International Journal of Game Theory (1993) 22: 1-11

Game Theory

Asymptotic Properties in Dynamic Programming

DOV MONDERER

Department of Economics, Queen's University, Kingston, Canada, K7L 3N6, Canada and Faculty of Industrial Engineering and Management, the Technion, Haifa, Israel

SYLVAIN SORIN

DMJ (URA CNRS 762), Ecole Normale Supérieure, 45 rue d'Ulm, 75230 Paris, France

Abstract: In the framework of dynamic programming we provide two results:

– An example where uniform convergence of the T-stage value does not imply equality of the limit and the lower infinite value.

– Generalized Tauberian theorems, that relate uniform convergence of the T-stage value to uniform convergence of values associated with a general distribution on stages.

1 Introduction

Let S be a state space. For each $s \in S$ let $\emptyset \neq \Gamma(s) \subseteq S$, and let f be a real bounded function on S. Consider the dynamic programming problem where the decision maker on day t, at stage s_t, has to choose a new state $s_{t+1} \in \Gamma(s_t)$, and receives a payoff $f(s_t)$. A *play* at $s \in S$ is a sequence $(s_t)_{t=0}^{\infty}$ with $s_0 = s$ and $s_{t+1} \in \Gamma(s_t)$ for all $t \geq 0$. One traditionally considers the λ-discounted value $V_\lambda(s)$:

$$V_\lambda(s) = \sup_{(s_t)_{t=0}^{\infty}} (1-\lambda) \sum_{t=0}^{\infty} \lambda^t f(s_t),$$

or the T-stage value $V_T(s)$:

$$V_T(s) = \sup_{(s_t)_{t=0}^{\infty}} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

where in both cases the supremum ranges over all plays at s.

One can also consider other evaluations: Let $\theta = (\theta(t))_{t=0}^{\infty}$ be a probability on the set of non-negative integers and define:

$$V_\theta(s) = \sup_{(s_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \theta(t) f(s_t).$$

Lehrer and Sorin (1992) proved that if either one of the limits $\lim_{\lambda \to 1} V_\lambda(s)$, or $\lim_{T \to \infty} V_T(s)$ exists uniformly in $s \in S$, then the other limit also exists uniformly, and the limit functions coincide.

In Section 3 we give sufficient conditions on linearly ordered families $(\Theta, <)$ of probabilities on the integers to get analogous results for $(V_\theta)_{\theta \in \Theta}$ and $(V_T)_{T \geq 0}$.

This research was supported by the fund for the promotion of research in the technion.

0020-7276/93/1/1-11 $ 2.50

There are other natural ways of evaluating streams of payoffs in dynamic programming (except for those discussed above):

The lower (long-run average) value,

$$\underline{V}(s)=\sup_{(s_t)_{t=0}^{\infty}}\liminf_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}f(s_t),$$

and the upper (long-run average) value,

$$\overline{V}(s)=\sup_{(s_t)_{t=0}^{\infty}}\limsup_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}f(s_t),$$

where, again, the supremum is taken on all plays at s.

Lehrer and Monderer (1989) proved that uniform convergence of $(V_\lambda)_{\lambda\in[0,1)}$ to some V implies $V=\overline{V}$, and showed in an example that it does not imply the equality $V=\underline{V}$. If one allows the decision maker to use mixed strategies, i.e., to choose a play in random, and then defines the payoff of each state as the expectation, one obtains new evaluations. It is clear that the evaluations V_λ, V_T, V_θ, and $\overline{V}$ will not change by allowing mixed strategies, but $\underline{V}$ will change in general. Let

$$\underline{U}(s)=\sup_{\mu\in\Delta}\liminf_{T\to\infty}E_\mu\left(\frac{1}{T+1}\sum_{t=0}^{T}f(s_t)\right),$$

where Δ is the set of all probabilities on the set of plays, endowed with the cylinder σ-field, and E_μ stands for the expectation operator with respect to μ.

