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"Big Match" with Lack of Information on One Side (Part II)

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Abstract: This is the second paper on a class of stochastic games with incomplete information. As in *Sorin* [1984] we prove the existence of the asymptotic value ($\lim v_n$) of the maxim and of the minmax although the infinite value may not exist. Nevertheless the results and the tools used are rather different from the previous case.

1 Introduction

As in the previous paper [*Sorin,* 1984] we consider a two-person zero-sum infinitely repeated game with incomplete information and absorbing states.

We are given two states of nature, hence two payoff matrices

$$A = \begin{bmatrix} a_{11}^* & a_{12} \\ a_{21}^* & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11}^* & b_{12} \\ b_{21}^* & b_{22} \end{bmatrix}$$

with the left column absorbing (i.e. once any entry with a star (*) is reached, all payoffs in the future will be equal to that entry. See *Blackwell/Ferguson*, and *Kohlberg*). Now one of these two matrices is chosen once and for all by the referee (with probability p for A) and this choice is told to player I. The game is then played in stages. After each stage n the players are told the previous moves i_n, j_n by the referee, but the current payoff q_n is not stated. The description of the game, including this sentence, is common knowledge. A player's (behavioral) strategy is the choice of a probability over his set of moves, at each stage, conditional on his information on the state and on the history (i.e. the sequence of moves) up to that stage.

We shall denote by H_m the set of m-stage histories. Given the state such a history

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determines a payoff at stage m, q_m, and an average payoff $\bar{q}_m$ which is the Cesaro mean of the payoffs up to stage m. Its expectation with respect to p, σ and τ (strategies of the players) is denoted by $\bar{\gamma}_m$. γ_m is the expected payoff at stage m.

v_n (p) is the value of the n repeated game G_n (p) with payoff $\bar{\gamma}_n$.

In order to study G_∞ (p) we recall the following definitions [*Mertens/Zamir*].

$\underline{v}$ (p) is the maxmin of G_∞ (p) if

i) $\forall\ \epsilon > 0, \exists\ \sigma$ and $\exists\ N$ such that

$$\bar{\gamma}_n\ (\sigma, \tau) \geq \underline{v}\ (p) - \epsilon \text{ for all } \tau \text{ and all } n \geq N$$

ii) $\forall\ \epsilon > 0, \forall\ \sigma, \exists\ \tau$ and $\exists\ N$ with

$$\bar{\gamma}_n\ (\sigma, \tau) \leq \underline{v}\ (p) + \epsilon \text{ as soon as } n \geq N.$$

We shall refer to these conditions by saying that player I (*PI*) can guarantee $\underline{v}$, (i), and that he cannot expect more, (ii).

The minmax $\bar{v}$ is defined in a dual way. G_∞ (p) has a value v_∞ iff $\bar{v}\ (p) = \underline{v}\ (p)$.

The "Big Match" of Blackwell and Ferguson is G_∞ (0), and they proved the existence of v_∞.

In *Sorin* [1984] the payoff matrices have the first row absorbing and the existence of lim v_n, $\bar{v}$ and $\underline{v}$ is proved. Nevertheless there are games without a value. For the present class we obtain similar results, but the tools used are rather different. The main difficulty being for the minmax where *PII* faces a "stochastic game with vector payoffs" [*Blackwell*].

2 Maxmin

If Δ_n (p), $n \in \mathbf{N} \cup \{+\infty\}$, is the repeated game where none of the players is informed, we recall [*Kohlberg*] that its value u_n (p) exists and is constant w.r.t. n. This value will be denoted by u.

$\mathcal{H}_n$ is the σ-field induced by H_n on H_∞ and p_n is the posterior induced by σ, i.e. $p_n = \text{Prob}_\sigma\ (A \mid \mathcal{H}_{n-1})$.

If f is a real function on [0, 1], Cav$f$ is the smallest concave function greater than f on [0, 1].

Finally we introduce some notations.

L is the maximum absolute value of the payoff entries. If α is a probability distribution on the moves, $\tilde{\alpha}$ denotes the associated strategy identically independently distributed. If $x \in [0, 1]$, $\hat{x}$ denotes $1 - x$.

Proposition 1

$\underline{v}(p)$ *exists and* $\underline{v}(p) = \text{Cav}\, u(p)$ *on* $[0, 1]$.

Proof

1) Let us first prove that *PI* can guarantee Cav $u(p)$. A general result for games with incomplete information states that if *PI* can guarantee some payoff $f(p)$ in $G_\infty(p)$ he can also guarantee Cav $f(p)$ [see e.g. *Sorin,* 1979, 2.17]. Since *PI* can guarantee $u(p)$ by playing non separating, i.e., by ignoring his information, the result follows.

2) It remains to show that *PI* cannot expect more. The idea of the proof is now standard [see e.g. *Mertens/Zamir,* p. 205]: given *PI's* strategy, *PII* can compute the posteriors p_n and, using the convergence of this martingale, can determine a stage N after which *PI* is essentially playing without using his information. From this stage on, *PII* can obtain $u(p_N)$ as a payoff, hence his expected average payoff will be at most Cav $u(p)$.

So let σ be a strategy for *PI* and denote by $\widetilde{R}$ the strategy of *PII* defined by always playing Right. Given σ and $\widetilde{R}$, the sequence $\{p_n\}$ is a martingale in [0, 1], hence its quadratic variation is bounded. It follows that, given $\epsilon > 0$, we can define N such that

$$E_{\sigma,\widetilde{R}} \sum_{n=1}^{\infty} (p_{n+1} - p_n)^2 \leqslant E_{\sigma,\widetilde{R}} \sum_{n=1}^{N} (p_{n+1} - p_n)^2 + \epsilon \tag{1}$$

Let us define

$$g_n(\sigma, \widetilde{R}) = E(q_n \mid H_{n-1}) \text{ and } \bar{\sigma} = E(\sigma \mid H_{n-1}).$$

Then we have [e.g. see *Sorin* [1979], 2.11]

$$| g_n(\sigma, \widetilde{R}) - g_n(\bar{\sigma}, \widetilde{R}) | \leqslant 2 L E_{\sigma,\widetilde{R}}(| p_{n+1} - p_n \| H_{n-1}). \tag{2}$$

Moreover since $\bar{\sigma}$ is non separating, there exists a pure strategy τ^* of *PII* such that

$$g_n(\bar{\sigma}, \tau^*) \leqslant u(p_n) \tag{3}$$

We can now describe the strategy τ for *PII* in G_∞:

– play according to $\widetilde{R}$ up to stage N;

– from stage $N + 1$ on, play according to τ^*.

In order to compute the payoff induced by σ and τ, we first define the stopping time X by:

$$N + X = \min [\{m; j_m = \text{Left}\} \cup \{+\infty\}] \text{ and } X_k = X \wedge k.$$

Note that X_k is $\mathcal{H}_{N+k-1}$ — measurable.

It follows then from (2) and (3) that

$$(N+n)\,\bar{\gamma}_{N+n}(\sigma,\tau) \leqslant NL + E_{\sigma,\tau}\Big[\sum_{m=1}^{X_n} u(p_{N+m}) + (n-X_n)\,u(p_{N+X_n})\Big] \tag{4}$$

$$+\,2\,LE_{\sigma,\tau}\Big[\sum_{m=1}^{X_n} |p_{N+m+1}-p_{N+m}| + (n-X_n)\,|p_{N+X_n+1}-p_{N+X_n}|\Big].$$

But we have

$$E_{\sigma,\tau}\Big[\sum_{m=1}^{X_n} u(p_{N+m}) + (n-X_n)\,u(p_{N+X_n})\Big] \leqslant E_{\sigma,\tau}\Big[\sum_{m=1}^{X_n} \operatorname{Cav} u\,(p_{N+m})$$

$$+\,(n-X_n)\operatorname{Cav} u\,(p_{N+X_n})\Big]$$

$$\leqslant n\,E_{\sigma,\tau}\operatorname{Cav} u\left[\frac{1}{n}\Big(\sum_{m=1}^{X_n} p_{N+m} + (n-X_n)\,p_{N+X_n}\Big)\right]$$

$$\leqslant n\operatorname{Cav} u\left[\frac{1}{n}E_{\sigma,\tau}\Big(\sum_{m=1}^{X_n} p_{N+m} + (n-X_n)\,p_{N+X_n}\Big)\right] \tag{5}$$

(by Jensen's inequality).

Define, for $1 \leqslant k \leqslant n$, $Y_k = \sum_{m=1}^{X_k} p_{N+m} + (n-X_k)\,p_{N+X_k}$.

