Distribution equilibrium I: Definition and examples

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Abstract

In a 2-person Nash equilibrium, any pure strategy is evaluated against the same mixed strategy of the opponent. The equilibrium condition says that all pure strategies used at equilibrium are best replies, hence they give the same payoff. We keep here the last two requirements (best reply and same payoff) but relax the first one (facing the same opponent's strategy) in the spirit of correlated distributions. We obtain a concept that has a natural interpretation in terms of equilibrium of populations: the various active genotypes have specific fitness and present also different norms of behavior.

Résumé

Dans un équilibre de Nash d'un jeu à deux joueurs, chaque stratégie pure est évaluée contre la même stratégie mixte de l'adversaire. La condition d'équilibre dit que toutes les stratégies pures utilisées à l'équilibre sont des meilleures réponses, donc donnent le même paiement. Nous conservons ici les deux dernières propriétés (meilleure réponse et même paiement) mais nous nous affranchissons de la première (confrontation à une stratégie mixte identique), dans l'esprit des distributions corrélées. Nous obtenons un concept, équilibre en distribution, qui a une interprétation naturelle en termes d'équilibre de populations: les différents génotypes présentent des particularités au niveau aussi bien de la "fitness" que des normes de comportement.

1 Presentation

Consider a strategic form game defined by a finite set I of players and for each $i \in I$, a finite strategy set S^i and a payoff function g^i from the product $S = \times_{i \in I} S^i$ to $I\!R$. Let $\Sigma^i = \Delta(S^i)$ be the corresponding set of mixed strategies and denote $\Sigma = \times_{i \in I} \Sigma^i$.
Let σ be a Nash equilibrium profile. If $\sigma^i(s^i) > 0$, s^i is best reply for player i to the strategy used by the other players, when she is playing s^i. Call this property (A).
Since in fact the strategy she is facing, namely $\sigma^{-i} \in \Sigma^{-i} = \prod_{j \neq i} \Sigma^j$, is independent of her own move s^i (Property (B)), a consequence of (A) is: the payoff of player i, $g^i(s^i, \sigma^{-i})$ is constant (and maximal) on the support of σ^i. Call this last property (C).
When dealing with a correlated equilibrium distribution (Aumann, 1974) $Q \in \Delta(S)$, condition (A) is kept: if $Q(s^i \times S^{-i}) > 0$, then s^i is a best reply for player i to the conditional distribution $Q(.|s^i)$ on S^{-i}. However usually the corresponding payof:

$$\gamma^i(s^i; Q) \equiv \frac{1}{Q(s^i \times S^{-i})} \sum\nolimits_{s^{-i} \in S^{-i}} Q(s^i, s^{-i}) g(s^i, s^{-i})$$

does vary with s^i.
We consider here a concept of equilibrium that does not require (B), but keeps both Properties (A) and (C).

2 Definition

The formal definition is as follows:

Definition
A **distribution equilibrium** is a correlated equilibrium distribution inducing for each player a payoff independent of her move. Formally:
$DE = \{Q \in \Delta(S)$; Q is a correlated equilibrium distribution and for all i and all s_i, t_i in S_i, $Q(s^i \times S^{-i}) Q(t^i \times S^{-i}) > 0$ implies $\gamma^i(s^i; Q) = \gamma^i(t^i; Q)\}$

The above requirements can be written in several ways like:
$\exists c^i \in I\!R$, such that:

$$\sum_{s^{-i}\in S^{-i}} Q(s^i,s^{-i})g^i(s^i,s^{-i}) = c^i \sum_{s^{-i}\in S^{-i}} Q(s^i,s^{-i}), \quad \forall s^i$$
$$\sum_{s^{-i}\in S^{-i}} Q(s^i,s^{-i})g^i(t^i,s^{-i}) \le c^i \sum_{s^{-i}\in S^{-i}} Q(s^i,s^{-i}), \quad \forall s^i, t^i$$

where c^i is the equilibrium payoff of player i and $Q(s^i,.)$ is the (unnormalized) strategy that she is facing, given s^i.