Obviously $\underline{U}\geq\underline{V}$. As for the relationship between $\underline{U}$ and the limit V of the discounted value functions, Mertens and Neyman (1981) provided sufficient conditions, stronger than the uniform convergence of $(V_\lambda)_{\lambda\in[0,1)}$ (and satisfied in every finite setup), that ensure the equality $\underline{U}=V$ (even for stochastic games). In Section 2 we show that uniform convergence alone is not sufficient by providing a counter example. See Mertens (1987) for related conjectures, hints, and comments. Other type of necessary conditions, for specific types of dynamic programming problems, are discussed in Dutta (1991).

2 The Counter Example

Every rooted directed tree without terminal nodes naturally defines a dynamic programming problem when we attach payoffs to the nodes. Our dynamic programming problem will be defined as a tree, constructed inductively in the spirit of Lehrer and Monderer (1989).

Given two decreasing vanishing sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, define for every real number x the tree $T(x)$ as follows:

Every node of $T(x)$ except for the root has an outdegree one, and the root itself has countably many branches. On the n^{th} branch of the root the payoff, $g(s)$, is 0 until node $[\varepsilon_n n]+1$, it then equals $x-\delta_n$ until node n, and from then on it equals 0. Define a valuation φ at each node s different from the root as follows: $\varphi(s)=x-\delta_n$ for every s in the n^{th} branch appearing before the n^{th} node in this branch, and $\varphi(s)=0$ for every node thereafter. Set $T_1=T(1)$. T_2 is obtained from T_1 by attaching the tree $T(\varphi(s))$ to each node s of T_1, different from the root, and keeping the old payoff of s (i.e., its payoff in T_1). One can continue naturally and define inductively the trees T_3, T_4, ... and finally define $T=\bigcup_{n=1}^{\infty} T_n$. Denote the root of T by s_0, and the payoff function by g.

Note that although g is bounded from above by 1, it is not necessarily bounded from below. Therefore we replace g with a new bounded payoff function f, defined by: $f(s)=\max(g(s), 0)$ for every node s of T.

It is clear that $\lim_{T\to\infty} V_T(s)=\varphi(s)$ uniformly on all nodes s of T. In particular $V(s_0)=1$.

We will show that for a specific choice of the sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, $\underline{U}(s_0)=0$.

Let then $\alpha>0$ and let us prove that $\underline{U}(s_0)<\alpha$. Assume in negation that there exists $\mu\in\Delta$ such that for some integer M, $T\geq M$ implies

$$E_\mu\left(\frac{1}{T+1}\sum_{t=0}^{T} f(s_t)\right)\geq\alpha. \tag{2.1}$$

We remark that we can assume that all plays in the support of μ belong to the following set Ω:

If a play in Ω is on the n^{th} branch of some $T(.)$, it remains in this branch until exactly node n. In fact, if some play leaves the branch before node $[\varepsilon_n n]+1$, the decision maker will increase his payoff by leaving the branch at its root, and if a play leaves the n^{th} branch after node n, it is better for the decision maker to leave it at precisely node n. In particular, a play in Ω never remains in a branch of some $T(.)$ and is thus characterized by a sequence $(m_i)_{i=1}^{\infty}$ of integers inducing the path: Branch m_1 of $T(1)$ until the m_1th node, s_{m_1}, branch m_2 of $T(\varphi(s_{m_1}))$ until node m_2 of this branch (with the valuation $1-\delta_{m_1}-\delta_{m_2}$), etc. ... Finally, for every play in Ω,

$$\sum_{i=1}^{\infty}\delta_{m_i}\leq 1. \tag{2.2}$$

Having done the above reduction, we can now replace any strictly positive payoff on any play in Ω by 1.