Write

$$E_{\sigma,\tau}\,Y_n = E_{\sigma,\tau}\,(1_{X_n<n}\,Y_n) + E_{\sigma,\tau}\,(1_{X_n=n}\,Y_n).$$

But on $X_n < n$, $X_n = X_{n-1}$, hence $Y_n = Y_{n-1}$, so that

$$E_{\sigma,\tau}\,(1_{X_n<n}\,Y_n) = E_{\sigma,\tau}\,(1_{X_n<n}\,Y_{n-1}). \tag{6}$$

Furthermore on $X_n = n$, we have $X_{n-1} = n - 1$, hence (with $E_{\sigma,\tau} = E$)

$$E(1_{X_n=n} Y_n) = E(E\, 1_{X_n=n} Y_n \mid H_{N+n-1})$$

$$= E\,[1_{X_n=n} (\sum_{m=1}^{n-1} p_{N+m} + E(p_{n+N} \mid H_{N+n-1}))]$$

$$= E\,[1_{X_n=n} (\sum_{m=1}^{n-1} p_{N+m} + p_{N+n-1})]$$

$$= E\,[1_{X_n=n} (\sum_{m=1}^{X_{n-1}} p_{N+m} + p_{N+X_{n-1}})]$$

$$= E(1_{X_n=n} Y_{n-1}). \tag{7}$$

It follows from (6) and (7) that

$$E(Y_n) = E(Y_{n-1}) = E(Y_1) = E(n\, p_{N+1}) = n\, p. \tag{8}$$

As for the last term in the right member of (4) we obtain

$$M = E_{\sigma,\tau} (\sum_{m=1}^{X_n} |p_{N+m+1} - p_{N+m}| + (n - X_n)\, |p_{n+X_n+1} - p_{N+X_n}|)$$

$$= E_{\sigma,\tau} (\sum_{m=1}^{n} |p_{N+m+1} - p_{N+m}|\, Z_m)$$

with $Z_m = \begin{cases} 0 & \text{if } m > X_n \\ n + 1 - X_n & \text{if } m = X_n \\ 1 & \text{if } m < X_n. \end{cases}$

We obviously have $\sum_{m=1}^{n} Z_m = n$.

Now, note that the laws of the Z_m are the same under τ and $\widetilde{R}$ since τ and $\widetilde{R}$ coincide up to stage $X - 1$, and moreover the posteriors p_{N+m} and p_{N+m+1} are the same under τ and $\widetilde{R}$ on $Z_m \neq 0$. It follows that

$$M = E_{\sigma,\widetilde{R}} (\sum_{m=1}^{n} |p_{N+m+1} - p_{N+m}|\, Z_m).$$

Using the Cauchy–Schwartz inequality and (1) we now obtain

$$M \leqslant E_{\sigma,\tilde{R}} [(\sum_{m=1}^{n} (p_{N+m+1} - p_{N+m})^2 (\sum_{m=1}^{n} Z_m^2))^{1/2}]$$

$$\leqslant [E_{\sigma,\tilde{R}} \sum_{m=1}^{n} (p_{N+m+1} - p_{N+m})^2 \, E_{\sigma,\tilde{R}} \sum_{m=1}^{n} Z_m^2]^{1/2}$$

$$\leqslant \sqrt{\epsilon} \cdot n. \tag{9}$$

From (4), (5), (8) and (9) we finally get

$$(n+N)\, \bar{\gamma}_{n+N} (\sigma, \tau) \leqslant NL + n \operatorname{Cav} u(p) + 2\, Ln \sqrt{\epsilon}.$$

Thus $n \geqslant \dfrac{N}{\sqrt{\epsilon}}$ implies

$$\bar{\gamma}_{n+N} (\sigma, \tau) \leqslant \operatorname{Cav} u(p) + 4\, L \sqrt{\epsilon}.$$ Q.E.D.

Remark 1

It is easy to see that the above proof remains true for any finite sets $K, I, J,$ as long as all of the matrices $A^k, k \in K$, are of the following type: the first column is absorbing and there is no other absorbing payoffs. The only modification is in the definition of $\tilde{R}$. Let $T = \{\tau \mid \forall m, \forall h_{m-1}, \tau(h_{m-1})$ is supported by $J - \{1\}\}$. Then choose $\tilde{\tau}$ and N such that

$$\sup_{\tau \in T} E_{\sigma,\tau} \sum_{n=1}^{\infty} (p_{n+1} - p_n)^2 \leqslant E_{\sigma,\tilde{T}} \sum_{n=1}^{N} (p_{n+1} - p_n)^2 + \epsilon.$$

3 Lim v_n

Proposition 2

$\lim_{n\to\infty} v_n(p)$ *exists and equals* Cav $u(p)$ *on* $[0, 1]$.

Proof

1) We know that $v_n(p) \geqslant u_n(p) = u(p)$ and that v_n is concave, hence we have $v_n(p) \geqslant \operatorname{Cav} u(p)$ for all n.

2) Let σ be a strategy for *PI*.

Since $E_{\sigma,\tilde{R}} (\sum_{m=1}^{n} (p_{m+1} - p_m)^2)$ is bounded by some M (uniformly in σ and n)

there are at most $n^{3/4}$ stages m with

$$E_{\sigma,\tilde{R}}\,(p_{m+1}-p_m)^2 \geqslant \frac{M}{n^{3/4}}. \tag{10}$$

We denote by $R(n)$ the set of such stages, and define $S(n)=\{1,\ldots,n\}\setminus R(n)$. Given $m\in S(n)$, it follows that the probability of the set H'_{m-1} of histories h_{m-1} such that

$$E_{\sigma,\tilde{R}}\,((p_{m+1}-p_m)^2 \mid h_{m-1}) > \frac{M}{n^{1/2}} \tag{11}$$

is less than $n^{-1/4}$.

We can now describe the strategy τ of *PII*:

– play $\tilde{R}$ if $m\in R(n)$, or if $h_{m-1}\in H'_{m-1}$ with $m\in S(n)$;

– play according to τ^* otherwise, where as in the previous proof τ^* is a pure strategy satisfying (3).

We introduce the following stopping time

$$X=\min(\{m; j_m = L,\ 1\leqslant m\leqslant n\}\cup\{n\}).$$

It follows then, using (2), (10) and (11) that

$$\begin{aligned} n\bar{\gamma}_n(\sigma,\tau) &\leqslant 2\,n^{3/4}L + 2\,n\cdot n^{-1/4}L \\ &+ E_{\sigma,\tau}\Big[\sum_{m=1}^{X} u(p_m) + (n-X)\,u(p_X)\Big] \\ &+ 2\,L\,E_{\sigma,\tau}\Big[\sum_{m=1}^{X} |p_{m+1}-p_m| + (n-X)\,|p_{X+1}-p_X|\Big] \end{aligned} \tag{12}$$

First, as in the previous proof, it is easy to see that

$$E_{\sigma,\tau}\Big(\sum_{m=1}^{X} u(p_m) + (n-X)\,u(p_X)\Big) \leqslant \operatorname{Cav} u(p).$$

As for the last term in the right member of (12) we can take the expectation with respect to σ, $\tilde{R}$ since τ and $\tilde{R}$ coincide up to stage $X-1$, and then majorize by

$$E_{\sigma,\tilde{R}}\left(\sum_{m=1}^{n} |p_{m+1}-p_m|\right) + n\,E_{\sigma,\tilde{R}}\,(|p_{X+1}-p_X|\,1_{X<n}).$$

But we have

$$E_{\sigma,\tilde{R}}\left(\sum_{m=1}^{n} |p_{m+1} - p_m|\right) \leqslant \sqrt{Mn}$$

by Cauchy Schwartz inequality and

$$E_{\sigma,\tilde{R}}(|p_{X+1} - p_X| \,|\, H_{X-1}) \leqslant E_{\sigma,\tilde{R}}(p_{X+1} - p_X)^2 \,|\, H_{X-1})^{1/2}$$

$$\leqslant \frac{\sqrt{M}}{n^{1/4}} \text{ if } X < n, \text{ by (11).}$$

Coming back to (12) we obtain

$$n\,\bar{\gamma}_n(\sigma,\tau) \leqslant 2\,L\,n^{3/4} + 2\,L\,n^{3/4} + n\,\mathrm{Cav}\,u(p) + 2\,L\sqrt{M}\,n^{1/2} + 2\,L\sqrt{M}\,n^{3/4}$$

hence there exists some $K \in \mathbf{R}^+$ such that

$$\bar{\gamma}_n(\sigma,\tau) \leqslant \mathrm{Cav}\,u(p) + \frac{K}{n^{1/4}} \text{ for all } n\,. \qquad \text{Q.E.D.}$$

Remark 2

The previous proof still holds for the games described in Remark 1. But now $\tilde{R}$ has to be replaced by a "stage by stage" best reply in T to $\bar{\sigma}$.

Remark 3

If $v_\lambda(p)$ is the value of the game $G_\lambda(p)$ with payoff $\sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1}\gamma_m$, it is easy to see that $\lim_{\lambda\to 0} v_\lambda(p) = \mathrm{Cav}\,u(p)$. The first part of the proof is exactly like that in Proposition 2. The second half uses the same kind of strategy, defining first $N = \lambda^{-3/4}$ and a set of exceptional stages where $E(p_{m+1} - p_m)^2 \geqslant \frac{M}{N}$. Now for each "regular" stage m the probability of exceptional histories, i.e. such that $E((p_{m+1} - p_m)^2 \,|\, h_{m-1}) \geqslant \frac{M}{N^{2/3}}$ is less than $N^{1/3}$. We thus obtain a majorization of the payoff by some $\mathrm{Cav}\,u(p) + 0(1)\,.\,(1-(1-\lambda)^{N+1} + N^{-1/3} + (1/N^{2/3})^{1/2} + \lambda^{1/2})$, hence $\mathrm{Cav}\,u(p) + 0(\lambda^{1/4})$.

4 Minmax

In this section we shall prove the following

Theorem 3

$\bar{v}(p)$ exists.

In order to get this result we shall first assume

$$v(A) = v(B) = 0$$

by subtracting $p\,v(A) + \hat{p}\,v(B)$ from all the payoffs.

We shall split the games into several cases for each of which optimal minmax strategies for *PII* and best responses for *PI* will be constructed and an explicit formula for $\bar{v}(p)$ will be given.

