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NOTE

A Note on Repeated Extensive Games

SYLVAIN SORIN*

Département de Mathématiques et d'Informatique, URA CNRS 762, Ecole Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France; and UFR SEGMI, Université Paris X, 200 Ave de la République, 92001 Nantene Cedex, France

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We compare results on repeated games when dealing with an extensive form stage game or its strategic form. © 1995 Academic Press, Inc.

PRESENTATION

In a recent paper, Rubinstein and Wolinsky (1995) consider a game in extensive form Γ and its reduced strategic (normal) form G. They then study the associated repeated discounted games. First, they show in examples that the corresponding sets of perfect equilibrium payoffs may differ (even for long games, i.e., with a discount factor close to one). Then they remark that given the usual dimensional conditions, the "Folk theorem" also holds in the extensive case.

We will consider basically the same framework—Γ is a finite game with perfect recall—but from another point of view.

In fact when reducing the game Γ to its strategic form, two consequences are obtained. One is the disappearance of "subgames" and this is the main reason for the phenomena observed in Rubinstein and Wolinsky (1995). The other is related to the information at the end of the play of the stage game, namely the signaling structure.

More precisely given a strategic form game, when one speaks about its repetition without specification, it is usually assumed that standard signal-

* E-mail: sorin@dmi.ens.fr.

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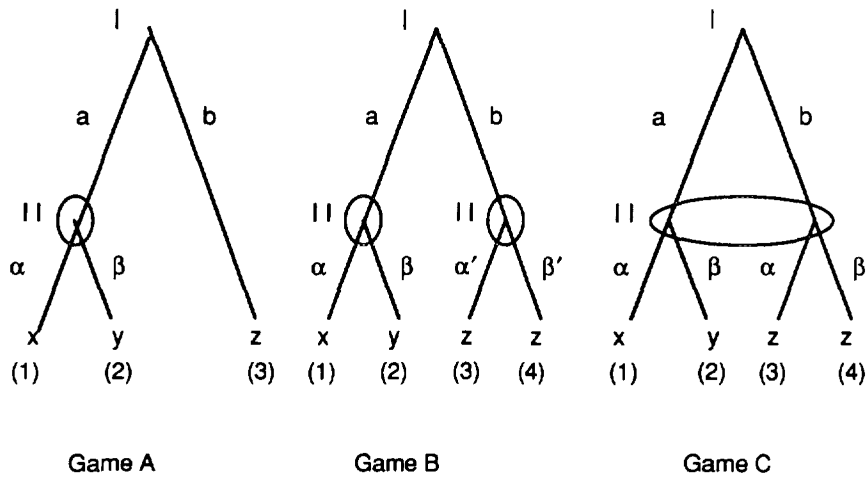
ing holds, i.e., that at the end of each stage, the choice of the N-tuple of pure strategies in the one-stage game, corresponding to the realization at that stage of the mixed strategies used by the N players, is public knowledge.

On the other hand it is also implicitly assumed when dealing with an extensive form game that each terminal node is public knowledge (while a priori no information set contains any terminal node). It is clear that this assumption is inconsistent with equivalent representation of the game (namely, inducing the same reduced strategic form). In the following examples the reduced strategic form is the same while the signaling structure, namely the information partitions of the terminal nodes for each player, even though compatible with perfect recall, may differ.



In game A one can have the partition ({(1), (2)}, (3)) on nodes (corresponding to the partition ({x, y}, z) on outcomes) or ((1), (2), (3)) for player I and the partition ((1), (2), (3)) for player II.

In game B one has to add ((3), (4)) to player II, and one can add it to player I too or keep ({(1), (2)}, {(3), (4)}).

In game C, the information of player I has only to refine ({(1), (2)}, {(3), (4)}), hence may or may not include II's move. For example the action b may be revealing, leading to the partition ({(1), (2)}, (3), (4)). Similarly player II's information can be reduced to its move hence: ({(1), (3)}, {(2), (4)}); or he can get more.

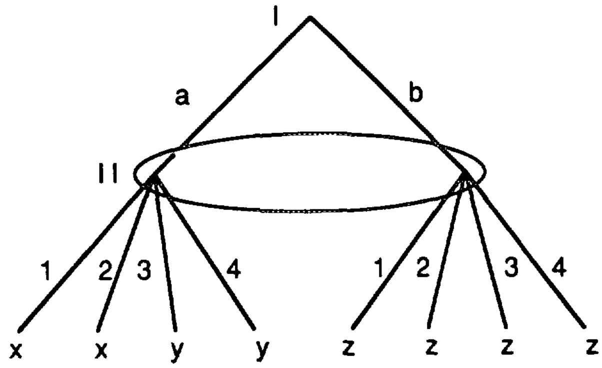
(Note that the discussion was developed in the framework of a game but obviously only the game form was used.)

