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BEST RESPONSE DYNAMICS FOR CONTINUOUS ZERO–SUM GAMES

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Abstract. We study best response dynamics in continuous time for continuous concave-convex zero-sum games and prove convergence of its trajectories to the set of saddle points, thus providing a dynamical proof of the minmax theorem. Consequences for the corresponding discrete time process with small or diminishing step-sizes are established, including convergence of the fictitious play procedure.

1. **Introduction.** This paper contributes to the literature studying game theoretical problems through dynamical tools. A first example was a proof of the minmax theorem for finite games through a differential equation by Brown and von Neumann [7]. Another one is fictitious play, a discrete time process also introduced by Brown [6] and shown to converge to the set of optimal strategies for zero-sum games by Robinson [17]. Recently, continuous time dynamics have been used to give alternative proofs of this convergence result ([11], [12]). This approach is in the spirit of numerical dynamics [20] or more generally, stochastic approximation theory [2]: fictitious play is nothing but an Euler discretization procedure with diminishing stepsizes of a certain continuous time process. However, one technicality arises from the fact that the approximating continuous time dynamics, the best response dynamics studied below, is not given by a smooth differential equation, but by a discontinuous and even multivalued one. Thus we extend the basic result on numerical approximation (convergence to a global attractor) to the more general multivalued setting of a differential inclusion (section 5). More refined results involving chain recurrent components are established in [3].

In this paper we generalize the above mentioned convergence result of the best reply dynamics on the strategy space, from the bilinear case to the setup of concave–convex functions. We thus obtain a purely dynamic proof of the minmax theorem

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in the saddle case. Another consequence is a simple proof of Brown–Robinson's convergence result in this framework.

2. **The main result.** Let X, Y be compact, convex subsets of some finite dimensional Euclidean spaces and $U : X \times Y \to \mathbb{R}$ be a continuous saddle function, i.e., concave in x and convex in y. This defines a two-person zero-sum game where U is the payoff and X (resp. Y) is the strategy set of the maximizer (resp. minimizer). Denote

$$A(y) = \max_{x \in X} U(x, y), \qquad B(x) = \min_{y \in Y} U(x, y), \tag{1}$$

A is convex and continuous on Y, B is concave and continuous on X, using the Maximum theorem (Berge [4], p. 123). One has $B(x) \leq U(x, y) \leq A(y)$, for all x, y in $X \times Y$ hence

$$\underline{w} = \max_{x \in X} B(x) = \max_{x \in X} \min_{y \in Y} U(x, y) \leq \min_{y \in Y} \max_{x \in X} U(x, y) = \min_{y \in Y} A(y) = \overline{w}$$

Let

$$V(x, y) = A(y) - B(x). \tag{2}$$

Then $V : X \times Y \to \mathbb{R}$ is convex, nonnegative and continuous and its minimum, $\overline{w} - \underline{w} \geq 0$, is reached on a product of compact convex sets, $X(U) \times Y(U)$. If this minimum is 0, then the game is said to have a value $w = \overline{w} = \underline{w}$. More precisely, $V(x, y) = 0$ iff $A(y) = \overline{w} = \underline{w} = B(x)$ or equivalently $U(x', y) \leq U(x, y) \leq U(x, y')$ for all x', y' in $X \times Y$: (x, y) belongs to $X(U) \times Y(U)$, the set of saddle points of U on $X \times Y$.

Under the above assumptions on U, the minmax theorem ([21], [9]) applies and the game is known to have a value. In fact weaker assumptions suffice (quasiconcave and u.s.c. in x, quasiconvex and l.s.c. in y, X and Y convex compact subsets of a locally convex Hausdorff topological vector space [19]).

