VALUE AND EQUILIBRIUM

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ABSTRACT.
We compare the two basic concepts of minmax value and equilibrium points in terms of :
- definitions, properties and interpretation,
- tools used in the existence proofs,
- developments and applications, in particular for repeated games.
This will lead us to question the approach asserting the following viewpoint:
Equilibrium as extension of the value paradigm to non-zero sum games,
Value as a special case of equilibrium payoff for zero-sum games.
We claim that many important properties apply only to one of these concepts.

1. BASIC CONCEPTS

We first recall the framework and the definition of these fundamental notions: value and equilibria.

1.1. Value.

The min max theorem allows to evaluate some interactive situations of pure conflict by a *number*: the *value* of the game.
Formally a function f from a product set $X \times Y$ to $\mathbb{R}$ defines a *two-person zero-sum game*, with *strategy* set X and *payoff* function f for player 1 (resp. Y and $-f$ for player 2).
$\underline{v} = \sup_{x\in X} \inf_{y \in Y} f(x,y)$ is the largest amount that player 1 can guarantee and similarly $\overline{v} = \inf_{y\in Y} \sup_{x\in X} f(x,y)$ for player 2.

The *minmax theorem* gives conditions under which the game has a *value*, v, namely:

$$\underline{v} = \overline{v} = v \tag{1}$$

see Borel (1921) [14].
In this case v is a natural and unique evaluation - in the payoff space- of the interaction.
Associated notions are the *duality gap*: $\delta = \overline{v} - \underline{v}$ and the *gap function* $\phi(x,y) = \sup_X f(.,y) - \inf_Y f(x,.)$.
An equivalent formulation of (1) takes the duality form, for any $a \in \mathbb{R}$:

$$\forall y \in Y, \exists x \in X, \quad f(x,y) \geq a \Rightarrow \exists x \in X, \forall y \in Y, \quad f(x,y) \geq a. \tag{2}$$

1.2. Equilibria.

An equilibrium describes some joint behavior of players in a game exhibiting a robustness property.
Consider a *strategic game* G defined by a set I of *players*, a set S^i of *strategies* for each $i \in I$, a *payoff mapping* g from $S = \Pi_{i=1}^I S^i$ to $\mathbb{R}^I$.

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A *Nash equilibrium*, Nash (1950) [61], is a profile of strategies $s \in S$ where no player can gain by changing her strategy:

$$(3) \qquad g^i(t^i, s^{-i}) \leq g^i(s), \quad \forall t^i \in S^i, \quad \forall i \in I.$$

Alternatively, a profile t *eliminates* a profile s if there exists a player $i \in I$ with $g^i(t^i, s^{-i}) > g^i(s)$. Let $E(t) \subset S$ be the set of profiles *not eliminated* by t. An equilibrium is then a profile in $\cap_{t \in S} E(t)$. This formulation is in the spirit of an equilibrium being a "rational" rule of behavior.

The minimal hypotheses used to sustain this concept are:
- s is public knowledge,
- each player i knows her best reply correspondence BR^i (where $BR^i(s^{-i}) = \{s^i \in S^i; g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}), \forall t^i \in S^i\}$),
- each player i is rational.

The associated scenario is the following: a profile s is suggested to all players, each player i plays the corresponding component s^i if and only if $s^i \in BR^i(s^{-i})$.
At this level, no player knows whether the suggestion s is an equilibrium or not, and no notion of "credible threat" applies since i does not know g^j, for $j \neq i$.
This leads to two interpretations, see the discussion in [45]:
the first one (proposal) is that the only acceptable public proposals are equilibria;
the second one (norm) involves a norm of behavior, namely a profile of strategies used repeatedly in the past, under the hypotheses: 1) that the history available to the players includes the strategies and is public knowledge, 2) that the structure of the interaction is stationary (repeated game with short-lived players). Note that in this case the players, assuming a rational behavior of their opponents, will deduce that s is an equilibrium.
A related procedure with a dynamical aspect was suggested by Nash in his thesis [62], under the name "mass action": the players are facing a stationary process (play generated by a population) and can identify their best replies hence rest points will be equilibria.

The standard hypotheses under which the equilibrium concept is currently used are:
- common knowledge of the game parameters,
- common knowledge of rationality of the players.

The main issue is then a choice of equilibrium.
This interpretation (rational prediction) was also suggested by Nash in his thesis [62], but he added a uniqueness requirement to justify the play of an equilibrium.

