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AN OPERATOR APPROACH TO ZERO-SUM REPEATED GAMES

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ABSTRACT

We consider two person zero-sum stochastic games. The recursive formula for the values v_λ (resp. v_n) of the discounted (resp. finitely repeated) version can be written in terms of a single basic operator $\Phi(\alpha, f)$ where α is the weight on the present payoff and f the future payoff. We give sufficient conditions in terms of $\Phi(\alpha, f)$ and its derivative at 0 for $\lim v_n$ and $\lim v_\lambda$ to exist and to be equal.

We apply these results to obtain such convergence properties for absorbing games with compact action spaces and incomplete information games.

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Introduction

We study asymptotic properties of zero-sum repeated games. In such a framework there exist several possible evaluations of the stream of payoffs collected stage after stage by the players. Two classical definitions correspond to the λ-discounted game in which the overall payoff is the expectation of the discounted average of the stage payoffs and the n-stage game in which it is the expectation of the Cesaro mean of the first n-stage payoffs. Under standard assumptions both games have a value, denoted by v_λ and v_n, respectively. The main problem is to study the asymptotic behavior of these quantities as λ goes to 0 and as n goes to infinity.

In various classes of games the existence of the limit of v_λ and v_n has been proved using specific tools. In the case of games with incomplete information, first on one side (Aumann and Maschler [1]) then on both sides (Mertens and Zamir [11]) the limit is characterized by a functional equation and the proof is based on the explicit construction of best reply strategies and on martingale properties.

For finite stochastic games (Bewley and Kohlberg [2] and [3]) the proof relies on properties of solutions of a finite set of algebraic equations. In addition, in the cases above, these values converge to the same limit. In the set up of dynamic programming (one player stochastic game) Lehrer and Sorin [7] proved that uniform convergence of v_λ is equivalent to uniform convergence of v_n and that moreover the limits are the same.

For general zero-sum games, one approach to get such a result would be to construct a "compactification" of the repeated game as a continuous time game on $[0,1]$ for which the λ-discounted game and the n-stage game would correspond to two different time discretizations. This has been done by Sorin in the case of Big Match with incomplete information on one side [17], and by Laraki [6] in the case of the dual of a game with incomplete information on one side.

A very large class of games (including stochastic games and incomplete information games) exhibits a recursive structure, [10] Chapter IV.3. This implies the existence of recursive formulas satisfied by v_n and v_λ, extending the usual ones occurring in stochastic games (Shapley, [16]) or incomplete information games [1], [11]. They can be written using a single basic operator $\Phi(\alpha, .)$ as follows:

$$\Phi(\lambda, v_\lambda) = v_\lambda \quad \text{and} \quad \Phi\Big(\frac{1}{n+1}, v_n\Big) = v_{n+1}.$$

The current approach relies directly on these formulas to get the existence and the equality of the limits of v_n and v_λ.

A first result of convergence of v_n along these lines is due to Kohlberg [4] in the case of absorbing games. The idea is that the asymptotic analysis of v_n can be derived from an asymptotic analysis of the mappings $\Phi(\alpha, .)$ with α near 0. The limit v of v_n will be characterized by a functional system defined through Φ. The first equation of the system is $\Phi(0, v) = v$. But such a fixed point formula (meaning that the players should be able to maintain their future payoffs above the level v) does not characterize the limit. Indeed it does not depend on the current payoffs. Kohlberg shows how v is determined by an equation in terms of $\varphi(.)$, the derivative of $\Phi(\alpha, .)$ with respect to α evaluated at 0.

Kohlberg and Neyman [5] pointed out that the recursive formulas correspond to the n-th iterate (resp. the λ-perturbation) of a non-expansive mapping. A general result in this framework [5] implies that both norms $\|v_n\|$ and $\|v_\lambda\|$ converge to the same limit. In the case of games with incomplete information on one side, this is enough to prove the convergence of v_n and v_λ [10], Ex. 5, p. 298.

In this paper we generalize Kohlberg's approach. We first consider a general stochastic game and study some properties of the operator $\Phi(\alpha, .)$. In section 2 we derive sufficient conditions in terms of these mappings for v_n and v_λ to converge to a function v. In section 3 we characterize the derivative φ of $\Phi(\alpha, f)$ at 0 as the value of the derived game. We deduce then sufficient conditions, expressed in terms of φ, for the convergence of the values. We finally apply these results first to absorbing games with compact action sets and then to incomplete information games. In both cases we prove convergence of v_n and v_λ to the same function v. In the latter class we characterize v by means of functional inequalities involving $\Phi(0, .)$ and φ. Moreover, we prove that this characterization is equivalent to the previous one through two functional equations obtained by Mertens and Zamir [11].

1. Recursive formula

We consider a two person zero-sum stochastic game Γ on a state space Ω with current payoff g from $X \times Y \times \Omega$ to $[-1, 1]$ and transition ρ from $X \times Y \times \Omega$ to $\Delta(\Omega)$ (the set of probabilities on Ω). At each stage m, given the state ω, player 1 (resp. 2) plays in X (resp. Y), the stage payoff is $g_m = g(x, y, \omega)$, the new state ω' is selected according to the distribution $\rho(x, y, \omega)$ and announced to the players.

Let $\mathcal{F}$ be the set of bounded functions on Ω:

$$\mathcal{F} = \{f \colon \Omega \to \mathbb{R}, \|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)| < \infty\}.$$

To each $f \in \mathcal{F}$ and α in $[0,1]$, one associates a game $\Gamma(\alpha, f)$ with strategy spaces X and Y and payoff function in state ω:

$$\Phi_{xy}(\alpha, f)(\omega) = \alpha g(x, y, \omega) + (1-\alpha) E_{\rho(x,y,\omega)}(f).$$

Assume that this game is well defined, has a value on a subset $\mathcal{F}_\Gamma$ of $\mathcal{F}$ and that this value itself belongs to $\mathcal{F}_\Gamma$. One introduces then a one parameter family of operators $\Phi(\alpha, .)$ from $\mathcal{F}_\Gamma$ to itself by

$$\begin{aligned}(1)\qquad \Phi(\alpha, f)(\omega) &= \mathtt{val}_{X\times Y}\Phi_{xy}(\alpha, f)(\omega)\\ &= \sup_X \inf_Y \Phi_{xy}(\alpha, f)(\omega) = \inf_Y \sup_X \Phi_{xy}(\alpha, f)(\omega)\end{aligned}$$

and another operator Ψ by

$$(2)\qquad \Psi(f)(\omega) = \mathtt{val}_{X\times Y}\{g(x, y, \omega) + E_{\rho(x,y,\omega)}(f)\}.$$

The game $\Gamma(\alpha, f)$ is the one shot game where the present, with current payoff g, has weight α and the future, with weight $(1-\alpha)$, is evaluated through the function f which depends only on the state tomorrow. Ψ is the primitive non-expansive "Shapley operator" [16] from which one can derive $\Phi(\alpha, .)$ for $\alpha > 0$.

The main basic properties of these operators are the following:

LEMMA 1: (a) *For any constant a,*

$$\Phi(\alpha, f + a) = \Phi(\alpha, f) + (1-\alpha)a$$

and

$$\Psi(f + a) = \Psi(f) + a.$$

(b) Φ *and* Ψ *are monotonic w.r.t.* f.
(c) $|\Phi(\lambda, f) - \Phi(\mu, f)| \le |\lambda - \mu| \max(\|f\|_\infty, 1)$.
(d) *For* $\alpha > 0$,

$$\Phi(\alpha, f) = \alpha\Psi\Big(\frac{(1-\alpha)}{\alpha} f\Big).$$

COROLLARY 1: (a) *On the set* $(\mathcal{F}_\Gamma, \|\ \|_\infty)$, Φ *is contracting with coefficient* $(1-\alpha)$ *and* Ψ *is non-expansive.*
(b) Φ *is jointly continuous on* $[0,1]\times\mathcal{F}_\Gamma$.