The basic idea of the proof is to choose a sequence $(\delta_n)_{n=1}^{\infty}$ converging very slowly to zero, implying by (2.2), that for every play in Ω, for a set of integers i with positive density, $\varepsilon_{m_i}m_i$ is much larger than $\sum_{k<i} m_k$. Hence, every play in Ω has "many" large blocks of zeros.

More precisely, let $M_1=2$, and define inductively $n_i=\sum_{k\leq i} M_k$ and $M_{i+1}=n_i^4$ for every $i\geq 1$. Define $\delta_{n_i}=\frac{1}{i}$ for all i and extend δ by monotonicity to all other integers. Choose $\varepsilon_n = \frac{1}{\sqrt{n}}$. We say that a play w is good in the i^{th} block $I_i = [n_{i-1}, n_i]$ if a sequence of ones starts in this block. That is, if w is determined by $m_1, m_2, \ldots$, there exists m_k adapted to I_i in the sense that

$$\sum_{j<k} m_j + \varepsilon_{m_k} m_k \in I_i. \tag{2.3}$$

Set $S_n(w) = \frac{1}{n} \sum_{k=1}^{n} w_k$. We claim that there exists i_0 such that for every $i > i_0$ and for every $w \in \Omega$, if $S_{n_i}(w) \geq \alpha$, then w is good in the i^{th} block. Otherwise, denote by k the largest integer such that the k^{th} sequence of ones in w starts before the i^{th} block. Then $\varepsilon_{m_k} m_k \leq n_{i-1}$, and hence $m_k \leq n_{i-1}^2 = \frac{1}{n_{i-1}^2} M_i$. This implies that this sequence of ones ends very early in the i^{th} block, and that $w_t = 0$ for $\left(1 - \frac{1}{n_{i-1}^2}\right) M_i$ t's in this block. As $\frac{n_{i-1}}{M_i} \to 0$ as $i \to \infty$, then $S_{n_i}(w)$ must be very small contradicting our assumption.

Define $J_i(w)$ to be one if $S_{n_i}(w) \geq \alpha$ and 0 otherwise. If $J_i(w) = 1$, one can by the above claim, define $k(w, i)$ as the smallest k that satisfy (2.3). Denote $\theta_i(w) = \delta_{k(w,i)}$ if $J_i(w) = 1$ and 0 otherwise.

Using the monotone convergence theorem we have:

$$1 \geq E_\mu\left(\sum_{i \geq i_0} J_i(w)\theta_i(w)\right) \geq \sum_{i \geq i_0} E_\mu(J_i(w)\theta_i(w) \geq \sum_{i \geq i_0} E_\mu(J_i(w))\delta_{n_i}.$$

Since (2.1) at n_i implies $E_\mu(J_i(w)) \geq \alpha$, we obtain, recalling that $\delta_{n_i} = \frac{1}{i}$,

$$1 \geq \left(\sum_{i \geq i_0} \frac{1}{i}\right) \alpha,$$

a contradiction. ■

3 Uniform Convergence

We first establish a few notations. Let D denote the set of all probability distributions θ on the set $N = \{0, 1, 2, \ldots\}$ of non-negative integers, that are non-increasing. That is,

$$\theta(t+1) \leq \theta(t) \quad \text{for all } t \in N. \tag{A}$$

For real numbers $\alpha \leq \beta$ and for a distribution θ,

$$\theta[\alpha, \beta] = \sum_{\alpha \leq t \leq \beta} \theta(t).$$

For $\theta \in D$, define $\hat{\theta}$ on N as follows:

$$\hat{\theta}(t) = (\theta(t) - \theta(t+1))(t+1) \quad \text{for all} \quad t \in N. \tag{3.1}$$

Note that

$$\sum_{t=0}^{T} \hat{\theta}(t) = \sum_{t=0}^{T} \theta(t) - (T+1)\,\theta(T+1) \quad \text{for all} \quad T \geq 0. \tag{3.2}$$

Because of (A), $\lim_{t\to\infty} t\theta(t) = 0$, and therefore $\hat{\theta}$ is a probability distribution on N.

