THE OPERATOR APPROACH TO ZERO-SUM STOCHASTIC GAMES

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Abstract. This chapter studies the recursive operator arising in stochastic games.

1. Introduction

This chapter develops an asymptotic analysis of stochastic games through the recursive operator. Given a two-person zero-sum stochastic game (with state space S, action spaces X and Y, payoff r and transition p) and a real-valued function f defined on S, one introduces the associated game $\Gamma(f)(z)$: the stochastic game is played once and one adds to the payoff the evaluation of f at the new state. The **Shapley operator** is specified by $\mathbf{\Psi} : f \mapsto \mathbf{\Psi}(f)$ with:

$$\mathbf{\Psi}(f)(z) = \mathtt{val}_{X\times Y}\{r(z,x,y) + E(f|z,x,y)\}$$

where the expectation is with respect to the transition p on the state space. Recall [18] that in defining

$$\Phi(\alpha, f) = \alpha\mathbf{\Psi}(\frac{(1-\alpha)}{\alpha}f)$$

one has

$$v_{n+1} = \Phi(\frac{1}{n+1}, v_n) \quad \text{and} \quad v_\lambda = \Phi(\lambda, v_\lambda).$$

We study the asymptotic behavior of v_λ and v_n (assuming existence).

In the framework of stochastic dynamic programming (or MDP) Lehrer and Sorin [7] have provided an example (with S countable and A finite) where $\lim_{n\to\infty} v_n$ and $\lim_{\lambda\to 0} v_\lambda$ exist and differ (and are also different from the infinite value with lim inf or lim sup payoff). On the other hand they proved that, for the most general case, uniform convergence of v_n is equivalent to uniform convergence of v_λ and that the limits are the same.

For stochastic games the equality $\lim_{n\to\infty} v_n = \lim_{\lambda\to 0} v_\lambda$ is obtained in the finite case ([1], [2]) as a consequence of the algebraic aspect (see [13]) or in general under a bounded variation condition (and then the uniform value v_∞ even exist); see [14].

Our aim here is to obtain properties on these families only through the analysis of the family of operators $\Phi(\alpha, .)$. The material described below is mainly derived from [16] and follows the approach of [5] for (finite) absorbing games.

2. Regular Operators

In this section we only assume that $\mathbf{\Psi}$ maps a complete cone $\mathcal{F}$ of bounded real-valued functions defined on S, with the uniform norm, into itself and satisfies

$$0\in\mathcal{F} \quad \text{and} \quad f\in\mathcal{F}\Rightarrow f+c\in\mathcal{F}, \quad \forall c\in\mathbb{R}.$$

$\mathbf{\Psi}$ is **monotonic**:

$$f \geq g \quad \Rightarrow \quad \mathbf{\Psi}(f) \geq \mathbf{\Psi}(g). \tag{1}$$

$\mathbf{\Psi}$ **reduces the constants**:

$$\forall c > 0, \qquad \mathbf{\Psi}(f+c) \leq \mathbf{\Psi}(f) + c. \tag{2}$$

Clearly, (1) and (2) imply that $\mathbf{\Psi}$ is nonexpansive since

$$g - \|f-g\|_\infty \leq f \leq g + \|f-g\|_\infty$$

gives

$$\mathbf{\Psi}(g) - \|f-g\|_\infty \leq \mathbf{\Psi}(f) \leq \mathbf{\Psi}(g) + \|f-g\|_\infty.$$

Hence the following operator

$$\Phi(\alpha, f) = \alpha\mathbf{\Psi}(\frac{(1-\alpha)}{\alpha} f) \tag{3}$$

is contracting with coefficient $(1-\alpha)$, so that v_λ is well defined with

$$\Phi(\lambda, v_\lambda) = v_\lambda \qquad \text{or} \qquad \frac{v_\lambda}{\lambda} = \mathbf{\Psi}((1-\lambda)\frac{v_\lambda}{\lambda}). \tag{4}$$

Finally, by induction, one introduces

$$v_1 = \mathbf{\Psi}(0) = \Phi(1,0) \quad \text{and} \quad v_{n+1} = \frac{\mathbf{\Psi}^{n+1}(0)}{n+1} = \Phi(\frac{1}{n+1}, v_n). \tag{5}$$

The basic property is expressed by the domination by approximately superharmonic functions in the following sense.

