)

$$\tilde{\gamma}_n^A(\sigma,\tau_r) \leqslant S_A(r) + 4\epsilon L \tag{21}$$

with

$$
\begin{aligned}
S_A(r) = P_A(0)\,a_1\left(\frac{r_0}{R}\right) + \sum_{l=1}^{r}(P_A(l)-P_A(l-1))\,a_1\left(\frac{r_0+l}{R}\right) \\
+ (1-P_A(r))\,a_2\left(\frac{r_0+r}{R}\right)
\end{aligned}
$$

and similarly for $\tilde{\gamma}_n^B(\sigma,\tau_r)$.

Define now $\bar{\mu}_C \in Q$ by

$$
\begin{aligned}
\bar{\mu}_C(t) &= P_C(l) \text{ for } \frac{r_0+l}{R} \leqslant t \leqslant \frac{r_0+l+1}{R} \\
&= 0 \qquad \text{for } t < \frac{r_0}{R},\ C = A, B.
\end{aligned}
$$

But then

$$S_C(r) = \int_{r_0/R}^{(r_0+r)/R} c_1(y)\,d\mu(y) + c_2\left(\frac{r_0+r}{R}\right)\left(1-\mu_C\left(\frac{r_0+r}{R}\right)\right), \qquad C = A, B.$$

By (15) and (17) it is easy to see that there exists r^*, $r^* + r_0 \leqslant R$ such that

$$|Z^+(p, \bar{\mu}_A, \bar{\mu}_B, t_1) - (p\, S_A(r^*) + \hat{p}\, S_B(r^*))| \leqslant \frac{2L}{R}.$$

Hence $R \geqslant (1/\epsilon)$ and $n \geqslant N_{r^*}$ imply by (21) that

$$\bar{\gamma}_n(\sigma, \tau_{r^*}) \leqslant Z^+(p, \bar{\mu}_A, \bar{\mu}_B, t_1) + 6\epsilon L$$

which means that

$$\forall\, \delta > 0, \forall\, \sigma, \forall\; t \in [0, 1], \exists\, \mu_A \text{ and } \mu_B \text{ in } Q, \exists\, \tau, \exists\, N \text{ such that for } n \geqslant N, \quad (22)$$

$$\bar{\gamma}_n(\sigma, \tau) \leqslant Z^+(p, \mu_A, \mu_B, t) + \delta.$$

This prove that PI cannot guarantee more than

$$Z^+(p) = \inf_{0 \leqslant t \leqslant 1} \sup_{\substack{\mu_A \in Q \\ \mu_B \in Q}} Z^+(p, \mu_A, \mu_B, t)$$

Q.E.D.

Obviously starting from t_1, PII can also decrease his frequency, hence it is natural to introduce:

$$\psi_C^-(\mu, t_1, t_2) = c_1(\hat{t}_1)\, \mu(t_1) + \int_{t_1^+}^{t_2} c_1(\hat{y})\, d\mu(y) + c_2(\hat{t}_2)(1 - \mu(t_2)), \qquad C = A, B$$

$$\psi^-(p, \mu_A, \mu_B, t_1, t_2) = p\, \psi_A^-(\mu_A, t_1, t_2) + \hat{p}\, \psi_B^-(\mu_B, t_1, t_2)$$

$$Z^-(p, \mu_A, \mu_B, t_1) = \inf_{1 \geqslant t_2 \geqslant t_1} \psi^-(p, \mu_A, \mu_B, t_1, t_2).$$

$$Z^-(p) = \inf_{0 \leqslant t \leqslant 1} \sup_{\substack{\mu_A \in Q \\ \mu_B \in Q}} Z^-(p, \mu_A, \mu_B, t)$$

and to define

$$Z(p) = \min(Z^+(p), Z^-(p)).$$

As an immediate consequence of the previous lemma we now obtain:

Corollary 21: PI *cannot guarantee more than* $Z(p)$.

Lemma 22: $\forall\ \delta > 0, \exists\ f \in F_0$ *δ-optimal for* PII *in* $\Gamma(p)$ *satisfying either*

(i) *f is strictly increasing on* $[0, 1]$ *or*
(ii) *f is strictly decreasing on* $[0, 1]$.

Proof: By Proposition 13 there exists $f \in F_1$ optimal in $\Gamma(p)$ and monotonic (since the solutions of (E_C) are monotonic). Moreover f is strictly monotonic on $[0, x_0]$ and then constant. Assume f increasing. If $f(x) = 1$ on $[x_0, 1]$ let $f_\epsilon(x) = \min(f(x), g_1(x))$ with $g_1(x) = (1 - \epsilon) + \epsilon x$ on $[0, 1]$. Otherwise define:

$$g_2(x) = f(x_0) + \epsilon_1 \left(\frac{x - x_0}{1 - x_0}\right) \text{ on } [x_0, 1]$$

$$= 0 \qquad\qquad \text{on } [0, x_0] \text{ with } \epsilon_1 = \min(\epsilon, 1 - f(x_0))$$

and let $f_\epsilon(x) = \max(f(x), g_2(x))$.

In both cases it follows that:

$$|M_A(f, x) - M_A(f_\epsilon, x)| \leqslant (1 - x) L\, |f(x) - f_\epsilon(x)| + L \int_0^x |f(y) - f_\epsilon(y)|\, dy$$

hence if $x < 1 - \epsilon$, since $|f(y) - f_\epsilon(y)| < \epsilon$ on $[0, 1 - \epsilon]$ we have

$$|M_A(f, x) - M_A(f_\epsilon, x)| \leqslant L\, \epsilon$$

and if $x > 1 - \epsilon$,

$$|M_A(f, x) - M_A(f_\epsilon, x)| \leqslant 3L\epsilon$$

so that f_ϵ is δ-optimal for $\epsilon < \delta/3L$. Q.E.D.

Lemma 23: $Z(p) \leqslant v(p)$.

Proof: Let $\delta > 0$. By Lemma 22 there is f in F δ-optimal for PII in $\Gamma(p)$ that is strictly monotonic. Assume that f is a one to one increasing function from $[0, 1]$ to $[\underline{t}, \bar{t}]$. Now we shall prove that if $Z^+(p, \underline{t}) = \sup_{\substack{\mu_A \in Q \\ \mu_B \in Q}} Z^+(p, \mu_A, \mu_B, \underline{t})$ then

$Z^+(p, \underline{t}) \leqslant v(p) + \delta$.

Given $t \in [\underline{t}, \bar{t}]$ let us define $G_t(y) = 1_{]t,1]}(y)$. For any increasing function G from $[0, 1]$ to $[0, 1]$ we introduce

$$W(G) = p\,[a_1(\underline{t})\,\mu_A(\underline{t}) + \int_{\underline{t}^+}^{1} a_1(y)\,d\mu_A(y)\,(1 - G(y)) + \int_{\underline{t}}^{1} a_2(y)\,(1 - \mu_A(y))\,dG(y)]$$
$$+ \hat{p}\,[b_1(\underline{t})\,\mu_B(\underline{t}) + \int_{\underline{t}^+}^{1} b_1(y)\,d\mu_B(y)\,(1 - G(y)) + \int_{\underline{t}}^{1} b_2(y)\,(1 - \mu_B(y))\,dG(y)].$$

It follows that for any μ_A, μ_B in Q we have by (15)

$$Z^+(p, \mu_A, \mu_B, \underline{t}) = \inf_{\underline{t} \leq t \leq 1} W(G_t). \tag{23}$$

But now G defined by

$$\begin{aligned} G(t) &= 0 && \text{on } [0, \underline{t}] \\ &= f^{-1}(t) && \text{on } [\underline{t}, \bar{t}] \\ &= 1 && \text{on } [\bar{t}, 1] \end{aligned}$$

is in the closed convex hull of $\{G_t;\, t \in [\underline{t}, 1]\}$. Hence (23) implies

$$Z^+(p, \mu_A, \mu_B, \underline{t}) \leq W(G). \tag{24}$$

Let us define $\rho_C \in Q$ by

$$\rho_C([0, t]) = \mu_C([0, f(t)]) \quad \forall\, t \in [0, 1], \quad \forall\, C = A, B.$$

The change of variable $x = G(y)$ now gives in $W(G)$

$$\begin{aligned} W(G) &= p\,a_1(f(0))\,\rho_A(0) + \int_{0^+}^{1} a_1(f(x))\,(1 - x)\,d\rho_A(x) \\ &\quad + \int_{0}^{1} a_2(f(x))\,(1 - \rho_A(x))\,dx \\ &\quad + \hat{p}\,b_1(f(0))\,\rho_B(0) + \int_{0^+}^{1} b_1(f(x))\,(1 - x)\,d\rho_B(x) \end{aligned} \tag{25}$$

$$+\int_0^1 b_2\,(f(x))\,(1-\rho_B(x))\,dx$$
$$=\varphi\,(f,\rho_A,\rho_B)$$

hence, since f is δ-optimal in $\Gamma(p)$, (24) and (25) imply

$$Z^+(p,\mu_A,\mu_B,\bar{t})\leqslant v(p)+\delta$$

which achieves the proof. Q.E.D.