First case

$$(a_{11} - a_{21})(b_{11} - b_{21}) \geqslant 0.$$

By changing the name of the lines if necessary, we can assume

$$a_{21} \geqslant a_{11} \text{ and } b_{21} \geqslant b_{11} \tag{13}$$

Let us introduce the following notations and definitions.

$$x^+ = \max(x, 0) \text{ for } x \in \mathbf{R}.$$

$$c_j(t) = c_{1_j} t + c_{2_j} \hat{t}, j = 1, 2, c = a, b \text{ for } t \in [0, 1].$$

$$Q = \{\rho; \rho \text{ positive Borel measure on } [0, 1] \text{ with total mass} \leqslant 1\}.$$

$$\rho(x) = \rho([0, x]).$$

$$w(p) = \inf_{\rho \in Q} \sup_{\substack{x \in [0,1] \\ y \in [0,1]}} [p \int_0^x a_1(t)\, d\rho(t) + (1 - \rho(x))\, a_2(x)^+ \tag{14}$$

$$+ \hat{p} \int_0^y b_1(t)\, d\rho(t) + (1 - \rho(y))\, b_2(y)^+].$$

Then we have

Proposition 4

If (13) holds, $\bar{v}(p)$ exists and equals $w(p)$.

The proof of this proposition will follow from the two next lemmas.

Lemma 5

PII cannot expect less than $w(p)$.

Proof

The idea of the proof is the same as in *Sorin* [1984], Lemma 21. Knowing τ, *PI* starts by playing $\widetilde{\text{B}}$ottom until he reaches the maximum of the probability of getting an absorbing payoff at this level. From this time on he increases his frequency slowly (i.e. he will use $(\epsilon, 1-\epsilon)$) until the maximum of the "absorbing" probability is reached and so on up to some level x. Then he will get $c_2(x)$ if he stays at $\tilde{x}$ or 0 by playing optimally. This strategy obviously induces a probability $d\rho(t)$ of getting an absorbing payoff $c_1(t)$, and it follows by (14) that the payoff will be at least $w(p)$.

First let $\underset{\sim}{m}$ be the stopping time min $\{m; j_m = L\} \cup \{+\infty\}$ and choose a large N in $\mathbf{N}$.

Given $\epsilon > 0$ and τ a strategy for *PII*, define

$$\sigma_0 = \widetilde{\text{B}}\text{ottom}$$

$$P^*(0) = \text{Prob}_{\sigma_0,\tau}(\underset{\sim}{m} < +\infty)$$

then n_0 and $P(0)$ such that

$$P(0) = \text{Prob}_{\sigma_0,\tau}(\underset{\sim}{m} \leqslant n_0) > P^*(0) - \epsilon.$$

Given σ_{r-1}, n_{r-1}, define inductively

$$\sigma_r : \text{play according to } \sigma_{r-1} \text{ up to stage } n_{r-1}, \text{then } \widetilde{\left(\frac{r}{N}, \widehat{\left(\frac{r}{N}\right)}\right)}$$

$$P^*(r) = \text{Prob}_{\sigma_r,\tau}(\underset{\sim}{m} < +\infty)$$

then $n_r \geqslant n_{r-1}$ and $P(r)$ with

$$P(r) = \text{Prob}_{\sigma_r,\tau}(\underset{\sim}{m} \leqslant n_r) > P^*(r) - \epsilon.$$

Now if *PI* uses σ_r up to stage n_r in game A, and then plays $\widetilde{\left(\frac{r}{N}, \widehat{\left(\frac{r}{N}\right)}\right)}$ if $a_2(r/N) \geqslant 0$, or optimally in game A otherwise, the expected payoff in game A for $n \geqslant n_r$ will satisfy

$$\bar{\gamma}_n^A(\sigma_r, \tau) \geqslant P(0)\left[a_1(0) - 2L\frac{n_0}{n}\right]$$

$$+ (P(1) - P(0)) \left[a_1 \left(\frac{1}{N} \right) - 2L \frac{n_1}{n} \right]$$

$$+ \dots$$

$$+ (P(r) - P(r-1)) \left[a_1 \left(\frac{r}{N} \right) - 2L \frac{n_r}{n} \right]$$

$$+ (1 - P(r)) \left[a_2 \left(\frac{r}{N} \right)^+ - 2L \frac{n_r}{n} \right]$$

$$- 2 (P^*(r) - P(r)) L.$$

If $\mu \in Q$ is the atomic measure with mass $P(\ell) - P(\ell - 1)$ at point ℓ/N, then for n large enough we obtain

$$\bar{\gamma}_n^A (\sigma_r, \tau) \geqslant \int_0^{r/N} a_1 (t)\, d\mu (t) + \left(1 - \mu\left(\frac{r}{N} \right) \right) a_2 \left(\frac{r}{N} \right)^+ - 4 \epsilon L.$$

Now there exists $r^* \in \mathbf{N}$, $0 \leqslant r^* \leqslant N$ which realizes the supremum over all reals $r \in [0, N]$ of the right member within $\frac{2L}{N}$.

A similar construction for game B induces a strategy σ for PI such that for n large enough

$$\bar{\gamma}_n (\sigma, \tau) \geqslant p \sup_{0 \leqslant x \leqslant 1} \int_0^x a_1 (t)\, d\mu (t) + (1 - \mu (x))\, a_2 (x)^+$$

$$+ \hat{p} \sup_{0 \leqslant y \leqslant 1} \int_0^y b_1 (t)\, d\mu (t) + (1 - \mu (y))\, b_2 (y)^+$$

$$- 4 \epsilon L - \frac{2L}{N}$$

hence the result is obtained by choosing N large enough. Q.E.D.

In order to prove that PII can guarantee $w(p)$ we shall use "Big Match" strategies, hence we need the following definitions and results. Let Γ_s^+ be the zero sum two person infinitely repeated game with payoff matrix

$$\begin{bmatrix} -(1-s)^* & (1-s) \\ s^* & -s \end{bmatrix}$$

(The "Big Match" of *Blackwell/Fergusson* is precisely $\Gamma^+_{1/2}$.) As above we define the stopping time $\underset{\sim}{m}$ and the payoff q^s_n at stage.

We also introduce $\tilde{t}_n = \frac{1}{n} \# \{m; i_m = T, 1 \leqslant m \leqslant n\}$ which is the frequency of Top up to stage n.

Then we have:

Proposition 6

[*Blackwell/Fergusson, Kohlberg*].

$\forall \epsilon > 0, \forall \delta > 0, \exists N_s$ *and* τ_s *strategy of PII in* Γ^+_s *such that for any* σ

$$\text{Prob}_{\sigma,\tau_s} (\underset{\sim}{m} \leqslant n) E_{\sigma,\tau_s} (q^s_m \mid \underset{\sim}{m} \leqslant n) \leqslant \epsilon \quad \forall n \tag{15}$$

$$\text{Prob}_{\sigma,\tau_s} (\underset{\sim}{m} < n \mid \tilde{t}_n \geqslant s + \delta) \geqslant 1 - \epsilon \qquad \forall n > N_s \tag{16}$$

Using this result we shall prove

Lemma 7

PII can guarantee $w(p)$.

Proof

The idea of the proof there is also similar to *Sorin* [1984], Propositon 26.

Let ρ be ϵ-optimal in (14). Then *PII* uses τ_s with probability $d\rho(s)$. It follows from (13) that a best response of *PI* is to increase his frequency, starting from 0, in order to achieve the greatest absorbing payoff, and then to decrease it if necessary, which gives (14).

Let us start with $\bar{\rho}$, $\theta/2$ optimal in (14) and choose ρ to be a discrete "θ / 2 approximation" of $\bar{\rho}$ as in *Sorin* [1984], Lemma 28, i.e. such that

$$p \left[\int_0^x a_1(t)\, d\rho(t) + (1-\rho(x))\, a_2(x)^+ \right] + \tag{17}$$
$$+ \hat{p} \left[\int_0^y b_1(t)\, d\rho(t) + (1-\rho(y))\, b_2(y)^+ \right] \leqslant w(p) + \theta$$

for all x, y in $[0, 1]$.

Let $\{s_r : r = 0, \ldots, R\}$ be the finite support of ρ. We can assume by selecting a refinement if necessary that $s_0 = 0$, $s_R = 1$, and $s_r - s_{r-1} < \eta$, where R is bounded by some $R(\theta, \eta)$ uniformly in $\bar{\rho}$.

We shall use the following notations.

$$\tau_{s_r} = \tau_r, \rho(\{s_r\}) = d\rho_r, \sum_0^r \rho(\{s_r\}) = \rho_r, N_{s_r} = N_r.$$

Also let $N = \max N_r$.

The strategy for *PII* is as follows: First choose $r^* \in R$ according to the distribution defined by Prob $(r^* = r) = d\,\rho_r, r = 0, \ldots, R-1$ and Prob $(r^* = R) = 1 - \rho_{R-1}$. If $r^* = 0$, play Left at the first stage and define $Y_0 = 1$. If $0 < r^* < R$ play Right until stage $Y_{r^*} - 1$, and Left at stage Y_{r^*}, where the stopping times Y_r are defined inductively by

$Y_1 = \min\{m : j_m = L\}$ is induced by σ and τ_1;

$Y_2 = \min\{m : j_m = L\}$ is induced by σ and τ_1 up to stage $Y_1 - 1$

and then τ_2;

...