We now relate these operators to the repeated game. In the n-stage game, Γ_n ($n \in \mathbb{N}$), the payoff is $\frac{1}{n}\sum_{m=1}^{n} g_m$ and in the λ-discounted game, Γ_λ ($\lambda \in (0,1]$), it is $\sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1} g_m$. If these games have values, respectively denoted by v_n and v_λ, they obey recursive formulas and can be characterized by means of the mappings defined above.

PROPOSITION 1: (a)

$$v_n = \Phi(1/n, v_{n-1}).$$

(b)

$$nv_n = \Psi^n(0).$$

(c)

$$v_\lambda = \Phi(\lambda, v_\lambda).$$

(d)

$$v_\lambda = \lambda\Psi\Big(\frac{(1-\lambda)v_\lambda}{\lambda}\Big).$$

(e) *If in addition* $(\mathcal{F}_\Gamma, \| \ \|_\infty)$ *is complete,*

$$v_\lambda = (\Phi(\lambda, .))^\infty(f) \quad \textit{for any } f \textit{ in } \mathcal{F}_\Gamma.$$

The current approach aims at deducing asymptotic properties of v_n (as n goes to infinity) and of v_λ (as λ goes to 0) from a direct study of the mapping Φ. We describe now explicitly how the previous framework applies to specific classes. A much more general analysis can be found in [10], Chapter IV.3.

I. FINITE STOCHASTIC GAMES [16]. The move sets, say I and J, are finite as well as the state space Ω. A real payoff function g on $I\times J\times\Omega$ and, for each (i,j), a transition kernel $\rho(i,j;.)$ from Ω to $\Delta(\Omega)$ are given. Let $X = \Delta(I)$, $Y = \Delta(J)$ be the mixed moves sets. Then one has

$$v_\lambda(\omega) = \mathtt{val}_{X\times Y}\{\lambda\sum\nolimits_{ij} x_i y_j g(i,j,\omega) + (1-\lambda)\sum\nolimits_{ij\omega'} x_i y_j \rho(i,j;w)(\omega')v_\lambda(\omega')\}$$

and similarly

$$(n+1)v_{n+1}(\omega) = \mathtt{val}_{X\times Y}\{\sum\nolimits_{ij} x_i y_j g(i,j,\omega) + n\sum\nolimits_{ij\omega'} x_i y_j \rho(i,j;w)(\omega')v_n(\omega')\}.$$

If we still denote by ρ and g the multilinear extensions from $I\times J$ to $X\times Y$ of the corresponding functions, the operator is written as

$$\Phi(\alpha, f) = \mathtt{val}_{X\times Y}\{\alpha g(x,y,\omega) + (1-\alpha)\sum\nolimits_{\omega'} \rho(x,y;w)(\omega')f(\omega')\}.$$

(Note that we did not specify the signalling structure; we only assumed ω to be known at each stage by both players.)

II. GAMES WITH INCOMPLETE INFORMATION AND STANDARD SIGNALLING [1] [11]. For player 1, I is the finite move set, K the finite type set and p the initial distribution on K. The corresponding notions for player 2 are denoted by J, L and q.

The payoff matrix in state k,l is A^{kl}. The game is played as follows: a couple (k,ℓ) in $K\times L$ is chosen according to the product probability $p\otimes q$. Player 1 (resp. 2) is informed upon k (resp. ℓ) and the same game $A^{k\ell}$ is played repeatedly. At each stage the moves chosen are announced but the payoff is not revealed. Then one has

$$v_\lambda(p,q) = \mathrm{val}_{\Delta(I)^K\times\Delta(J)^L}\{\lambda\sum\nolimits_{ijkl}p^kq^ls_i^kA_{ij}^{kl}t_j^l + (1-\lambda)\sum\nolimits_{ij}\bar{s}_i\bar{t}_jv_\lambda(p(i),q(j))\},$$

$$(n+1)v_{n+1}(p,q)=\mathrm{val}_{\Delta(I)^K\times\Delta(J)^L}\{\sum\nolimits_{ijkl}p^kq^ls_i^kA_{ij}^{kl}t_j^l+n\sum\nolimits_{ij}\bar{s}_i\bar{t}_jv_n(p(i),q(j))\},$$

where $s\in\Delta(I)^K, \bar{s} = \sum_k p^ks^k$ and $p(i)$ is the conditional probability on K given i (and similarly for $t,\bar{t}$ and $q(j)$):

$$p(i)^k = \frac{p^ks_i^k}{\bar{s}_i}.$$

Note that in the true play of the incomplete information game, none of the players is able to compute both $p(i)$ and $q(j)$. However, the above formulas show that the values would not change if these posteriors were publicly announced at each stage. This property allows us to consider these games as stochastic games (as far as only the values v_n or v_λ are concerned). Explicitly X is the set of type-dependent mixed moves of player 1, $X = \Delta(I)^K$ and similarly $Y = \Delta(J)^L$. The state space is $\Omega = \Delta(K)\times\Delta(L)$. The transition $\rho(x,y,(p,q))$ gives probability $\bar{s}_i\bar{t}_j$ to the new state $(p(i),q(j))$.

III. FINITE PUBLIC CASE. This corresponds to a finite stochastic game (I and J are the finite pure action sets) on a finite parameter space Z, with lack of information on both sides, K being the set of types of player 1 and L of player 2, p and q being the initial distributions according to which k and ℓ are chosen, once and for all. Assume move/parameter/types dependent payoff A and transition π on Z. Hence for each (k,ℓ) a stochastic game on Z is defined. In addition, the players get random signals after each stage and we assume that the signal of each player contains the new parameter z' (and his own move) and allows us to compute the posterior probability of the opponent on his unknown state variable,

$$v_\lambda(p,q,z) = \mathrm{val}_{\Delta(I)^K\times\Delta(J)^L}\{\lambda\sum\nolimits_{ijkl}p^kq^ls_i^kA_{ij}^{klz}t_j^l + (1-\lambda)E_{\pi,p,q,s,t,z}v_\lambda(p(\alpha,\beta),q(\alpha,\beta),z(\alpha,\beta))\},$$

where α and β are the signals to the players, $z(\alpha,\beta)$ the corresponding new parameter in Z, and $p(\alpha,\beta)$ and $q(\alpha,\beta)$ the new conditional distributions on K and L given the signals. The public hypothesis means that, for any couple of signals (α, β) and (α, β') having positive probability under (s,t), the conditional probability on K is the same given β or β' (and is $p(\alpha, \beta)$) and similarly for q.

Here $X = \Delta(I)^K$ and $Y = \Delta(J)^L$, $\Omega = \Delta(K) \times \Delta(L) \times Z$. The transition on Ω is defined by the distribution on the signals.

Remark: The "public" hypothesis implies that one can keep Ω as "universal belief space" (see [10], Chapter III): at each stage, player 1 can compute the new beliefs of player 2 on his unknown variable k. He does not have to introduce private beliefs on those. A same property holds for player 2.