The usual interpretation of game A would be: player II is asked to play iff Player I played a. So that perfect recall implies that he remembers whether he played or not (in addition to what he played). While in the representation corresponding to game C, player II remembers only his past actions.

It follows that the fact "the players know whether they play in the game (or not)" has to be explicitly assumed when dealing with the repeated game.

This aspect is related to the two alternative interpretations of "information sets" in extensive form games: one corresponds indeed to the effective information the players obtain during the play, the other being more in the spirit of a decision tree with (conditional) "choice sets."

Finally, the standard normal form signaling structure can sometimes not be represented on the tree itself, as in game B above, where the node x should give some information on α' or β'. One should go to the following tree (where the moves $\{1, 2, 3, 4\}$ correspond to $\{\alpha\alpha', \alpha\beta', \beta\alpha', \beta\beta'\}$).



Game D

In the same spirit, the fact that duplicating strategies may have a tremendous impact on equilibria in repeated games is well known in the framework of signaling games or more general games with incomplete information.

To conclude, there is some ambiguity in the notion of an extensive repeated game, and the analysis of the reduced strategic form with standard signaling structure is not sufficient.

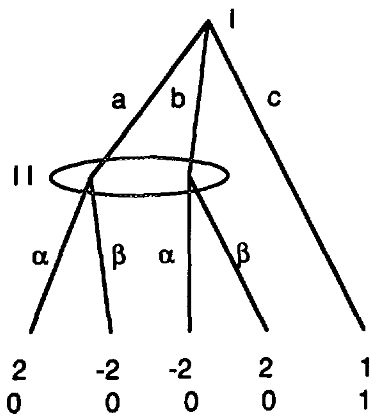
We will find it useful to introduce a specific class of strategies in standard repeated extensive form games; they depend only on the past sequence of "public" one stage terminal nodes and we call them "public."

1. Public Strategies and Equilibria

Given an extensive game Γ, consider any equivalent larger (in terms of inclusion of tree and refinement of information sets) representation Γ' and the associated signaling structures where all terminal nodes are public knowledge. So that by construction any strategy in Γ induces a strategy in Γ'.

The first remark of this note aims at comparing the equilibria of the n-stage games Γ_n and Γ'_n.

We first observe that an equilibrium in Γ_n may not induce an equilibrium in Γ'_n (for $n > 1$). In fact in the following example, consider in Γ_2 the strategy cc for player I and for player II play $(\frac{1}{2}, \frac{1}{2})$ at stage one, then at stage two again play $(\frac{1}{2}, \frac{1}{2})$ if a or b was played at stage one and otherwise play the move selected (by himself) at stage one. (In words, player II plays at stage 2 the action he chose at stage 1 if it was not used.)



This forms an equilibrium in the initial game but no longer in the corresponding standard strategic form where at stage two, player I, given his information, could anticipate player II's move in case of deviation and get 2 by playing a or b.

Note, however, that player II could have replaced his strategy by an equivalent one (i.e., inducing the same distribution on the terminal nodes against any kind of strategy of the opponents) by the following procedure:

at stage one if c, use the initial strategy to obtain a probability on past (private) histories given c, then choose a past history at random and use the initial strategy given this history.

This procedure is general. In fact let us introduce the following:

DEFINITION. A behavioral strategy is public if it depends only on the previous path.

Then any strategy in Γ is equivalent to a public one. Moreover one has:

PROPOSITION. (a) *An equilibrium in the game restricted to public strategies is an equilibrium in the original game.*

(b) *One obtains the same set of equilibria (in terms of distributions on the terminal nodes).*

(c) *Any public equilibrium of Γ_n induces an equilibrium of Γ'_n (similarly for subgame perfect equilibria).*

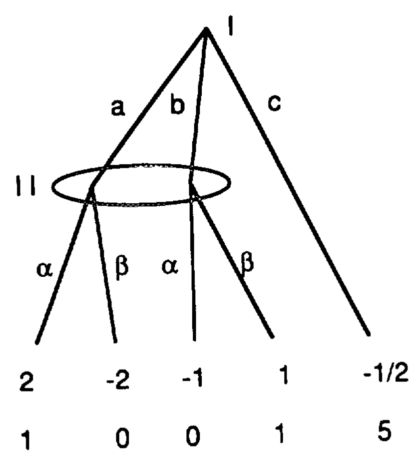
Proof. Points (a) and (b) follow from the comment after the definition.

(c) Consider a public equilibrium profile in Γ_n. It is feasible in Γ'_n. Let us consider a deviation in Γ'_n. If it induces the same path, we can ignore it. Otherwise it will also be observed in Γ_n. Finally, the fact that the deviator may have more information is irrelevant concerning the future behavior of his opponents, hence the result. ∎

Obviously the same property holds for the discounted game or the infinitely repeated game.

(A related result for pure strategies equilibria in general extensive form games was obtained by Dubey and Shubik (1981).)