Let $a = (a_t)_{t=0}^{\infty}$ be a bounded sequence. For every $T \geq 0$, denote

$$S_T(a) = \frac{1}{T+1} \sum_{t=0}^{T} a_t,$$

and denote $S(a) = (S_t(a))_{t=0}^{\infty}$. For every probability θ, set,

$$S_\theta(a) = \sum_{t=0}^{\infty} \theta(t) a_t.$$

Observe that by (3.1), similarly to the way (3.2) was obtained, we have $S_\theta(a) = S_{\hat{\theta}}(S(a))$ for all sequences a and probabilities θ, that is,

$$\sum_{t=0}^{\infty} \theta(t) a_t = \sum_{t=0}^{\infty} \hat{\theta}(t) S_t(a). \tag{3.3}$$

We consider linearly ordered families $(\Theta, >)$, where $\Theta \subseteq D$, and "$>$" is a linear (complete) order on Θ, satisfying:

$$\forall \varepsilon > 0,\ \forall N \geq 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \sum_{t=0}^{N} \theta(t) < \varepsilon, \tag{B}$$

which is obviously equivalent to:

$$\forall \varepsilon > 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \theta(0) < \varepsilon. \tag{B^*}$$

Note that Condition (B) implies that for every $\theta \in \Theta$, there exists $\bar{\theta} \in \Theta$, with $\theta < \bar{\theta}$. Therefore, the notions of lim, lim inf, lim sup, etc. ... are naturally defined for real-valued function on Θ. An increasing sequence $(\theta_n)_{n=0}^{\infty}$ in Θ, is *increasing to* ∞, if for every $\theta \in \Theta$, there exists an integer N such that $\theta_n > \theta$ for all $n \geq N$. For the equivalence results we will need the next properties:

(C) $\exists \varepsilon_0 > 0$ and $\varphi: (0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists J(\varepsilon)$, and a sequence $(\theta_{n,\varepsilon})_{n=J(\varepsilon)}^{\infty}$, that increases to ∞ and satisfies:

$$\hat{\theta}_{n,\varepsilon}[(1-\varepsilon)n, n] > \varphi(\varepsilon) \quad \text{for all } n \geq J(\varepsilon).$$

(*D*) There exists a sequence $(\bar{\theta}_n)_{n=0}^{\infty}$, that increases to ∞, and $\exists \varepsilon_0 > 0$ and $\psi:(0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists I(\varepsilon)$,

$$\hat{\bar{\theta}}_n[\psi(\varepsilon)n, n] \geq 1-\varepsilon \quad \text{for all } n \geq I(\varepsilon).$$

3.1 Preliminary Results

We will assume without loss of generality that the payoff function in our dynamic programming satisfies $0 \leq f \leq 1$.

Lemma 3.1. $\forall \varepsilon > 0$, $\forall N$, $\exists \theta_0$ *such that* $\forall \theta > \theta_0$, $\forall s_0 \in S$, $\exists n \geq N$ *satisfying* $V_n(s_0) \geq V_\theta(s_0) - \varepsilon$.

Proof: By condition (*B*) and by (3.2), there exists θ_0, such that $\sum_{t=0}^{N} \hat{\theta}(t) < \frac{\varepsilon}{2}$ for all $\theta > \theta_0$. Let $\theta > \theta_0$, and let $s_0 \in S$. Let $s = (s_t)_{t=0}^{\infty}$ be an $\frac{\varepsilon}{2}$-optimal play for θ in s_0. Then by (3.3),

$$\sum_{t=N+1}^{\infty} \hat{\theta}(t) S_t(f(s)) \geq V_\theta(s_0) - \varepsilon,$$

where $f(s) = (f(s_t))_{t=0}^{\infty}$.