2. Properties of Φ

The basic property is expressed by the domination of fixed points v_λ ($\lambda \in (0,1]$) by approximately superharmonic functions in the following sense:

PROPOSITION 2: *Assume that for some constant δ:*

$$\mathbf{\Phi}(\lambda, f)(\omega) \leq f(\omega) + \delta, \quad \forall \omega \in \Omega;$$

then

$$v_\lambda(\omega) \leq f(\omega) + \delta/\lambda, \quad \forall \omega \in \Omega.$$

Proof: Let $\rho = \sup_\omega (v_\lambda - f - \delta/\lambda)^+(\omega)$. Then

$$(v_\lambda - f - \delta/\lambda) \leq \mathbf{\Phi}(\lambda, v_\lambda) - \mathbf{\Phi}(\lambda, f) + \delta - \delta/\lambda$$

but $v_\lambda \leq f + \delta/\lambda + \rho$. Thus by Lemma 1 (a) and (b)

$$\mathbf{\Phi}(\lambda, v_\lambda) \leq \mathbf{\Phi}(\lambda, f + \delta/\lambda + \rho) \leq \mathbf{\Phi}(\lambda, f) + (1-\lambda)(\delta/\lambda + \rho),$$

hence

$$(v_\lambda - f - \delta/\lambda) \leq (1-\lambda)\rho,$$

a contradiction, if $\rho > 0$. ∎

Remark: The same bound holds on a subset Ω' if for each $\omega \in \Omega'$ and for each $x \in X$ there exists $y = y(\omega, x)$ such that Ω' is stable under x, y at ω and $\mathbf{\Phi}_{xy}(\lambda, f)(\omega) \leq f(\omega) + \delta$.

We let now the "per stage error" $\delta\varepsilon$ be proportional to the "per stage weight" ε.

Definition: C_δ^+ is the set of functions f for which there exists a positive ε_0 such that

$$f+\delta\varepsilon \geq \Phi(\varepsilon,f), \quad \forall\varepsilon\in(0,\varepsilon_0), \tag{3}$$

and similarly C_δ^- for the set of f such that there exists $\varepsilon_0>0$ with

$$f-\delta\varepsilon \leq \Phi(\varepsilon,f), \quad \forall\varepsilon\in(0,\varepsilon_0); \tag{4}$$

$f\in C_\delta^+$ is approximately (up to δ) superharmonic for all maps $\Phi(\varepsilon,.)$ for ε small enough. We will call such f, δ superuharmonic (u is for uniform). Then one obtains

COROLLARY 2: *If f belongs to C_δ^+, then*

$$f+\delta \geq \limsup_{\lambda\to 0} v_\lambda.$$

Similarly, if f belongs to C_δ^-, then

$$f-\delta \leq \liminf_{\lambda\to 0} v_\lambda.$$

Proof: f belongs to C_δ^+ implies that for all λ small enough $\Phi(\lambda,f)\leq f+\delta\lambda$, thus by Proposition 2, $v_\lambda\leq f+\delta$. ∎

A related result holds for the value v_n of the finite game.

LEMMA 2: *If f belongs to C_δ^+, then*

$$f+\delta \geq \limsup_{n\to\infty} v_n.$$

If f belongs to C_δ^-, then

$$f-\delta \leq \liminf_{n\to\infty} v_n.$$

Proof: Let $f\in C_\delta^+$ and $\varepsilon_0>0$ be such that (3) holds. Choose $N\geq 1/\varepsilon_0$. We show by induction that

$$v_n \leq f+\frac{(n+1-N)}{n}\delta+\frac{(N-1)}{n}\|f-v_{N-1}\|_\infty.$$

In fact by Lemma 1

$$v_N=\Phi\Big(\frac{1}{N},v_{N-1}\Big)\leq\Phi\Big(\frac{1}{N},f\Big)+\frac{(N-1)}{N}\|f-v_{N-1}\|_\infty$$

and

$$\Phi\Big(\frac{1}{N},f\Big)\leq f+\frac{\delta}{N}.$$

Now at step $n+1$

$$\begin{aligned} v_{n+1} &= \Phi\Big(\frac{1}{n+1}, v_n\Big) \\ &\leq \Phi\Big(\frac{1}{n+1}, f\Big) + \frac{n}{n+1}\Big(\frac{(n+1-N)}{n}\delta + \frac{(N-1)}{n}\|f - v_{N-1}\|_\infty\Big) \\ &\leq f + \frac{(n+2-N)}{n+1}\delta + \frac{(N-1)}{(n+1)}\|f - v_{N-1}\|_\infty. \qquad \blacksquare \end{aligned}$$

Remark: Given a distribution θ on the positive integers, let v_θ be the value of the game with payoff $\sum_{m=1}^{\infty}\theta_m g_m$. The same proof as above shows that $v_\theta \leq f+\delta$ as soon as $\theta_m/\sum_{\ell=m}^{\infty}\theta_\ell \leq \varepsilon_0$, for all m.

Definition: Let $\mathcal{N}_\delta(f) = \{f' : \|f' - f\|_\infty \leq \delta\}$ be a δ neighborhood of f in the uniform norm. $\mathcal{C}$ is the set of $f \in \mathcal{F}_\Gamma$ satisfying

$$\forall \delta > 0, \quad \exists f', f'' \in \mathcal{F}_\Gamma \cap \mathcal{N}_\delta(f), \quad f' \in \mathcal{C}_\delta^-, \quad f'' \in \mathcal{C}_\delta^+.$$

This means that in each δ neighborhood of $f \in \mathcal{C}$, there exists both a sub and a super δ uharmonic function.

Then one obtains

PROPOSITION 3: *If f belongs to $\mathcal{C}$, then*

$$f = \lim_{\lambda\to 0} v_\lambda = \lim_{n\to\infty} v_n.$$

Proof: The proof follows from Corollary 2 and Lemma 2. $\blacksquare$

COROLLARY 3: *$\mathcal{C}$ has at most one element.*

3. The derived game

We first prove the following extension of Mills' result ([13], see also [10] pp. 12–13).

PROPOSITION 4: *Let X and Y be compact sets, f and g real functions on $X \times Y$. Assume that for any $\alpha \geq 0$, the functions g and $f + \alpha g$ are u.s.c. in x and l.s.c. in y and that the game $(f + \alpha g; X, Y)$ has a value, $\mathtt{val}_{X\times Y}(f + \alpha g)$. Denote the optimal strategy sets by $X(f + \alpha g)$ and $Y(f + \alpha g)$. Then*

$$\mathtt{val}_{X(f)\times Y(f)}(g) = \lim_{\alpha\to 0^+} \frac{\mathtt{val}_{X\times Y}(f + \alpha g) - \mathtt{val}_{X\times Y}(f)}{\alpha}.$$

Proof: Let $\alpha > 0$, x_α in $X(f+\alpha g)$ and y in $Y(f)$. Then

$$\alpha g(x_\alpha, y) = (f+\alpha g)(x_\alpha, y) - f(x_\alpha, y) \geq \mathtt{val}_{X\times Y}(f+\alpha g) - \mathtt{val}_{X\times Y}(f).$$

Hence

$$\inf_{Y(f)} g(x_\alpha, y) \geq \frac{\mathtt{val}_{X\times Y}(f+\alpha g) - \mathtt{val}_{X\times Y}(f)}{\alpha}$$

and

$$\limsup_{\alpha\to 0^+} \inf_{Y(f)} g(x_\alpha, y) \geq \limsup_{\alpha\to 0^+} \frac{\mathtt{val}_{X\times Y}(f+\alpha g) - \mathtt{val}_{X\times Y}(f)}{\alpha}.$$

Let x be an accumulation point of a family $\{x_\alpha\}$, as α goes to 0 along a sequence realizing the lim sup. Since g is u.s.c. in x, one obtains

$$\inf_{Y(f)} g(x,y) \geq \limsup_{\alpha\to 0^+} \inf_{Y(f)} g(x_\alpha, y).$$

Note also that x is in $X(f)$ (X is compact and $f+\alpha g$ is u.s.c. in x), hence

$$\sup_{X(f)} \inf_{Y(f)} g(x,y) \geq \limsup_{\alpha\to 0^+} \frac{\mathtt{val}_{X\times Y}(f+\alpha g) - \mathtt{val}_{X\times Y}(f)}{\alpha}$$

and the result follows from the dual inequality. ∎

Remark: The derivative of the value of f in the direction g is the value of g played on the optimal subsets for f.

Before applying this result we assume from now on the following hypotheses:

(i) *X and Y are compact Hausdorff.*

The function $(x,y) \mapsto \Phi_{xy}(\alpha, f)(\omega)$ is, for each α, f, ω, upper semi-continuous in x and lower semi-continuous in y (u.s.c./l.s.c.).