5.3 Other Results on $\Gamma(p)$

In this section we state some corollaries which follow from the proof of Proposition 13 and which will be needed for the construction of an optimal strategy of PI in the next section.

We first prove that for some optimal strategies of both players in $\Gamma(p)$ the payoff during the play is constant from time 0 to time 1.

Corollary 24: Let $g(x)$ be the payoff "at time x" in $\Gamma(p)$, hence

$$g(x)=p\,[\int_0^x a_1\,(f(y))\,d\rho_A(y)+a_2\,(f(x))\,(1-\rho_A(x))]$$

$$+\hat{p}'\,[\int_0^x b_1\,(f(y))\,d\rho_B(y)+b_2\,(f(x))\,(1-\rho_B(x))].$$

Then there exists f in F_1 and ρ optimal strategies in $\Gamma(p)$ such that

$$g(x)=v(p)\quad\forall\,x\in[0,1].$$

Proof: If $f\equiv t$, then for each ρ optimal we must have:
- if $a_1(t)>a_2(t)$, $\rho_A(0)=1$,
- if $a_1(t)<a_2(t)$, $\rho_A([0,1[)=0$ (hence we can choose $\rho_A\equiv 0$).

Now if $a_1(t)=a_2(t)$, the payoff in game A is independent of ρ_A. The similar result for B proves that $g(x)$ is constant, hence equals $v(p)$, $\forall\,x\in[0,1]$.

Assume now that f satisfy (ii) and is strictly increasing on $[0,x_B]$. Using (10) we obtain that:

either $N_B(f)\subset\{0\}\cup[x_B,1]$, or (26)

if $x\in N_B(f)\cap\,]0,x_B[$ then f satisfies (E_B) at x.

Integrating by parts, write

$$g(x) = g(0) + \int_{0^+}^{x} p\,[(a_1(f(y)) - a_2(f(y)))\,d\rho_A(y)$$
$$+ (a_{21} - a_{22}) f'(y)(1 - \rho_A(y))\,dy]$$
$$+ \hat{p}\,[(b_1(f(y)) - b_2(f(y)))\,d\rho_B(y)$$
$$+ (b_{21} - b_{22}) f'(y)(1 - \rho_B(y))\,dy]. \tag{27}$$

Thus, using the fact that f satisfies E_A on $[0, x_B[$ and (26) we obtain, replacing in (27)

$$g(x) = g(0) + \int_{0^+}^{x} f'(y)\,dm(y) \tag{28}$$

with

$$dm(x) = p\,[(1-x)(a_{11} - a_{12})\,d\rho_A(x) + (a_{21} - a_{22})(1 - \rho_A(x))\,dx] \tag{29}$$
$$+ \hat{p}\,[(1-x)(b_{11} - b_{12})\,d\rho_B(x) + (b_{21} - b_{22})(1 - \rho_B(x))\,dx].$$

On the other hand, since by definition 8

$$\varphi(f, \rho_A, \rho_B) = \int_0^1 f(x)\,dm(x) + \int_0^1 p\,((1-x)\,a_{12}\,d\rho_A(x) + a_{22}(1 - \rho_A(x))\,dx)$$
$$+ \hat{p}\,((1-x)\,b_{12}\,d\rho_B(x) + b_{22}(1 - \rho_B(x))\,dx)$$

the fact that f is a best reply to ρ_A and ρ_B implies

$$m \equiv 0 \qquad \text{on } \{x; 1 > f(x) > 0\} \tag{30}$$
$$m \text{ positive} \qquad \text{on } \{x; f(x) = 0\}$$
$$m \text{ negative} \qquad \text{on } \{x; f(x) = 1\}.$$

In particular m is zero on $]0, x_B[$ hence by (28)

$$g(x) = g(0) \text{ on } [0, x_B[. \tag{31}$$

If $x_B < 1$ then f is constant on $[x_B, 1]$, hence $g(x)$ is also constant.

Hence it remains to prove the continuity of g at x_B, but either $D_B(f, x_B) = 0$ and we use (27) to get the result, or ρ_A and ρ_B are absolutely continuous at x_B and the result follows from continuity. Q.E.D.

We now use the fact that an optimal strategy for PI is a best response to any optimal strategy for PII to obtain the following.

Corollary 25: There exist f and ρ optimal strategies in $\Gamma(p)$ satisfying Corollary 24 such that either

$$p(a_{11} - a_{12})\, d\rho_A(x) + \hat{p}(b_{11} - b_{12})\, d\rho_B(x) \textit{ is positive as long as } f(x) < 1 \tag{32}$$

$$p(a_{21} - a_{22})(1 - \rho_A(x)) + \hat{p}(b_{21} - b_{22})(1 - \rho_B(x)) \leqslant 0 \quad \textit{on } [0, x_B[\tag{33}$$

$$p(a_{21} - a_{22})(1 - \rho_A(1)) + \hat{p}(b_{21} - b_{22})(1 - \rho_B(1)) = 0 \textit{ if } f(x_B) < 1, \text{ and } \leqslant 0 \textit{ if } f(x_B) = 1 \tag{34}$$

or the dual statements hold.

Proof: If $f \equiv t$ is optimal with $t \neq y_C$ for some C, then $\rho_C(0) = 1$ or $\rho_C(1) = 0$ is the only best response and it follows easily, using (29), that the above statement holds.

Now if $y_A = y_B$, either $a_1 \cdot b_1 > 0$ or $a_2 \cdot b_2 > 0$ and the result follows, or again $\rho_C(0) = 1$ or $\rho_C(1) = 0$ for some C.

Finally assume that f in F_1 satisfies (ii) and is increasing on $[0, x_B]$. If $N_B(f) \supset [0, x_B]$ this implies that $y_A = y_B$ as above. Otherwise ρ_B is atomic on $[0, x_B]$. Hence by (29) for all x in the support of ρ_B such that $f(x) < 1$, we have

$$p(a_{11} - a_{12})\, \rho_A(\{x\}) \geqslant \hat{p}(b_{11} - b_{12})\, \rho_B(\{x\})$$

which gives (32) since $a_1 > 0$.

Using again (2.9) this gives

$$p(a_{21} - a_{22})(1 - \rho_A(x)) + \hat{p}(b_{21} - b_{22})(1 - \rho_B(x)) \leqslant 0 \ \forall x \text{ such that } f(x) < 1.$$

Now assume that $f(x_B) < 1$, then using (29) it follows that as x goes to 1

$$p\,(a_{21} - a_{22})\,(1 - \rho_A\,(x)) + \hat{p}\,(b_{21} - b_{22})\,(1 - \rho_B\,(x)) \text{ goes to } 0.$$

It remains to see that if $f(x_B) = 1$, either $y_B = 1$ and the same result holds, or $\rho\,(1) = \rho\,(x_B)$ thus using (33), (34) holds true.

Q.E.D.

5.5 PI can Guarantee $v\,(p)$

Using the previous results we are now in position to construct a strategy for PI in $G_\infty\,(p)$ proving the following.

Proposition 26: PI *can guarantee* $v\,(p)$ *in* $G_\infty\,(p)$, $\forall\, p \in [0, 1]$.

Proof: Let f and ρ satisfy Corollaries 24 and 25, and assume that (32), (33) and (34) holds. A sketch of the proof is as follows.

