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Asymptotic Properties of Monotonic Nonexpansive Mappings

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Abstract. Let **T** be a nonexpansive monotonic mapping from C to itself where C is a closed subset of a space of bounded real functions, with the supremum norm. We study asymptotic properties of several average iterates of **T**, related to the cycle time.

Keywords: nonexpansive mappings, asymptotic properties, cycle time, dynamic programming, Shapely operator

1. Presentation

The study of asymptotic behavior of iterates of nonexpansive maps has been developed in several directions. General results have been obtained in connection with evolution equations and geometric properties of Banach spaces. Applications in infinite dimensional spaces include financial mathematics (Kolokoltsov, 1998) and the special case of monotonic nonexpansive mappings from a subset of $\mathbb{R}^n$ to itself is of fundamental importance in the analysis of discrete event systems (Baccelli et al., 1992; Gunawardena, 2003).

The purpose of this paper is to extend arguments in terms of operators, used in Rosenberg and Sorin (2001) for the study of two-person zero-sum repeated games, to the general framework of monotonic nonexpansive mappings acting on a closed subset of a space of real bounded functions. We will provide conditions on such mappings for some average itereates to converge. This in particular covers the time cycle—see Section 3.1. The main idea is that the existence of (generalized) time cycle extends to a class of operators much larger than the one usually covered and that the tools presented will be useful to prove it.

1.1. General Properties and Notations

Let X be a Banach space, C be a closed subset of X, star-shaped from 0 ($0 \in C$ and $tx \in C$ for $0 \leq t \leq 1$ as soon as $x \in C$) and **T** be a nonexpansive mapping from C to itself, namely such that:

$$\|\mathbf{T}(x) - \mathbf{T}(y)\| \leq \|x - y\|, \quad \forall x, y \in C \tag{1}$$

Define the sequence

$$V_n = \mathbf{T}^n(0) \tag{2}$$

and given $0 < \lambda < 1$, denote by V_λ the fixed point of the contracting operator $x \mapsto \mathbf{T}((1-\lambda)x)$ from C to itself:

$$V_\lambda = \mathbf{T}((1-\lambda)V_\lambda) \tag{3}$$

The main purpose of this paper is to study the asymptotic behavior of $v_n = V_n/n$ as n goes to ∞ and of $v_\lambda = \lambda V_\lambda$ as λ goes to 0. (For more general orbits see Neyman, 2003, and Neyman and Sorin, 2001.)

The previous equations can be written as:

$$v_n = \frac{1}{n}\mathbf{T}((n-1)v_{n-1}) \tag{4}$$

$$v_\lambda = \lambda \mathbf{T}\left(\left(\frac{1}{\lambda} - 1\right)v_\lambda\right) \tag{5}$$

It is thus natural to introduce, for $1 > \varepsilon > 0$, a one parameter family of operators $\Phi(\varepsilon, \cdot)$ satisfying:

$$\Phi(\varepsilon, x) = \varepsilon\, \mathbf{T}\left(\left(\frac{1}{\varepsilon} - 1\right)x\right) \tag{6}$$

and note that

$$\|\Phi(\varepsilon, x) - \Phi(\varepsilon, y)\| \leq (1-\varepsilon)\|x - y\| \tag{7}$$

v_n is defined inductively by: $v_0 = 0$ and

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right) \tag{8}$$

and for $\lambda > 0, v_\lambda$ satisfies:

$$v_\lambda = \Phi(\lambda, v_\lambda) \tag{9}$$

A first general result is the following:

THEOREM 1 *Kohlberg and Neyman (1981)*
Assume C convex. There exists f, a linear functional of norm 1 on X, such that:

$$\lim_{n\to\infty} f(v_n) = \lim_{n\to\infty} \|v_n\| = \lim_{\lambda\to 0} f(v_\lambda) = \lim_{\lambda\to 0} \|v_\lambda\| = \inf_{x\in C} \|\mathbf{T}(x) - x\| (= \rho) \tag{10}$$

In particular if X is reflexive and strictly convex, there is weak convergence and if X^* has a Frechet differentiable norm, the convergence is strong. Further extensions can be found in Plant and Reich (1983).