As $\sum_{t=N+1}^{\infty} \hat{\theta}(t) \leq 1$, the above inequality implies that a convex combination of $\{S_t(f(s)) \mid t \geq N+1\}$ is greater or equals $V_\theta(s_0) - \varepsilon$. Therefore there exists $t \geq N+1$ with $S_t(f(s)) \geq V_\theta(s_0)$, implying $V_t(s_0) \geq V_\theta(s_0) - \varepsilon$. ■

Corollary 3.2.

$$\limsup_{n \to \infty} V_n \geq \limsup_{\theta \to \infty} V_\theta.$$

Lemma 3.3. lim sup V_θ *is non-increasing in plays. That is,*

$$\limsup V_\theta(s_0) \geq \limsup V_\theta(s_1) \quad \textit{for every } s_1 \in \Gamma(s_0).$$

Proof: Note that if $(s_t)_{t=1}^{\infty}$ is ε-optimal in s_1 for θ, then $s = (s_t)_{t=0}^{\infty}$ is a play in s_0. Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \theta(t) f(s_{t+1}) - f(s_t) < \varepsilon.$$

By rearranging terms and by (3.3), the last inequality can be proved by showing that

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{f(s_{t+1})-f(s_0)}{t+1} < \varepsilon.$$

Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{1}{t+1} < \varepsilon,$$

which follows easily from Condition (B). ■

Lemma 3.4 (Lehrer and Sorin (1992)). $\forall \varepsilon > 0$, $\forall n > \frac{2}{\varepsilon}$, *and* $\forall s_0 \in S$, *there exist a play* $s = (s_t)_{t=0}^{\infty}$ *and a stage* L *such that*

$$\frac{1}{T+1} \sum_{t=0}^{T} f(s_{L+t}) \geq V_n(s_0) - \varepsilon \quad \textit{for every } 0 \leq T \leq \frac{\varepsilon}{2} n.$$

3.2 From V_θ to V_n

Proposition 1. Assume $\lim_{\theta \to \infty} V_\theta = V$, *uniformly.*

$\forall \varepsilon > 0 \; \exists N$, *such that* $\forall n \geq N$, $V_n \leq V + \varepsilon$.

Proof: Set $\varepsilon_1 = \frac{\varepsilon}{3}$. By the uniform convergence assumption, there exists θ_0, such that

$$| V_\theta(s_0) - V(s_0) | < \varepsilon_1 \quad \text{for all } s_0 \in S. \tag{3.4}$$

Let M be an integer satisfying

$$\sum_{t=0}^{M} \hat{\theta}_0(t) > 1 - \varepsilon_1, \tag{3.5}$$

and let N be an integer satisfying $N > \frac{2}{\varepsilon_1}$. We now show that N satisfies the assertion of the proposition. Indeed, let $n \geq N$, and let $s_0 \in S$. By Lemma 3.4, there exists a play $s = (s_t)_{t=0}^{\infty}$ and an integer L that satisfy the assertion of Lemma 3.4 for ε_1. By (3.3) and (3.5), this implies, $V_{\theta_0}(s_L) \geq V_n(s_0) - 2\varepsilon_1$. Therefore $V(s_L) \geq V_n(s_0) - 3\varepsilon_1$, by (3.4). Hence, by Lemma 3.3, and because $3\varepsilon_1 = \varepsilon$,

$V(s_0) \geq V_n(s_0) - \varepsilon.$ ■

Proposition 2. Assume $(\Theta, >)$ *satisfies Condition* (*C*), *and uniform convergence of* $(V_\theta)_{\theta \in \Theta}$ *to* V.

$$\forall \varepsilon > 0,\ \exists N, \text{ such that } \forall n \geq N,\ V_n \geq V - \varepsilon.$$

Proof: Otherwise, there exists $\varepsilon > 0$ such that for every N, there exists $n \geq N$ and $s_0 \in S$ with $V_n(s_0) < V(s_0) - \varepsilon$. We now choose a particular integer N as follows: set $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$, and choose ε_3, ε_4, ε_5 in a way that will be described later. Choose an integer K satisfying the following 4 properties.