We first use f to construct measures μ_C on $[0, 1]$ (considered as PII's move space in $G_\infty\,(p)$) starting from the measures ρ_C on $[0, 1]$ (time space in $\Gamma\,(p)$). It will follow (Lemma 27) that if PII increases his frequency from 0 to some t, and if PI plays Top with probability $d\mu_C\,(s)$ when PII plays s, the payoff in $G_\infty\,(p)$ will majorize the payoff induced by ρ_A, ρ_B and f from time 0 to $x = f^{-1}\,(t)$ in $\Gamma\,(p)$. By Corollary 24 the last one is $v\,(p)$. We shall then introduce a mixture of "Big Match" strategies (see Proposition 29) to show that PI can in fact "block" at level s with probability $d\mu_C\,(s)$. Finally we shall prove, using Corollary 25, that PII cannot do better than using the above "monotonic" strategy.

Let us denote by $[\underline{t}, \bar{t}]$ the range of f and define μ_A, μ_B in Q by:

$$\mu_C\,(0) = \rho_C\,(0),\ \mu_C\,(f(x)) = \rho_C\,(x) \text{ on }]0, x_B[, \tag{35}$$

$$\mu_C\,(t) = \mu_C\,(\bar{t}) = \rho_C\,(1) \qquad \text{for } t \geq \bar{t} = f\,(x_B).$$

Lemma 27: Let $w = W\,(\mu_A, \mu_B)$ *be defined by*

$$w\,(t) = p\,[\int_0^t a_1\,(y)\,d\mu_A\,(y) + a_2\,(t)\,(1 - \mu_A\,(t))] \tag{36}$$

$$+ \hat{p}\,[\int_0^t b_1\,(y)\,d\mu_B\,(y) + b_2\,(t)\,(1 - \mu_B\,(t))]$$

then

$$\psi^+\,(p, \mu_A, \mu_B, t_1, t) \geq w\,(t) \geq v\,(p) \quad \forall\, t \geq t_1 \geq 0. \tag{37}$$

Proof: First if $\underline{t} > 0$, then by (29) $p\, a_1(y)\, \mu_A(0) + \hat{p}\, b_1(y)\, \mu_B(0)$ is independent of y and $p\, a_2(y)(1 - \mu_A(0)) + \hat{p}\, b_2(y)(1 - \mu_B(0))$ is decreasing with respect to y. Hence

– if $t \leqslant \underline{t}$

$$w(t) \geqslant p\,[a_1(\underline{t})\, \mu_A(t) + a_2(\underline{t})(1 - \mu_A(\underline{t}))]$$
$$+ \hat{p}\,[b_1(\underline{t})\, \mu_B(\underline{t}) + b_2(\underline{t})(1 - \mu_B(\underline{t}))]$$

– if $t \geqslant \underline{t}$

$$w(t) = p\,[a_1(\underline{t})\, \mu_A(\underline{t}) + \int_{\underline{t}^+}^{t} a_1(y)\, d\mu_A(y) + a_2(t)(1 - \mu_A(t))] \tag{38}$$
$$+ \hat{p}\,[b_1(\underline{t})\, \mu_B(\underline{t}) + \int_{\underline{t}^+}^{t} b_1(y)\, d\mu_B(y) + b_2(t)(1 - \mu_B(t))].$$

Moreover if $\bar{t} < 1$, since $\mu_A(\bar{t}) = \rho_A(1)$ and $\mu_B(\bar{t}) = \rho_B(1)$, we use (34) to conclude that

$$w(t) = w(\bar{t}), \text{ for } t > \bar{t}.$$

Hence it is enough to minorize $w(t)$ on $[\underline{t}, \bar{t}]$.

But on $[\underline{t}, \bar{t}[$ the change of variable $x = f^{-1}(t)$ in (38) gives $w(t) = g(x)$. Finally, if $\bar{t} = y_B$, then $b_1(\bar{t}) = b_2(\bar{t})$ and otherwise $\rho_B(1) = \rho_B(x_B)$ hence in both cases $w(\bar{t}) = g(1)$. Thus Corollary 24 gives $w(t) \geqslant v(p)$.

Moreover by (29)

$$p\, a_1(t_1)\, \mu_A(t_1) + \hat{p}\, b_1(t_1)\, \mu_B(t_1) \geqslant \int_0^{t_1} p\, a_1(y)\, d\mu_A(y) + \hat{p}\, b_1(y)\, d\mu_B(y)$$

hence

$$\psi^+(p, \mu_A, \mu_B, t_1, t) \geqslant w(t) \quad \forall\, t \geqslant t_1 \geqslant 0 \qquad \text{Q.E.D.}$$

In order to get later a uniform majorization we need a discrete approximation of μ_C.

Lemma 28: $\forall\, \eta > 0$ there exists $\bar{\mu}_A$ and $\bar{\mu}_B$ in Q with common finite support such that if $\bar{w} = W(\bar{\mu}_A, \bar{\mu}_B)$ (see (36)) then

$$|\, w(t) - \bar{w}(t)\, | < \eta \quad \forall\, t \in [0, 1].$$

Proof: We first define $x_0 = 0$, and inductively (x_i), such that

$$\mu_A(x_{i+1}) - \mu_A(x_i) \geqslant \frac{\eta}{3L} \text{ and } \mu_A(x_{i+1}^-) - \mu_A(x_i) \leqslant \frac{\eta}{3L}$$

and similarly (y_i), for μ_B.

Let now (z_r), $r \in \bar{R}$, be a common finite refinement of (x_i) and (y_i) such that moreover

$$|z_{r+1} - z_r| < \frac{\eta}{3L}.$$

We introduce now $\bar{\mu}_C$, $C = A, B$, with support included in (z_r), $r \in \bar{R}$, defined by

$$\bar{\mu}_C(z_r) = \mu_C(z_r) \quad \forall\, z_r, \quad \forall\, C = A, B.$$

It follows that if $t \in [z_r, z_{r+1}[$

$$\begin{aligned}
|w(t) - w(\bar{t})| \leqslant p\Big[&\sum_{i=0}^{r-1} \int_{z_i^+}^{z_{i+1}} |a_1(y) - a_1(z_{i+1})|\, d\mu_A(y) \\
&+ \int_{z_r^+}^{t} |a_1(y)|\, d\mu_A(y) + |a_2(t)|\,(\mu_A(t) - \mu_A(z_r))\Big] \\
+ \hat{p}\Big[&\sum_{i=0}^{r-1} \int_{z_i^+}^{z_{i+1}} |b_1(y) - b_1(z_{i+1})|\, d\mu_B(y) \\
&+ \int_{z_r^+}^{t} |b_1(y)|\, d\mu_B(y) + |b_2(t)|\,(\mu_B(t) - \mu_B(z_r))\Big]
\end{aligned}$$

hence

$$|w(t) - w(\bar{t})| \leqslant \frac{\eta}{3L} \cdot L + L \cdot \frac{\eta}{3L} + L \cdot \frac{\eta}{3L}. \qquad \text{Q.E.D.}$$

Note that by the definition of μ_C and $\bar{\mu}_C$, using (32), (33) and (34) we have the following

$$p(a_{11} - a_{12})(\bar{\mu}_A(z_{r+1}) - \bar{\mu}_A(z_r)) + \hat{p}(b_{11} - b_{12})(\bar{\mu}_B(z_{r+1}) - \bar{\mu}_B(z_r)) \geqslant 0, \tag{39}$$

$$p(a_{21} - a_{22})(1 - \bar{\mu}_A(z_r)) + \hat{p}(b_{21} - b_{22})(1 - \bar{\mu}_B(z_r)) \leqslant 0. \tag{40}$$

Before introducing the strategy PI we need the following result.

Let Γ_z^+ be the zero-sum infinitely repeated game with payoff matrix

$$\begin{bmatrix} (1-z)^* & -z^* \\ -(1-z) & z \end{bmatrix} \quad \text{where } z \in [0, 1]$$

(The "Big Match" of *Blackwell/Ferguson* is $\Gamma_{1/2}^+$.)

We define as in part II the stopping time m, the payoff at stage n, q_n^z and the average payoff up to stage n, $\bar{q}_n^z$.