Even in the case of uniqueness the argument leading to an equilibrium may be problematic. A basic example is the following game due to Aumann and Maschler (1968) [3]:

	L	R
T	$(2,0)$	$(0,1)$
B	$(0,1)$	$(1,0)$

where for both players equilibrium payoff and value coincide.
The only equilibrium is $x^1 = (1/2, 1/2), x^2 = (1/3, 2/3)$ with payoff $(2/3, 1/2)$. For player 1 the minmax strategy $y^1 = (1/3, 2/3)$ is also a best reply to x^2 but in addition

it guarantees $2/3$. An analogous property holds for player 2 and no strong argument supports the use of x in a one-shot game.
A condition like "unique best reply" is needed to enforce the equilibrium play.

On the other hand if the players share information on their payoffs, some equilibria are non-justifiable, see e.g. Laraki, Renault and Sorin (2019) [46]:

	L	R
T	$(0,0)$	$(10,0)$
B	$(1,1)$	$(0,1)$

If player 2 (a dummy player) knows player 1's payoff and knows that player 1 knows this fact, he will play L (to enforce B) and the (T,R) equilibrium does not make sense. The only remaining equilibrium is (B,L).

2. Tools

We describe now briefly the different tools used in the existence proofs.

2.1. **Value.**

The initial proof of the minmax theorem is due to Von Neumann (1928) [80]. f is a bilinear function on a product of simplices corresponding to the mixed extension of a game with finitely many choices for each player.
Actually Von Neumann shows the existence of a *saddle point* for f, $(x^*,y^*) \in X \times Y$ with:

$$f(x,y^*) \leq f(x^*,y^*) \leq f(x^*,y) \quad \forall (x,y) \in X \times Y$$

(x^*,y^*) being a fixed point of the best reply correspondence or a zero of the gap function ϕ.
The first proof using the separation property of disjoint convex sets is due to Ville (1938) [79], see von Neumann and Morgenstern (1944) [81] third ed. 1953, p. 154, note 1.

2.1.1. *Alternative proofs of the minmax theorem.*

As already noticed by Kuhn (2003) [44] in his lecture notes from 1953, a large variety of proofs are available, among which:
- induction on the size of the game, Loomis (1946) [49],
- Fourier elimination procedure (in finite dimension) and alternative lemma; duality for Linear Programming,
- separation in $\mathbb{R}^n$ (or more generally projection in Hilbert spaces),
- unilateral procedure in discrete time, following Blackwell (1956) [12],
- continuous time dynamics, Brown and von Neumann (1950) [16],
- fictitious play in discrete and continuous time, Brown (1951) [15], Robinson (1951) [65],
- replicator dynamics, or more generally no-regret dynamics, see e.g. Hofbauer (2018) [37].

2.1.2. *General formulation of the minmax theorem.*
A useful and standard statement of the minmax theorem is the following:

Theorem 1 (Sion, 1958)**.** [73]
Let $G = (X, Y, f)$ be a zero-sum game satisfying:
(i) X and Y are convex subsets of a topological vector space,
(ii) X or Y is compact,
(iii) for each y in Y, $f(., y)$ is quasi-concave upper semi-continuous in x, and for each x in X, $f(x, .)$ is quasi-convex lowser semi-continuous in y.
Then G has a value.

Note that the hypotheses involve geometrical and topological properties on the strategy sets and the payoff function.
An optimal strategy $x^* \in X$ for player 1, which satisfies a unilateral property, namely guaranteeing the value, $f(x^*, y) \geq v, \forall y \in Y$, exists under a compactness assumption on X, and similarly for player 2.
The basic tool used in the proof is the Hahn-Banach separation theorem via the intersection lemma, see Berge (1966) [11] p. 172.

2.2. **Equilibria.**
The first proof of existence for finite games was provided by Nash (1950) [61]; it used the Nash map and Brouwer's fixed point theorem.
A general version is as follows:

Theorem 2 (Nash, 1951 [63], Glicksberg, 1952 [27], Fan, 1952 [20])**.**
Let S^i be a compact convex subset of a topological vector space, g^i be continuous on S and quasi concave w.r.t. s^i, for all $i \in I$, then the set of equilibria is compact and non empty.