There are also deep connections with the analysis of accretive operators. In fact let $J_t = (I - tA)^{-1}$ be the resolvent associated to the accretive operator $A = I - \mathbf{T}$, and $S(t)$ the associated semi group, then one has $v_\lambda = J_t/t$ with $t = \lambda^{-1} - 1$ and $\|S(n)x - \mathbf{T}^n(x)\| \leq \sqrt{n}\|\mathbf{T}(x) - x\|$. Related results are in Reich (1981, 1982).

An easy consequence of Theorem 1 is the following:

COROLLARY 2 *Any nonexpansive map* **T** *from a closed convex subset of* $\mathbb{R}$ *to itself satisfies:*

$$\lim_{\lambda\to 0} v_\lambda = \lim_{n\to\infty} v_n (= \pm\rho)$$

Proof: $\|V_{n+1} - V_n\| \leq \|\mathbf{T}(0)\|$ implies that $\|v_{n+1} - v_n\|$ goes to 0 as n increases to ∞ and the result follows since by Theorem 1 $\|v_n\|$ converges.

Similarly one obtains $\|v_\lambda - v_\mu\| \leq 2|1 - \lambda/\mu|\, \|v_\mu\|$, hence the convergence of the absolute value implies convergence of v_λ as λ goes to 0. ■

A second general property is due to Neyman (2003). First introduce a definition.

DEFINITION 1 *v_λ is of bounded variation if for any sequence λ_i decreasing to 0*

$$\sum_i \|v_{\lambda_{i+1}} - v_{\lambda_i}\| < \infty \tag{11}$$

Note in particular that this condition implies uniform convergence of v_λ, but is strictly stronger.

PROPOSITION 3 *Neyman (2003)*
If v_λ is of bounded variation, then:

$$\lim_{\lambda\to 0} v_\lambda = \lim_{n\to\infty} v_n$$

To state the next result we first introduce generalized iterates.

DEFINITION 2 *A sequence u_n is admissible if: $u_0 \in C$ and for $n \geq 1, u_n = \Phi(\alpha_n, u_{n-1})$ where $\{\alpha_n\}$ is a real sequence decreasing to 0, with $\sum_{n=1}^{\infty} \alpha_n = +\infty$.*

In the same spirit as above, one obtains:

PROPOSITION 4 *Assume v_λ of bounded variation. Then, for any admissible sequence $\{u_n\}$:*

$$\lim_{\lambda\to 0} v_\lambda = \lim_{n\to\infty} u_n$$

Proof: Let w_n denote v_λ, for $\lambda = \alpha_n$. Then one obtains:

$$\|u_{n+1} - w_{n+1}\| = \|\Phi(\alpha_{n+1}, u_n) - \Phi(\alpha_{n+1}, w_{n+1})\| \leq (1 - \alpha_{n+1})\|u_n - w_{n+1}\|$$

so that:

$$\|u_{n+1} - w_{n+1}\| \leq (1 - \alpha_{n+1})(\|u_n - w_n\| + \|w_n - w_{n+1}\|) \tag{12}$$

Inductively, letting $\ell_k^n = \prod_{j=k}^n (1 - \alpha_j)$, this gives:

$$\|u_{n+1} - w_{n+1}\| \leq \ell_2^{n+1}\|u_1 - w_1\| + \sum_{m=1}^{m=n} \ell_{m+1}^{n+1}\|w_m - w_{m+1}\| \tag{13}$$

Since, for any k, ℓ_k^n goes to 0 as n goes to ∞ and $\sum_{m=1}^{\infty} \|w_m - w_{m+1}\|$ is bounded this implies that $\|u_{n+1} - w_{n+1}\|$ goes to 0 as well. ∎

1.2. Monotonic Mappings

We consider a subclass of nonexpansive mappings specified by their domain and their properties.