Given the strategy τ for PII let

$$\tilde{t}_n = \frac{1}{n} \# \{m; j_m = L, \quad 1 \leqslant m \leqslant n\}$$

be the frequency of Left up to stage n. Then we have

Proposition 29: [*Blackwell/Ferguson; Kohlberg*] $\forall\, \epsilon > 0, \forall\, \delta > 0, \exists\, N_z$ *and* $\exists\, \sigma_z$ *strategy of* PI *in* Γ_z^+ *such that for any* τ

$$\mathrm{Prob}_{\sigma_z, \tau} (\mathrm{m} \leqslant n) E(q_n^z \mid \mathrm{m} \leqslant n) \geqslant -\epsilon \quad \forall\, n = 1, 2, \ldots, \text{and} \tag{41}$$

$$\mathrm{Prob}_{\sigma_z, \tau} (\mathrm{m} < n \mid \tilde{t}_n > z + \delta) \geqslant 1 - \epsilon \qquad \forall\, n > N_z. \tag{42}$$

Proof: (41) follows from the existence of v_∞ for Γ_z^+ (with $v_\infty = 0\; \forall\, z \in [0, 1]$) and the existence of ϵ-optimal strategies for PI [see *Blackwell/Ferguson; Kohlberg*, or more generally *Mertens/Neyman*]. If (41) is not satisfied, then $\exists\, \tau_0$, $\exists\, \epsilon_0$, $\exists\, n_0$ such that

$$\mathrm{Prob}_{\sigma_z, \tau_0} (\mathrm{m} \leqslant n_0) E(q_{n_0} \mid \mathrm{m} \leqslant n_0) < -\epsilon_0.$$

Hence if PII use τ_0 up to stage n_0 and plays after $(z, 1-z)$ i.i.d. the payoff will satisfy for n large enough

$$\bar{\gamma}_n(\sigma_z, \tau) = \mathrm{Prob}_{\sigma_z, \tau} (\mathrm{m} \leqslant n_0) E(\bar{q}_n^z \mid \mathrm{m} \leqslant n_0)$$

$$+ \mathrm{Prob}_{\sigma_z, \tau} (\mathrm{m} > n_0) E(\bar{q}_n \mid \mathrm{m} > n_0) < -\frac{\epsilon_0}{2}$$

contradicting the existence of v_∞.

As for (43), it follows from the existence of ϵ-optimal strategy of the following kind:

$$s_{z,n}(T) = \varphi(y_n^z)$$

where φ is strictly positive and decreasing and $y_n^z = n\,[-(1-z)\,\tilde{t}_n + z\,(1-\tilde{t}_n)]$ is the non absorbing cumulative payoff up to stage n. Thus $y_n^z \leqslant 0$ infinitely often implies Prob $(\mathrm{m} < \infty) = 1$ and (42) follows. Q.E.D.

Note that we may and shall assume that σ_0 is always Top and σ_1 is always Bottom.

Coming back to the proof of Proposition 26 we now introduce the strategy for PI.

Given $\epsilon_0 > 0$, let $\bar{\mu}_A$ and $\bar{\mu}_B$ satisfy Lemma 28 with $\eta = (\epsilon_0)/8$ and denote their finite support by $z_r; r \in \bar{R} = \{0, \ldots, R\}$ with $z_0 = 0$ and $z_R = 1$. Let $\tilde{\mu}_C$, $C = A, B$ be the probabilities on $(z_r)_{i \in \bar{R}}$ induced by $\bar{\mu}_C$ and the additional mass $1 - \bar{\mu}_C(1)$ on $z_R = 1$.

For each z_r let σ_{z_r} (denoted by σ_r) and N_{z_r} satisfying (41), (42) with $\epsilon = (\epsilon_0/8LR)$ and $\delta = (\epsilon_0/4L)$. Let $N = \max_r N_{z_r}$. PI chooses $r_C^* \in \bar{R}$ according to $\tilde{\mu}_C$, $C = A, B$.

Let m_0 be the stopping time given σ_0 and τ (here $\mathrm{m}_0 \equiv 1$) and define inductively m_r, $r \in \bar{R}$ in the following way:

Given m_r, use σ_{r+1} in the game starting at stage $\mathrm{m}_r + 1$ to realize m_{r+1}.

Let $\mathrm{l}_r = \sum_{k=0}^{r} \mathrm{m}_k$, then if C, PI plays Bottom up to stage $\mathrm{l}_{r_C^*} - 1$ and plays Top at stage $\mathrm{l}_{r_C^*}$.

Since we want to minorize the payoff of PI given σ we shall assume that PII is using a pure strategy, i.e., a sequence of L and R.

Let us now compute $\tilde{\gamma}_n(\sigma, \tau)$. We introduce the following notations:

$t_m(R) = t_m \qquad m \geqslant 1$ describes the strategy τ for PII.

$$
\begin{array}{ll}
\mathrm{x}_0 = 1 & \mathrm{y}_0 = 0 \\
\mathrm{x}_1 = \min(n, \mathrm{m}_1) & \mathrm{y}_1 = \mathrm{x}_1 \\
\mathrm{x}_2 = \min(n - \mathrm{y}_1, \mathrm{m}_2) & \mathrm{y}_2 = \mathrm{x}_2 + \mathrm{y}_1 \\
\cdots & \\
\mathrm{x}_r = \min(n - \mathrm{y}_{r-1}, \mathrm{m}_r) & \mathrm{y}_r = \mathrm{x}_r + \mathrm{y}_{r-1} \\
\cdots & \\
\mathrm{x}_R = n - \mathrm{y}_{R-1} & \mathrm{y}_R \equiv n
\end{array}
$$

Note that from stage $\mathrm{y}_r + 1$ to y_{r+1} the payoff is absorbing in game C iff $r_C^* \leqslant r$.

Hence we have

$$n\,\bar{\gamma}_n(\sigma,\tau) = E\sum_{r=0}^{R-1}\Big[\mathrm{x}_{r+1}\Big(\sum_{i=0}^{r} p\,[\mu_A(z_i)-\mu_A(z_{i-1})]\,a_1(t_{\mathrm{y}_i+1}) \tag{43}$$

$$+\hat{p}\,[\mu_B(z_i)-\mu_B(z_{i-1})]\,b_1(t_{\mathrm{y}_i+1})\Big)$$

$$+p\,(1-\mu_A(z_r))\sum_{m=\mathrm{y}_r+1}^{\mathrm{y}_{r+1}} a_2(t_m)$$

$$+\hat{p}\,(1-\mu_B(z_r))\sum_{m=\mathrm{y}_r+1}^{\mathrm{y}_{r+1}} b_2(t_m)\Big].$$

Now by (43), since PI is using σ_{r+1} from y_r+1 to y_{r+1} we have

$$E\left(\sum_{m=\mathrm{y}_r+1}^{\mathrm{y}_{r+1}}(t_m)\right) \leqslant N + (z_{r+1}+\delta)\,E(\mathrm{x}_{r+1}) + \epsilon\, n \tag{44}$$

by taking first the conditional expectation given y_r+1 and then integrating with respect to y_r+1.

Since, by the construction of $\bar{\mu}_C$, $z_{r+1} \leqslant z_r + (\eta/L)$, using (40), it follows from (43) and (44) that

$$n\,\bar{\gamma}_n(\sigma,\tau) \geqslant \sum_{r=0}^{R-1} E(\mathrm{x}_{r+1})\,(\bar{w}_r - \eta - \delta L) - NLR - \epsilon nLR + \Delta \tag{45}$$

where

$$\bar{w}_r = \sum_{i=0}^{r} p\,(\mu_A(z_i)-\mu_A(z_{i-1}))\,a_1(z_i) + \hat{p}\,(\mu_B(z_i)-\mu_B(z_{i-1}))\,b_1(z_i)$$

$$+\,p\,(1-\mu_A(z_r))\,a_2(z_r) + \hat{p}\,(1-\mu_B(z_r))\,b_2(z_r)$$

hence by definition (36)

$$\bar{w}_r = \bar{w}(z_r) \tag{46}$$

and

$$\Delta = \sum_{i=0}^{R-1}\big[p\,(a_{11}-a_{12})\,(\mu_A(z_i)-\mu_A(z_{i-1})) \tag{47}$$

$$+\,\hat{p}\,(b_{11}-b_{12})\,(\mu_B(z_i)-\mu_B(z_{i-1}))\big]\,\alpha_i$$

with

$$\alpha_i = E\,[(t_{y_i+1} - z_i) \sum_{r=i+1}^{R} x_r]. \tag{48}$$

From (46) we deduce using Lemma 28 and (37) that

$$\bar{w}_r \geqslant v\,(p) - \eta. \tag{49}$$

Now it remains to minorize Δ, but by (39) it is enough to minorize α_i. Note that $\sum_{r=i+1}^{R} x_r = n - y_i$ and that $t - z_i = t\,(1 - z_i) - z_i\,(1 - t)$. Hence α_i is the expectation of the sum of the absorbing payoffs in $\Gamma^+_{z_i}$, using σ_i. Thus by (41) we obtain

$$\alpha_i \geqslant -\epsilon n \tag{50}$$

and replacing in (47) we get

$$\Delta \geqslant -\epsilon L n. \tag{51}$$

Since $E \sum_{r=0}^{R-1} x_{r+1} = n$, using (49) and (51) it follows from (45) that

$$n\,\bar{\gamma}_n\,(\sigma, \tau) \geqslant n\,(\bar{v}\,(p) - 2\eta - \delta L) - NLR - \epsilon LRn - \epsilon Ln. \tag{52}$$