Definition Given $c > 0$, $\mathcal{C}_c^+$ denotes the set of functions $f \in \mathcal{F}$ satisfying: there exists a positive constant L_0 such that

$$(L+1)f + c \geq \mathbf{\Psi}(Lf), \ \forall L \geq L_0 \tag{6}$$

and similarly $\mathcal{C}_c^-$ is the set of $f \in \mathcal{F}$ such that $\exists L_0 > 0$ with

$$(L+1)f - c \leq \mathbf{\Psi}(Lf), \ \forall L \geq L_0.$$

Note that (6) implies

$$\mathbf{\Psi}(L(f+c)) \leq (L+1)f + c + Lc = (L+1)(f+c)$$

$$\Phi(\varepsilon, f+c) = \varepsilon \mathbf{\Psi}((\frac{1}{\varepsilon} - 1)(f+c)) \leq f + c, \text{ for } \varepsilon \text{ small enough.}$$

Hence, if $f \in \mathcal{C}_c^+$ then $f+c$ is superharmonic for all maps $\mathbf{\Phi}(\varepsilon, .)$ with ε small enough. Such f are called c-superuharmonic (u is for uniform). Then one deduces

Proposition 1 *If f belongs to $\mathcal{C}_c^+$, then*

$$f + c \geq \limsup_{\lambda \to 0} v_\lambda$$

$$f + c \geq \limsup_{n \to \infty} v_n.$$

Proof. For v_λ use the fact that $\Phi^m(\lambda, .)(g)$ converges to v_λ as m goes to ∞, for any initial g in $\mathcal{F}$ and apply it at $f + c$, for λ small enough:

$$v_\lambda = \lim_{m \to \infty} \Phi^m(\lambda, f+c) \leq f + c.$$

For v_n we write

$$n v_n = \mathbf{\Psi}^n(0) \text{ and } \mathbf{\Psi}^n(Lf) \leq (L+n)f + nc,$$

hence

$$v_n \leq (f+c) + 2\frac{L}{n}\|f\|.$$

∎

The previous result implies

Proposition 2 *If f belongs to the intersection of the closure of $\cap_{c>0}\mathcal{C}_c^+$ and $\cap_{c>0}\mathcal{C}_c^-$, then*

$$f = \lim_{\lambda \to 0} v_\lambda = \lim_{n \to \infty} v_n.$$

3. The Derived Operator

In this section we explicitly use the fact that $\mathbf{\Psi}$ is the value of a game and we relate condition (6) and its dual to variational inequalities.

The asymptotic properties of the game are studied through the behavior around 0 of the operator $\Phi(\alpha,.)$. We use the following extension of Mills's Theorem [12] (see also [10], pp. 12-13):

Proposition 3 *Let X and Y be compact sets, f and g bounded real functions on $X{\times}Y$. Assume that for any $\alpha \geq 0$, the functions g and $f+\alpha g$ are u.s.c. in x and l.s.c. in y and that the game $(f+\alpha g; X, Y)$ has a value, $\mathtt{val}_{X\times Y}(f+\alpha g)$. Denote the sets of optimal strategies in the game f by $X(f)$ and $Y(f)$. Then*

$$\mathtt{val}_{X(f)\times Y(f)}(g) = \lim_{\alpha\to 0^+} \frac{\mathtt{val}_{X\times Y}(f+\alpha g) - \mathtt{val}_{X\times Y}(f)}{\alpha}.$$

To apply this result in our framework, let $\Gamma(\alpha, f)(z)$ be the game associated to the α discounted Shapley operator. We assume:
1) X and Y are compact,
2) the mappings $(x,y)\mapsto\alpha r(z,x,y)+(1-\alpha)E(f|z,x,y)$ and $(x,y)\mapsto r(z,x,y)-E(f|z,x,y)$ are for each (α,z,f) upper semicontinuous in x and lower semicontinuous in y,
3) the game $\Gamma(\alpha,f)(z)$ has a value.
Denote by $X(\alpha,f)(z)$ and $Y(\alpha,f)(z)$ the set of optimal strategies in $\Gamma(\alpha,f)(z)$.