The following theorem provides an alternative proof for the existence of a value. Introduce the best response correspondences

$$\mathrm{BR}_1(y) = \operatorname*{Argmax}_{x \in X} U(x, y), \qquad \mathrm{BR}_2(x) = \operatorname*{Argmin}_{y \in Y} U(x, y). \tag{3}$$

Since U is continuous, the Maximum theorem [4, p. 123] implies that BR_1 (resp. BR_2) is an upper semi-continuous correspondence from Y to X (resp. from X to Y) with nonempty closed convex values. Consider the *best response dynamics* ([10], [12], [13], [14]) on $X \times Y$

$$\dot{x} \in \mathrm{BR}_1(y) - x, \qquad \dot{y} \in \mathrm{BR}_2(x) - y. \tag{4}$$

A *solution* of this differential inclusion is an absolutely continuous function $t \mapsto (x(t), y(t))$ from $[0, +\infty)$ to $X \times Y$ satisfying (4) for almost all $t \geq 0$. Given a solution $(x(t), y(t))$ of (4), denote $v(t) = V(x(t), y(t))$, $\alpha(t) = x(t) + \dot{x}(t) \in \mathrm{BR}_1(y(t))$ and $\beta(t) = y(t) + \dot{y}(t) \in \mathrm{BR}_2(x(t))$.

Theorem. *(i) (4) has a solution for every initial condition $(x(0), y(0)) \in X \times Y$.*
(ii) Along every solution of (4), $v(t)$ is absolutely continuous and satisfies

$$\dot{v}(t) \leq -v(t) \quad \text{for almost all } t \tag{5}$$

hence

$$v(t) \leq e^{-t} v(0). \tag{6}$$

Thus the game has a value and every solution of (4) converges to the nonempty set of saddle points $X(U) \times Y(U)$, which is a uniform global attractor for (4).

Versions of this result have been shown by Brown [6], Harris [11] and Hofbauer [12] for finite zero-sum games: X and Y being simplices and U bilinear. In this case equality holds in (5): $\dot{v}(t) = -v(t)$.
The Theorem will be proven in Section 4.

3. **Properties of the trajectories.** The best reply correspondence $\mathrm{BR} = \mathrm{BR}_1 \times \mathrm{BR}_2$ is u.s.c. from the compact space $X \times Y$ to itself and has nonempty convex compact values. In addition the set X is convex and closed and its tangent cone $T_X(x)$ at x contains all vectors $x' - x$ with $x' \in X$. In particular $\mathrm{BR}_1(y) - x \subset T_X(x)$ and similarly for BR_2. The existence of solutions to (4) follows from general existence results (Aubin and Cellina [1, Chapter 4, Section 2, Theorem 1], or Clarke et al. [8, Section 4.2]). In addition one has a Lipschitz property: since X and Y are compact, the derivatives in (4) are uniformly bounded by the diameters of these sets.

3.1. **Boundary behavior.** The dimension of a convex set C is the dimension of the affine space it generates. A facet of a convex set C of dimension n is defined as $C \cap H$ where H is a closed half space of dimension n which intersects C and such that the corresponding open half space does not. $C \cap H$ is a convex set of dimension at most $n - 1$. The set of faces of C consists of C itself, all facets of C, facets of facets and so on. Let $F_C(x)$ be the intersection of the faces of C containing x, and hence the minimal face of C containing x. If x is in the relative interior of C then $F_C(x) = C$. The relative boundary of a convex set equals the union of all its facets. Hence x is in the relative interior of the minimal face $F_C(x)$. This is also true if $F_C(x) = \{x\}$ is a point.

Lemma 1. *Any trajectory $(x(t), y(t))$ that belongs to a face of $X \times Y$ at time $T > 0$ is included in this face on $[0, T]$.*

Proof. First consider a facet of $X \times Y$ with corresponding closed half space H. By applying a linear transformation one can assume that H corresponds to either $\{x_1 \leq 0\}$ or $\{y_1 \leq 0\}$ or $\{x_1 \leq 0 \text{ and } y_1 \leq 0\}$. Consider the first of these cases (the others are analogous). Suppose $x_1(T) = 0$, for some $T > 0$. Equation (4) and $\mathrm{BR}_1(y) \subseteq X \subseteq \{x_1 \geq 0\}$ implies that $\dot{x}_1(t) \geq -x_1(t)$, so that $x_1(t) \geq e^{-(t-s)} x_1(s)$, for all $t \geq s \geq 0$. Hence $x_1(t) = 0$ for $t \in [0, T]$. It follows that $\dot{x}_1(t) = 0$ as well. Let $X' = X \cap \{x_1 = 0\}$ and $\mathrm{BR}_1' = \mathrm{BR}_1 \cap X'$. Then $(x(t), y(t))$ is a solution of the differential inclusion

$$\dot{x} \in \mathrm{BR}_1'(y) - x, \qquad \dot{y} \in \mathrm{BR}_2(x) - y$$

on $X' \times Y$ for $t \in [0, T]$. The proof then proceeds by induction on the dimension of $X \times Y$. □