Finally, the by choise of η, ϵ and δ, $n \geqslant (4NLR)/\epsilon_0$ implies

$$\bar{\gamma}_n\,(\sigma, \tau) \geqslant v\,(p) - \epsilon_0.$$

This completes the proof of Proposition 26. Q.E.D.

Remark: Note that if the dual statements hold in Corollary 25, in particular if f is decreasing, we use the optimal strategies in Γ^-_z, $z \in [0, 1]$ with payoff matrix

$$\begin{bmatrix} -z^* & (1-z)^* \\ z & -(1-z) \end{bmatrix}$$

5.5 A New Formula for $v\,(p)$

Recall that

$$Z^+\,(p) = \inf_{t_1} \sup_{\mu_A, \mu_B} \inf_{t_2} \psi^+\,(p, \mu_A, \mu_B, t_1, t_2)$$

and let

$$W^{+}(p) = \sup_{\mu_A, \mu_B} \inf_{t_1} \inf_{t_2} \psi^{+}(p, \mu_A, \mu_B, t_1, t_2).$$

Z^{-} and W^{-} are defined similarly.

Proposition 30: If (32), (33), (34) *hold, then*

$$\nu(p) = Z(p) = Z^{+}(p) = W^{+}(p).$$

Proof: By Lemma 27 we have $W^{+}(p) \geqslant \nu(p)$ but then using Lemma 23, $Z^{+}(p) \leqslant \nu(p)$. Hence by Lemma 21, $Z(p) = Z^{+}(p)$, since $\nu(p) = \underline{\nu}(p)$.

Finally the fact that we always have $Z^{+}(p) \geqslant W^{+}(p)$ implies the result.

Q.E.D.

Now for each $t_1 \in [0, 1]$ define $\mathcal{F}_{t_1}$ to be the convex hull of $\{F_t; t \in [t_1, 1]$ with $F_t(y) = 1_{]t, +\infty[}(y)$. Hence $\mathcal{F}_{t_1}$ is the set of left continuous increasing functions F in $[0, 1]$ with $F(t_1) = 0$ and $F(1^{+}) = 1$. For F in $\mathcal{F}_{t_1}$ let

$$\begin{aligned}
\chi^{+}(p, \mu_A, \mu_B, t_1, F) = p\,[a_1(t_1)\mu_A(t_1) &+ \int_{t_1^{+}}^{1} (1 - F(y))\, a_1(y)\, d\mu_A(y) \\
&+ \int_{t_1}^{1} a_2(y)(1 - \mu_A(y))\, dF(y)] \\
+ \hat{p}\,[b_1(t_1)\mu_B(t_1) &+ \int_{t_1^{+}}^{1} (1 - F(y))\, b_1(y)\, d\mu_B(y) \\
&+ \int_{t_1}^{1} b_2(y)(1 - \mu_B(y))\, dF(y)]
\end{aligned}$$

then we have the following result:

Lemma 31: $Z^{+}(p) = \inf_{t_1} \inf_{\mathcal{F}_{t_1}} \sup_{Q \times Q} \chi^{+}(p, \mu_A, \mu_B, t_1, F)$.

Proof: By definition $\psi^{+}(p, \mu_A, \mu_B, t_1, t_2) = \chi^{+}(p, \mu_A, \mu_B, t_1, F_{t_2})$ hence

$$Z^{+}(p) = \inf_{t_1} \sup_{Q \times Q} \inf_{\mathcal{F}_{t_1}} \chi^{+}.$$

For a fixed t_1 we shall prove that

$$\sup_{Q\times Q}\ \inf_{F_{t_1}} \chi^+ = \inf_{F_{t_1}}\ \sup_{Q\times Q} \chi^+. \tag{53}$$

Remark first that

$$\sup_{Q\times Q} \chi^+ = p \sup_{\hat{Q}} G_A(F,\mu) + \hat{p} \sup_{\hat{Q}} G_B(F,\mu)$$

where $\hat{Q}$ are the extreme points of Q and

$$G_A(F,\mu) = a_1(t_1)\,\mu(t_1) + \int_{t_1^+}^{1} (1-F(y))\,a_1(y)\,d\mu(y)$$

$$+ \int_{t_1}^{1} a_2(y)\,(1-\mu_A(y))\,dF(y).$$

Now if $\mu \doteq \delta_t$ we have:

$$G_A(F,\mu) = a_1(t_1) \quad \text{for } t \leqslant t_1, \text{ and}$$

$$G_A(F,\mu) = (1-F(t))\,a_1(t) + \int_{t_1}^{t^-} a_2(y)\,dF(y) \qquad \text{for } t \in [t_1, 1]$$

and if $\mu \equiv 0$

$$G_A(F,\mu) = \int_{t_1}^{1} a_2(y)\,dF(y).$$

Let $QNA(t_1) = \{\mu \in Q \mid \mu \text{ is non-atomic on }]t_1, 1]\}$. Then since F is left continuous it follows from the above formulas that

$$\sup_{\hat{Q}} G_A(F,\mu) = \sup_{QNA(t_1)} G_A(F,\mu) = \sup_{QNA(0)} G_A(F,\mu).$$

We can now apply Sion's minmax Theorem to χ^+ on F_{t_1} and $QNA(0) \times QNA(0)$. In fact F_{t_1} is convex and weakly compact and $QNA(0)$ is convex.

Moreover χ^+ is affine in each variable and $\chi^+(\cdot, \mu_A, \mu_B)$ is continuous with respect to F.

This last statement follows from the fact that since $\mu \in QNA(0)$, $1-\mu_A(y)$ is continuous, hence

$$\int_{t_1}^{1} a_2(y)\,(1-\mu_A(y))\,dF(y)$$

is continuous for the weak topology on F_{t_1} and similarly for

$$\int_{t_1^+}^{1} (1 - F(y))\, a_1(y)\, d\mu(y)$$

since μ has no mass at the discontinuity points of F. Hence

$$\inf_{F_{t_1}} \sup_{QNA(0)^2} \chi^+ = \sup_{QNA(0)^2} \inf_{F_{t_1}} \chi^+$$

which implies (53). Q.E.D.

Thus we obtain:

Corollary 32: If $v(p) = W^+(p)$, then

$$v(p) = \underline{v}(p) = \sup_{Q^2} \inf_{t_1} \inf_{F_{t_1}} \chi^+ = \inf_{t_1} \sup_{Q^2} \inf_{F_{t_1}} \chi^+ = \inf_{t_1} \inf_{F_{t_1}} \sup_{Q^2} \chi^+.$$

6. Examples

For each example we shall give $v_1(p) = \bar{v}(p)$, $v(p) = \underline{v}(p)$, Cav $u(p)$ and describe the optimal strategies ρ and f in $\Gamma(p)$ and the measure μ which induces an optimal strategy for PI in the maxmin.