Y_r is induced by $\sigma\, \tau_1$ up to stage $Y_1 - 1, \ldots, \tau_{r-1}$ up to stage $Y_{r-1} - 1$ and then τ_r.

Finally if $r^* = R$, always play Right (i.e. $Y_R \equiv +\infty$).

We shall prove that $\forall\, \epsilon_0 > 0$ the average payoff in game A for n large enough will be majorized uniformly for any strategy σ^A by $\alpha + \epsilon_0$ where

$$\alpha = \sup_{0 \leq x \leq 1} \alpha(x) \text{ and } \alpha(x) = \int_0^x a_1(t)\, d\rho(t) + (1 - \rho(x))\, a_2(x)^+. \tag{18}$$

Given n, σ_A and τ we define

$$Z_r = \min(Y_r, n+1), r = 0, \ldots, R$$

and

$$X_0 = 0, X_r = Z_r - Z_{r-1}, \text{hence } \sum_r X_r = n$$

$$t_r = 1_{\{i_{Z_r} = T\}}, \bar{t}_r = \frac{1}{X_r} \#\{i_m = T; Z_{r-1} \leq m < Z_r\}.$$

Now since the strategy of *PII* is independent of r^*, up to stage Y_{r^*} we obtain

$$n\, \bar{\gamma}_n^A(\sigma, \tau) = E\Big(\sum_1^R X_r\, [d\,\rho_0\, a_1(t_0) + \ldots + d\,\rho_{r-1}\, a_1(t_{r-1}) + \tag{19}$$

$$+ (1 - \rho_{r-1}\, a_2(\bar{t}_r)]\Big).$$

Let us first consider the term with $a_2(\cdot)$.

i) If $a_{22} > a_{12}$ with $a_{22} > 0$, then $v(A) = 0$ implies $a_{21} \leqslant 0$, hence $n\,\bar{\gamma}_n^A(\sigma, \tau) \leqslant \alpha(0) = a_{21}\, d\,\rho_0 + (1 - \rho_0)\, a_{22}$ by (13), and the result follows.

ii) If $a_{22} > a_{12}$ with $a_{22} = 0$, then $a_2(\bar{t}_r) \leqslant 0 = a_2(s_{r-1})^+ \ \forall r$

iii) If $a_{22} \leqslant a_{12}$ we majorize the coefficient of $a_{12} - a_{22}$:

$$E(X_r\, \bar{t}_r) = E(1_{X_r < N}\, X_r\, \bar{t}_r) + E(1_{X_r \geqslant N}\, X_r\, \bar{t}_r).$$

For the second term, since during these X_r stages from Z_{r-1} up to $Z_r - 1$, *PII* is using σ_r, it follows from (16) that

$$E(X_r\, \bar{t}_r) \leqslant N + E(X_r)(s_r + \delta + \epsilon) \leqslant N + E(X_r)(s_{r-1} + \delta + \eta) + \epsilon\, n.$$

Coming back to (19) and using (13) we obtain in cases ii) and iii)

$$n\,\bar{\gamma}_n^A(\sigma, \tau) \leqslant E\Big(\sum_1^R X_r\, \alpha(s_{r-1})\Big) + R\,N\,L + n\,L(\delta + \eta) + R\,L\,\epsilon\, n + L\,\Delta \qquad (20)$$

with

$$\Delta = E\Big[\sum_1^R X_r\big(d\,\rho_0\,(s_0 - t_0) + \ldots + d\,\rho_{r-1}\,(s_{r-1} - t_{r-1})\big)\Big].$$

Hence

$$\Delta \leqslant \sum_{i=1}^R d\,\rho_{r-1}\, E\Big[\sum_{\ell=r}^R X_\ell\,(s_{r-1} - t_{r-1})\Big].$$

Note that $s_{r-1} - t_{r-1} = -(1 - s_{r-1})\, t_{r-1} + s_{r-1}(1 - t_{r-1})$ is the absorbing payoff in $\Gamma_{s_{r-1}}$ and $\sum_{\ell=r}^R X_\ell = (n + 1 - Y_r)^+$ is the number of stages during which σ_{r-1} induces such an absorbing payoff. It follows then from (15) that

$$E\Big(\sum_{\ell=r}^R X_\ell\,(s_{r-1} - t_{r-1})\Big) \leqslant \epsilon\, n.$$

Substituting in (20) we obtain

$$n\,\bar{\gamma}_n^A(\sigma, \tau) \leqslant n\,\alpha + R\,N\,L + n\,L(\delta + \eta) + R\,L\,\epsilon\, n + L\,\epsilon\, n.$$

Obviously, a same result holds for $\bar{\gamma}_n^B$.

Given ϵ_0, we take $\theta = \frac{\epsilon_0}{2}$. Then $\eta = \frac{\epsilon_0}{8L}$, which determines some $R(\theta, \eta)$. We define τ_s according to $\delta = \frac{\epsilon}{8L}$ and $\epsilon = \frac{\epsilon_0}{8L(R+1)}$. This defines N_s, hence N. It follows then that $n \geqslant \frac{8RNL}{\epsilon_0}$ implies $\bar{\gamma}_n(\sigma, \tau) \leqslant w(p) + \epsilon_0$. Q.E.D.

This completes the proof in the first case.

For the other cases it is more convenient to work in the space of vector payoffs induced by A and B, and to determine the sets that *PII* can approach [see *Blackwell*, 1956]. Some definitions follow.

PII can approach $(x, y) \in \mathbf{R}^2$ if, $\forall \epsilon > 0$, $\exists\, \tau$ and $\exists\, N$ such that, $\forall\, \sigma$, $\forall\, n \geqslant N$,

$$\bar{\gamma}_n^A(\sigma, \tau) \leqslant x + \epsilon$$

$$\bar{\gamma}_n^B(\sigma, \tau) \leqslant y + \epsilon$$

where $\bar{\gamma}_n^A$ is the average expected payoff in game A.

D_{II} is the set of vector payoffs that *PII* can approach, and note that D_{II} is closed, convex, and $D_{II} = D_{II} + (\mathbf{R}^+)^2$.

Given an half space $D(p, \alpha) = \{x, y; p\,x + p'\,y \geqslant \alpha\}$, $p \in [0, 1]$, $\alpha \in \mathbf{R}$, we say that *PI* can force $D(p, \alpha)$ if

$$\forall\, \tau, \forall\, \epsilon, \exists\, \sigma \text{ and } \exists\, N \text{ such that for all } n \geqslant N \quad p\,\gamma_n^A(\sigma, \tau) + \hat{p}\,\gamma_n^B(\sigma, \tau) \geqslant \alpha - \epsilon.$$

Note that if *PI* can force $D(p, f(p))$, he can also force $D(p, \operatorname{Cav} f(p))$.

In fact, let p_1 and p_2 be such that $\operatorname{Cav} f(p) \leqslant \lambda f(p_1) + (1-\lambda) f(p_2) + \frac{\epsilon}{2}$ and $\lambda p_1 + (1-\lambda) p_2 = 1$.

Given τ and $\frac{\epsilon}{2}$ note σ_1 for $\sigma\left(p_1, f(p_1), \tau, \frac{\epsilon}{2}\right)$ as defined above and likewise for σ_2.

Let σ^A be σ_1^A with probability $\frac{\lambda \hat{p}_1}{p}$, σ_2^A otherwise, and let σ^B be σ_1^B with probability $\frac{\lambda p_1}{\hat{p}}$, σ_2^B otherwise. Then σ forces $D(p, \operatorname{Cav} f(p))$.

Denote by D_I the intersection of the sets $D(p, \alpha)$ that *PI* can force. The existence of $\bar{v}$ is now equivalent to the fact that $D_I = D_{II}$, denoted by D, and then

$$\bar{v}(p) = \min_{(x,y) \in D} \{p\,x + (1-p)\,y\}.$$

Redefining the games if necessary we can assume

(I) $a_{11} > a_{21}$ and $b_{11} < b_{21}$

(and since $v(A) = v(B) = 0$ we have $a_{11} \geqslant 0$ and $b_{21} \geqslant 0$).

We introduce some notation

$$T^* = (a_{11}, b_{11}),\ B^* = (a_{21}, b_{21})$$

$$X = (a_{22}^+, 0),\ Y = (0, b_{12}^+),\ Z = (a_{22}^+, b_{12}^+).$$

If P_1 and P_2 are two points in $\mathbf{R}^2$ on a line $p\,x + (1-p)\,y = \alpha$, $p \in [0, 1]$, then $H(P_1, P_2)$ is $D(p, \alpha)$.

Finally, $H_x = \{(x, y) : x \geqslant 0\}$ and similarly for H_y.

Second case

We now assume

(II) $\max(a_{12}, a_{22}, 0) = a_{22}^+$, $\max(b_{12}, b_{22}, 0) = b_{12}^+$

(III) $Z \in H(T^*, B^*)$.

Lemma 8

PI can force $H(T^*, Y)$, $H(B^*, X)$ and $H(B^*, T^*)$.

Proof

1) We show first that *PI* can force $H(T^*, Y)$ (hence $H(B^*, X)$ by symmetry). Given τ, let $\tilde{T} = \widetilde{\text{T}}\text{op}$ and define

$$\theta = \text{Prob}_{\tilde{T}, \tau}(\underset{\sim}{m} < +\infty).$$

By playing always $\tilde{T}$, or by switching after a large number of stages to an optimal strategy in the corresponding game, *PI* will reach

$$\theta\, a_{11} + (1-\theta) \max(a_{12}, v(A)) \text{ in game } A$$

$$\theta\, b_{11} + (1-\theta) \max(b_{12}, v(B)) \text{ in game } B.$$

and this vector payoff dominates weakly $\theta\, T^* + (1-\theta)\, Y$.