Definition The **derived game** $\mathcal{G}(f)(z)$ is the game with payoff $r(z,x,y)-E(f|z,x,y)$ played on $X(0,f){\times}Y(0,f)$.

The interpretation is as follows. From (4) and (5) it is clear that any accumulation point (for the uniform norm) g of the family $\{v_\lambda\}$ or $\{v_n\}$ will satisfy

$$g = \Phi(0,g).$$

However, this condition is not sufficient to characterize the limit. For example, any $c{\in}[0,1]$ will be such a fixed point for the following absorbing game $(1 \geq b \geq 0)$:

$$\begin{pmatrix} a & 0^* \\ 1^* & b \end{pmatrix}$$

The derived game expresses the fact that each player has to play optimally in the "projective game" corresponding to the "shift" operator $\Phi(0,.)$ (i.e., by taking care of the transitions; compare with [8], [9] where it corresponds to the one-day game with operator $\mathcal{A}$) and under this constraint he optimizes his current reward.

Proposition 3 translates as

Proposition 4 *$\mathcal{G}(f)(z)$ has a value and optimal strategies. Moreover, its value, denoted by $\varphi(f)(z)$, satisfies*

$$\varphi(f)(z) = \lim_{\alpha \to 0^+} \frac{\Phi(\alpha, f)(z) - \Phi(0, f)(z)}{\alpha} \ .$$

One deduces immediately

Corollary 1

$$\varphi^*(f)(z) := \lim_{\alpha \to 0} \frac{\Phi(\alpha, f)(z) - f(z)}{\alpha} \quad \textit{exists (in } \mathbb{R} \cup \{\pm\infty\}) \ .$$

We now use this functional to introduce the following:

Definition Let $\mathcal{D}^+$ be the set of functions such that:

$$\varphi^*(f) \ \leq \ 0 \tag{7}$$

or equivalently satisfying the following system:

$$\Phi(0, f) \leq f \qquad \text{and}$$

$$\Phi(0, f)(z) = f(z) \Rightarrow \varphi(f)(z) \leq 0 \ .$$

$\mathcal{D}^-$ is defined similarly with $\varphi^*(f) \ \geq \ 0$ or $\Phi(0, f) \geq f$ and $\Phi(0, f)(z) = f(z) \Rightarrow \varphi(f)(z) \geq 0$.

Note that f in $\mathcal{D}^+$ is a natural candidate to majorize $\limsup v_\lambda$ or $\limsup v_n$ since the above conditions indicate that Player 2 can on the one hand control the level f and on the other one obtain a daily reward less than the new expected level.

The next step is to relate explicitly these new families of functions to the ones introduced in Part 2. First, one deduces easily from the definition that $\bigcap_{c>0} \mathcal{C}_c^+ \subset \mathcal{D}^+$. In fact, for S finite the converse holds.

Proposition 5 *Assume S finite. Then*

$$\mathcal{D}^+ \subset \mathcal{C}_c^+, \quad \forall c > 0$$

(and similarly $\mathcal{D}^- \subset \mathcal{C}_c^-$).

From Propositions 2 and 6 we thus obtain in the finite case a "variational" condition:

Corollary 2 *Assume S finite. If f belongs to the intersection of the closures $\overline{\mathcal{D}^+} \cap \overline{\mathcal{D}^-}$, then $f = \lim_{\lambda \to 0} v_\lambda = \lim_{n \to \infty} v_n$.*

The next proposition extends a property proved by [5] for the case of constant functions.

Proposition 6 *(Maximum principle) Let f_1 and f_2 in $\mathcal{F}$ and z in S satisfy*

$$f_2(z) - f_1(z) = c = \max_{z' \in S} \ (f_2 - f_1)(z') \ > \ 0 \, .$$

Then

$$\varphi^*(f_1)(z) - \varphi^*(f_2)(z) \ \geq \ c.$$

Proof. For any $z' \in S$:

$$\begin{aligned} \Phi(\alpha, f_2)(z') - \Phi(\alpha, f_1)(z') &\leq \Phi(\alpha, f_1 + c)(z') - \Phi(\alpha, f_1)(z') \\ &\leq (1-\alpha)c \\ &\leq (1-\alpha)(f_2(z) - f_1(z)). \end{aligned}$$

So that in particular:

$$(\Phi(\alpha, f_1)(z) - f_1(z)) - (\Phi(\alpha, f_2)(z) - f_2(z)) \geq \alpha(f_2(z) - f_1(z)).$$

Hence, dividing by α, letting α go to 0 and using Corollary 1, one has

$$\varphi^*(f_1)(z) - \varphi^*(f_2)(z) \geq c.$$

∎

This result allows us to compare functions in $\mathcal{D}^+$ and $\mathcal{D}^-$ in the continuous case.