1. $$A = \begin{bmatrix} 1^* & 0^* \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0^* & 0^* \\ 0 & 1 \end{bmatrix}$$

(This was the example studied in *Sorin* [1980].)

$$\operatorname{Cav} u(p) = p(1-p)$$

$$v_1(p) = \min(p, 1-p)$$

$$v(p) = (1-p)(1 - \exp(-p/(1-p)))$$

$$f(x) = \frac{k}{1-x} \text{ on } [0, 1-k]$$

$$= 1 \qquad \text{on } [1-k, 1] \text{ with } k = \exp(-p/(1-p))$$

$$\rho_A(x) = -\frac{\hat{p}}{p}\operatorname{Log}(1-x) \quad \text{on } [0, 1-k]$$

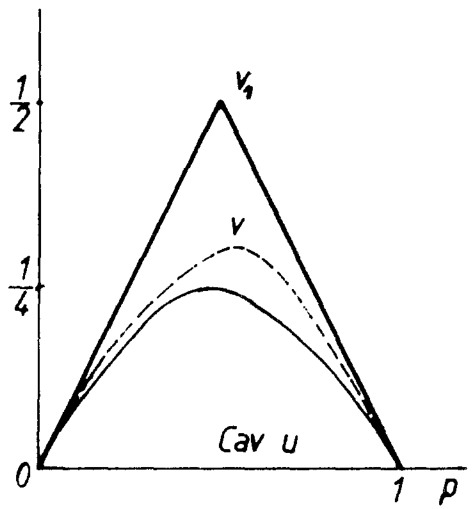
$$\quad = 1 \quad \text{on } [1-k, 1]$$

$$\rho_B(x) = 0 \quad \text{on } [0, 1]$$

$$\mu_A(t) = 0 \quad \text{on } [0, k]$$

$$\quad = 1 + \frac{\hat{p}}{p}\operatorname{Log} t \quad \text{on } [k, 1]$$

$$\mu_B(t) = 0 \quad \text{on } [0, 1].$$



If we define $p(t)$ to be the probability of $\{C = A\}$ given the fact that the payoff is non-absorbing when PII plays $(t, 1-t)$ i.i.d. we obtain

$$p(t) = p \quad \text{on } [0, k]$$

$$p(t) = \frac{-\operatorname{Log} t}{-\operatorname{Log} t + 1} \quad \text{on } [k, 1]$$

which is independent of p on the range of f.

Note that, in this example, $\nu(p)$ and $\underline{\nu}(p)$ are transcendental functions.

2. $$A = \begin{bmatrix} 1^* & 0^* \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0^* & 1^* \\ -1 & 3 \end{bmatrix}$$

$$\operatorname{Cav} u(p) = \frac{5}{4} p \qquad \text{on } \left[0, \frac{4}{7}\right]$$

$$= \frac{5}{3} \hat{p} \qquad \text{on } \left[\frac{4}{7}, 1\right]$$

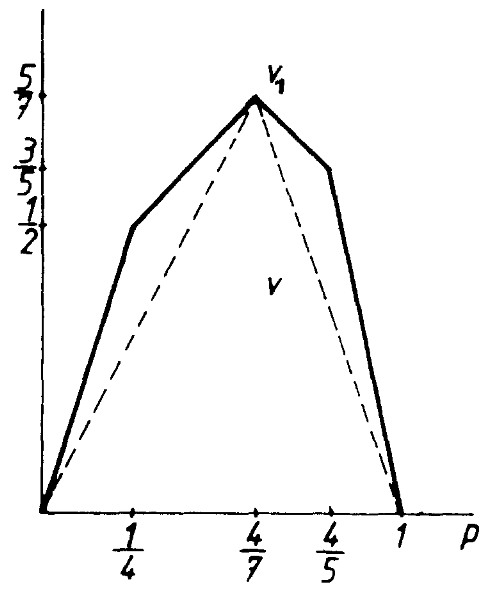
$$\nu_1(p) = 2p \qquad \text{on } \left[0. \frac{1}{4}\right]$$

$$= \frac{2p+1}{3} \qquad \text{on } \left[\frac{1}{4}, \frac{4}{7}\right]$$

$$= \frac{2-p}{2} \qquad \text{on } \left[\frac{4}{7}, \frac{4}{5}\right]$$

$$= 3\hat{p} \qquad \text{on } \left[\frac{4}{5}, 1\right]$$

$$\nu(p) = \operatorname{Cav} u(p)$$



$$f(x) = \frac{1}{2}[1-(x-1)^2] \quad \text{on } [0,1] \quad \text{if } p \in \left[\frac{4}{7}, 1\right]$$

$$= \frac{1}{3}[2-(x-1)^3] \quad \text{on } [0,1] \quad \text{if } p \in \left[0, \frac{4}{7}\right]$$

$$\left.\begin{aligned} \rho_A(x) &= \frac{7p-4}{3p} && \text{on } [0,1] \\ \rho_B(x) &= 0 && \text{on } [0,1] \end{aligned}\right\} \text{if } p \in \left[\frac{4}{7}, 1\right]$$

$$\left.\begin{aligned} \rho_A(x) &= 0 && \text{on } [0,1] \\ \rho_B(x) &= \frac{4-7p}{4-4p} && \text{on } [0,1] \end{aligned}\right\} \text{if } p \in \left[0, \frac{4}{7}\right]$$

and μ is the same as ρ.

PI is playing a mixture of always Top, always Bottom such that the posterior given Top is 1 (if $p \geqslant 4/7$) or 0 (if $p \leqslant 4/7$) and given Bottom is $4/7$.

Note that PII cannot obtain Cav $u(p)$ in $\Gamma(p)$ by playing a strategy i.i.d.

3. $$A = \begin{bmatrix} 2^* & -1^* \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1^* & 3^* \\ 0 & 1 \end{bmatrix}$$

Cav $u(p)$ and $v_1(p)$ are obviously the same as in Example 2. We shall give the results for $p \in [4/7, 1]$, the analysis is similar on $[0, 4/7]$.

Let $K_0 \in [0, 2/3]$ satisfying $2/3 + 2 \cdot ((3K_0)/2)^{4/3} = 1 + K_0$ and $p_0 \in [4/7, 1]$ such that $((3K_0)/2)^{1/3} = (2p_0 - 1)/(p_0)$.

– if $p \geqslant p_0$:

$$f(x) = \frac{2}{3} - K(1-x)^{-3/4} \quad \text{on } [0, x_K] \text{ with } \left(\frac{3K}{2}\right)^{1/3} = \frac{2p-1}{p}$$

$$= 0 \quad \text{on } [x_K, 1] \text{ and } x_K = 1 - \left(\frac{2p-1}{p}\right)^4$$

$$\rho_A(x) = 0 \quad \text{on } [0,1]$$

$$\rho_B(x) = \frac{1-2p}{1-p}(1-(1-x)^{-1/4}) \quad \text{on } [0, x_K]$$

$$= 1 \quad \text{on } [x_K, 1]$$

and

$$v(p) = \frac{p+1}{3} - \frac{2}{3}\frac{(2p-1)^4}{p^3}.$$

– <u>if $p \leqslant p_0$:</u>

$$f(x) = \frac{2}{3} - K_0 (1-x)^{-3/4} \qquad \text{on } [0, x_{K_0}]$$

$$= 0 \qquad \text{on } [x_{K_0}, 1]$$

$$\rho_A(x) = 1 - \frac{7p-4}{p} \cdot \frac{p_0}{7p_0 - 4} \qquad \text{on } [0, 1]$$

$$\rho_B(x) = 1 - \frac{7p-4}{\hat{p}} \cdot \frac{p_0}{7p_0-4} + \frac{7p-4}{7p_0-4}\left(\frac{2p_0-1}{1-p}\right)(1-x)^{-1/4} \quad \text{on } [0, x_{K_0}]$$

$$= 1 \qquad \text{on } [x_{K_0}, 1]$$

and

$$\nu(p) = \frac{1+2p}{3} + (4p-7)K_0.$$

As for the maxmin we have (note that f is decreasing hence $\mu_C(t) = \mu_C[1, t]$)

– <u>if $p \geqslant p_0$:</u>

$$\mu_A(t) \equiv 0 \qquad \text{on } [0, 1]$$

$$\mu_B(t) = \frac{1-2p}{1-p} + \frac{p}{1-p}\left(1 - \frac{3}{2}t\right)^{1/3} \quad \text{on} \left[\frac{2}{3}\left(1 - \left(\frac{2p-1}{p}\right)^3\right), 0\right]$$

and

$$p(t) = \frac{1}{2 - (1 - (3/2)t)^{1/3}}$$

– <u>and if $p \leqslant p_0$:</u>

$$\mu_A(t) = 1 - \frac{7p-4}{p} \cdot \frac{p_0}{7p_0-4} \text{ on } [t_0, 0] \text{ with } t_0 = \frac{2}{3}\left[1 - \left(\frac{2p_0-1}{p_0}\right)^3\right]$$