Let $\mathscr{F}_0$ denote the Banach space of real bounded functions on a set Ω, endowed with the uniform norm: $\mathscr{F}_0 = \{f : \Omega \to \mathbb{R}, \|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)| < \infty\}$. From now on $X = \mathscr{F}_0$ and $\mathbf{T}$ is defined on a closed convex cone $\mathscr{F} \subset \mathscr{F}_0$ containing the constants.

The previous Theorem 1 implies thus convergence of the norm but usually not more; see, however, an application for the case of concave functions in Mertens et al. (1994, Chapter V, Exercise 6.5).

DEFINITION 3 *A monotonic operator T is a map from $\mathscr{F}$ to itself that satisfies:*
(A) Monotonicity:

$$f \geq g \Rightarrow \mathbf{T}(f) \geq \mathbf{T}(g) \tag{14}$$

For any $a \in \mathbb{R}$ and $f \in \mathscr{F}$, let $f + a$ denote the function $\omega \mapsto f(\omega) + a$.
Let us introduce several conditions.

(B) Reduction of constants:

$$\mathbf{T}(f + a) \leq \mathbf{T}(f) + a, \quad \forall a \geq 0 \tag{15}$$

(C) *Homogeneity*:

$$\mathbf{T}(f + a) = \mathbf{T}(f) + a, \quad \forall a \in \tag{16}$$

(NE) *Nonexpansiveness.*

An immediate property is the next result.

PROPOSITION 5 *Under (A), conditions (B) and (NE) are equivalent.*

In the case $\mathscr{F} = \mathbb{R}^n$, mappings satisfying (A) and (C) have been intensively studied under the name topical functions, see Gunawardena and Keane (1995), Gaubert and Gunawardena (2001).

Crandall and Tartar (1980) proved that under (C), (A) and (NE) are equivalent.

We consider from now on mappings satisfying (A) and (B). These are called MNE operators (monotonic nonexpansive).

1.3. An Example

A typical example of such an operator $\mathbf{T}$ is obtained in the framework of stochastic games: Ω is a measurable state space, X and Y are move spaces, g is a real bounded payoff function on $\Omega \times X \times Y$, ρ is a transition probability from $\Omega \times X \times Y$ to Ω. $\mathbf{T}$ corresponds to the Shapley operator expressing the value of the one shot game with terminal payoff f on Ω.

$$\mathbf{T}(f)(\omega) = \sup_{x \in X} \inf_{y \in Y} \left\{ g(x, y, \omega) + \int_{\Omega} f(\omega')\rho(x, y; \omega)(d\omega') \right\} \tag{17}$$

More generally $\rho(x, y; \omega)$ could be a positive measure on Ω with total mass less than one and then T is a generalized Shapley operator.

Note that this provides a generalization of Markov and dynamic programming operators as well as a generalization of matrices over the max-plus semi ring.

Such topical functions and their extensions arise from the modeling of discrete event systems: communication networks, digital circuits, manufacturing processes.... The temporal evolution of the repeated occurence of events in a given set I, leads to a discrete event system.

Typically if x_i denotes the time of last occurrence of event i, the next one will be, starting from the configuration x, at time $\mathbf{T}(x)_i$. Hence $\mathbf{T}(x) - x$ measures the delay between successive occurences and $1/n(\mathbf{T}^n(x) - x) = 1/n\sum_{k=1}^{n}(\mathbf{T}^k(x) - \mathbf{T}^{k-1}(x))$ the average delay. Note that, by nonexpansiveness, this performance measure has an asymptotic behavior independent of the initial state x, hence v_n, as defined in (4) leads asymptotically to the cycle time.

More examples and explicit relations with discrete event systems can be found in Cochet-Terrasson et al. (1999), Gunawardena (1994, 1995, 2003), and Vincent (1997).