Corollary 2. *On any trajectory $(x(t), y(t))$, for all but finitely many t, $(x(s), y(s))$ belongs to the relative interior of the minimal face $F_X(x(t)) \times F_Y(y(t)) \subset X \times Y$ that contains $(x(t), y(t))$, for s close to t.*

Proof. Given any trajectory $(x(t), y(t))$, there are strictly increasing sequences of times t_i and of natural numbers $m_i \geq 0$ such that on $]t_i, t_{i+1}]$, $x(t)$ belongs to the relative interior of a face of dimension m_i, and similarly for $y(t)$. □

3.2. **Directional derivatives on trajectories.** In the following lemma on convex functions, the one sided right and left derivatives of a function $g : \mathbb{R} \to \mathbb{R}$ are

$$\frac{d}{dt}^{+} g(t) = \dot{g}(t^{+}) = \lim_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}$$

$$\frac{d}{dt}^{-} g(t) = \dot{g}(t^{-}) = -\lim_{h \downarrow 0} \frac{g(t-h) - g(t)}{h}$$

and the directional derivative of a function $G : \mathbb{R}^n \to \mathbb{R}$ at x in direction v is

$$DG(x)(v) = \lim_{h \downarrow 0} \frac{G(x+hv) - G(x)}{h}.$$

Lemma 3. *Let $t \mapsto y(t) \in \mathbb{R}^n$ be Lipschitz and $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be a convex function where $\mathcal{U}$ is an open convex set containing $y(t)$. Then $t \mapsto f(y(t))$ is locally Lipschitz and its one-sided derivatives are given by*

$$\frac{d}{dt}^{\pm} f(y(t)) = \pm Df(y(t))(\pm \dot{y}(t^{\pm})) \tag{7}$$

whenever $\dot{y}(t^{\pm})$ exists.

Proof. First $t \mapsto f(y(t))$ is locally Lipschitz since f is locally Lipschitz (Rockafellar [18, Thm. 10.4]). If $\dot{y}(t^{+})$ exists then $y(t+h) = y(t) + h\dot{y}(t^{+}) + o(h)$. This implies

$$\begin{aligned} \frac{d}{dt}^{+} f(y(t)) &= \lim_{h \downarrow 0} \frac{f(y(t+h)) - f(y(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{f(y(t) + h\dot{y}(t^{+}) + o(h)) - f(y(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{f(y(t) + h\dot{y}(t^{+})) - f(y(t))}{h}. \end{aligned}$$

Hence

$$\frac{d}{dt}^{+} f(y(t)) = Df(y(t))(\dot{y}(t^{+}))$$

which exists by convexity [18, Thm. 23.1]. Similarly

$$\frac{d}{dt}^{-} f(y(t)) = -\lim_{h \downarrow 0} \frac{f(y(t-h)) - f(y(t))}{h} = -Df(y(t))(-\dot{y}(t^{-})).$$

□

The following is a version of the envelope theorem, compare [5, Section 4.3.1].

Lemma 4. *Let $(x(t), y(t))$ be a solution of (4). Then the function $t \mapsto A(y(t))$ is locally Lipschitz except for finitely many t and satisfies for almost all $t > 0$*

$$\frac{d}{dt} A(y(t)) = D_y U(\alpha(t), y(t))(\dot{y}(t)) \tag{8}$$

where the right hand side means the directional partial derivative with respect to y in direction $\dot{y}(t)$.