The game $\Gamma(\alpha, f)(\omega)$ has a value. (This is in particular the case if in addition the sets X and Y are convex and $(x,y) \mapsto \Phi_{xy}(\alpha, f)(\omega)$ is quasi-concave in x and quasi-convex in y.)

We denote by $X(\alpha, f)(\omega)$ and $Y(\alpha, f)(\omega)$ the corresponding sets of optimal strategies, for $\alpha \in [0,1]$.

(ii) *The function*

$$(x,y) \mapsto \varphi_{xy}(f)(\omega) = g(x,y,\omega) - E_{\rho(x,y,\omega)}(f)$$

is u.s.c./l.s.c. on $X \times Y$.

Following [13], see also [10] pp. 12–13, we consider the “derivative” of the value of the game $\Gamma(\alpha, f)$ at 0 and associate to it a new game as follows:

Definition: The **derived game** $\mathcal{G}(f)(\omega)$ is the game with payoff $\varphi_{xy}(f)(\omega)$ played on $X(0,f)(\omega)\times Y(0,f)(\omega)$.

The main property is the following:

PROPOSITION 5: *$\mathcal{G}(f)(\omega)$ has a value and optimal strategies. Moreover, its value, denoted by $\varphi(f)(\omega)$, satisfies*

$$\varphi(f)(\omega) = \lim_{\alpha\to 0^+} \frac{\Phi(\alpha,f)(\omega) - \Phi(0,f)(\omega)}{\alpha}.$$

Proof: Recall that $\Phi_{xy}(0,f)(\omega) = E_{q(x,y,\omega)}(f)$ so that $\Phi_{xy}(\alpha,f)(\omega) = \Phi_{xy}(0,f)(\omega) + \alpha\varphi_{xy}(f)(\omega)$.

For ω given, apply the previous Proposition 4 to the game $\Gamma(\alpha,f)(\omega)$. ∎

Remark: Note that $\varphi(f+c) = f - c$ for any constant c, but φ is not monotonic since the set of optimal strategies may change.

$\Gamma(0,f)$ appears as the **projective game**: if f represents the "level" on the state space, the payoff today is the expectation of the level tomorrow, $E_{\rho(x,y,\omega)}(f)$. The payoff φ in the derived game measures the difference between the current payoff g and the expected future "level" when both players play optimally in the projective game.

A useful consequence of the previous Proposition is the following property: if a strategy x is good in the projective game, it cannot guarantee more than the value in the derived game.

COROLLARY 4: *For any ω, for any $\beta > 0$, there exists $\eta > 0$ such that, for all $x\in X$, either*

(a) *there exists $y\in Y$ with $\Phi_{xy}(0,f)(\omega)\le\Phi(0,f)(\omega) - \eta$,*

or

(b) *for any y optimal in $\mathcal{G}(f)(\omega)$, $\varphi_{xy}(f)(\omega)\le\varphi(f)(\omega) + \beta$.*

Proof: By contradiction, otherwise for a specific ω and some $\beta > 0$ one can find, for each positive integer m, x_m in X with

$$\Phi_{x_m y}(0,f)(\omega) > \Phi(0,f)(\omega) - 1/m, \quad \forall y\in Y$$

and

$$\varphi_{x_m y_m}(f)(\omega) > \varphi(f)(\omega) + \beta, \quad \text{for some } y_m \text{ optimal in } \mathcal{G}(f)(\omega).$$

If x^* (resp. y^*) is an accumulation point of the sequence x_m (resp. y_m), the first inequality

$$\Phi_{x^*y}(0,f)(\omega)\geq\Phi(0,f)(\omega)\quad\forall y\in Y$$

shows that x^* belongs to $X(0,f)(w)$ which, together with the second inequality

$$\varphi_{x^*y^*}(f)(\omega)\geq\varphi(f)(\omega)+\beta,\quad \text{for } y^* \text{ optimal in } \mathcal{G}(f)(\omega),$$

contradicts the definition of $\varphi(f)(\omega)$. ∎

From Proposition 1, it is clear that any uniform limit of v_n or v_λ will satisfy $\Phi(0,f)=f$. It is thus natural to consider also the ratio $\{\Phi(\alpha,f)(\omega)-f(\omega)\}/\alpha$.

PROPOSITION 6:

$$\varphi^*(f)(w)=\lim_{\alpha\to 0}\frac{\Phi(\alpha,f)(\omega)-f(\omega)}{\alpha}$$

exists in $\mathbb{R}\cup\{-\infty,+\infty\}$.

Proof: If $\Phi(0,f)(\omega)<f(\omega)$, then $\varphi^*(f)(\omega)=-\infty$.
If $\Phi(0,f)(\omega)>f(\omega)$, then $\varphi^*(f)(\omega)=+\infty$.
If $\Phi(0,f)(\omega)=f(\omega)$, then $\varphi^*(f)(\omega)=\varphi(f)(\omega)$. ∎

These operators are useful to define new subsets of functions.

Definition: Let $\mathcal{S}^+$ be the set of functions f satisfying the following system:

$$\Phi(0,f)\leq f,$$

$$\Phi(0,f)(\omega)=f(\omega)\Rightarrow\varphi(f)(\omega)\leq 0, \tag{5}$$

or equivalently

$$\varphi^*(f)\leq 0.$$

Let similarly $\mathcal{S}^-$ be the set defined by

$$\Phi(0,f)\geq f,$$

$$\Phi(0,f)(\omega)=f(\omega)\Rightarrow\varphi(f)(\omega)\geq 0, \tag{6}$$

or

$$\varphi^*(f)\geq 0.$$

We now relate these new sets to the ones introduced in Part 2.

PROPOSITION 7:

$$\bigcap_{\delta>0} C_\delta^+ \subset \mathcal{S}^+.$$

Proof: Let $f \in \bigcap_{\delta>0} C_\delta^+$. Then for any δ there is an $\varepsilon(\delta)$ such that for all $\varepsilon \le \varepsilon(\delta)$ and all $\omega \in \Omega$,

$$\Phi(\varepsilon, f)(\omega) \le f(\omega) + \delta\varepsilon.$$

By letting ε go to 0, one gets

$$\varphi^*(f)(\omega) \le \delta. \qquad \blacksquare$$

In fact the reverse inclusion holds in the finite case.

PROPOSITION 8: *Assume Ω is finite.*

$$\mathcal{S}^+ \subset \mathcal{C}_\delta^+, \quad \forall \delta > 0$$

(and similarly for $\mathcal{S}^-$ and $\mathcal{C}_\delta^-$).

Proof: Let $f \in \mathcal{S}^+$. Fix $\omega \in \Omega$. Hence $\varphi^*(f)(\omega) \le 0$, so that by Proposition 6, for any $\delta > 0$ there exists $\varepsilon_0(\omega) > 0$ such that $\varepsilon \le \varepsilon_0(\omega)$ implies

$$\Phi(\varepsilon, f)(\omega) - f(\omega) \le \delta\varepsilon.$$

Since Ω is finite the conclusion follows. $\blacksquare$

We obtain in this case a "local" condition implying that f belongs to $\mathcal{C}$, namely, with $\bar{\mathcal{S}}$ being the closure of $\mathcal{S}$,

COROLLARY 5: *Assume Ω finite. If f is in the (uniform) closure of both $\mathcal{S}^+$ and $\mathcal{S}^-$, then f belongs to $\mathcal{C}$. In particular, $\overline{\mathcal{S}}^+ \cap \overline{\mathcal{S}}^-$ contains at most one point.*

In the remainder of this section, we provide conditions that generalize to some extent such a result to compact state space Ω.

The next proposition compares the variation $\Phi(\alpha, f) - f$ for two functions at a point ω maximizing their difference. A similar property is proved by Kohlberg [4] for the case of constant functions.