(1) K is large enough such that at every play $s = (s_t)_{t=0}^{\infty}$, $\forall n \geq K$, if $V_n(s_0) < V(s_0) - \varepsilon$, then

$$S_T(f(s)) \leq V(s_0) - \varepsilon_1 \quad \text{for all } (1 - \varepsilon_1) n \leq T \leq n.$$

(2) Let $J(\varepsilon_2)$ and the sequence $(\theta_{n,\varepsilon_2})_{n \geq J(\varepsilon_2)}$ satisfy the property stated in Condition (*C*). Choose $K \geq J(\varepsilon_2)$. That is,

$$\hat{\theta}_n[(1 - \varepsilon_2) n, n] > \varphi(\varepsilon_2) \quad \text{for every } n \geq K,$$

where $\theta_n = \theta_{n,\varepsilon_2}$.

(3) As $(\theta_n)_{n=k}^{\infty}$ is increasing to ∞, and $V_\theta \to V$, we can choose K large enough such that

$$-\varepsilon_4 \leq V_{\theta_n} - V \leq \varepsilon_4 \quad \text{for all } n \geq K.$$

(4) By Proposition 1, we can choose K large enough, such that for every $n \geq K$,

$$V_n \leq V + \varepsilon_3 \quad \text{for all } n \geq K.$$

Finally, choose $N > K$ satisfying

$$\sum_{t=0}^{K} \hat{\theta}_n(t) < \varepsilon_5 \quad \text{for al } n \geq N.$$

By our initial assumption there exists $n \geq N$ and s_0 with $V_n(s_0) < V(s_0) - \varepsilon$. Let $s = (s_t)_{t=0}^{\infty}$ be any play at s_0. Set $a_t = \hat{\theta}_n(t) S_t(f(s))$. Then

$$S_{\theta_n}(f(s)) = \sum_{t=0}^{K} a_t + \sum_{K < t < (1-\varepsilon_2)n} a_t + \sum_{(1-\varepsilon_2)n \leq t \leq n} a_t + \sum_{t > n} a_t.$$

Therefore, by the way we chose N,

$$S_{\theta_n}(f(s)) \leq V(s_0) + \Delta,$$

where

$$\Delta = \varepsilon_3 + \varepsilon_5 - \varphi(\varepsilon_2)\varepsilon_1 .$$

As the last inequality holds for every play at s_0, then

$$V_{\theta_n}(s_0) \leq V(s_0) + \Delta .$$

Hence, by property (3), satisfied by K and hence by N, and recalling that $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, we have

$$\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} \leq \varepsilon_3 + \varepsilon_4 + \varepsilon_5 .$$

Thus we can have a contradiction by choosing ε_i, $i = 3, 4, 5$, to be less than $\frac{1}{3}\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}$. ■

3.3 From V_n to V_θ

Proposition 3. Assume $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists \theta_0, \text{ such that } \forall \theta > \theta_0,\ V_\theta \leq W + \varepsilon .$$

Proof: The proof is an immediate consequence of Lemma 3.1. ■

Lemma 3.5 (Lehrer and Sorin (1992)). *Assume* $\lim_{n\to\infty} V_n = W$ *uniformly. Then for every* ε *small enough, there exists an integer* N*, such that for every* $n \geq N$ *and* $s_0 \in S$*, there is a play* $s = (s_t)_{t=0}^{\infty}$ *at* s_0 *satisfying:*

$$\frac{1}{T+1}\sum_{t=0}^{T} f(s_t) \geq W(s_0) - \varepsilon \quad \text{for every } \varepsilon n \leq T \leq (1-\varepsilon)n .$$

Proposition 4. Assume $(\Theta, >)$ *satisfies condition* (*D*), *and* $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists N, \text{ such that } \forall n \geq N,\ V_{\theta_n} \geq W - \varepsilon,$$

where $(\bar{\theta}_n)_{n=0}^{\infty}$ *is defined in Condition* (*D*).