Proof. Start with

$$\begin{aligned} A(y(s)) - A(y(t)) &= U(\alpha(s), y(s)) - U(\alpha(t), y(t)) \\ &\geq U(\alpha(t), y(s)) - U(\alpha(t), y(t)) \end{aligned} \tag{9}$$

By Corollary 2 and Lemma 3, with $\mathcal{U}$ being the relative interior of the minimal face $F_Y(y(t))$, the function $t \mapsto A(y(t))$ is locally Lipschitz, except for finitely many t, and hence the limits

$$\mathcal{A}^+ = \lim_{s \downarrow t} \frac{A(y(s)) - A(y(t))}{s-t}, \quad \mathcal{A}^- = \lim_{s \uparrow t} \frac{A(y(s)) - A(y(t))}{s-t} \tag{10}$$

exist and coincide for almost all $t > 0$. Again by Lemma 3, (7), for each $t > 0$ for which $\dot{y}(t)$ exists,

$$\mathcal{B}^+ = \lim_{s \downarrow t} \frac{U(\alpha(t), y(s)) - U(\alpha(t), y(t))}{s-t} = D_y U(\alpha(t), y(t))(\dot{y}(t)), \tag{11}$$

and

$$\mathcal{B}^- = \lim_{s \uparrow t} \frac{U(\alpha(t), y(s)) - U(\alpha(t), y(t))}{s-t} = -D_y U(\alpha(t), y(t))(-\dot{y}(t)). \tag{12}$$

The convexity of U in y implies $\mathcal{B}^- \leq \mathcal{B}^+$, while [18, Thm. 23.1] and (9) show $\mathcal{B}^+ \leq \mathcal{A}^+ = \mathcal{A}^- \leq \mathcal{B}^-$. Hence $\mathcal{B}^- = \mathcal{B}^+ = \mathcal{A}^+ = \mathcal{A}^-$. □

Similar to (8), one obtains

$$\frac{d}{dt} B(x(t)) = D_x U(x(t), \beta(t))(\dot{x}(t)). \tag{13}$$

4. **Convergence theorems.**

4.1. **Trajectories.**

Proof of the Theorem. Along any solution of (4), (8) and (13) give, for almost all t

$$\begin{aligned}
\frac{d}{dt} v(t) &= D_y U(\alpha(t), y(t))(\dot{y}(t)) - D_x U(x(t), \beta(t))(\dot{x}(t)) \\
&= D_y U(\alpha(t), y(t))(\dot{y}(t)) - U(\alpha(t), \beta(t)) \\
&\quad + U(\alpha(t), \beta(t)) - D_x U(x(t), \beta(t))(\dot{x}(t)) && (14) \\
&\leq -U(\alpha(t), y(t)) + U(x(t), \beta(t)) && (15) \\
&= -A(y(t)) + B(x(t)) \\
&= -V(x(t), y(t)) \\
&= -v(t) && (16)
\end{aligned}$$

where (15) follows from (14) by convexity (resp. concavity) of U in y (resp. x). (Note that equality holds if U is bilinear, compare the remark after the Theorem.) By Lemma 4, v is locally Lipschitz except for finitely many t. Since v is continuous, it is absolutely continuous on $[0, +\infty)$. Thus (16) implies (6) and $v(t)$ converges to 0 on each trajectory. This shows the existence of a value: $\overline{w} = \underline{w} = w$.
Since $v(0)$ is uniformly bounded, (6) implies that for any $\epsilon > 0$, there exists T such that along every solution, $t \geq T$ implies

$$B(x(t)) \geq w - \epsilon. \tag{17}$$

B being u.s.c. and X compact there exists T' such that for $t \geq T'$

$$d(x(t), X(U)) \leq \epsilon \tag{18}$$

where d denotes the usual Euclidean distance. A dual result holds for $y(t)$. □

We define M to be an *invariant set* of (4) if for each point m in M, there exists a solution $m(t)$ of (4), defined for all positive and negative $t \in \mathbb{R}$ with $m(t) \in M$ and $m(0) = m$. The maximal invariant set is the union of all such complete trajectories.

Corollary 5. *The maximal invariant set of* (4) *is the set* $X(U) \times Y(U)$ *of saddle points.*

Proof. (6) and boundedness of v imply that $v(t) \equiv 0$ along any complete solution. □

4.2. **Payoffs.** In this section we prove convergence of a certain average payoff along the trajectories.