PROPOSITION 9 (Maximum principle): (i) *Let f_1, f_2 and ω satisfy*

$$f_2(\omega) - f_1(\omega) = \delta = \max_{\omega' \in \Omega} (f_2 - f_1)(\omega') > 0;$$

then

$$(\Phi(\alpha, f_1)(\omega) - f_1(\omega)) - (\Phi(\alpha, f_2)(\omega) - f_2(\omega)) \ge \alpha(f_2(\omega) - f_1(\omega)),$$

hence

$$\varphi^*(f_1)(\omega) - \varphi^*(f_2)(\omega) \geq \delta.$$

(ii) *If moreover*

$$\Phi(0, f_2)(\omega) \geq f_2(\omega), \quad \Phi(0, f_1)(\omega) \leq f_1(\omega),$$

then

$$\Phi(0, f_i)(\omega) = f_i(\omega), \quad i = 1, 2,$$

hence

$$\varphi(f_1)(\omega) - \varphi(f_2)(\omega) \geq \delta.$$

Proof: For any $\omega' \in \Omega$

$$\begin{aligned}\Phi(\alpha, f_2)(\omega') - \Phi(\alpha, f_1)(\omega') &\leq \Phi(\alpha, f_1 + \delta)(\omega') - \Phi(\alpha, f_1)(\omega') \\ &\leq (1-\alpha)\delta \\ &\leq (1-\alpha)(f_2(\omega) - f_1(\omega)).\end{aligned}$$

Hence, in particular,

$$(\Phi(\alpha, f_1)(\omega) - f_1(\omega)) - (\Phi(\alpha, f_2)(\omega) - f_2(\omega)) \geq \alpha(f_2(\omega) - f_1(\omega)).$$

Hence dividing by α and letting α go to 0, using Proposition 6 one has

$$\varphi^*(f_2)(\omega) - \varphi^*(f_1)(\omega) \geq \delta.$$

For (ii), taking $\alpha = 0$ in the previous inequality implies $\Phi(0, f_i)(\omega) = f_i(\omega), i = 1, 2$ so that one obtains as well

$$(\Phi(\alpha, f_2)(\omega) - \Phi(0, f_2)(\omega)) - (\Phi(\alpha, f_1)(\omega) - \Phi(0, f_1)(\omega)) \geq \alpha(f_1(\omega) - f_2(\omega)),$$

hence

$$\varphi(f_2)(\omega) - \varphi(f_1)(\omega) \geq \delta. \qquad \blacksquare$$

This result allows one to compare functions in $\mathcal{S}^+$ and $\mathcal{S}^-$ in the continuous case.

PROPOSITION 10: *Assume Ω compact. For all continuous functions $f_1 \in \mathcal{S}^+$ and $f_2 \in \mathcal{S}^-$ one has:*

$$f_1(\omega) \geq f_2(\omega), \quad \forall \omega \in \Omega.$$

Proof: Otherwise, let ω be a point realizing the maximum of $(f_2 - f_1)$ on Ω and assume $(f_2 - f_1)(\omega) = \delta > 0$. By Proposition 9, $\varphi^*(f_1)(\omega) - \varphi^*(f_2)(\omega) \geq \delta$, hence a contradiction to $\varphi^*(f_1) \leq 0$ and $\varphi^*(f_2) \geq 0$ on Ω. $\blacksquare$

Hence we obtain also uniqueness in this case.

COROLLARY 6: *Assume Ω compact. Let $\mathcal{S}_0^+$ (resp. $\mathcal{S}_0^-$) be the subset of continuous functions on Ω belonging to $\mathcal{S}^+$ (resp. $\mathcal{S}^-$). Then the closures of $\mathcal{S}_0^+$ and $\mathcal{S}_0^-$ have at most one common element.*

In the same spirit as Proposition 9 one has

PROPOSITION 11: *Let f, v_λ and ω satisfy*

$$f(\omega) - v_\lambda(\omega) = \max_{w' \in \Omega}(f - v_\lambda)(w') = \delta > 0.$$

Then

$$\Phi(\lambda, f)(\omega) \leq f(\omega).$$

Proof:

$$\begin{aligned}\Phi(\lambda, f)(\omega) - \Phi(\lambda, v_\lambda)(\omega) &\leq \Phi(\lambda, v_\lambda + \delta)(\omega) - \Phi(\lambda, v_\lambda)(\omega)\\ &\leq (1-\lambda)\delta \quad = (1-\lambda)(f(\omega) - v_\lambda(\omega)),\end{aligned}$$

hence

$$\Phi(\lambda, f)(\omega) \leq f(\omega). \quad \blacksquare$$

In particular one deduces

COROLLARY 7: *Assume that ω maximizes $(f - v_{\lambda_n})$ and satisfies $f(\omega) - v_{\lambda_n}(\omega) \geq 0$ on a sequence λ_n going to 0; then $\varphi^*(f)(\omega) \leq 0$.*

Proof: Follows from Proposition 11. ∎

The next property expresses the fact that under the condition of Proposition 9 one has a full interval of fixed points.

PROPOSITION 12 (Tightness): *Let f_1, f_2 and ω satisfy*

$$f_2(\omega) - f_1(\omega) = \delta = \max_{w' \in \Omega}(f_2 - f_1)(\omega') > 0$$

and

$$\Phi(0, f_2)(\omega) \geq f_2(\omega), \quad \Phi(0, f_1)(\omega) \leq f_1(\omega).$$

Then, for any $\theta \in [0, 1]$,

$$\Phi(0, \theta f_2 + (1-\theta) f_1)(\omega) = \theta f_2(\omega) + (1-\theta) f_1(\omega)$$

and

$$X(0, f_2)(\omega) \subset X(0, \theta f_2 + (1-\theta) f_1)(\omega),$$

$$Y(0,f_1)(\omega)\subset Y(0,\theta f_2+(1-\theta)f_1)(\omega).$$

Proof: One has

$$\Phi(0,\theta f_2+(1-\theta)f_1)\leq\Phi(0,f_1+\theta\delta)\leq f_1+\theta\delta,$$

hence at ω

$$\Phi(0,\theta f_2+(1-\theta)f_1)(\omega)\leq\theta f_2(\omega)+(1-\theta)f_1(\omega)$$

and a dual inequality holds.

Let $x\in X(0,f_2)(\omega)$;

$$\begin{aligned}\Phi_{xy}(0,\theta f_2+(1-\theta)f_1)(\omega)&\geq\Phi_{xy}(0,f_2-(1-\theta)\delta)(\omega)\\&\geq\Phi_{xy}(0,f_2)(\omega)-(1-\theta)\delta\geq f_2(\omega)-(1-\theta)\delta\\&\geq\theta f_2(\omega)+(1-\theta)f_1(\omega)=\Phi(0,\theta f_2+(1-\theta)f_1)(w),\end{aligned}$$

hence the result. ∎

In the next two sections, we apply these results to various classes of repeated games and deduce from the operator approach the convergence of v_n and v_λ to the same limit.

4. Absorbing games

We first consider a subclass of stochastic games.

Definition: An **absorbing state** ω satisfies $\rho(\{\omega\}|\omega,i,j)=1$ for all i,j. An **absorbing game** is a stochastic game where all states except one, ω_0, are absorbing.

It is thus enough to describe the game starting from ω_0 and we drop the references to this state. I and J are compact sets and the payoff g is separately continuous on $I\times J$. $(\Omega,\mathcal{A})$ is a measurable space and for each $A\in\mathcal{A}$, $\rho(A|i,j)$ is separately continuous on $I\times J$. Finally, there is a bounded and measurable absorbing payoff, r, defined on $\Omega\setminus\{\omega_0\}$. Let $X=\Delta(I)$ and $Y=\Delta(J)$. In this set up the domain of the recursive operator can be reduced to the payoff in state ω_0. Hence one considers the operator on $\mathbf{R}$ defined by

$$\Phi(\alpha,f)=\mathrm{val}_{X\times Y}\{\alpha g(x,y)+(1-\alpha)E_{\rho(x,y)}(\tilde{f}(\tilde{\omega}))\},$$

where $\tilde{f}$ is equal to f on the active state ω_0 and equal to the absorbing payoff r elsewhere.