2) Now let us introduce σ_0 and N_0 such that

$$\text{Prob}_{\sigma_0, \tau}(\underset{\sim}{m} \leqslant N_0) \geqslant \sup_{\sigma} \text{Prob}_{\sigma, \tau}(\underset{\sim}{m} < +\infty) - \epsilon.$$

Then *PI* uses σ_0 up to stage N_0, and then plays Bottom or optimally in game A (and symmetrically for B). It follows that for large enough n that the vector payoff will be at least some

$$\theta\, W^* + (1-\theta)\, Z + 3\, \epsilon\, L$$

where W^* is an absorbing payoff on the segment $[B^*, T^*]$. Now (III) implies the result. Q.E.D.

We are now in position to state

Proposition 9

If (I), (II), (III) *hold, then*

$$D_I = D_{II} = D = H_x \cap H_y \cap H(B^*, T^*) \cap H(T^*, Y) \cap H(B^*, X).$$

Proof

Since $v(A) = v(B) = 0$, we obviously have $D_I \subset H_x \cap H_y$, hence $D_I \subset D$ by Lemma 9.

Hence it remains to prove $D \subset D_{II}$, and for this it is sufficient to show that the extreme points of the (strict) Pareto boundary $\bar{D}$ of D belongs to D_{II}.

Let us denote by $\bar{X} = (\bar{x}, 0)$ and $\bar{Y} = (0, \bar{y})$ the points on the axes of $\bar{D}$, and let us first prove that *PII* can approach $\bar{X}$ and $\bar{Y}$.

Let τ_B an η_B-optimal strategy for *PII* in the infinitely repeated game with payoff matrix B satisfying

$$n \geqslant N_B \Rightarrow \bar{\gamma}_n^B(\sigma, \tau_B) \leqslant \eta_B \tag{21}$$

$$\forall\, n \;\; \mathrm{Prob}_{\tau_B}(\underset{\sim}{m} = n) \leqslant \beta \tag{22}$$

where β is a parameter to be specified later.

We shall exhibit a strategy for *PII* which approaches $\bar{X}$. It is enough to consider the case where *PI* uses a pure strategy, hence a sequence of moves $\{i_1, \ldots, i_n, \ldots\}$. We shall still write h_n for the n-stage history corresponding to these moves. (Note that here h_n does not include the moves of *PII*.)

We now introduce

$$p_n^* = \mathrm{Prob}_{\tau_B}(\underset{\sim}{m} \leqslant n \mid H_n)$$

$$t_n^* = E_\tau(t_{\underset{\sim}{m}} \mid \underset{\sim}{m} \leqslant n,\, H_n) \text{ with } t_k = 1_{\{i_k = \mathrm{Top}\}}.$$

Hence p_n^* is the "absorbing" probability up to stage n and t_n^* is the corresponding "absorbing" frequency.

The strategy τ is as follows.

- First *PII* uses $\tau(\bar{X})$ which is:

— play τ_B at stage $n + 1$ if either

(*) $p_n^* \leqslant \beta$

or

(**) $b_1(t_n^*) \geqslant -\frac{\beta L}{d}$

with $d = \max\left(\lambda, \frac{1}{\lambda}\right)$ where $-\lambda$ is the slope of the line $T^* B^*$.

— if (*) and (**) fail, play Left with probability β if

(***) $p_n^* b_1(t_n^*) + (1 - p_n^*)\bar{y} > 0$.

- If at some stage $\theta_1 + 1$, (*), (**) and (***) fail, *PII* uses $\tau(\bar{Y})$ in the game starting at that stage, which is defined symmetrically (starting with τ_A, with parameters η_A and β).

Let us denote by $\theta_1, \theta_1 + \theta_2, \ldots,$ the "reversing times" defined by

on $\{1, \ldots, \theta_1\}$	*PII* plays $\tau(\bar{X})$
on $\{\theta_1 + 1, \ldots, \theta_1 + \theta_2\}$	*PII* plays $\tau(\bar{Y})$
on $\{\theta_1 + \theta_2 + 1, \ldots, \theta_1 + \theta_2 + \theta_3\}$	*PII* plays $\tau(\bar{X})$, and so on,

and let $K_1, K_2, \ldots,$ be the corresponding blocks of stages.

On each block K_i we define the exceptional stages to be such that (*) and (**) fail and (***) holds. It follows then from the definitions, that there is at most a finite number $N(\alpha)$ or $N(\beta)$ of such stages.

Given $\epsilon > 0$, we want to show that *PII* can approach $\bar{X}$ within ϵ. We obviously can assume $\bar{x} > 0$ $\left(\text{otherwise } PII \text{ will approach } \bar{x} + \frac{\epsilon}{2} \text{ within } \frac{\epsilon}{2}\right)$, and similarly $\bar{y} > 0$.

Now this implies that for (***) and (**) to fail the absorbing probability has to be greater than some $p_B > 0$ (resp. p_A). It follows that after a finite number of "reversing times", the total absorbing probability $M(\epsilon)$ will be greater than $1 - \frac{\epsilon}{3L}$ hence the remaining payoff will be bounded by $\frac{\epsilon}{3}$.

Thus we denote by M the finite number of blocks, and τ will approach $\bar{X}$ within $\frac{2\epsilon}{3}$. Let

$$\eta_A = \eta_B = \frac{\epsilon}{6}, \beta = \frac{\epsilon}{6Ld} p_B, \alpha = \frac{\epsilon}{6Ld} p_A$$

$$N_0 = \max(N(\alpha), N(\beta)), N_1 = \max(N_A, N_B) \text{ and } N = N_0 + N_1.$$

For each block of small length (i.e. less than N) we majorize the payoff per stage by L. For each other block K_i we majorize the payoffs corresponding to the exceptional stage, and we denote by $\overline{\rho_i}$ the average vector payoff of the λ_i other stages (i.e. where PII is using τ_A or τ_B).

Then we obtain

$$n\,\bar{\gamma}_n(\sigma, \tau) \leqslant MNL + \sum_{i \in M} \lambda_i \overline{\rho_i} \tag{23}$$

where $\bar{\gamma}_n$ is the vector payoff $(\bar{\gamma}_n^A, \bar{\gamma}_n^B)$ and MNL is written for $MNL(1, 1)$. Since $\Sigma\,\lambda_i + MN \geqslant n$ it is enough to prove

$$\overline{\rho_i} \leqslant \bar{X} + \frac{\epsilon}{3} \tag{24}$$

and from (23) we obtain for large enough n

$$\bar{\gamma}_n(\sigma, \tau) \leqslant \bar{X} + \frac{2\epsilon}{3}.$$

Now we write, with $p^i = p^*_{\theta_i}$ and $t^i = t^*_{\theta_i}$

$$\overline{\rho_i} = p^1 \begin{pmatrix} a_1(t^1) \\ b_1(t^1) \end{pmatrix} + \ldots + (1-p^1)(1-p^2)\ldots(1-p^{i-2})\,p^{i-1} \begin{pmatrix} a_1(t^{i-1}) \\ b_1(t^{i-1}) \end{pmatrix}$$
$$+ (1-p^1)\ldots(1-p^{i-1})\,\bar{g}_i$$

where the first terms correspond to the absorbing payoffs obtained during the preceding blocks, and $\bar{g}_i$ is the new average payoff on block K_i during the regular stages.

Assume i odd. Then we shall first prove

$$\bar{g}_i \leqslant \bar{X} + \frac{\epsilon}{3} \tag{25}$$

and then that (25) implies

$$p^{i\text{-}1}\begin{pmatrix} a_1\,(t^{i\text{-}1}) \\ b_1\,(t^{i\text{-}1}) \end{pmatrix} + (1-p^{i\text{-}1})\,\bar{g}_i \leqslant Y + \frac{\epsilon}{3} \tag{26}$$

hence (24) by induction.

Let us now majorize $\bar{g}_i$.

Since *PII* is using τ_B and the number of stages is greater than N_B, the payoff in game B is at most

$$\bar{g}_i^B \leqslant \eta_B = \frac{\epsilon}{6}.$$

As for the payoff in game A, we have at the last regular stage: either

(*) $p^* \leqslant \beta$, hence

$$\bar{g}_i^A \leqslant \beta_0\, L + (1-\beta_0)\max(a_{12}, a_{22}) \text{ for some } \beta_0 \leqslant \beta$$

$$\leqslant x + \beta L \qquad\qquad \text{(using III)}$$

$$\leqslant \bar{x} + \beta L \quad \text{by definition of } \bar{X}$$

$$\leqslant \bar{x} + \frac{\epsilon}{6}$$

or

(**) $b_1\,(t^*) \geqslant -\dfrac{\beta L}{d}$ and then :

$$\bar{g}_i^A \leqslant p^*(x_0 + \beta L) + (1-p^*)\,x$$

where $(x_0, 0)$ is on $[T^*, B^*]$, hence by definition of $\bar{X}$

$$\bar{g}_i^A \leqslant \bar{x} + \beta L \leqslant \bar{x} + \frac{\epsilon}{6}$$

This proves (25).