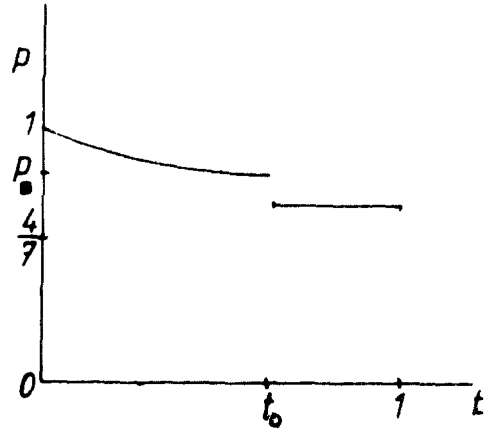
$$\mu_B(t) = 1 - \frac{7p-4}{(1-p)} \cdot \frac{p_0}{7p_0-4} + \frac{p_0}{1-p} \cdot \frac{7p-4}{7p_0-4}\left(-\frac{3}{4} + \left(1 - \frac{3}{2}t\right)^{1/3}\right)$$

$$\text{on } [t_0, 0]$$

and

$$p(t_0) = p_0 \text{ (and Prob } (C = A \mid \text{PI plays Top at } t_0) = \frac{4}{7}\text{)}$$

$$p(t) = \frac{1}{2 - (1 - (3/2)t)^{1/3}} \quad \text{on } [t_0, 0]$$

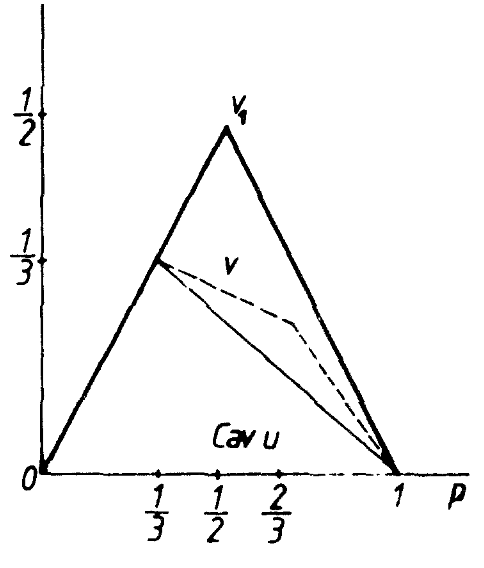


4. $$A = \begin{bmatrix} 3^* & -1^* \\ 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1^* & 1^* \\ 0 & -1 \end{bmatrix}$$

$$\operatorname{Cav} u\,(p) = p \qquad \text{on } \left[0, \frac{1}{3}\right]$$

$$= \frac{1-p}{2} \qquad \text{on } \left[\frac{1}{3}, 1\right]$$

$$v_1\,(p) = \min\,(p, \hat{p}) \qquad \text{on } [0, 1]$$



– if $p \in \left[0, \frac{1}{3}\right]$:

$$\nu(p) = p$$

$$f(x) = \frac{1}{2} \text{ on } [0, 1]$$

$$\rho_A(x) = \rho_B(x) = 1 - (1 - x)^{1/(1-3p)} \text{ on } [0, 1]$$

$$\mu_A(t) = \mu_B(t) = \delta_{1/2}(t) \quad \text{(i.e., PI plays } \sigma_{1/2} \text{ optimal in } B\text{).}$$

– if $p \in \left[\frac{1}{3}, \frac{2}{3}\right]$:

$$\nu(p) = \frac{1}{9}(4 - 3p)$$

$$f(x) = \frac{1}{2} - \frac{2}{9(1-x)^2} \qquad \text{on } \left[0, \frac{1}{3}\right]$$

$$= 0 \qquad \text{on } \left[\frac{1}{3}, 1\right]$$

$$\rho_A(x) = \frac{1}{p}\left(\frac{2}{3} - p\right) \qquad \text{on } [0, 1]$$

$$\rho_B(x) = \frac{1}{\hat{p}}\left[2\left(\frac{2}{3} - p\right) + (3p - 1)x\right] \qquad \text{on } \left[0, \frac{1}{3}\right]$$

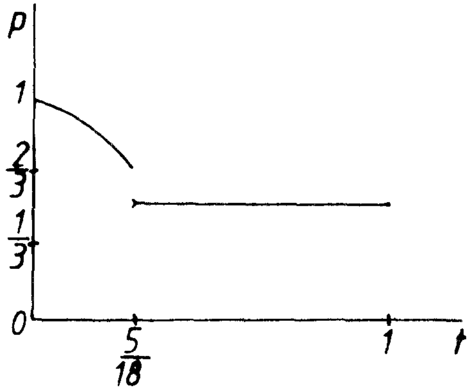
$$= 1 \qquad \left[\frac{1}{3}, 1\right]$$

$$\mu_A(t) = \frac{1}{p}\left(\frac{2}{3} - p\right)\delta_{5/18}(t)$$

$$\mu_B(t) = 0 \qquad \text{on } \left[1, \frac{5}{18}\right[$$

$$= \frac{1}{\hat{p}}\left[p + \frac{1}{3} - \frac{2}{3}\frac{(3p-1)}{\sqrt{1-2t}}\right] \qquad \text{on } \left[\frac{5}{18}, 0\right]$$

$$p(t) = \sqrt{1 - 2t} \qquad \left[\frac{5}{18}, 0\right]$$



– if $p \in \left[\frac{2}{3}, 1\right]$:

$$\nu(p) = p \cdot \hat{p}$$

$$f(x) = \frac{1}{2} - \frac{p^2}{2(1-x)^2} \qquad \text{on } [0, 1-p]$$

$$= 0 \qquad \text{on } [1-p, 1]$$

$$\rho_A(x) = 0 \qquad \text{on } [0, 1]$$

$$\rho_B(x) = \frac{x}{\hat{p}} \qquad \text{on } [0, \hat{p}]$$

$$= 1 \qquad \text{on } [\hat{p}, 1]$$

$$\mu_A(t) = 0 \qquad \text{on } [1, 0]$$

$$\mu_B(t) = \frac{1}{\hat{p}}\left[1 - \frac{p}{\sqrt{1-2t}}\right] \text{on } \left[\frac{1-p^2}{2}, 0\right]$$

$$= 0 \qquad \text{on } \left[1, \frac{1-p^2}{2}\right]$$

$$p(t) = \sqrt{1-2t} \qquad \text{on } \left[\frac{1-p^2}{2}, 0\right]$$

5. $$A = \begin{bmatrix} 4^* & -2^* \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0^* & 1^* \\ -1 & 2 \end{bmatrix}$$

$$u(p) = 4p \qquad \text{on } \left[0, \frac{1}{7}\right]$$

$$= (1-p)\frac{5p+1}{4p+2} \qquad \text{on } \left[\frac{1}{7}, 1\right]$$

and $u = \operatorname{Cav} u$

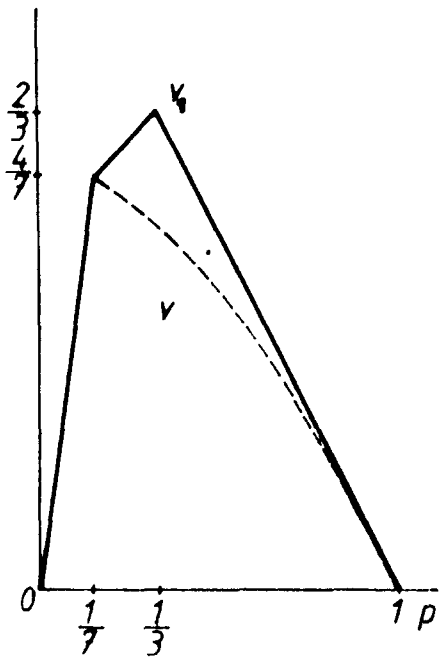
$$\nu_1(p) = \min\left(4p, \frac{1}{2}(p+1), 1-p\right).$$

Obviously we have

$$\nu(p) = \nu_1(p) = \operatorname{Cav} u(p) \quad \text{on } \left[0, \frac{1}{7}\right].$$