Proposition 6. *Define*

$$C(t_0, T) = \frac{1}{T}\int_{\ln(t_0)}^{\ln(t_0+T)} U(\alpha(t), \beta(t))e^t dt.$$

Then $C(t_0, T) \to w$ *as* $T \to \infty$.

Proof. Concavity implies

$$U(\alpha(t), \beta(t)) \le U(x(t), \beta(t)) + D_x U(x(t), \beta(t))(\dot{x}(t)).$$

Using (13) this gives, for almost all t

$$U(\alpha(t), \beta(t)) \le B(x(t)) + \frac{d}{dt}B(x(t)).$$

This implies

$$\begin{aligned} C(t_0, T) &\le \frac{1}{T}\int_{\ln(t_0)}^{\ln(t_0+T)} (B(x(t)) + \frac{d}{dt}B(x(t)))e^t dt \\ &= \frac{1}{T}\int_{\ln(t_0)}^{\ln(t_0+T)} \frac{d}{dt}(B(x(t))e^t)dt \\ &= \frac{1}{T}\Big(B(x(\ln(t_0+T))(t_0+T) - B(x(\ln(t_0))t_0\Big). \end{aligned} \tag{19}$$

Hence $\limsup_{T\to\infty} C(t_0, T) \le w$. A dual inequality gives the result. □

5. **Discrete counterpart.**

5.1. **Vanishing stepsizes.** Recall that a fictitious play process (Brown [6]) associated to the game U satisfies

$$p_{n+1} \in \mathrm{BR}_1(Q_n), \quad q_{n+1} \in \mathrm{BR}_2(P_n) \tag{20}$$

with initial values $p_1 = P_1 \in X$, $q_1 = Q_1 \in Y$ and for $n \in \mathbb{N}$,

$$P_n = \frac{1}{n}\sum_{k=1}^{n} p_k, \quad Q_n = \frac{1}{n}\sum_{k=1}^{n} q_k.$$

Hence (20) gives first

$$\begin{aligned} (n+1)P_{n+1} - nP_n &\in \mathrm{BR}_1(Q_n) \\ (n+1)Q_{n+1} - nQ_n &\in \mathrm{BR}_2(P_n) \end{aligned} \tag{21}$$

and finally the difference inclusion

$$\begin{aligned} P_{n+1} - P_n &\in \frac{1}{n+1}[\mathrm{BR}_1(Q_n) - P_n] \\ Q_{n+1} - Q_n &\in \frac{1}{n+1}[\mathrm{BR}_2(P_n) - Q_n]. \end{aligned} \tag{22}$$

The corresponding equation in continuous time writes

$$\dot{P}(t) \in \frac{1}{t}[\mathrm{BR}_1(Q(t)) - P(t)]$$
$$\dot{Q}(t) \in \frac{1}{t}[\mathrm{BR}_2(P(t)) - Q(t)]. \tag{23}$$

Changing time scale

$$x(t) = P(e^t), \quad y(t) = Q(e^t)$$

leads to the best response dynamics (4)

$$\dot{x} \in \mathrm{BR}_1(y) - x, \quad \dot{y} \in \mathrm{BR}_2(x) - y.$$

From (6), one obtains, as in [11], that any solution of (23) satisfies

$$v(P(t), Q(t)) \leq \frac{1}{t} v(P(1), Q(1))$$

hence convergence to 0 at a rate $1/t$. Similarly $d(P(t), X(U))$ and $d(Q(t), Y(U))$ go to 0 as $t \to \infty$. In addition, in the finite case, the same rate of convergence holds [11].