An alternative formulation is, with $g^i(Q) = \sum_{s\in S} Q(s)g^i(s)$:

$$\sum_{s^{-i}\in S^{-i}} Q(s^i,s^{-i})(g^i(t^i,s^{-i}) - g^i(Q)) \le 0, \quad \forall s^i, t^i \tag{1}$$

3 Interpretation

The interpretation is deeply related to the population (mass action) interpretation of Nash equilibria (Nash (1950), see also Leonard (1994), the Nobel seminar 1994 (1996) and Aumann (1997)).
Let us first consider the case of a two person symmetric game described by an $I\times I$ matrix A: A_{ij} is the fitness of type i meeting type j.
A mixed strategy represents the composition of the population and a symmetric Nash equilibrium satisfies simultaneously both requirements for long term stability:
1) no mutation is advantageous (if type i present in equilibrium mutates to type j he will not increase his (expected) fitness).
2) the expected fitness of each present type is equal, so that the composition of the population does not change from one generation to the next one.
Note again that under the independence condition, property 1) implies property 2). They are equivalent iff the equilibrium has full support: in this case no "strategical" argument (like mutation, experimentation, ...) is needed. Stability of the population implies equal expected fitness.
In our framework a symmetric distribution $Q\in\Delta(I\times I)$ describes the interaction of the population as a whole. More precisely we interpret the row $Q(i,.)$ as the (unnormalized) behavior of type i: he will match a type j with frequency $Q(i,j)/\sum_{k\in I}Q(i,k)$.
In case of a Nash equilibrium the "matching is uniform", since every type meets the average population; the difference between the types is thus only

in the fitness characteristics.
In a distribution equilibrium the two previous requirements: stability versus mutation and equal expected fitness are satisfied however the distribution of matchings may be correlated. (In Mailath, Samuelson and Shaked (1997) the fitness is not equal hence the composition of the population cannot be stable).
Note however that the interpretation is very far from a correlated equilibrium (and from its use by Cripps (1991) or Shmida and Peleg (1997)). In the latter case, once a signal is received, the only criteria is to maximize the expected payoff and the payoff one could have obtained given another signal is irrelevant. To be more explicit consider a canonical representation: the private information to player i, namely her signal i hence the corresponding conditional distribution $Q(.|i)$, represents some information on the opponent's behavior that cannot be changed. A deviation would be a move i' different from the signal i.
On the other hand, in a distribution equilibrium the payoff of each type i is specified by its own fitness function and the quantity $Q(.|i)$ corresponding to the matching behavior of type i.
Two kinds of deviations could be considered: in both cases the result is of the form $(i', Q(.|i"))$ with $i' \neq i"$ but the reference point, namely the initial situation could be either $(i', Q(.|i'))$ or $(i", Q(.|i"))$.
Explicitely for each type i one obtains:
a) a deviation in terms of "imitation": i joins the subpopulation of type i' and follows their matching behavior $Q(.|i')$ (same fitness, new matching)
b) a deviation in term of "mutation": i switches to type i' while staying in the i's subpopulation (new fitness, same matching).
The distribution equilibrium condition (1) implies that none of these deviation is profitable.
In addition condition (1) forces the composition of the population to be stable . This is not a consequence of the two previous requirements (see Example 1). On the other hand under equal fitness property, robustness with respect to deviations of kind a) or of kind b) are equivalent.
An correlated equilibrium distribution is robust with respect to deviations of type b). (One could call "dual correlated equilibrium distributions" those imune to deviation of type b)).

The extension of this idea to the general case is standard: it represents the interaction of I different populations and given the fitness characteristics describes a "stable collection of behavior": a $\boxed{\text{distribution equilibrium}}$.