Now for $p \geqslant \frac{1}{7}$

$$\nu(p) = \hat{p}\left(1 - \frac{1}{3}\exp\frac{1-7p}{3\hat{p}}\right)$$



$$f(x) = \frac{1}{3} + \frac{K}{1-x} \qquad \text{on } [0, 1-6K]$$

$$= \frac{1}{2} \qquad \text{on } [1-6K, 1] \text{ with } K = \frac{1}{9}\exp\frac{1-7p}{3\hat{p}}$$

$$\rho_A(x) = -\frac{1}{2}\frac{\hat{p}}{p}\,\mathrm{Log}\,(1-x) \quad \text{on } [0, 1-9K[$$

$$= 1 \quad \text{on } [1-9K, 1]$$

$$\rho_B(x) = \delta_{1\text{-}9K}(x)$$

and for the maxmin

$$\mu_A(t) = \frac{7p-1}{6p} + \frac{\hat{p}}{2p}\,\mathrm{Log}\,[9t-3] \quad \text{on } \left[\frac{1}{3}+K, \frac{4}{9}\right[$$

$$= 1 \quad \text{on } \left[\frac{4}{9}, 1\right]$$

$$\mu_B(t) = \delta_{4/9}(t).$$

Hence the posterior is given by

$$p(t) = \frac{1 - 3\,\mathrm{Log}\,(9t-3)}{7 - 3\,\mathrm{Log}\,(9t-3)} \quad \text{on } \left[\frac{1}{3}+K, \frac{4}{9}\right].$$

(Note that $p(4/9) = 1/7$.)

7. Concluding Remarks

The relation between this study and the previous results can be summarized as follows:

1. For games with lack of information on one side and for stochastic games the asymptotic value, the minmax and the maxmin do exist. We proved that it is also the case for the class of games under consideration.
2. Similarly it was shown that $\lim v_n = \lim \tilde{v}_\lambda$ which is also true for stochastic games and holds up to now for all games with incomplete information where the existence of $\lim v_n$ has been proved.
3. Nevertheless the infinite value which always exists, either for stochastic games or for games with lack of information on one side, may not exist in this case (i.e., the maxmin and the minmax are different, see examples).
4. Note that the asymptotic value may be a transcendental function (see example 1) which cannot be the case neither for games with lack of information on one side where $v = \mathrm{Cav}\, u$ und u is algebraic, nor for stochastic games, by the result of *Bewley/Kohlberg.*
5. Moreover the maxmin itself (which equals the asymptotic value in the games studied here) may be transcendental. This cannot happen for games with lack of information on one side, and even with lack of information on both sides [*Mertens/Zamir*].

6. Finally we found that the maxmin was equal to the asymptotic value. It is conjectured that this will be the case for stochastic games with lack of information on one side.
7. In *Sorin* [1982] games with the first column absorbing are considered and results corresponding to 1, 2, 3 and 6 are proved. In this class the minmax may be a transcendental function.
   Nevertheless the tools used are rather different, the main difficulty being for the (minmax) strategy of PII.

8. Appendix: Relation with Games with Signalling Matrices

Aumann/Maschler [1968] have introduced a more general class of games with incomplete information described as follows. In addition to the initial probability on the state spaces and to the payoff matrices, we are given two families of "signalling matrices", H_I^k and H_{II}^k, with entries in some alphabet. After each stage, if the state is k and the moves of the players are (i, j), PI (resp. PII) is told $H_I^k\,(i, j)$ (resp. $H_{II}^k\,(i, j)$). The case where $H_I^k\,(i, j) = H_{II}^k\,(i, j) = (i, j)$, $\forall\, k, i, j$ is called standard signalling.

It is clear that such games may be similar to games with incomplete information and absorbing states. For example

$$
\begin{cases}
p: & A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad H_I^A = H_{II}^A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\[2ex]
\hat{p}: & B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad H_I^B = H_{II}^B = \begin{bmatrix} \alpha' & \beta \\ \gamma & \delta \end{bmatrix}
\end{cases}
\quad \text{and} \quad
\begin{cases}
p: & \hat{A} = \begin{bmatrix} \nu\,(A)^* & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\[2ex]
\hat{p}: & \hat{B} = \begin{bmatrix} \nu\,(B)^* & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
\end{cases}
$$

Kohlberg/Zamir used this fact to prove the existence of the infinite value in the "symmetric case": both players have the same information about the state's choice and the same signalling matrices, which satisfies $\forall\, k, k', H^k\,(i, j) \neq H^{k'}\,(i', j')$ as soon as $i \neq i'$ or $j \neq j'$.

Another class was studied by *Mertens/Zamir* [1976] [see also *Waternaux*] where they proved the existence of the maxmin and of the minmax. These games can be desribed as follows:

$$\begin{array}{ll} p & \begin{bmatrix} \nu(A)^* & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ \hat{p} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & \nu(B)^* \end{bmatrix} \end{array} \qquad H_x^C = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \quad C = A, B, \quad x = I, II.$$

The example below shows that special kind of games with incomplete information and signalling matrices are similar to the games studied here.

Let $J(p)$ be described as follows: There are six payoff matrices (A_i) $i = 1, 2, 3$; (B_i) $i = 1, 2, 3$ and Prob $(C = A_i) = (p/3)$; Prob $(C = B_i) = (\hat{p}/3)$. PI knows whether $C \in \{A_i; i = 1, 2, 3\}$ or not. PII knows only the initial probabilities. The signalling matrices are the same for both players and $H^{A_i} = H^{B_i} = H_i$. The explicit data are

$$\begin{array}{lccc} & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\ p & A_1 = \begin{bmatrix} 8 & 2 \\ -4 & -4 \end{bmatrix} & A^2 = \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} & A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{p} & B_1 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} & B_2 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} & B_3 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \\ & H_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & H_2 = \begin{bmatrix} \alpha' & \beta \\ \gamma & \delta \end{bmatrix} & H_3 = \begin{bmatrix} \alpha' & \beta' \\ \gamma & \delta \end{bmatrix} \end{array}$$

We claim that this game is similar to the following $G(p)$:

$$\begin{array}{ll} p & \begin{bmatrix} 2^* & 0^* \\ 0 & 0 \end{bmatrix} \\ \hat{p} & \begin{bmatrix} 0^* & 1^* \\ 0 & 2 \end{bmatrix} \end{array} \quad \text{standard signalling}$$

which belongs to the class under consideration in this paper.

In fact if the moves are Bottom Left or Bottom Right, the signalling matrices do not reveal anything about the state and the payoff is the average payoff, i.e., (0, 0) if $C \in \{A_i; i = 1, 2, 3\}$ and (0, 2) otherwise.

Now if TL is played, then either α or α' is told.

– Given α both players now play the game

$$\begin{array}{ll} p & \begin{bmatrix} 8 & 2 \\ -4 & -4 \end{bmatrix} \\ \hat{p} & \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \end{array} \qquad \text{standard signalling}$$

where PII can guarantee at each step the vector payoff (2, 0) by playing R, and PI can guarantee similarly (2, 0) by playing TT.

– Given α' the situation is the following

$$\begin{array}{lcc} & \frac{1}{2} & \frac{1}{2} \\ p & \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{p} & \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \\ & \begin{bmatrix} \alpha' & \beta \\ \gamma & \delta \end{bmatrix} & \begin{bmatrix} \alpha' & \beta' \\ \gamma & \delta \end{bmatrix} \end{array}$$

where PII can guarantee (2, 0) by playing L and similarly for PI by playing BT (note that if TR the true game will be revealed but $1/2\,(\nu(B_2) + \nu(B_3)) = 0$). Finally if TR is played we obtain

– Given β

$$\begin{array}{ccc} & \frac{1}{2} & \frac{1}{2} \\ p & \begin{bmatrix} 8 & 2 \\ -4 & -4 \end{bmatrix} & \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} \\ \hat{p} & \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \\ & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & \begin{bmatrix} \alpha' & \beta \\ \gamma & \delta \end{bmatrix} \end{array}$$

where PII can guarantee (0, 1) by playing R and the same for PI by playing BB.

– Given β'

$$\begin{array}{cc} p & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{p} & \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \end{array} \qquad \text{standard signalling}$$

where PII can still guarantee (0, 1) by playing L and the same for PI by using TT.

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