To study the asymptotic properties of a solution of (22) it is enough to show that they are analogous to those of (23) or of (4). In fact the analysis applies to a much more general framework. Consider a differential inclusion of the form

$$\dot{z} \in \Phi(z) - z \tag{24}$$

where Z is a compact convex subset of an Euclidean space, Φ is an u.s.c. compact convex valued correspondence from Z to itself and $z(0) \in Z$.
A discrete counterpart can be written as

$$P_{n+1} = \alpha_{n+1} p_{n+1} + (1 - \alpha_{n+1}) P_n \tag{25}$$

with $P_1 \in Z$, $p_{n+1} \in \Phi(P_n)$, $\alpha_n \in [0,1]$ decreasing to 0 as $n \to \infty$ and $\sum_n \alpha_n = +\infty$.
Then the following comparison result holds:

Proposition 7. *Assume that $Z_0 \subset Z$ is a global uniform attractor of (24) in the sense that for any $\varepsilon > 0$, there exists T such that for any solution z of (24) with $z(0) \in Z$ and any $t \geq T$*

$$d(z(t), Z_0) \leq \varepsilon.$$

Then for any $\varepsilon > 0$, there exists N such that any solution of (25) with $P_1 \in Z$ satisfies, for all $n \geq N$,

$$d(P_n, Z_0) \leq \varepsilon.$$

Proof. The proof relies on the following approximation result [1, Ch. 2, sect. 2]: *Given $T_1 > 0$, let $\mathcal{A}(\Phi, T_1, u)$ be the set of solutions z of (24) on $[0, T_1]$ with $z(0) = u$, endowed with the topology of uniform convergence. Let $D(x, y) = \max_{0 \leq t \leq T_1} \|x(t) - y(t)\|$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that, given a correspondence Ψ with graph included in a δ-neighborhood of the graph of Φ*

$$\min\{D(y, z) : z \in \mathcal{A}(\Phi, T_1, u)\} \leq \varepsilon \tag{26}$$

for any u in Z and for any solution y of

$$\dot{y} \in \Psi(y) - y \tag{27}$$

with $y(0) = u$ (or even $\|y(0) - u\| \leq \delta$).

Consider a linear interpolation of the points P_n thus defining a path $y(t)$ with $t_1 = 0$, $t_n = \sum_{m=2}^{n} \alpha_m$ and

$$y(t_n) = P_n, \quad \frac{y(t) - y(t_n)}{t - t_n} = \frac{y(t_{n+1}) - y(t_n)}{t_{n+1} - t_n} \in \Phi(y(t_n)) - y(t_n), \quad t \in [t_n, t_{n+1}].$$

The divergence of the sum of the sequence α_m implies that y is defined for all $t \geq 0$. Moreover, since α_m decreases to 0, for any $\delta > 0$, there exists T_2 such that for $t \geq T_2$, y satisfies the inclusion (27) with $\Psi(y) = N^\delta(\Phi(N^\delta(y))) \cap Z$ where N^δ stands for δ-neighborhood. Alternatively the graph of Ψ is the intersection of a δ-neigborhood of the graph of Φ with $Z \times Z$.

Given $\varepsilon > 0$, let T be as in the statement of Proposition 7. Put $T_1 = T$ thus defining δ and finally choose T_2 as above adapted to this δ. Let N such that $t_N \geq T_1 + T_2$. Then $n \geq N$ implies that $d(P_n, Z_0) \leq 2\varepsilon$. In fact, on the trajectory of y after time $t_n - T_1 \geq T_2$ the approximation property applies. Along this time interval of length T_1, any z solution of (24) starting from $y(T_1)$ reaches Z_0 within ε at time t_n and y remains ε close to the set of such z during this period. □

While the above analysis is similar to that of Harris [11], the following alternative approach from Hofbauer [12] uses the explicit construction of a solution of the differential inclusion ([1, Ch. 2, sect. 1] or [8, Section 4.1]).