The definition of the cycle time can be extended to the situation where the number of occurrences is itself a random time, Neyman (2003), Neyman and Sorin (2001). In particular, the evaluation v_λ introduced in (5) corresponds to the simple model of a geometric distribution with parameter λ.

In the case $\mathscr{F} = \mathbb{R}^n$, a characterization of generalized Shapley operators is available. It is very similar to the one obtained by Kolokoltsov and Maslov (1997) for topical functions, see also Rubinov and Singer (2000).

PROPOSITION 6 *Any MNE operator* $\mathbf{T}$ *from* $\mathbb{R}^n$ *to itself is a generalized Shapley operator.*

Proof: From (NE), $\mathbf{T}$ is almost everywhere differentiable. From (NE) and (A), for each $i = 1, \ldots, n$, the range of $\text{grad}\mathbf{T}^i$ is included in $P_n = \{f \in \mathbb{R}^n; f_j \geq 0, j = 1, \ldots, n, \sum_{j=1}^n f_j \leq 1\}$.

Now one uses the min-max representation for $\mathscr{C}^1$ functions, see, for example, Evans (1984):

$$\mathbf{T}^i(f) = \max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \{\langle \Pi^i(x,y), f - x\rangle + \mathbf{T}^i(x)\} \tag{18}$$

$$= \min_{y \in \mathbb{R}^n} \max_{x \in \mathbb{R}^n} \{\langle \Pi^i(x,y), f - x\rangle + \mathbf{T}^i(x)\} \tag{19}$$

with $\Pi^i(x,y) = \int_0^1 \text{grad}\mathbf{T}^i(x + t(y-x))dt$.

In fact, for $x = f$, the right side is $\mathbf{T}^i(f)$ and for $y = f$, $\langle \Pi^i(x,f), f - x\rangle = \int_0^1 \left[d/dt\, \mathbf{T}^i(x + t(f-x))\right]dt = \mathbf{T}^i(f) - \mathbf{T}^i(x)$, hence the right hand side is again $\mathbf{T}^i(f)$.

Note then that:

$$\mathbf{T}^i(f) = \max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \{-\langle \Pi^i(x,y), x\rangle + \mathrm{T}^i(x) + \langle \Pi^i(x,y), f\rangle\}$$

which is of the form

$$\mathrm{T}^i(f) = \max_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \left\{ g^i(x,y) + \sum_j \Pi^i_j(x,y) f_j \right\}$$

and recall that $\Pi^i(x,y)$ belongs to P_n. ■

2. Properties of MNE Operators

The basic property is the domination of v_n and v_λ by approximately superharmonic functions in the following sense:

Notation 4: $\mathscr{L}^+$ denotes the set of functions $f \in \mathscr{F}$ satisfying: there exists a positive constant L_0 such that:

$$\mathbf{T}(Lf) \leq (L+1)f, \quad \forall L \geq L_0 \tag{20}$$

and similarly $\mathscr{L}^-$ is the set of $f \in \mathscr{F}$ such that $\exists L_0 > 0$ with:

$$(L+1)f \leq \mathbf{T}(Lf), \quad \forall L \geq L_0$$

Using (6), (20) is equivalent to:

$$\Phi(\varepsilon, f) \leq f, \quad \text{for } \varepsilon \text{ small enough} \tag{21}$$

Hence $f \in \mathscr{L}^+$ is superharmonic for all maps $\Phi(\varepsilon, \cdot)$ with ε small enough. Such functions are called superuharmonic (u is for uniform). From this property one deduces:

PROPOSITION 7 *If f belongs to $\mathscr{L}^+$, then:*

$$f \geq \limsup_{\lambda \to 0} v_\lambda \tag{22}$$

$$f \geq \limsup_{n \to \infty} v_n \tag{23}$$

More generally, for any admissible sequence $\{u_n\}$:

$$f \geq \limsup_{n \to \infty} u_n \tag{24}$$

Proof: For v_λ, we use the fact that the sequence $\Phi^m(\lambda, g)$ converges, as m goes to ∞, to the fixed point v_λ of $\Phi(\lambda, \cdot)$, for any initial g in $\mathscr{F}$ and we apply it at f:

$$v_\lambda = \lim_{m \to \infty} \Phi^m(\lambda, f) \leq f, \quad \text{for } \lambda \text{ small enough}$$

For v_n, we write that:

$$nv_n = \mathbf{T}^n(0) \quad \text{and} \quad \mathbf{T}^n(L_0 f) \leq (L_0 + n)f$$

to deduce from:

$$\mathbf{T}^n(0) \leq \mathbf{T}^n(L_0 f) + \|\mathbf{T}^n(0) - \mathbf{T}^n(L_0 f)\|$$

and nonexpansiveness that:

$$v_n \leq f + 2\frac{L_0}{n}\|f\|$$

For u_n, we compute inductively for n large enough so that (21) holds with $\varepsilon = \alpha_n$:

$$u_n = \Phi(\alpha_n, u_{n-1}) \leq \Phi(\alpha_n, f) + (1 - \alpha_n)\|u_{n-1} - f\| \leq f + (1 - \alpha_n)\|u_{n-1} - f\|$$

hence for $m \geq n$:

$$\begin{aligned} u_m &= \Phi(\alpha_m, u_{m-1}) \\ &\leq \Phi\left(\alpha_m, f + \left(\prod_{k=n}^{m-1}(1-\alpha_k)\right)\|u_{n-1} - f\|\right) \\ &\leq \Phi(\alpha_m, f) + \left(\prod_{k=n}^{m}(1-\alpha_k)\right)\|u_{n-1} - f\| \\ &\leq f + \left(\prod_{k=n}^{m}(1-\alpha_k)\right)\|u_{n-1} - f\| \end{aligned}$$

and the result follows since $\Sigma\alpha_k$ diverges. ■

One is thus led to introduce the following sets.

Notation 5: $\mathscr{C}^+$ is the set of functions $f \in \mathscr{F}$ such that: $\forall c > 0, \exists L_c$ such that $L \geq L_c$ implies:

$$\mathbf{T}(Lf) \leq (L+1)f + c \tag{25}$$

In particular, one obtains:

$$\mathbf{T}(L(f+c)) \leq \mathbf{T}(Lf) + Lc \leq (L+1)(f+c), \qquad \text{for } L \geq L_c \tag{26}$$

so that $f + c \in \mathscr{L}^+, \forall c > 0$. (Conditions (24) and (25) in fact equivalent if $\mathbf{T}$ satisfies (C).)

Alternative useful formulations for $f \in \mathscr{C}^+$ are:

$$\forall c > 0, \exists \varepsilon_c > 0 \text{ such that } \varepsilon \leq \varepsilon_c \text{ implies } \Phi(\varepsilon, f) \leq f + \varepsilon c \tag{27}$$

or with $g^+ = \max(g, 0)$:

$$\limsup_{L \to \infty} \|(\mathbf{T}(Lf) - (L+1)f)^+\| = 0 \tag{28}$$

The previous Proposition 7 implies thus, denoting by $\overline{A}$ the closure, for the uniform norm $\| \ \|_\infty$, of a set A in $\mathscr{F}$:

PROPOSITION 8 *If f belongs to the intersection $\overline{\mathscr{C}^+} \bigcap \overline{\mathscr{C}^-}$ then:*

$$f = \lim_{\lambda \to 0} v_\lambda = \lim_{n \to \infty} v_n$$

with the obvious consequence:

COROLLARY 9 *$\overline{\mathscr{C}^+}\bigcap\overline{\mathscr{C}^-}$ contains at most one element.*

3. Non Uniform Condition

We use again the specific structure of $\mathscr{F}$ to build on (28) and consider the weaker condition of simple convergence.