(Note that the only relevant parameters are for each (i,j), the probability of absorption $(1-q(\{\omega_0\}|i,j))$ and the absorbing part of the payoff

$(\int_{\Omega\setminus\{\omega_0\}} r(\omega)q(d\omega|i,j))$. By rescaling, one could assume that there are only two absorbing states, with payoff 0 and 1.)

Clearly the conditions of section 3 are satisfied. In the current framework Proposition 9 has the following simple form [4]:

LEMMA 3: *Assume* $f_2 > f_1$. *Then*

$$(\Phi(\alpha, f_1) - f_1) - (\Phi(\alpha, f_2) - f_2) \geq \alpha(f_2 - f_1)$$

and

$$\varphi^*(f_1) - \varphi^*(f_2) \geq (f_2 - f_1).$$

Then one deduces that φ^* is strictly decreasing. Clearly $\varphi^*(f) \leq 0$ for f large enough and similarly $\varphi^*(f) \geq 0$ for f small enough and therefore:

COROLLARY 8: *There exists a unique real number* W *such that:*

$$W' < W \Rightarrow \varphi^*(W') > 0,$$

$$W'' > W \Rightarrow \varphi^*(W'') < 0.$$

Note that this W satisfies $W = \Phi(0, W)$, hence $\varphi(W) = \varphi^*(W)$.

THEOREM 1:

$$\lim_{\lambda\to 0} v_\lambda = \lim_{n\to\infty} v_n = W.$$

Proof: Let $W' > W$ and consider the associated function $\tilde{W}'$ on Ω. It belongs to S^+, hence by Proposition 8 to C_δ^+, for $\delta > 0$. The result then follows from Proposition 3. ∎

5. Incomplete information repeated games

We consider incomplete information games as defined in section 1, subsection II.

Recall that with the previous notations one has $\omega = (p, q)$, $X = \Delta(I)^K$, $Y = \Delta(J)^L$ and $g(x, y, \omega) = \sum_{k\ell} p^k q^\ell s^k A^{k\ell} y^\ell$ with $x = \{s^1, \ldots, s^K\}$, $s^k \in \Delta(I)$, and similarly for y.

Then the operator is defined by

$$\Phi(\alpha, f)(p, q) = \mathtt{val}_{X\times Y}(\alpha g(x, y, p, q) + (1-\alpha)\sum_{ij} \bar{s}(i)\bar{t}(j) f(p(i), q(j))),$$

which we write as

$$\Phi(\alpha, f)(p, q) = \mathtt{val}(\alpha g(p, q) + (1-\alpha)E(f(\tilde{p}, \tilde{q}))).$$

Notations: $\mathcal{F}$ denotes the set of bounded real functions on $\Delta(K)\times\Delta(L)$, concave on $\Delta(K)$ and convex on $\Delta(L)$. $\mathcal{F}_c$ denotes the subset $\mathcal{F}$ of separately continuous functions.

Let us also denote by $NR^1(p)$ the set of non-revealing strategies of player 1, i.e. such that $p^k p^{k'} > 0$ implies $s^k = s^{k'}$, and similarly for player 2.

$u(p,q)$ is the value of the non-revealing game

$$u(p,q) = \mathtt{val}_{\Delta(I)\times\Delta(J)} \sum_{k\ell} s(p^k q^\ell A^{k\ell})t.$$

Given a function h on $\Delta(K)\times\Delta(L)$, $\mathtt{Cav}h$ denotes the smallest function on $\Delta(K)\times\Delta(L)$, greater than h, and such that for each q, $h(.,q)$ is concave.

Similarly, $\mathtt{Vex}h$ denotes the greatest function on $\Delta(K)\times\Delta(L)$, smaller than h, and such that for each p, $h(p,.)$ is convex.

PROPOSITION 13: *The operator $\Phi(\alpha, f)$ is well defined on $\mathcal{F}_c$ and maps $\mathcal{F}_c$ to itself.*

Proof: (1) The game $\Gamma(\alpha,f)(p,q)$ has a value whenever $f\in\mathcal{F}_c$. In fact in this case $\Phi_{xy}(\alpha,f)$ is continuous and concave in x, continuous and convex in y and Sion's theorem applies.

(2) This value $\Phi(\alpha,f)(p,q)$ belongs to $\mathcal{F}$. In fact let $p = \mu p_1 + (1-\mu)p_2$ be a convex combination. Given x_m optimal for Player 1 in $\Gamma(\alpha,f)(p_m,q)$, $m=1,2$, consider x which chooses, if k, x_1 with probability $\mu p_1^k/p^k$. The corresponding payoff, given some y, is

$$\begin{aligned}\Phi_{xy}(\alpha,f)(p,q) =& \alpha\sum_{k\ell} p^k q^\ell s^k A^{k\ell} t^\ell + (1-\alpha)\sum_{ij} \bar{s}(i)\bar{t}(j) f(p(i),q(j))\\ =& \alpha(\mu \sum_{k\ell} p_1^k q^\ell s_1^k A^{k\ell} t^\ell + (1-\mu)\sum_{k\ell} p_2^k q^\ell s_2^k A^{k\ell} t^\ell)\\ &+ (1-\alpha)\sum_{ij} \bar{s}(i)\bar{t}(j) f(\mu \frac{\bar{s}_1(i)}{\bar{s}(i)} p_1(i) + (1-\mu)\frac{\bar{s}_2(i)}{\bar{s}(i)} p_2(i), q(j)),\end{aligned}$$

hence from the concavity of f in p

$$\begin{aligned}\Phi_{xy}(\alpha,f)(p,q) \geq& \mu \Phi_{x_1 y}(\alpha,f)(p_1,q) + (1-\mu)\Phi_{x_2 y}(\alpha,f)(p_2,q)\\ \geq& \mu\Phi(\alpha,f)(p_1,q) + (1-\mu)\Phi(\alpha,f)(p_2,q).\end{aligned}$$

(3) Moreover, the value $\Phi(\alpha,f)(p,q)$ is separately continuous. Indeed, since f is continuous there is a η such that $\|p-p'\| \leq \eta$ implies $|f(p,q)-f(p',q)| \leq \varepsilon$. Let $\|p-p'\| \leq \eta.\varepsilon$. Then, $\bar{s}_i \geq \varepsilon$ implies $\|p_i - p_i'\| \leq \eta$. Moreover, f being bounded by M,

$$|\Phi(\alpha,f)(p,q) - \Phi(\alpha,f)(p',q)| \leq \alpha\|A\|(\|p-p'\|) + (1-\alpha)(2M\varepsilon + \sum_i \bar{s}_i\varepsilon),$$

which is the result. ∎

A first property shows that any function in $\mathcal{F}_c$ is invariant by $\Phi(0,.)$.

LEMMA 4:

$$\forall f \in \mathcal{F}_c, \quad \Phi(0,f) = f.$$

Proof: Recall that $\Phi(0,f)(p,q) = \mathtt{val}_{X\times Y} E(f(\tilde{p},\tilde{q}))$ and that $\tilde{p}$ and $\tilde{q}$ are martingales. For any y non-revealing, one has by Jensen's inequality

$$\Phi_{xy}(0,f) = E_x(f(\tilde{p},q)) \leq f(p,q).$$

The dual inequality implies the result. ∎

Definition: Given $f \in \mathcal{F}$, let $\mathcal{E}_f(q)$ denote the set of p such that $(p, f(p,q))$ is an extreme point of the hypograph of the function $f(.,q)$ defined on $\Delta(K)$.