As for (26), first note that (***) implies, by definition of $\theta_{i\text{-}1}$, that

$$p^{i\text{-}1}\, a_1\,(t^{i\text{-}1}) + (1-p^{i\text{-}1})\,\bar{x} \leqslant 0.$$

Hence it remains to majorize the second component. Denoting the stage $\theta_{i-1} - 1$ by k, there are three cases:

– if $p_k^* \leqslant \alpha$, then $p^{i-1} \leqslant 2\alpha$, thus $p^{i-1}\, b_1\,(t^{i-1}) \leqslant 2\,\alpha L$

– if $a_1\,(t_k^*) \geqslant -\dfrac{\beta L}{d}$, then $b_1\,(t_k^*) \leqslant y_0 + \alpha L$

where $(0, y_0)$ is on the line $B^* X$. It follows that $p^{i-1}\, b_1\,(t^{i-1}) \leqslant y_0 + 2\,\alpha L$.

– finally, if $p_k^*\, a_1\,(t_k^*) + (1 - p_k^*)\,\bar{x} > 0$, this implies $p_k^*\, b_1\,(t_k^*) < y_0$

hence $p^{i-1}\, b_1\,(t^{i-1}) \leqslant y_0 + \alpha L$.

In both cases we obtain

$$p^{i-1}\, b_1\,(t^{i-1}) + (1 - p^{i-1})\,\bar{g}_i^B \leqslant \bar{y} + 2\,\alpha L + (1 - p_A)\,\frac{\epsilon}{3} \leqslant \bar{y} + \frac{\epsilon}{3}$$

which gives (26) and achieves the proof that *PII* can approach $\bar{X}$ (or $\bar{Y}$).

It remains to show the following

Lemma 10

Assume that PII can approach U and V with

(i) *U and $V \in H(B^*, T^*)$, $u_1 < v_1$, $u_2 > v_2$.*

(ii) *$U \in H(B^*, V)$, $V \in H(T^*, U)$.*

Then PII can approach W which is the intersection of VB^* and UT^*.

Proof

Since D_{II} is closed and convex, we can define $U(\lambda_1)$ and $v(\lambda_2)$ where

$$U(\lambda) = \lambda T^* + (1 - \lambda)\, U$$

$$V(\lambda) = \lambda B^* + (1 - \lambda)\, V$$

and $\lambda_1 = \max\,\{\lambda;\, U(\lambda) \in D_{II}\}$; $\lambda_2 = \max\,\{\lambda;\, V(\lambda) \in D_{II}\}$. If $W \notin D_{II}$ we can redefine U to be $U(\lambda_1)$ and V to be $V(\lambda_2)$, and i), ii) still hold.

Introduce λ_0 such that $u_1\,(\lambda_0) = v_1\,(\lambda_0)$, and note that $\lambda_0 > 0$.

PII uses the following strategy τ: play Left with probability λ_0 at stage 1; from stage 2 on play according to τ_U (resp. τ_V) if i_1 = Top (resp. Bottom), where τ_U (resp. τ_V) approach U (resp. V). The vector payoff that τ approaches is now:

$$\text{if } i_1 = \text{Top}, \qquad \lambda_0 T^* + (1-\lambda_0) U = U(\lambda_0)$$

$$\text{if } i_1 = \text{Bottom}, \quad \lambda_0 B^* + (1-\lambda_0) V = V(\lambda_0).$$

But by the choice of λ_0, $U(\lambda_0)$ dominates $V(\lambda_0)$ or reciprocally, hence *PII* can approach either $U(\lambda_0)$ or $V(\lambda_0)$, contradicting the definition of U or V. Q.E.D.

It follows that *PII* can approach the extreme points of $\bar{D}$ and this finishes the proof of Proposition 9. Q.E.D.

Third case

Here we assume (I), (II) and (III'): $Z \notin H(T^*, B^*)$.

We first remark that *PII* can approach Z by playing Right and (see Lemma 8) that *PI* can force a payoff $\theta W^* + (1-\theta) Z$ with $W^* \in [B^*, T^*]$. It follows that if $Z = (0, 0)$ then $D_I = D_{II} = (\mathbf{R}_+)^2$ and obviously $\bar{v} = 0$.

Hence we can assume $z_1 > 0$. Thus $z_1 = a_{22}$ and we shall determine the points $(k, \varphi(k))$, $k \in [0, a_{22}]$ that *PII* can approach. (Obviously the analysis is similar if $z_2 > 0$, for the points $(\psi(k), k)$, $k \in [0, b_{12}]$.) Given $k \in [a_{12}^+, a_{22}]$, let $t_k \in [0, 1]$ be such that $a_2(t_k) = k$ and define $S_k = t_k T^* + (1 - t_k) B^*$.

Let us now introduce φ on $[a_{12}^+, a_{22}]$ such that, if $C_k = (k, \varphi(k))$, the line $S_k C_k$ is tangent to the graph $g(\varphi)$ of φ at C_k and $\varphi(a_{22}) = b_{12}^+$ (i.e. $Z \in g(\varphi)$). Formally we obtain for the line $S_k C_k$

$$x_2 - \varphi(k) = \lambda(k)(x_1 - k).$$

Thus

$$\lambda(k) = \varphi'(k)$$

which gives

$$\varphi(k) = \frac{b_{11} a_{22} - b_{21} a_{12}}{a_{22} - a_{12}} + \frac{b_{11} - b_{21}}{a_{11} - a_{21}} k + K k^{\frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}}$$

with K such that $\varphi(a_{22}) = b_{12}^+$.

Now if $a_{12} > 0$, then $a_{11} = 0$, and we define $g(\varphi)$ on $[0, a_{12}]$ to be the segment $T^* C_{a_{12}}$.

We first prove the following

Lemma 11

PII can approach $(k, \varphi(k))$, $\forall k \in [0, a_{22}]$.

Proof

Let us assume $k \in [a_{12}^+, a_{22}]$.

The idea of the proof is the following: we shall define a finite sequence of vector payoffs on $g(\varphi)$, $C(r), r = 0, \ldots, R$, starting from $C(0) = C_k$ and reaching $C(R) = Z$.

The strategy for *PII* will be such that if the absorbing probability is small the non absorbing payoff approaches $C(0)$. If not, the absorbing payoff will be such that from some stage n, it will be enough for *PII* to approach $C(1)$, and so on. Since *PII* can approach $C(R) = Z$, the induction will be complete.

Given R large in N we introduce

$$k_r = k + \frac{r}{R}(a_{22} - k), r = 0, \ldots, R$$

and we denote t_{k_r} by x_r, S_{k_r} by $S(r)$, and C_{k_r} by $C(r)$.

Let τ_r be an α-optimal strategy for *PII* in the game

$$\begin{bmatrix} (1-x_r)^* - (1-x_r) \\ \\ -x_r^* \qquad x_r \end{bmatrix}$$

with $\text{Prob}_{\sigma,\tau_r}(\underset{\sim}{m} = n) \leqslant \alpha, \forall \sigma, \forall n$; and let N_r be such that

$$E_{\sigma,\tau_r}(\bar{q}_n \mid \underset{\sim}{m} \geqslant n) \leqslant \alpha, \forall n \geqslant N_r, \forall \sigma.$$

We also define, as in the second case,

$$p_n^* = \text{Prob}(\underset{\sim}{m} \leqslant n) \text{ and } t_n^* = E[i_{\underset{\sim}{m}} \mid \underset{\sim}{m} \leqslant n].$$

Now *PII* starts by playing τ_0 as long as

$$(*) \quad p_n^* \left[a_1 \left(t_n^* - \frac{1}{R} \right) \right] - \alpha L + (1 - p_n^*) c_1(1) > k.$$

If (*) fails for the first time after stage θ_1, *PII* uses τ_1 in order to approach $C(1)$, in the game starting at stage $\theta_1 + 1$, conditionally on $\underset{\sim}{m} > \theta_1$. Hence he will play τ_1 as long as

$$p_n^* \left[a_1 \left(t_n^* - \frac{1}{R} \right) \right] - \alpha L + (1 - p_n^*) c_1(2) > k_1.$$

and so on until occasionally reaching θ_R, and then *PII* will play $\widetilde{\text{Right}}$ and approach Z.

Thus it is enough to prove that *PII* approaches $C(0)$ within ϵ on the first block (i.e. from stage 1 to θ_1), if θ_1 is large enough, and that, if *PII* approaches $C(1)$ within ϵ on the second block conditionally on $\underset{\sim}{m} > \theta_1$, its total vector payoff on this block will be $C(0)$ within ϵ.

Assume then that (*) holds. Since *PII* is using τ_0, its average payoff in game A is at most $k + L\,\alpha$.