Proof: Let $\varepsilon > 0$. Let $\delta > 0$ satisfies $\dfrac{\delta}{1-\delta} < \min(\psi(\varepsilon), \varepsilon)$. Then by Lemma 3.5 there exists N such that for every $n \geq N$ and $s_0 \in S$, there is a play $s = (s_t)_{t=0}^{\infty}$ at s_0 satisfying:

$$\frac{1}{T+1} \sum_{t=0}^{T} f(s_t) \geq W(s_0) - \delta \quad \text{for every } \delta n \leq T \leq (1-\delta) n.$$

Without loss of generality we can choose $N \geq I(\varepsilon)$. Note that if $m \geq N$ (assuming that N was chosen large enough), there exists $n \geq N$, with

$$[\psi(\varepsilon) m, m] \subseteq [\delta n, (1-\delta) n].$$

Hence, $\hat{\theta}_m [\psi(\varepsilon) m, m] \geq 1 - \varepsilon$, and $S_T(f(s)) \geq 1 - \delta \geq 1 - \varepsilon$, for $T \in [\psi(\varepsilon) m, m]$. Therefore,

$$V_{\theta_m}(s_0) \geq W(s_0) - 2\varepsilon \quad \text{for all } m \geq N \text{ and all } s_0 \in S. \qquad \blacksquare$$

Remark 1.

If the sequence $(\hat{\theta}_n)_{n=0}^{\infty}$, given in Condition (D) is dense in $(\Theta, >)$ (in the sense that its uniform convergence implies the uniform convergence of $(V_\theta)_{\theta \in \Theta}$), then under conditions (C) and (D), uniform convergence of $(V_n)_{n=0}^{\infty}$ implies uniform convergence of $(V_\theta)_{\theta \in \Theta}$ to the same limit function. As it was proved in Lehrer and Sorin (1992), such is the case when $\Theta = \{\theta_\lambda : \lambda \in [0, 1)\}$, where $\theta_\lambda(t) = (1-\lambda)\lambda^t$, and "$>$" is the natural order on real numbers.

Remark 2.

Let $(\Theta, >)$ be a linearly ordered set of distributions on N satisfying (B), (C^*), and (D^*), where (C^*) and (D^*) are obtained from (C) and (D) respectively, by replacing $\hat{\theta}$ with θ everywhere. Define,

$$U_\theta(s_0) = \sup_{(s_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \theta(t) S_t(f(S)).$$

It is obvious that our proofs yield the equivalence theorem for this solution concept as well. E.g., for every $0 < \lambda < 1$ define

$$U_\lambda(s_0) = \sup_{(s_t)_{t=0}^{\infty}} (1-\lambda) \sum_{t=0}^{\infty} \lambda^t S_t(f(s)).$$

Then (U_λ) converges uniformly if and only if (V_n) converges uniformly, and both share the same limit function.

References

1. Dutta PK, What Do Discounted Optima Converge to? A Theory of Discount Rate Asymptotics In Economic Models, Journal of Economic Theory 55 (1991), 64–94.
2. Lehrer E and Monderer D, Discounting Versus Averaging in Dynamic Programming, Games and Economic Behavior (to appear) (1989).
3. Lehrer E and Sorin S, A Uniform Tauberian Theorem in Dynamic Programming, Mathematics of Operations Research 17 (1992), 303–307.
4. Mertens J-F, Repeated Games, Proceeding of the International Congress of Mathematicians (Berkeley 1986) (1987), 1528–1577.
5. Mertens J-F and Neyman A, Stochastic games, International Journal of Game Theory 10, 2 (1981), 53–66.

Received June 1992
Revised version February 1993