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A Note on the Value of Zero-Sum Sequential Repeated Games with Incomplete Information

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Abstract: We consider repeated two-person zero-sum games with lack of information on both sides. If the one shot game is played sequentially, it is proved that the sequence v_n is monotonic, v_n being the value of the n shot game. Moreover the speed of convergence is bounded by K/n, and this is the best bound.

Introduction

The class of games considered in this note are those introduced and studied by *Aumann/Maschler* [1966, 1967, 1968]. Later it was proved by *Mertens/Zamir* [1971] that, if v_n is the value of an n repeated zero-sum game with lack of information on both sides, $\lim v_n$ exists and $d_n = | v_n - \lim v_n |$ is bounded by $K/\sqrt{n}$. Then *Zamir* [1972] has proved that it is the best bound. Now, if we consider the "independent" case, where moreover the moves are made sequentially, we can compute v_n by using an explicit formula proved by *Ponssard* [1975] for "games with almost perfect information". Then we prove that the sequence v_n is monotonic and that d_n is bounded by K/n. We also give an example of a game with $d_n = 1/2n$.

1. The Game

Let (A^{rs}), $r \in \{1, \ldots, R\}$, $s \in \{1, \ldots, S\}$, be $m \times n$ matrices viewed as payoff matrices of two-person zero-sum games, with elements a_{ij}^{rs}, $i \in I = \{1, \ldots, m\}$, $j \in J = \{1, \ldots, n\}$. Let P (resp. Q) be the simplex of $\mathbf{R}^R$ (resp. $\mathbf{R}^S$).

For each $p \in P$, $q \in Q$, $n \in \mathrm{N}$, $G_n(p, q)$ is the n-times repeated game played as follows.

Stage 0. Chance chooses some r (resp. s) according to the probability distribution p (resp. q). Then player I is informed of r and player II of s. (All of the above description is common knowledge.)

Stage 1. Player I chooses $i_1 \in I$, player II is told which i_1 was choosen and chooses $j_1 \in J$; then player I is informed of j_1.

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Now this stage is repeated again and again. After the n-stage player I receives from player II the following amount:

$\frac{1}{n} \sum_{h=1}^{n} a^{rs}_{i_h j_h}$, where (i_h, j_h) are the strategies used at the h-th stage. We denote by $v_n(p, q)$ the value of $G_n(p, q)$.

Le us define, for each $p \in P$, $q \in Q$:

$A(p, q) = \sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s A^{rs}$ and let $u(p, q)$ be the value of the zero-sum sequential game with matrix $A(p, q)$. (The game where both players play non revealing strategies.)

For each real function f on $P \times Q$ we denote by Cf the smallest real function g on $P \times Q$ such that:

$$g(\cdot, q) \text{ is concave on } P \text{ for each } q \in Q$$

$$g(p, q) \geq f(p, q) \text{ on } P \times Q.$$

Similarly we define $Vf = \underset{Q}{\text{Vex}}\, f$.

We can now state the main result of *Mertens/Zamir* [1971]:

$\lim_{n \to \infty} v_n(p, q)$ exists and is the only solution of the system:

$$\begin{aligned} x(p, q) &= V \max \{u(p, q), x(p, q)\} \\ x(p, q) &= C \min \{u(p, q), x(p, q)\}. \end{aligned} \tag{1}$$

In order to symplify the notations we shall denote by:

M the maximum over i, $i \in I$,

m the minimum over j, $j \in J$,

Σ the expression $\sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s a^{rs}_{ij}$. (In fact Σ is some $\Sigma(i, j)$ but no confusion will result.)

2. The Results

Proposition 1: For each $p \in P$, $q \in Q$, the sequence $v_n(p, q)$ is increasing.

Proof: Using Theorem 1 of *Ponssard* [1975] we have:

$$nv_n(p, q) = CMVm(\Sigma + (n-1)\, v_{n-1}(p, q)) \tag{2}$$

for all $n \geq 1$, where $v_0(p, q) = 0$ on $P \times Q$. Then

$$\begin{aligned} 2\, v_2(p, q) &= CMVm(\Sigma + v_1(p, q)) \\ &\geq CM(Vm\, \Sigma + v_1(p, q)) \end{aligned}$$

since Vex $(a + b) \geqslant$ Vex (a) + Vex (b) and $v_n(p, q)$ is convex w.r.t. q, for all n. Now we have

$$2 v_2(p, q) \geqslant C(MVm\,\Sigma + v_1(p, q)).$$

Let us denote by $g_1(p, q)$ the function $MVm\,\Sigma$, then

$$v_1(p, q) = Cg_1(p, q).$$

But Cav $(a$ + Cav $(a))$ = 2 Cav (a), so that

$$2 v_2(p, q) \geqslant 2 v_1(p, q).$$

Let us assume now that

$$v_n(p, q) \geqslant v_{n-1}(p, q) \text{ on } P \times Q.$$

We have

$$\begin{aligned}(n + 1) v_{n+1}(p, q) &= CMV(m\,\Sigma + n\, v_n(p, q)) \\ &\geqslant CMV(m\,\Sigma + (n-1) v_{n-1}(p, q) + v_n(p, q)).\end{aligned}$$