Define:

$$\theta^+(f)(\omega) = \limsup_{L\to\infty}\{\mathbf{T}(Lf)(\omega) - (L+1)f(\omega)\} \tag{29}$$

and similarly

$$\theta^-(f)(\omega) = \liminf_{L\to\infty}\{\mathbf{T}(Lf)(\omega) - (L+1)f(\omega)\} \tag{30}$$

Notation 6: $\mathscr{S}^+$ (resp. $\mathscr{S}^-$) is the set of functions $f\in\mathscr{F}$ satisfying $\theta^+(f)\leq 0$ (resp. $\theta^-(f)\geq 0$).

By definition $\mathscr{C}^+\subset\mathscr{S}^+$ and equality holds whenever Ω is finite.

The next proposition compares the variation $\mathbf{T}(Lf)(\omega) - (L+1)f(\omega)$ for two functions at a point maximizing their difference. This variant of the maximum principle was used by Kohlberg (1974) on $\mathbb{R}$.

PROPOSITION 10 *Let f_1 and f_2 in $\mathscr{F}$ and $\omega\in\Omega$ satisfy:*

$$(f_2 - f_1)(\omega) = \delta = \max_{\omega'\in\Omega}(f_2 - f_1)(\omega') > 0$$

Then, for all $L\geq 0$:

$$[\mathbf{T}(Lf_1)(\omega) - (L+1)f_1(\omega)] - [\mathbf{T}(Lf_2)(\omega) - (L+1)f_2(\omega)] \geq f_2(\omega) - f_1(\omega)$$

and in particular:

$$\theta^+(f_1)(\omega) - \theta^-(f_2)(\omega) \geq \delta$$

Proof: For any $\omega'\in\Omega$, we use (A) and (B) to obtain:

$$\begin{aligned}\mathbf{T}(Lf_2)(\omega') - \mathbf{T}(Lf_1)(\omega') &\leq \mathbf{T}(L(f_1+\delta))(\omega') - \mathbf{T}(Lf_1)(\omega')\\ &\leq L\delta = (L+1)(f_2(\omega) - f_1(\omega)) - \delta\end{aligned}$$

So that in particular:

$$[\mathbf{T}(Lf_1)(\omega) - (L+1)f_1(\omega)] - [\mathbf{T}(Lf_2)(\omega) - (L+1)f_2(\omega)] \geq \delta$$

hence

$$\theta^{+}(f_1)(\omega) - \theta^{-}(f_2)(\omega) \geq \delta \qquad \blacksquare$$

The previous result allows to compare the sets $\mathscr{S}^{+}$ and $\mathscr{S}^{-}$ in the compact/continuous case.

PROPOSITION 11 *Assume Ω compact. For all continuous functions $f^{+} \in \mathscr{S}^{+}$ and $f^{-} \in \mathscr{S}^{-}$ one has:*

$$f^{+} \geq f^{-}$$

Proof: Otherwise, let ω be a point realizing the maximum of $(f^{-} - f^{+})$ on Ω with $(f^{-} - f^{+})(\omega) = \delta > 0$. By Proposition 10, $\theta^{+}(f^{+})(\omega) - \theta^{-}(f^{-})(\omega) \geq \delta$, hence a contradiction to $\theta^{+}(f^{+}) \leq 0$ and $\theta^{-}(f^{-}) \geq 0$ on Ω. $\blacksquare$

Hence we obtain also uniqueness in this case, in the following sense:

COROLLARY 12 *Assume Ω compact. Let $\mathscr{S}_0^{+}$ (resp. $\mathscr{S}_0^{-}$) be the subset of continuous functions on Ω belonging to $\mathscr{S}^{+}$ (resp. $\mathscr{S}^{-}$). The closure of $\mathscr{S}_0^{+}$ and $\mathscr{S}_0^{-}$ have at most one common element.*

A natural conjecture is thus: Assume $\overline{\mathscr{S}_0^{+}} \bigcap \mathscr{S}_0^{-} \neq \emptyset$, hence reduced to some f, then $\lim v_\lambda = \lim v_n = f$.