Proposition 8. *The set of limit points of a solution of (25) is an invariant set for the dynamics (24) and hence is contained in its maximal invariant set.*

Proof. Consider a solution P_n of (25) and its linear interpolation y as above. Let L be the set of limit points of $y(t)$ as $t \to \infty$, which equals the set of limit points of P_n as $n \to \infty$, since α_n goes to 0 and its sum diverges. Let $z \in L$. Then there exists a sequence $T_n \to \infty$ such that $y(T_n) \to z$. Given any $T > 0$, the sequence of trajectories $y(t+T_n)$ on $[-T, T]$ is equicontinuous and hence contains a subsequence that converges uniformly to a function $z(t)$ from $[-T, T]$ to L with $z(0) = z$. In addition, this subsequence can be chosen so that $\dot{y}(t + T_n)$ converges weakly to $\dot{z}(t)$ and since Φ is convex valued, $\dot{z}(t) \in \Phi(z(t)) - z(t)$. This being true for any T, one obtains a complete solution of (25) through z. □

Proposition 7 together with the Theorem, or alternatively Proposition 8 and Corollary 5, show the convergence of fictitious play (22) to the set of saddle points. The same approximation implies the convergence of average payoff (Rivière [16], Monderer et al. [15]).

Proposition 9.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=n}^{n+N} U(p_k, q_k) = w$$

Proof. Similar computations as above show that $p_n = P_n + (n+1)(P_{n+1} - P_n)$ satisfying (22) will have trajectories uniformly close to $P(s) + s\dot{P}(s)$ satisfying (23), hence $\frac{1}{N} \sum_{k=n}^{n+N} U(p_k, q_k)$ will be near

$$\frac{1}{N} \int_n^{n+N} U(s\dot{P}(s) + P(s), s\dot{Q}(s) + Q(s)) ds$$

which is, with $x(t) = P(e^t)$

$$\begin{aligned}\frac{1}{N}\int_{\ln(n)}^{\ln(n+N)} U(x(t)+\dot{x}(t), y(t)+\dot{y}(t))e^t dt &= \frac{1}{N}\int_{\ln(n)}^{\ln(n+N)} U(\alpha(t),\beta(t))e^t dt \\ &= C(n,N) \qquad (28)\end{aligned}$$

and Proposition 6 applies. □

5.2. **Small stepsizes.** We consider here alternative discrete procedures satisfying

$$P_{n+1} = \alpha_{n+1} p_{n+1} + (1-\alpha_{n+1})P_n \qquad (29)$$

with $P_1 \in Z$, $p_{n+1} \in \Phi(P_n)$, but where the step size $\alpha_n \in [0,1]$ does not necessarily goes to 0.

Proposition 10. *Assume that $Z_0 \subset Z$ is a global uniform attractor of (24) as in Proposition 7. Then for any $\varepsilon > 0$, there exists α and N such that any solution of (29) with $P_1 \in Z$ and $\alpha_n \le \alpha$ satisfies, for all $n \ge N$*

$$d(P_n, Z_0) \le \varepsilon.$$

Proof. The proof is similar to the proof of Proposition 7, using the same approximation argument. □

As an example we consider the following version of geometric fictitious play where the past is discounted at a rate $\rho < 1$. Explicitly

$$p_{n+1} \in \mathrm{BR}_1(Q_n), \quad q_{n+1} \in \mathrm{BR}_2(P_n) \qquad (30)$$

with initial values $p_1 = P_1 \in X$, $q_1 = Q_1 \in Y$ and

$$P_n = \frac{\sum_{k=0}^{n-1}\rho^k p_{n-k}}{\sum_{k=0}^{n-1}\rho^k}, \quad Q_n = \frac{\sum_{k=0}^{n-1}\rho^k q_{n-k}}{\sum_{k=0}^{n-1}\rho^k}.$$

This gives the difference inclusion

$$\begin{aligned}P_{n+1} - P_n &\in \frac{1-\rho}{1-\rho^{n+1}}[\mathrm{BR}_1(Q_n) - P_n] \\ Q_{n+1} - Q_n &\in \frac{1-\rho}{1-\rho^{n+1}}[\mathrm{BR}_2(P_n) - Q_n]. \qquad (31)\end{aligned}$$

Hence for any $\varepsilon > 0$, there exists $\overline{\rho} < 1$ and N such that $\overline{\rho} \le \rho < 1$ and $n > N$ imply

$$d((P_n, Q_n), X(U)\times Y(U)) \le \varepsilon.$$

So for discount rates close to 1, geometric fictitious play will converge to a small neighborhood of the set of saddle points. However, in general, the set of saddle points itself is unstable for (31).

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