4 Examples

The next first three examples deal with two person symmetric games.

4.1 Example 1

The game is described by:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

We will represent correlated distribution in tables. The Nash equilibria are:

1	0
0	0

0	0
0	1

4/9	2/9
2/9	1/9

A correlated equilibrium which is not a distribution equilibrium is given by the classical public correlation:

1/2	0
0	1/2

In fact clearly type B meeting only type B will have a fitness of 2 hence will invade the whole population.
(Note that this distribution is nevertheless also imune to deviations of type a)).

A distribution equilibrium is given by the following probability distribution:

12/20	3/20
3/20	2/20

Type A is facing:

4/5	1/5

hence is getting an average fitness of 4/5, while type B would obtain 2/5.
Similarly type B has the following norm of behavior:

3/5	2/5

and get the same fitness 4/5 while type A would realize 3/5.
The composition of the population is 3/4 of type A and 1/4 of type B.
The overall behavior process can be described as follows:

The interpretation is a follows. A large population consists of types A and B with proportion $(3/4, 1/4)$. The fitness of both types is described by the initial matrix. At the distibution equilibrium a fraction 1/5 of individuals of type A will match individuals of type B and a fraction 4/5 will match individuals of type A. For type B the proportions are 3/5 to match A and 2/5 to match B.
For an individual of type A a deviation in terms of imitation would be to keep the same fitness but to join the subpopulation B, hence to be matched with a type A with probability 3/5 and B with probability 2/5. A deviation in terms of mutation would be to have the fitness of type B while still following the matching behavior of type A.

In this example all distribution equilibria are symmetric and either BB or of the form:

$2t$	$t(t-1)$
$t(t-1)$	$t-1$

with $1 \leq t \leq 2$, see Appendix.
A non-symmetric correlated equilibria is given by

2/3	0
1/6	1/6

4.2 Example 2

This example is due to Moulin and Vial (1978) and corresponds to the payoff:

$$\begin{pmatrix} 0 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$$

The only Nash equilibrium is the symmetric uniform distribution $(1/3, 1/3, 1/3)$ with payoff 4/3 and the following is a distribution equilibrium:

0	1/6	1/6
1/6	0	1/6
1/6	1/6	0

which induces a payoff 2 that dominates the previous one.

4.3 Example 3

The next game is defined by:

$$\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$$

A distribution equilibrium is given by:

1/2	1/6
1/6	1/6

4.4 Example 4

We consider here zero-sum games.
Every correlated equilibrium payoff being equal to the value of the game, all correlated equilibria are distribution equilibria.
The next example is due to Forges (1990):

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

The following distribution equilibrium is not a product of optimal strategies:

1/3	1/3	0
1/3	0	0
0	0	0

Like example 2, example 4 was produced to illustrate a specific correlated equilibrium property, which was actually obtained by a distribution equilibrium.

4.5 Example 5

This is an non symmetric example with payoffs:

$$\begin{pmatrix} (1,1) & (0,0) \\ (1,0) & (2,1) \end{pmatrix}$$

All disribution equilibria are given by $(Bottom, Left)$ or the (unnormalized) distribution:

$t(t^2-2)$	t
t^2	t^2-2

with $t \in [2, +\infty]$.

5 Properties

Theorem

Any game has a distribution equilibrium

Any symmetric game has a symmetric distribution equilibrium

The result immediately follows from the similar one for Nash equilibria.

It is clear from equation (1) that the set of distribution equilibria is compact.
However examples 1 and 5 show that, unlike for correlated equilibria the set is not convex, not even connected.
In addition example 5 shows that it is not a finite union of convex polyhedra.
Other features including generic properties are of interest (Sorin, 1998).