4. Concluding Comments

4.1. Conditions in Terms of Φ

The approach using the ε-weighted operators $\Phi(\varepsilon, \cdot)$ is as follows.

DEFINITION 7 $\underline{\Phi}(0, \cdot)$ *(resp.* $\overline{\Phi}(0, \cdot)$*) is defined by*

$$\underline{\Phi}(0,f)(\omega) = \liminf_{\varepsilon\to 0+} \Phi(\varepsilon,f)(\omega) = \liminf_{\varepsilon\to 0+} \varepsilon\mathbf{T}\left(\frac{(1-\varepsilon)}{\varepsilon}f\right)(\omega) = \liminf_{\varepsilon\to 0+} \varepsilon\mathbf{T}\left(\frac{f}{\varepsilon}\right)(\omega)$$

(*resp.* $\limsup_{\varepsilon\to 0+} \Phi(\varepsilon, \cdot)$*)*.

It appears as the lower recession operator (resp. upper recession operator) associated to $\mathbf{T}$.

$\mathbf{T}$ *is of class 0 when both $\underline{\Phi}(0, f)(\omega)$ and $\overline{\Phi}(0, f)(\omega)$ coincide for all f in $\mathscr{F}$ and ω in Ω. We then write $\Phi(0, \cdot)$.*

The condition for f to belong to $\mathscr{S}^+$ reads:

$$\limsup_{\varepsilon\to 0}\frac{\Phi(\varepsilon,f)(\omega)-f(\omega)}{\varepsilon}\leq 0,\quad \forall\omega$$

If **T** is of class 0, since any (uniform) accumulation point of v_n, as n goes to ∞ or of v_λ, as λ goes to 0, will satisfy $\Phi(0,w)=w$, it is natural to consider the quantity

$$\frac{\Phi(\varepsilon,f)(\omega)-\Phi(0,f)(\omega)}{\varepsilon}\tag{31}$$

DEFINITION 8: **T** *is of class 1 if it is of class 0 and moreover*

$$\lim_{\varepsilon\to 0}\frac{\Phi(\varepsilon,f)(\omega)-\Phi(0,f)(\omega)}{\varepsilon}=\varphi(f)(\omega)$$

exists. In this case:

$$\overline{\theta^+}(f)=\overline{\theta^-}(f)=\begin{cases}\varphi(f) & \text{if } \Phi(0,f)=f\\ +\infty & \text{if } \Phi(0,f)>f\\ -\infty & \text{if } \Phi(0,f)<f\end{cases}$$

Hence, if there is an element in the intersection $\overline{\mathscr{S}_0^+}\bigcap\overline{\mathscr{S}_0^-}$, it appears as the point where φ changes sign.

4.2. The Case of the Shapley Operator

Explicitly, in the case of the Shapley operator, $\Phi(\varepsilon,f)$ is given by

$$\Phi(\varepsilon,f)(\omega)=\max_{x\in X}\min_{y\in Y}\left\{\varepsilon g(x,y,\omega)+(1-\varepsilon)\int_\Omega f(\omega')\rho(x,y;\omega)(d\omega')\right\}\tag{32}$$

Hence clearly **T** is of class 0. $\Phi(0,\cdot)$ is the projective operator $\mathscr{P}$ expressing the value today of the payoff f tomorrow:

$$\mathscr{P}(f)=\max_{x\in X}\ \min_{y\in Y}\int_\Omega f(\omega')\rho(x,y;\omega)(d\omega')\tag{33}$$

Note that $\Phi(\varepsilon,f)$ appears then as an ε perturbation of $\Phi(0,f)$ in the direction $g-\int fd\rho$ and the limit in (31) as a derivative.