From the previous proof we deduce

COROLLARY 9: $\forall f \in \mathcal{F}_c$, $\forall (p,q) \in \Delta(K) \times \Delta(L)$,

$$NR^1(p) \subset X(0,f)(p,q),$$

$$p \in \mathcal{E}_f(q) \Rightarrow NR^1(p) = X(0,f)(p,q).$$

Proof: For x non-revealing one has

$$\Phi_{xy}(0,f) = E_y(f(p,\tilde{q})) \geq f(p,q) = \Phi(0,f),$$

hence the first inclusion.

On the other hand, if player 2 plays non-revealing,

$$\Phi_{xy}(0,f) = E_x(f(\tilde{p},q)) \leq f(p,q)$$

and the last inequality is strict whenever $p \in \mathcal{E}_f(q)$ and x is revealing at p. ∎

This result means in particular that in an incomplete information game, a non-revealing behavior is an asymptotically optimal strategy in order to preserve the level f. It is the only optimal one whenever f is strictly concave at the current state. The incentive for using the information comes from the current payoff.

Definition: $\mathcal{A}^+$ is the set of functions f in $\mathcal{F}_c$ such that, for any function h positive, concave and continuous on $\Delta(K)$, with $f+h$ strictly concave,

$$\varphi(f+h)(p,q) \leq 0, \quad \forall (p,q) \in \Delta(K) \times \Delta(L).$$

A dual definition holds for $\mathcal{A}^-$, which is thus the set of functions f in $\mathcal{F}_c$ such that, for any function h positive, concave and continuous on $\Delta(L)$ and with $f-h$ strictly convex,

$$\varphi(f-h)(p,q)\geq 0, \quad \forall (p,q)\in\Delta(K)\times\Delta(L).$$

PROPOSITION 14: *$\mathcal{A}^+\cap\mathcal{A}^-$ contains at most one function.*

Proof: Note simply that $\mathcal{A}^+$ is included in the closure of $\mathcal{S}^+\cap\mathcal{F}_c$ and apply Corollary 6. ∎

We now turn to the study of the asymptotic behavior of the game. We first deal with the discounted case. Let W be an accumulation point of the family $\{v_\lambda\}$, which is uniformly Lipschitz, hence relatively compact, and let v_{λ_n} converge (uniformly) to W. Note that $W\in\mathcal{F}_c$.

PROPOSITION 15: $W\in\mathcal{A}^+$.

Proof: Assume by contradiction $\varphi(W+h)(p,q)\geq\delta>0$ for some h positive, continuous and concave on $\Delta(K)$ with $W+h$ strictly concave. We now use Corollary 4 (or rather its dual) at (p,q) with $\beta=\delta/2$.

Thus given $y\in Y$:

(a) Either there exists $x\in X$ and $\eta>0$ with

$$\boldsymbol{\Phi}_{xy}(0,W+h)(p,q)\geq\boldsymbol{\Phi}(0,W+h)(p,q)+\eta.$$

Hence, a fortiori, since h is concave

$$\boldsymbol{\Phi}_{xy}(0,W)(p,q)+h(p)\geq\boldsymbol{\Phi}(0,W+h)(p,q)+\eta=(W+h)(p,q)+\eta$$

by Lemma 4.

Hence by continuity, there exists N' such that for $n\geq N'$

$$\boldsymbol{\Phi}_{xy}(\lambda_n,v_{\lambda_n})(p,q)\geq v_{\lambda_n}(p,q)+\eta/2.$$

(b) Or there exists $x\in X(0,W+h)(p,q)$ with

$$\varphi_{xy}(W+h)(p,q)\geq\varphi(W+h)(p,q)-\beta\geq\delta/2.$$

Note that since $W+h$ is strictly concave, x is non-revealing (Corollary 9), hence the stage payoff satisfies

$$g(x,y,p,q)\geq E_{xy}((W+h)(\tilde{p},\tilde{q}))+\delta/2\geq W(p,q)+\delta/2.$$

Thus there exists N'' such that, since x is in $NR^I(p)$ and $v_{\lambda_n}\in\mathcal{F}_c$, $n \geq N''$ implies

$$\Phi_{xy}(\lambda_n, v_{\lambda_n})(p,q)\geq\lambda_n g(x,y,p,q) + (1-\lambda_n)v_{\lambda_n}(p,q)\geq v_{\lambda_n}(p,q) + \lambda_n\delta/4.$$

Thus from (a) and (b) we deduce that, for all $n\geq\max(N',N'')$, and all y there exists x such that

$$\Phi_{xy}(\lambda_n, v_{\lambda_n})(p,q) > v_{\lambda_n}(p,q),$$

a contradiction. ∎

Obviously one has also $W\in\mathcal{A}^-$.

We now consider the finitely repeated game G_n. Let us introduce $V^- = \liminf v_n$ and $V^+ = \limsup v_n$. Note that V^+ and V^- are, like all v_n, Lipschitz with constant $\|A\|$. We first prove

PROPOSITION 16: $V^- = V^+$

Proof: The proof goes by contradiction. We assume that $V^+ - V^-$ is non-0. Hence there exists h^+ and h^- both positive, continuous and strictly concave on $\Delta(L)$ (resp. $\Delta(K)$) such that the maximum of $(V^+ - h^+) - (V^- + h^-)$ on $\Delta(K)\times\Delta(L)$ is positive. Consider now an extreme point (p,q) of the set where $\mathtt{Cav}(V^+ - h^+) - \mathtt{Vex}(V^- + h^-)$ is maximum (and equal to $\delta > 0$). Since V^+ and $\mathtt{Cav}V^+$ are convex in q and $\mathtt{Cav}(V^+ - h^+) = \mathtt{Cav}V^+ - h^+$ (and dual properties for V^- and h^-), one has $\mathtt{Cav}V^+(p,q) = V^+(p,q)$ and $\mathtt{Vex}V^-(p,q) = V^-(p,q)$.

Note that V^- is concave and V^+ is convex, and both are continuous, and therefore both $\mathtt{Cav}(V^+ - h^+)$ and $\mathtt{Vex}(V^- + h^-)$ belong to $\mathcal{F}_c$. Hence Proposition 9 implies

$$\varphi(\mathtt{Vex}(V^- + h^-))(p,q) - \varphi(\mathtt{Cav}(V^+ - h^+))(p,q)\geq\delta.$$

Assume thus $\varphi(\mathtt{Vex}(V^- + h^-))(p,q)\geq\delta/2$. By the Lipschitz property of v_n and V^- one has: $\forall\varepsilon > 0, \exists N$ such that $N\geq n$ implies $v_n\geq V^- - \varepsilon$. Let $v_{n_m}(p,q)$ be a subsequence converging to $V^-(p,q)$ as m goes to ∞. As in Proposition 15 we obtain, using Corollary 4 with $\beta = \delta/4$, that given $y\in Y$:

(a) Either there exists $x\in X$ with

$$\begin{aligned}\Phi_{xy}(0, \mathtt{Vex}(V^- + h^-))(p,q) &\geq \Phi(0, \mathtt{Vex}(V^- + h^-))(p,q) + \eta \\ &= \mathtt{Vex}(V^- + h^-)(p,q) + \eta \\ &= V^-(p,q) + h^-(p,q) + \eta.\end{aligned}$$

Hence by continuity (Lemma 1), there exists M such that for $m \geq M$

$$\Phi_{xy}\left(\frac{1}{n_m+1}, v_{n_m}\right)(p,q) \geq v_{n_m}(p,q) + \eta/2.$$

(b) Or there exists $x \in X(0, \mathtt{Vex}(V^- + h^-))(p,q)$, hence in $NR^1(p)$, with $g(x,y)(p,q) \geq V^-(p,q) + \delta/4$.