6 Appendix

We consider Example 1.
A distribution equilibrium is a distribution of the form:

a	b
c	d

Either $d = 1$ and the payoff is 2 or the distribution equilibrium payoff, being the same for type A and B is less than 1 for both players.
We now prove that any distribution equilibrium is symmetric. Note that the equilibrium conditions are positively homogeneous hence one can work in the corresponding positive cone, $\mathbb{R}^4_+$.
Assume thus by contradiction $b \neq c$, hence e.g. $b = c+1$ (and $d < 1$). Remark that $b = 0$ and $d \neq 0$ is impossible since player 2 playing B would get 2 and similarly $b > 0$ and $d = 0$ is impossible.
The equalizing condition for player 1 gives now:

$$\frac{a}{a+b} = \frac{2d}{c+d}$$

hence:

$$ac = ad + 2cd + 2d$$

Similarly for player 2 one has:

$$\frac{a}{a+c} = \frac{2d}{b+d}$$

so that:

$$ac + a = ad + 2cd$$

hence the contradiction:

$$ad + 2cd + 2d \ > \ ad + 2cd = ac + a \ > \ ac.$$

We use thus the following parametrization:

$1-(2t+1)\varepsilon$	$t\varepsilon$
tε	ε

with $t \geq 0$ and $\varepsilon \geq 0$.
For $t\varepsilon \ > \ 0$ the equilibrium conditions are:

$$\frac{1-(2t+1)\varepsilon}{1-(t+1)\varepsilon} = \frac{2}{1+t}$$

$$\frac{t}{1+t} \leq \frac{2}{1+t}$$

$$\frac{2t\varepsilon}{1-(t+1)\varepsilon} \leq \frac{2}{1+t}$$

From the first inequality one deduces:

$$\varepsilon = \frac{t-1}{(t+1)(2t-1)}$$

and the two others give $t \leq 2$, hence the following representation, up to normalization:

$2t$	$t(t-1)$
$t(t-1)$	$t-1$

with $1 \leq t \leq 2$ and payoff $\frac{2}{1+t}$.
$t = 2$ corresponds to the mixed equilibrium and $t = 1$ to the homogeneous A distribution.

We consider now Example 5 with the same notation:

a	b
c	d

Either $d = 1$ and the payoff is $2, 1$ or the distribution equilibrium payoff, being the same for type A and B is less than 1 for both players.
It is easy to see that for $d < 1$ either $a = 1$ or the distribution has full support (first $c > 0$, then $b > 0$). The equalizing condition for player 1 gives now:

$$\frac{a}{a+b} = \frac{2d}{c+d}$$

hence:

$$ac = ad + 2bd$$

Similarly for player 2 one has:

$$\frac{a}{a+c} = \frac{d}{b+d}$$

so that:

$$ab = cd$$

If $b = c$, we obtain $a = d$ hence $c = a + 2b$, a contradiction.
Let us write $b = c + \theta$. If $\theta > 0$, one obtains $cd = ab = ac + a\theta > ac = ad + 2bd = ad + 2(c + \theta)d = 2cd + ad + 2\theta d > dc$. So that $c > b$.

We use thus the following parametrization: $c = tb$ with $t > 1$. The previous equations are now:

$$tab = ad + 2bd$$

and

$$ab = tbd$$

Thus we obtain:

$$t^2bd = ad + 2bd$$

hence:

$$c = tb, a = td, a = (t^2 - 2)b$$

One representation is given by:

$t(t^2-2)$	t
t^2	t^2-2

and the correlated equilibrium conditions imply $t \geq 2$.
The equilibrium payoffs are $\frac{t^2-2}{t^2-1}$ and $\frac{t^2-2}{(t+2)(t-1)}$ for player 1 and 2, respectively. The value $t = 2$ of the parameter corresponds to the completely mixed equilibrium $(2/3, 1/3); (1/2, 1/2)$ and $t = +\infty$ gives the pure one $(Top, Left)$. For $t = 3$ one obtains a payoff $(7/8, 7/10)$ wich is not a convex combination of $(1, 1)$ and $(2/3, 1/2)$.

7 References

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