But the right expression is greater than

$$CM(V(m\,\Sigma + (n-1) v_{n-1}(p, q)) + v_n(p, q)).$$

Denoting by $g_n(p, q)$ the function $MV(m\,\Sigma + (n-1) v_{n-1}(p, q))$, we obtain

$$(n + 1) v_{n+1}(p, q) \geqslant C(g_n(p, q) + \frac{1}{n} Cg_n(p, q))$$

and the right side is

$$\frac{n+1}{n} Cg_n(p, q) = (n + 1) v_n(p, q).$$ ∎

If we denote lim $v_n(p, q)$ by $v(p, q)$ we get obviously.

Corollary 1:

$$v_n(p, q) \leqslant v(p, q) \text{ on } P \times Q \text{ for all } n.$$

Proposition 2 (The Error Term):

$| v(p, q) - v_n(p, q) | \leqslant K/n$ on $P \times Q$, for some $K \in \mathbf{R}$, and this is the best bound.

Proof: We still suppose that player I is the maximizer. We shall write, for $i \in I$,

$$f_i(p, q) = -m\left(\sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s a_{ij}^{rs}\right).$$

Note that the f_i are convex and piecewise linear in both variables.

Let $f(p, q) = \sum_{i=1}^{m} f_i(p, q) - L$ where $L \in \mathbf{R}$ is choosen such that

$$v_1(p, q) \geqslant v(p, q) + f(p, q) \text{ on } P \times Q. \ (v_1 \text{ and } v \text{ are bounded on } P \times Q.)$$

Let us suppose now that:

$$n\, v_n(p, q) \geqslant n\, v(p, q) + f(p, q).$$

Using (2) we have

$$(n+1)\, v_{n+1}(p, q) \geqslant CMVm\,(\Sigma + f(p, q) + n\, v(p, q))$$
$$\geqslant CM\,(V(m\,\Sigma + f(p, q)) + n\, v(p, q))$$

since v is convex w.r.t. q.

But, by constructions, $m\,\Sigma + f(p, q)$ is convex w.r.t. q, for each $i \in I$, so that

$$(n+1)\, v_{n+1}(p, q) \geqslant C\,(u(p, q) + f(p, q) + n\, v(p, q)).$$

Now Cav $(a + b) \leqslant$ Cav (a) + Cav (b) so by letting $a = u(p, q) + f(p, q) + n\, v(p, q)$ and $b = -f(p, q)$, the right member is greater than

$$C\,(u(p, q) + n\, v(p, q)) - C\,(-f(p, q)).$$

Now $-f$ is concave w.r.t. p so we obtain

$$(n+1)\, v_{n+1}(p, q) \geqslant C\,(u(p, q) + n\, v(p, q)) + f(p, q)$$
$$\geqslant C\,((n+1) \min\,(u(p, q), v(p, q))) + f(p, q).$$

Using (1), the fact that f is bounded on $P \times Q$ and $v_n \leqslant v$ for all n (Cor. 1), we arrive at the proof. ∎

Example:

The following example shows that it is the best bound. Assume that $R = 1$ and $S = 2$ (there is lack of information on one side but player I is uninformed). The payoff matrices are given by:

$$A^{11} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad A^{12} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

The functions $u(q)$, $v_n(q)$, $v(q)$, are given in the diagrams below.
We note that $v(1/2) - v_n(1/2) = 1/2n$.

Remarks: If there is lack of information on one side, the informed player maximizing and moving first, we obviously have $v_1(p) \geqslant \text{Cav}\, u(p) = v(p)$ so that Prop. 1 and Cor. 1 imply $v_n(p) = v(p)$ for all n, a result which was already proved by *Ponssard/Zamir* [1973].