As for the absorbing payoff we have

$$p_n^* (t_n^* - x_0) \leqslant \alpha$$

hence

$$p_n^* a_1 \left(t_n^* - \frac{1}{R} \right) - \alpha L \leqslant p_n^* s_1 (1).$$

Since $C(0) \in H(S(1)\, C(1))$ (*) implies

$$p_n^* s_2 (1) + (1 - p_n^*) c_2 (1) \leqslant \varphi(k).$$

Now the non absorbing payoff in game B is at most $z_2 \leqslant c_2(1)$, thus

$$\gamma_n^B \leqslant p_n^* b_1 (t_n^*) + (1 - p_n^*) c_2 (1) \leqslant \varphi(k) + \alpha L D + \frac{L}{R}.$$

Assume now that *PII* approaches $C(1)$ within ϵ, in the second block, conditionally on $\underset{\sim}{m} > \theta_1$. Let $p = p_{\theta_1}^*$ and define S to be the absorbing vector payoff corresponding to $t_{\theta_1}^* = t$. Now if M is on $[B^*\, T^*]$ with $p\, m_1 = p\, a_1 \left(t - \frac{1}{R} \right) - \alpha L$ then $|\, p\, m_1 + (1-p) c_1 - k \,| < \alpha L$. Now $p\, m_1 < p\, s_1(1)$ implies as above $|\, p\, m_2 + (1-p) c_2 - \varphi(k) \,| < \alpha L\, d$. It follows that

$$p\, s_1 + (1-p)(c_1 + \epsilon) \leqslant k + 2\,\alpha L + p\,\frac{L}{R} + (1-p)\,\epsilon$$

$$p\, s_2 + (1-p)(c_2 + \epsilon) \leqslant \varphi(k) + 2\,\alpha L\, d + p\,\frac{L}{R} + (1-p)\,\epsilon.$$

Thus given ϵ we first choose $R > \frac{2L}{\epsilon}$. This gives a minorant q for the p_n^* where (*) fails (for all r) and we take $\alpha < \frac{\epsilon q}{4 L d}$.

If $k \in [0, a_{12}]$ with $a_{12} > 0$, then *PII* will play $\tilde{\alpha}$, where α is small, in order to reach an absorbing payoff S^* on $[B^*\, T^*]$ satisfying

$$|\, p_n^* s_1 + (1 - p_n^*) a_{12} - k \,| < \alpha L$$

and then he will approach $C_{a_{12}}$ as above. Q.E.D.

Let $\Delta = \{(x, y) \mid (x, y) \in M + (\mathbf{R}_+)^2$ for some $M \in g(\varphi) \cup g(\psi)\}$.

Thus it remains to prove

Lemma 12

$D_I = \Delta$.

Proof

We shall prove that *PI* can force $H(S_h\ C_h)$, $\forall\ C_h \in g(\varphi)$. (A similar proof works for $g(\psi)$) and we define p, α to be such that $D(p, \alpha) = H(S_h\ C_h)$.

Given τ, ϵ and R large in N, we define inductively, as in the first case

$$\sigma_0 = \widetilde{\text{Top}}$$

$$P^*(0) = \text{Prob}_{\sigma_0, \tau}(\underset{\sim}{m} < +\infty)$$

and n_0 and $P(0)$ with

$$P(0) = \text{Prob}_{\sigma_0, \tau}(\underset{\sim}{m} \leqslant n_0) > P^*(0) - \epsilon.$$

Now given (σ_{r-1}, n_{r-1}) we introduce

$$\sigma_r = \text{play according to } \sigma_{r-1} \text{ up to stage } n_{r-1}, \text{ and then } \widetilde{\left(1 - \frac{r}{R}, \frac{r}{R}\right)}$$

$$P^*(r) = \text{Prob}_{\sigma_r, \tau}(\underset{\sim}{m} < +\infty)$$

and finally $n_r \geqslant n_{r-1}$ and $P(r)$ such that

$$P(r) = \text{Prob}_{\sigma_r, \tau}(\underset{\sim}{m} \leqslant n_r) > P^*(r) - \epsilon.$$

Introducing $\rho \in Q$ with

$$\rho\left(\left[\ 0, \frac{r}{R}\right]\right) = P(r)$$

it follows, as in the first case, that *PI* can obtain for n large enough

$$\varphi_A(\rho, x) = \int_0^x \hat{a}_1(t)\, d\rho(t) + (1 - \rho(x))\, \hat{a}_2(x), \text{ where } \hat{a}_i(t) = a_i(\hat{t}) \qquad (27)$$

in game A, up to ϵ, by playing the relevant σ_r.

Note also that by playing σ_R up to some large stage and then as in Lemma 8.2, *PI* can obtain, up to some ϵ, a vector payoff

$$W = \rho(1)\,U + (1-\rho(1))\,V \tag{28}$$

where

$$\rho(1)\,U = \begin{pmatrix} \int_0^1 \hat{a}_1(t)\,d\rho(t) \\ \\ \int_0^1 \hat{b}_1(t)\,d\rho(t) \end{pmatrix}$$

and

$$V \in \Delta_0 = \{\alpha Z + (1-\alpha) S;\, \alpha \in (0,1], S \in [T^* B^*]\}.$$

Hence *PI* can force the following

$$\inf_{\rho \in Q} \Big[p \max\{ \sup_{0 \leqslant x \leqslant 1} \varphi_A(\rho, x), \rho(1) u_1 + (1-\rho(1)) v_1\}$$

$$+ \hat{p}' \max\{ \sup_{0 \leqslant y \leqslant 1} \varphi_B(\rho, x), \rho(1) u_2 + (1-\rho(1)) v_2\}.$$

Note first that $W \in \Delta_0$. Hence if $w_1 \geqslant z_1$, then $W \in D(p, \alpha)$. So it is enough to prove that for any $k \in [0, a_{22}]$

$$\varphi_A(\rho, x) \leqslant k \quad \forall\, x \in [0,1] \tag{29}$$

and

$$\rho(1) u_1 + (1-\rho_1) v_1 \leqslant k \tag{30}$$

which implies

$$\rho(1) u_2 + (1-\rho_1) v_2 \geqslant \varphi(k). \tag{31}$$

i) Assume $k \in [a_{12}^+, a_{22}]$.

We shall introduce a distribution $\bar{\rho}$ (and a corresponding $\bar{W}$) satisfying $\bar{w}_2 = \varphi(k)$, and we shall show first when $V = Z$, and then for $V \neq Z$, that $w_2 \geqslant \bar{w}_2$, hence the result will follow.

So let

$$\bar{\rho}(t) = 1 + K \left(t + \frac{a_{12} - a_{11}}{a_{11} + a_{22} - a_{12} - a_{21}} \right)^{\frac{a_{12} - a_{11}}{a_{11} + a_{22} - a_{12} - a_{21}}}$$

where K is such that $\bar{\rho}(\hat{t}_k) = 0$. It is now easy to see that

$$\varphi_A(\bar{\rho}, x) = k \quad \forall x \in [\hat{t}_k, 1] \tag{32}$$

$$\varphi_A(\bar{\rho}, x) \leqslant k \quad \forall x \in [0, 1]$$

$$\rho(1)\,\bar{u}_2 + (1 - \rho(1))\,z_2 = \varphi(k).$$

Let us prove that $\bar{\rho}$ is the best that *PII* can do.

Assume first that $V = Z$:

– if $\rho(1) \geqslant \bar{\rho}(1)$, then W belongs to $H(\bar{d})$ (by (III')), where $\bar{d}$ is the parallel to $B^* T^*$ through $\bar{W}$. Now (30) and (32) imply that $w_2 \geqslant \bar{w}_2$.

– if $\rho(1) \leqslant \bar{\rho}(1)$, (29) and (32) imply then that $\rho(x) \geqslant \bar{\rho}(x)$ for x in some neighbourhood of $\hat{t}_k$. (Note that the non absorbing payoff is greater than k if $x > \hat{t}_k$.)

Let x_0 be the last point where

$$\rho(x) = \bar{\rho}(x) \text{ with } \rho \geqslant \bar{\rho} \text{ on } [0, x].$$

By (29) and (32) we obtain

$$\int_0^{x_0} \hat{a}_1(t)\,(d\rho(t) - d\bar{\rho}(t)) \leqslant 0.$$

Thus integrating by parts

$$(a_{11} - a_{12}) \int_0^{x_0} (\rho(t) - \bar{\rho}(t)\,dt) \leqslant 0$$

which is a contradiction if $\rho \neq \bar{\rho}$.

Next assume $V \neq Z$.

Using (29), we can assume $v_1 > z_1$ (otherwise $v_2 \geqslant z_2$). We add some mass to $\bar{\rho}$ at point 1 in order to get $\tilde{\rho}$ with

$$\bar{\rho}(1)\,\bar{u}_1 + (\tilde{\rho}(1) - \bar{\rho}(1))\,a_{21} + (1 - \tilde{\rho}(1))\,v_1 = k.$$

By (32) it follows that the first component of $\tilde{\rho}(1) B^* + (1 - \tilde{\rho}(1))\, V$ is smaller than the first component of $\bar{\rho}(1) B^* + (1 - \bar{\rho}(1))\, Z$. Since $V \in \Delta_0$, we have the reverse the order on the second component, hence $(\bar{\rho}, Z)$ is better than $(\tilde{\rho}, V)$ for *PII*.

We now compare $\tilde{\rho}$ and ρ.

On the first component we have

$$\tilde{\rho}(1)\tilde{u}_1 + (1-\tilde{\rho}(1))v_1 = k \geqslant \rho(1)u_1 + (1-\rho(1))v_1$$

and we shall prove that $\tilde{\rho}(1) \leqslant \rho(1)$, hence as above $w_2 \geqslant \tilde{w}_2$. Otherwise we have

$$\tilde{\rho}(1)\tilde{u}_1 > \rho(1)u_1$$

but

$$\tilde{\rho}(1)\tilde{u}_1 = \bar{\rho}(1)u_1 + (\tilde{\rho}(1) - \bar{\rho}(1))a_{21}$$

$$\leqslant \bar{\rho}(1)\bar{u}_1 + (\rho(1) - \bar{\rho}(1))a_{21} \leqslant \rho(1)u_1$$

since as above $\rho \geqslant \bar{\rho}$ and $\hat{a}_1(.)$ is decreasing.