Thus one can choose α and γ positive such that in both cases, for m large enough $|v_{n_m}(p,q) - V^-(p,q)| \leq \gamma$ implies

$$(n_m+1)v_{n_m+1}(p,q) \geq (n_m+1)v_{n_m}(p,q) + \alpha.$$

This shows that the sequence $v_\ell(p,q)$ for $\ell \geq n_m$ leaves the band $|v_\ell(p,q) - V^-(p,q)| \leq \beta$ from above. Let M be such that $m \geq M$ implies $\|v_{m+1} - v_m\| \leq \gamma/2$ and $m > M$, a first index, where $|v_{m+1}(p,q) - V^-(p,q)| \leq \gamma/2$. It follows that already $|v_m(p,q) - V^-(p,q)| \leq \gamma$, hence $v_{m+1}(p,q) \geq v_m(p,q)$, a contradiction to the choice of m. ∎

We now obtain, denoting $V^* = V^+ = V^-$,

PROPOSITION 17: $V^* \in \mathcal{A}^+$.

Proof: The previous result shows that V^* belongs to $\mathcal{F}_c$. Assume by contradiction: $\varphi(V^* + h)(p,q) \geq \delta > 0$ for some positive concave function h on $\Delta(K)$ such that $V^* + h$ is strictly concave.

Exactly like in the previous proof of Proposition 15, one gets

$$\exists N,\ \forall n \geq N,\ v_{n+1}(p,q) \geq v_n(p,q) + \min\left(\frac{\eta}{2}, \frac{\delta}{4(n+1)}\right).$$

This contradicts Proposition 16. ∎

COROLLARY 10: *There exists $V \in \mathcal{A}^+ \cap \mathcal{A}^-$; $\lim v_\lambda$ and $\lim v_n$ exist and equal V with $\{V\} = \mathcal{A}^+ \cap \mathcal{A}^-$.*

Proof: From Proposition 15, any accumulation point W of the family v_λ belongs to $\mathcal{A}^+ \cap \mathcal{A}^-$ and Proposition 14 shows that this set contains at most one point. Hence v_λ converges to V. Similarly for v_n using Proposition 17. ∎

Definition: We recall the functional equations introduced by Mertens and Zamir [11] where f is defined on $\Delta(K) \times \Delta(L)$:

$$(I) \qquad f = \mathtt{Cav}\min(u, f),$$

$$(II) \qquad f = \mathtt{Vex}\max(u, f).$$

Mertens and Zamir [11] prove that this system has a unique solution which is the limit both of v_n and v_λ; see also [12]. Let us prove now that on $\mathcal{F}_c$ the systems $\{(I)$ and $(II)\}$ and $\{\mathcal{A}^+$ and $\mathcal{A}^-\}$ are equivalent.

PROPOSITION 18: *Let $f \in \mathcal{A}^+$ and $\mathcal{A}^-$; then f satisfies (I) and (II).*

Proof: The proof is based on the following

LEMMA 5: *Let $f \in \mathcal{A}^-$ and $p \in \mathcal{E}_f(q)$. Then*

$$f(p, q) \leq u(p, q).$$

Proof: Let h be strictly concave, continuous and positive on $\Delta(L)$ so that

$$\varphi(f - h)(p, q) \geq 0.$$

Note that $X(0, f-h)(p, q) = X(0, f)(p, q) = NR^1(p)$ and $Y(0, f-h)(p, q) = NR^2(q)$ by Corollary 9. Hence

$$\begin{aligned}\varphi(f-h)(p, q) &= \mathtt{val}_{NR^I(p)\times NR^{II}(q)}\{g(x, y; p, q) - E_{xy}((f-h)(\tilde{p}, \tilde{q}))\}\\ &= u(p, q) - (f-h)(p, q),\end{aligned}$$

so that $u(p, q) \geq f(p, q) - h(q)$ and the result follows. ∎

We now deduce

$$f = \mathtt{Cav}\ \min(u, f).$$

In fact, f being concave in p, $f \geq \mathtt{Cav}\min(u, f)$.

On the other hand, for each fixed q, $f(., q)$ is smaller than $\min(u, f)(., q)$ at each extreme point $p \in \mathcal{E}_f(q)$, by Lemma 5. Hence $f \leq \mathtt{Cav}\min(u, f)$ on $\Delta(K)$, for each fixed q, hence everywhere on $\Delta(K) \times \Delta(L)$.

This ends the proof of Proposition 18. ∎

Note that this provides an alternative proof of the existence of a solution to (I) and (II).

Finally, we deduce the unicity of the solution of (I) and (II) on $\mathcal{F}_c$ through the following:

PROPOSITION 19: *If f is separately continuous and satisfies (I) and (II) it belongs to $\mathcal{A}^+ \cap \mathcal{A}^-$.*

Proof: Let f satisfy (I) and (II) (hence f belongs to $\mathcal{F}_c$) and choose h positive, continuous concave on $\Delta(K)$ with $f + h$ strictly concave. Recall that

$X(0, f+h)(p,q)$ is reduced to $NR^1(p)$. On the other hand, $Y(0, f+h)(p,q) = Y(0,f)(p,q)$. From (II) there exists a finite family q_r, $r \in R$, in $\Delta(L)$ and $\beta \in \Delta(R)$ such that:

$$f(p,q) = \sum_r \beta_r \max(u,f)(p,q_r)$$

and

$$f(p,q_r) \geq u(p,q_r) \quad \forall r \in R.$$

Let y_r be a non-revealing optimal strategy of player 2 for $u(p,q_r)$ and y^* the "splitting strategy" generating the q_r through the y_r's: if ℓ is his type, play y_r with probability $\beta_r q_r^\ell / q^\ell$.

Note that y^* is optimal for $\Phi(0,f)(p,q)$ since, f being concave in p,

$$\sum_j \overline{y}^*(j) f(p,q(j)) \geq \Phi_{xy^*}(0,f)(p,q);$$

but f is convex in q and the $q(j)$'s can be decomposed as $q(j) = \sum_r y_r(j) q_r$, hence

$$f(p,q) \geq \sum_r \beta_r f(p,q_r) \geq \sum_j \overline{y}^*(j) f(p,q(j))$$

and the equality.

Let us now consider $\varphi(f+h)(p,q)$. For any x in $NR^1(p)$ one has

$$\varphi_{xy^*}(f+h)(p,q) = g(x,y^*;p,q) - E_{x,y^*}(f+h)(\tilde{p},\tilde{q}).$$

But

$$g(x,y^*;p,q) = \sum_r \beta_r g(x,y_r;p,q_r)$$

and

$$E_{x,y^*}(f+h)(\tilde{p},\tilde{q}) = \sum_j \overline{y}^*(j) f(p,q(j)) + h(p).$$

Using $g(x,y_r;p,q_r) \leq u(p,q_r)$ one obtains

$$\varphi_{xy^*}(f+h)(p,q) \leq \sum_r \beta_r (u-f)(p,q_r) - h(p) \leq -h(p)$$

and the result follows. ∎

6. Concluding remarks

This paper was devoted to the study of the asymptotic behavior of v_n and v_λ through the recursive operator Φ and its derivative φ. It allows one not only to show the convergence of v_n and v_λ but also to identify their common limit through functional inequalities. This approach underlines:

(1) the connection between the asymptotic behavior of v_n and v_λ in regular games (since the tools are the same),

(2) the relation between incomplete information and stochastic games since the former are studied as stochastic games on the belief spaces,

(3) the role of the derived game — in terms of asymptotic optimal strategies — and not only in terms of value.

Moreover, we obtain a convergence result in absorbing games with compact action sets, hence avoiding algebraic arguments, and shed new light on the functional equation in [11] by showing its equivalence to some variational inequalities.

Similar tools are successfully used in the framework of absorbing games with incomplete information on one side [15] and we conjecture that this perspective will apply to more general classes.

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