In the general case of game with lack of information on one side, the sequence v_n is monotonic, but decreases if the informed player is the maximizer, as already mentioned by *Aumann/Maschler* [1968]. This can be seen immediatly if one writes the recursion formula [*Zamir*] in the following manner:

$$(n+1)\, v_{n+1}(p) = \underset{s}{\text{Max}}\, \{\underset{t}{\text{Min}} \sum_k p^k s^k A^k t + n \sum_i \bar{s}_i v_n(p_i)\}$$

where $s = (s^1, \ldots, s^k, \ldots, s^r)$, s^k is a probability vector over I for all k, t is a probability vector over J, $\bar{s}_i = \sum_k s_i^k p^k$, and p_i is the conditional probability over K given i.
Assuming $v_n(p) \leqslant v_{n-1}(p)$ we have

$$(n+1)\, v_{n+1}(p) \leqslant \underset{s}{\text{Max}}\, \{\underset{t}{\text{Min}} \sum_k p^k s^k A^k t + (n-1) \sum_i \bar{s}_i v_{n-1}(p_i) + \sum_i \bar{s}_i v_n(p_i)\}$$

and since v_n is concave it follows that

$$(n+1)\, v_{n+1}(p) \leqslant \underset{s}{\text{Max}}\, \{\underset{t}{\text{Min}} \sum_k p^k s^k A^k t + (n-1) \sum_i \bar{s}_i v_{n-1}(p_i)\} + v_n(p).$$

Hence

$$(n+1)\, v_{n+1}(p) \leqslant (n+1)\, v_n(p).$$

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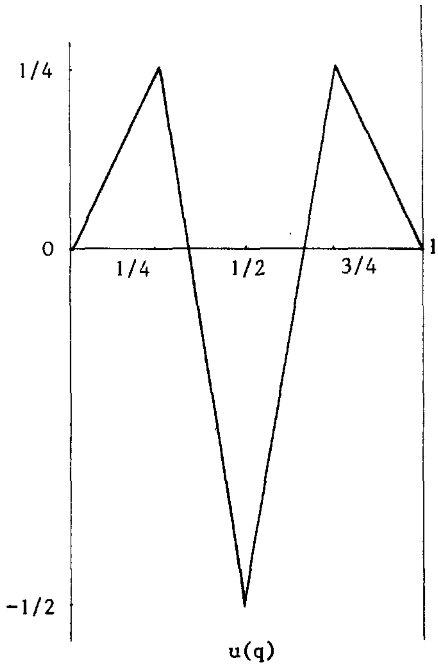
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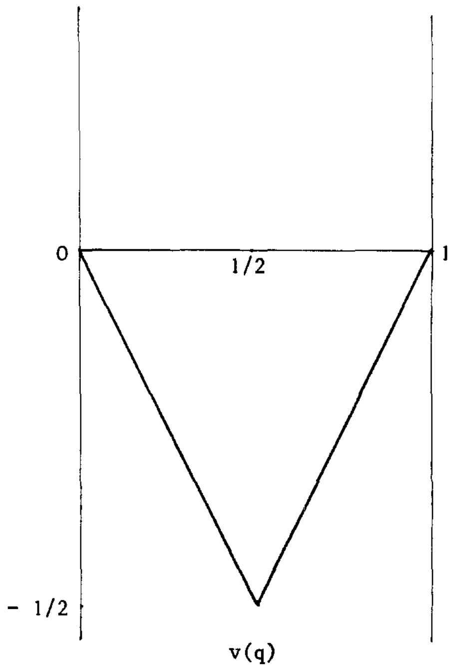
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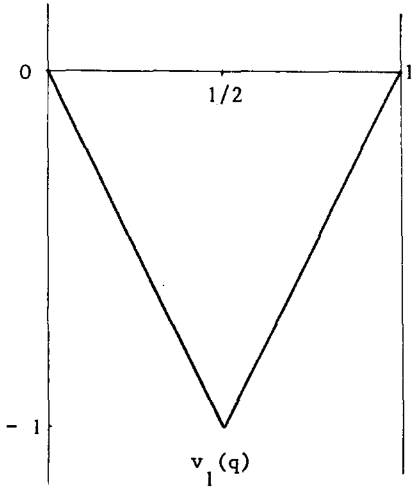
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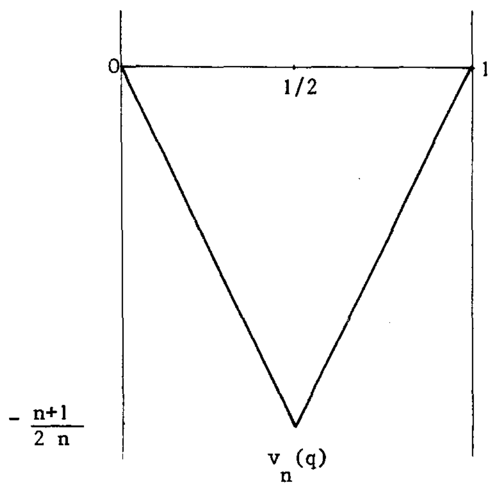
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