This completes the proof for case i).

ii) Assume $a_{12} > 0$ and $k \in [0, a_{12}]$.

Let λ be such that $(1-\lambda)a_{12} = k$. Then ρ_k is defined to be $\lambda\,\delta_0 + (1-\lambda)\,\bar{\rho}_{a_{12}}$, δ_0 being the Dirac mass at 0 and $\bar{\rho}_{a_{12}}$ corresponding to the $\bar{\rho}$ defined in i) for $k = a_{12}$.

It is straightforward to check that the analogue to (32) holds, and the proof is similar to i). Q.E.D.

Fourth case

It remains to study the games for which (I) holds and (II) fails. Note that

$$a_{12} > 0 \text{ and } b_{22} > 0 \text{ imply } a_{11} = 0 \text{ and } b_{21} = 0$$

hence *PII* can approach (0, 0) by playing $\widetilde{\text{Left}}$ and $\bar{v} = 0$. Thus we can restrict ourselves to the following games.

$$a_{11} = 0 > a_{21},\, a_{12} > a_{22}^{+}$$

$$b_{21} > b_{11}^{+},\, b_{12} \geqslant b_{22}^{+}.$$

The analysis is roughly the same as in the third case, the role of Z now being played by a point U on the x-axis.

Let $\bar{t} = \max\ \{t \mid b_1(t) > 0\} \wedge 1$ (hence $\bar{t} = 1$ if $b_{11} > 0$) and let $U = (a_2(\bar{t}), 0)$.

Lemma 13

PII can approach U and cannot expect less.

Proof

If PII desires a payoff 0 within ϵ^2 in game B, then the absorbing probability when PI plays $\widetilde{(\bar{t}-\epsilon)}$ has to be less than some $M\epsilon$. Hence the payoff in game A will be at least $a_2(\bar{t})$ within $K\epsilon$ where K depends on the a_{ij}, b_{ij}.

Now if PII uses an ϵ-optimal strategy in game B blocking at $\bar{t}$, his non absorbing payoff in game A will be at most $a_2(\bar{t})+\epsilon L$ and his absorbing payoff is negative, hence the result follows. Q.E.D.

Now for $k \in [a_{22}^+, a_2(\bar{t})]$ we define t_k with $a_2(t_k)=k$ and S_k to be the corresponding absorbing payoff. As in the third case $\varphi(k)$ is defined such that if $C_k=(k,\varphi(k))$ then $S_k C_k$ is tangent at C_k to be graph of φ. It is then easy to see that we have the analogue of Lemmas 11 and 12.

Finally for $k \in [0, a_{22}^+]$ it follows, like in case three that, denoting the point $(a_{22}, \varphi(a_{22}))$ by C, if $a_{22}>0$, the graph of φ is the line B^*C. This determines completely the Pareto boundary of D, hence $\bar{v}$.

It is straightforward to check that, due to the symmetry of the games, these four cases exhaust all the possibilities. Hence this complete the proof of Theorem 3.

Q.E.D.

5 Examples

1) $$A=\begin{bmatrix}1^* & 0\\ 0^* & 0\end{bmatrix} \quad B=\begin{bmatrix}0^* & 0\\ 0^* & 1\end{bmatrix}$$

This game was studied in *Sorin* [1980], and

$$\underline{v}(p)=\lim v_n(p)=\operatorname{Cav} u(p)=u(p)=p(1-p).$$

As for the minmax we have (see the first case)

$$\bar{v}(p)=\inf_{\rho\in Q}\ \sup_{t\in[0,1]}\ \{p\int_0^1(1-s)\,d\rho(s)+(1-p)\,t\,(1-\rho(t))\}$$

$$=p(1-\exp(1-(1-p)/p)).$$

2) $$A=\begin{bmatrix}1^* & 0\\ 0^* & 1\end{bmatrix} \quad B=\begin{bmatrix}0^* & 1\\ 1^* & 0\end{bmatrix}$$

This is in the second case.

The extreme points of D on its Pareto boundary are

$Y = (1/2, 1)$ (play optimally in game A)

$X = (1, 1/2)$ (play optimally in game B)

$U = \left(\frac{2}{3}, \frac{2}{3}\right)$ intersection of $B^* X$ and $T^* Y$

PII plays $\left(\frac{1}{3}, \frac{2}{3}\right)$ once and then approaches X (resp. Y) if $i_1 = B$ (resp. T). In this example, it can be shown, for $p = \frac{1}{2}$, that by using only a "mixture of Big Match strategies" as in the first case, *PII* cannot expect less than $\sqrt{3} - 1$.

3) $A = \begin{bmatrix} 1^* & 0 \\ 0^* & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0^* & \frac{3}{4} \\ 1^* & 0 \end{bmatrix}$ (second case)

$X = \left(1, \frac{3}{7}\right)$ $Y = \left(\frac{1}{2}, \frac{3}{4}\right)$ $U = \left(\frac{7}{13}, \frac{9}{13}\right)$

The strategy for *PII* to approach U can be described by the following diagram

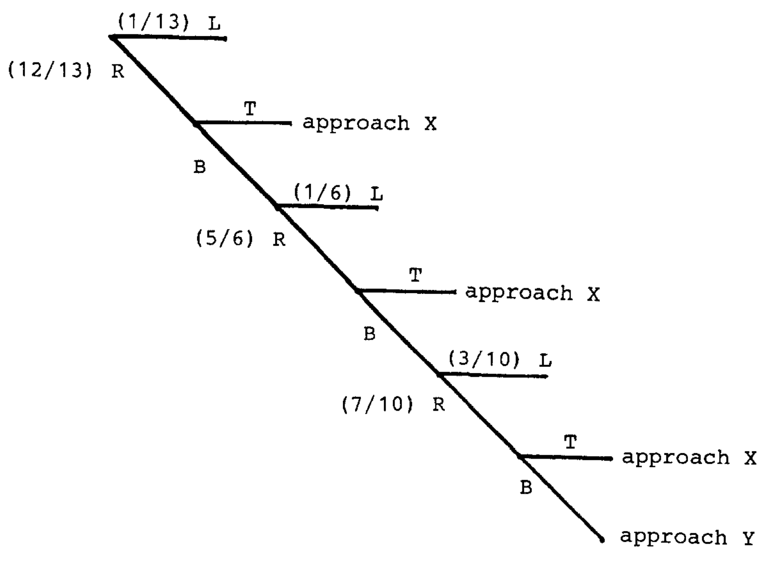

Abb.

4) $A = \begin{bmatrix} 8* & -2 \\ -4* & 1 \end{bmatrix}$ $B = \begin{bmatrix} -3* & 2 \\ 6* & -4 \end{bmatrix}$ (case 3)

For $k \in [0, 1]$, $C_k = (k, \varphi(k))$ with $\varphi(k) = 3 - \frac{3}{4} k - \frac{1}{4} k^{1/5}$.

For $\lambda \in [0, 2]$, $C_\lambda = (\psi(\lambda), \lambda)$ with $\psi(\lambda) = 4 - \frac{4}{3} \lambda - \frac{1}{3} \frac{\lambda}{2}^{2/5}$.

5) $A = \begin{bmatrix} 0* & 1 \\ -1* & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1* & 1 \\ 1* & -1 \end{bmatrix}$ (case 4)

For $k \in [0, \frac{1}{2}$, $C_k = (k, \varphi(k))$ with $\varphi(k) = -1 - 2k + 2e^{k-1/2}$.

6) $A = \begin{bmatrix} 0* & 2 \\ -1* & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1* & 0 \\ 2* & 0 \end{bmatrix}$

For $k \in [0, 2]$, $C_k = (k, \varphi(k))$ with $\varphi(k) = (1 - k) + e^{\frac{k}{2}-1}$ on $[1, 2]$ and φ is linear on $[0, 1]$ with C_k on $[B^*, C_1]$.

6 Concluding Remarks

1) We proved the existence of $\lim v_n$, $\underline{v}$ and $\bar{v}$ for the class of games under consideration. This results also hold for games with a lack of information, stochastic games, and the class studied in *Sorin* [1984].

2) However $\underline{v}$ and $\bar{v}$ may be different (see example 1), hence G_∞ may have no value, which is neither the case for stochastic games nor for games with a lack of information on one side.

3) Moreover $\bar{v}$ may be a transcendental function (i.e. given a game with parameters in **Q** (or algebraic) $\bar{v}(p)$ may be a transcendental number) which is not the case for stochastic games or games with a lack of information.

4) Note that if the results are similar to those in *Sorin* [1984] the tools used seem to be necessarily rather different. See Example 2.

5) As a consequence of Remark 3, Part 3 we know that $\lim v_n$ and $\lim v_\lambda$ exist and are equal. This is the case for all zero-sum games where $\lim v_n$ is known to exist. An open

problem is to check whether this equality can be obtained directly.

6) As in *Sorin* [1984] we proved here that lim v_n is equal to the Maxmin. It is conjectured that this property holds for all stochastic games with lack of information on one side. (This would be, in particular, a consequence of the extension of *Mertens-Neyman's* Theorem [1981] to games with compact state spaces.)

7) In a forthcoming paper [*Sorin*, 1984] we show how the tools introduced here can be used to compute the minmax and the maxmin of a game with lack of information on both sides, and state dependent signalling matrices.

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