Proposition 7 *Assume S compact. For all continuous functions $f_1 \in \mathcal{D}^+$ and $f_2 \in \mathcal{D}^-$ one has*

$$f_1(z) \geq f_2(z) \quad \forall z \in S.$$

Hence the following uniqueness result holds:

Corollary 3 *Assume S compact. Let $\mathcal{D}_0^+$ (resp. $\mathcal{D}_0^-$) be the subset of continuous functions on S belonging to $\mathcal{D}^+$ (resp. $\mathcal{D}^-$). The uniform closure of $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$ have at most one common element.*

4. Absorbing Games

We now apply the previous results to the case of two-person zero-sum "continuous" absorbing games.

Recall that an absorbing state z satisfies $p(z|z, a, b) = 1$ for all a, b and that an absorbing game is a stochastic game where all states except one, z_0, are absorbing.

Replacing the payoff in an absorbing state by an absorbing payoff equal to the value in that state, it is enough to describe the game starting from z_0 and we drop the references to this state. The action sets A and B are compact and the non-absorbing payoff r is separately continuous on $A{\times}B$. $(S,\mathcal{S})$ is a measurable space and for each $S'{\in}\mathcal{S}$, $p(S'|a,b)$ is separately continuous on $A{\times}B$. Finally, there is a bounded and measurable absorbing payoff, say ρ, defined on $S\setminus\{z_0\}$. As usual, write $X=\Delta(A)$ and $Y=\Delta(B)$ for the mixed actions.

In the current framework we can obviously reduce the domain of the Shapley operator to the payoff in state z_0. Hence $\mathbf{\Psi}$ is defined on $\mathbb{R}$ by

$$\mathbf{\Psi}(f)=\mathtt{val}_{X\times Y}\{g(x,y)+E_{p(.|x,y)}(\tilde{f})\}$$

where the function $\tilde{f}$ on S is equal to f on the non-absorbing state z_0 and equal to the absorbing payoff ρ elsewhere.

(Note that the only relevant parameters are, for each pair (a,b), the probability of absorption $(1-p(z_0|a,b))$, the non-absorbing payoff r and the absorbing part of the payoff $(\int_{s\setminus\{z_0\}}\rho(z)p(dz|a,b))$. By rescaling one could assume that there are only two absorbing states, with payoff 0 and 1.)

Clearly, the conditions of previous Sections 2 and 3 are satisfied. In the current framework Proposition 6 has the following simple form [5]:

Proposition 8 *Assume $f_2>f_1$. Then*

$$\varphi^*(f_1)-\varphi^*(f_2)\geq(f_2-f_1).$$

Hence the functional φ^* is strictly decreasing. It is easy to see that $\Phi(\alpha,f)-f$ becomes negative (resp. positive) as f goes to $+\infty$ (resp. $-\infty$) and therefore:

Corollary 4 *There exists a unique real number w such that*

$$w'<w\Rightarrow\varphi^*(w')>0$$

$$w''>w\Rightarrow\varphi^*(w'')<0.$$

Note that this w satisfies $w=\mathbf{\Phi}(0,w)$; hence $\varphi(w)=\varphi^*(w)$.