The converse statement of (c) above is clearly not true, as shown by the following example.



In the strategic form representation of the two-stage game the payoff $(-\frac{1}{2}, 5) + (\frac{1}{2})((2, 1) + (1, 1))$ can be obtained as equilibrium payoff by letting player II play $(\frac{1}{2}, \frac{1}{2})$ at stage one and then choosing a different one-shot equilibrium if α, resp. β, and c are played.

On the other hand in the extensive form representation, it is clear—using the fact that the second stage payoff is a one-shot equilibrium payoff and that the overall payoff is then maximal for II—that the first-stage strategy of I is pure, hence the previous payoff cannot be reached.

Note that a similar phenomenon may occur if one compares, for example, Γ_4 and the two-stage repetition of the reduced strategic form of Γ_2.

2. Extensive and Normal Form Repeated Games

The second purpose of this note is to study "Folk theorems"-type results in Γ and in G. In both cases we consider standard signaling structure, non-observable strategies (but remark that the structure of the proofs will imply that the result holds if we assume observability), and no public device. Denote by Δ the set of feasible individually rational payoffs and v the threat point.

Let us consider first Nash equilibrium payoffs.

The proof of the usual Folk theorem (i.e., the property: (P) the set of equilibrium payoffs is Δ) in the undiscounted game G_∞ is of the kind: follow a path and punish using i.i.d. strategies. Obviously this allows us to define similar strategies in Γ_∞ (where the strategies set is a priori smaller), hence the same result.

For the discounted version of G, (P) holds asymptotically if there exists $x \in \Delta$ with $x \gg v$ (Sorin, 1986). As above the strategies and the result extend to Γ.

Then for G_n, (P) holds asymptotically (Benoit and Krishna, 1987) if condition (C_1) holds: for each player i there exists an equilibrium payoff in the one-shot game, strictly greater than v^i. The equilibrium strategies are then to follow a path and play a sequence of one-shot equilibria or to use i.i.d. punishments. These properties clearly extend to Γ_n, hence again the same result.

(Note that the result in Benoit and Krishna (1987) is in fact more precise since it is enough to have an equilibrium payoff strictly greater than v^i in some G_m; the set of equilibrium payoffs in Γ_m may be smaller but I conjecture that the same property should hold.)

Let us now study the case of subgame perfect equilibria and the validity of the property: (P') the set of subgame perfect equilibrium payoffs is Δ.

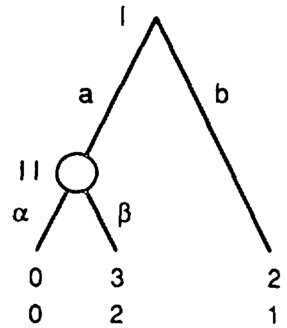
As remarked by Rubinstein and Wolinsky (1995) the validity of (P') for

G_∞ (Aumann and Shapley, 1994; Rubinstein, 1994) extends to Γ_∞: the proof uses i.i.d. punishments of finite length and the behavior of a player after a deviation at some stage and until the end of that stage is thus irrelevant.

Concerning the discounted game, (P') holds asymptotically for G if Δ has a non-empty interior (this is the dimension condition (C_1')). The proof (Fudenberg and Maskin, 1986, 1991; see also Sorin, 1990) uses strategies of the kind: follow a path or play i.i.d., hence they extend to Γ. It remains to check the conditions of subgame perfection after a deviation in the stage game itself. A simple way to fulfill them is to make the future payoff (after the stages of punishment) dependent on the payoff during that stage so that the players (except for the deviator) will be indifferent between their moves, in the stage game, after a deviation.

Finally, for G_n, (P') holds at the limit if in addition to (C_1) condition (C_2) is satisfied: there exist for each player two equilibria giving him different payoffs (Benoit and Krishna, 1985; and for non-observable mixed strategies, (Gossner, 1992). The proof is again based on a path and i.i.d. strategies; however, the play ends with a sequence of one-shot equilibria. Hence a sufficient condition for (P') to hold at the limit is in addition to (C_1) the analogy of (C_2) for subgame perfect equilibria in Γ.

In the next example, (P') holds for G_n but in any Γ_n the only subgame perfect equilibrium payoff is (3, 2).



A further observation is that all the strategies sketched above are public; hence under the "usual assumptions" the various Folk theorems hold for all representations of the game—meaning that for asymptotic results "to know the path is sufficient in repeated extensive games."

The natural extension is obviously to work in the framework of strategic form games with signals: in addition to the stage game data there is a signaling function defined on the product of the (pure) strategy spaces with value on the product of some signal spaces (one component for each player); see, e.g., Lehrer (1992a, 1992b). Note that in this framework a strategy may or may not be revealing or informative and new complex phenomena occur. In particular the above result on public strategies no longer holds if the actual moves used are unknown (even if the payoffs are public).

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