This aspect is used in Rosenberg and Sorin (2001) to prove that, under some topological assumptions on the spaces X, Y and on the maps g, ρ, the Shapley operator is of class 1.

Then for two kinds of games with Ω finite (absorbing games and recursive games) it is shown that the intersection of the closure of $\mathscr{C}^+$ and $\mathscr{C}^-$ is nonempty and Proposition 8 is used to obtain convergence. Notice that the argument is not algebraic (the strategy spaces X and Y are compact sets).

In a second class (games with lack of information on both sides) with Ω compact, one proves that any accumulation point of $\{v_\lambda\}$ as λ goes to 0 (resp. of $\{v_n\}$ as n goes to ∞) belongs to the closure of $\mathscr{S}_0^+$ and that a dual property holds. One then applies Corollary 12 to conclude on the convergence. Related tools are also used to show the convergence for absorbing games with incomplete information on one side in Rosenberg (2000).

Note finally that the operators used in the discrete event systems literature (min max functions, linear max plus ...) are special cases of operator of class one for which direct proofs of existence of the cycle time are available.

4.3. Open Problems

Gunawardena and Keane (1995) produced an example of a MNE mapping on $\mathbb{R}^2$, endowed with the maximum norm, for which v_n does not converge. Lehrer and Sorin (1992) proved in the framework of Markov Dynamic Programming that uniform convergence of v_λ is equivalent to uniform convergence of v_n, with the same limit. They also gave an example where both limits exist and differ.

General conditions under which average iterates of MNE (for the maximum norm) converge and also converge to the same limit are still to be found.

Appendix

Study of the Iterates

a. The same result as Proposition 8 holds if there exists an integer $m \geq 1$ such that:

$$\mathbf{T}^m(Lf) \leq (L+m)f$$

for L large enough.

Explicitly one requires in $\mathscr{C}^+$: $\forall c > 0, \exists m$ and L_c such that $L \geq L_c$ implies:

$$\mathbf{T}^m(Lf) \leq (L+m)f + c \tag{34}$$

b. Similarly if $\Phi^m(\lambda, f) \leq f + \alpha$ then using $\|\Phi^m(\lambda, f) - \Phi^m(\lambda, v_\lambda)\| \leq (1-\lambda)^m \|f - v_\lambda\|$, one has $v_\lambda = \Phi^m(\lambda, v_\lambda) \leq \Phi^m(\lambda, f) + (1-\lambda)^m \|f - v_\lambda\|$; thus $v_\lambda \leq (1-\lambda)^m \|f - v_\lambda\| + f + \alpha$, hence $\|f - v_\lambda\|(1-(1-\lambda)^m) \leq \alpha$ and finally $v_\lambda \leq f + (1-(1-\lambda)^m)^{-1}\alpha$.

Relations Between (B) and (C)

Gunawardena and Keane (1995) note that if $g : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies (A) and (B), then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \ldots, x_n) = g(x_1 - x_n, \ldots, x_{n-1} - x_n) + x_n$ satisfies (A) and (C).

Consider now G from $\mathbb{R}^{n-1}$ to itself satisfying (A) and (B).

Define F from $\mathbb{R}^n$ to itself by

$$F_i(x_1, \ldots, x_n) = G_i(x_1 - x_n, \ldots, x_{n-1} - x_n) + x_n, \quad i = 1, \ldots, n-1$$
$$F_n(x_1, \ldots, x_n) = x_n.$$

Then F satisfies (A) and (C) and moreover

$$F(x_1, \ldots, x_{n-1}, 0) = (G_1(x_1, \ldots, x_{n-1}), \ldots, G_{n-1}(x_1, \ldots, x_{n-1}), 0)$$

so that as well

$$F_i(x_1, \ldots, x_{n-1}, 0) = G_i(x_1, \ldots, x_{n-1}), \qquad 1 \leq i < n$$

and the study of the iterates of G reduces to the study of the iterates of F on $\mathbb{R}^{n-1} \times \{0\}$.

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