Theorem 1

$$\lim_{\lambda\to0}v_\lambda=\lim_{n\to\infty}v_n=w.$$

Proof. Let $w'>w$ and consider the associated function $\tilde{w}'$ on S. It belongs to $\mathcal{D}^+$. Similarly with any w" $<w$, $\tilde{w}$" belongs to $\mathcal{D}^-$. The result then follows from Corollary 2. ∎

5. Recursive Games

We consider here zero-sum recursive games: in all non-absorbing states the payoff is 0. We denote now by S the finite set of such states. The action sets

A and B are compact and the transition $p(z'|a,b,z)$ is separately continuous on $A\times B$ for each z. Let $X=\Delta(A)$ and $Y=\Delta(B)$. The recursive operator is defined on $\mathbb{R}^S$ by

$$\Phi(\alpha,w)(z)=\mathtt{val}_{X\times Y}\{(1-\alpha)(\rho(x,y;z)+\sum\nolimits_{z'\in S}p(z'|x,y,z)w(z'))\}$$

where $\rho(x,y;z)$ is the expected absorbing part of the payoff.

Following [4] we define
$\mathcal{E}^+=\{w\in\mathbb{R}^S:\Phi(0,w)\leq w$ and $w(z)<0$ implies $\Phi(0,w)(z)<w(z)\}$
and in a dual way
$\mathcal{E}^-=\{w\in\mathbb{R}^S:\Phi(0,w)\geq w$ and $w(z)>0$ implies $\Phi(0,w)(z)>w(z)\}$.

Proposition 9

$$\mathcal{E}^+=\mathcal{D}^+.$$

Proof. Let $w\in\mathcal{E}^+$. If $\Phi(0,w)(z)<w(z)$, then $\varphi^*(w)(z)=-\infty$. Otherwise $\varphi^*(w)(z)$ is the value of the derived game with payoff $-(\rho(x,y;z)+\sum_S p(z'|x,y,z)w(z'))$ (since $r\equiv 0$) played on $X(0,w)(z)\times Y(0,w)(z)$. Hence $\varphi^*(w)(z)=-\Phi(0,w)(z)=-w(z)$ and $w(z)\geq 0$ gives the result: $w\in\mathcal{D}^+$.

Let now $w\in\mathcal{D}^+$. Thus $\Phi(0,w)\leq w$. If equality holds at z, then $\varphi^*(w)(z)=\varphi(w)(z)$ is the value of the derived game with again $\varphi^*(w)(z)=-\Phi(0,w)(z)=-w(z)$, so that $w\geq 0$. ∎

Recall that in recursive games the average payoff converges on any play and that the game with such a payoff has an infinite value, $\tilde{v}$ [4].

Corollary 5

$$\lim_{\lambda\to 0} v_\lambda=\lim_{n\to\infty} v_n=\tilde{v}.$$

Proof. Everett's proof shows that $\overline{\mathcal{E}^+}\cap\overline{\mathcal{E}^-}\neq\emptyset$ and that the intersection is reduced to $\tilde{v}$. The result then follows from Corollary 2. ∎

Remark. The original proof of Everett does not imply Corollary 15. For a proof that the uniform value v_∞ actually exists, see the recent result by Rosenberg and Vieille [17] or [19].

A simple example where both $\lim_{\lambda\to 0} v_\lambda=\lim_{n\to\infty} v_n$ and $\tilde{v}$ exist and differ is given by the following dynamic programming framework: the set of non-absorbing states is $\mathbb{N}^*\cup\{\partial\}$. The state 0 is absorbing with payoff -1. On $\mathbb{N}$ the transition is deterministic from n to $n-1$. From state ∂ action n leads to state n. Clearly, any play eventually reaches state 0, hence $\tilde{v}\equiv -1$. However, for any positive integer n one can stay in the first n-stages (of the game with initial state ∂) in a non-absorbing state; hence $\lim_{\lambda\to 0} v_\lambda(\partial)=\lim_{n\to\infty} v_n(\partial)=0$.

6. Comments

The same approach in term of operators has been used to prove similar asymptotic results (namely existence and equality of $\lim v_\lambda$ and $\lim v_n$) in the following two frameworks:
- absorbing games with incomplete information on one side [15] (see also [19], level 4).
- repeated games with lack of information on both sides [16], leading to an alternative proof of Mertens and Zamir [11].

In both cases one uses more than the existence of a value for the derived game. The explicit description of the derived game in terms of strategies and payoffs plays a crucial role in the proof.

The main properties of this approach are:
- the same "limit game" is used to deal with $\lim v_\lambda$ and $\lim v_n$;
- it applies as soon as the recursive formula holds (see [3]);
- it does not rely on algebraic (hence finiteness) properties and identify the limit through variational inequalities.

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