The proof relies on Kakutani's fixed point property [40] applied to the best reply correspondence. Alternatively, equilibria are the zeroes of the Nikaido map $\Phi(s) = \sum_i [\sup_{S^i} g^i(., s^{-i}) - g^i(s)]$.
Notice that again the hypotheses involve geometrical and topological properties on strategy sets and payoff functions. The main difference with Theorem 1 is that joint continuity is required.
Conversely, McLennan and Tourky (2006)) [53] use the existence of Nash equilibria for two-person finite games to prove Kakutani's fixed point theorem.

3. Extensions and Applications

3.1. **Equilibrium.**
We identify here four different fields where research has been extremely active and productive.

3.1.1. *Equilibrium selection.*
The literature on this topic, which aims at reducing the equilibrium set, is huge and diverse by its approaches and domains of applications (extensive or normal form games). Let us mention a.o. :
- perfect, Selten (1975) [70], Harsanyi and Selten (1988) [33]

- proper, Myerson (1978) [60]
- sequential, Kreps and Wilson (1982) [43]
- persistent, Kalai and Samet (1984) [39]
- stable, Kohlberg and Mertens (1986) [41], Mertens (1989) [54, 55]
- divine, Banks and Sobel (1987) [7]
- essential, Govidan and Wilson (2005) [31]

and the surveys by Van Damme (2002) [78], Hillas and Kohlberg, (2002) [36].
A related area analyzes games with signals and the associated notion of conjectural equilibrium (where the information after a play is consistent with the anticipation), see e.g. Fudenberg and Levine (1993) [23].

3.1.2. *Manifold of equilibria.*
This area corresponds to the study of the graph E of the equilibrium correspondence over the space Γ of games (in the finite case),

$$E = \{(x, G);\ x \text{ equilibrium of } G, G \in \Gamma\}$$

and has been introduced by Kohlberg and Mertens (1986) [41].
A first property they prove is that E is homeomorphic to Γ.
Define a Nash field Φ as a continuous map from $\Gamma \times S$ to S such that the fixed points of $\Phi(G, .)$ are the equilibria of G. Given a component of equilibria, its index is constant among Nash fields and is equal to the local degree of the projection map from E to Γ, Demichelis and Germano (2000, 2002) [18, 19], Govidan and Wilson (1997a, 1997b) [29, 30].

3.1.3. *Supergames.*
This field is devoted to the analysis of the set of equilibria of a repeated game with complete information.
Basic results extend the "Folk theorem" asserting that "the set of equilibrium payoffs of the supergame are the feasible individually rational payoffs" to the framework of perfect equilibria, Aumann and Shapley (1994) [5], Rubinstein (1994) [67].
Then follows a large and important literature involving variations on the payoffs: finite/discounted/uniform, equilibrium selection, games with signals, various duration of the players, ... see a.o. Abreu,Pearce and Stacchetti (1990) [1], Fudenberg and Levine (1994) [24], Fudenberg and Maskin (1986) [26], Benoit and Krishna (1985) [10], Sorin (1992) [75], Gossner (1995) [28] , ... and the book by Mailath and Samuelson (2006) [50].
A closely related area studies the impact of perturbation/reputation effects on repeated interactions, Kreps, Milgrom, Roberts and Wilson (1982) [42], Aumann and Sorin (1989) [6].

3.1.4. *Games with a large number of players; population and non-atomic games.*
Extension of Nash equilibria appears in the analysis of congestion modelling involving a crowd of players, Wardrop (1952) [82] and for general non atomic games, Schmeidler (1973)[69], Mas Colell (1984)[51]. The model of population games is fundamental in biology, Maynard-Smith (1981)[52], with specific concepts like Evolutionary Stable Strategy, Taylor and Jonker (1978) [77].
The basic results in this field can be found in Hofbauer and Sigmund (1998) [38], a

more recent presentation is Sandholm (2011) [68].
Let us also mention Mean Field Games that study anonymous differential games with a continuum of players, Lasry and Lions (2007) [48].

3.2. **Value.**
We describe here three fields where the analysis in terms of value has been particularly efficient.

3.2.1. *Alternative/dual statements.*
This corresponds to properties that extends statement (2).
A typical example is approachability theory, Blackwell (1956)[12] for games with vector payoffs, where f maps $X \times Y$ to $\mathbb{R}^n$. The basic duality takes the form:

$$\forall y \in Y, \exists x \in X \quad f(x,y) \in W \Rightarrow \exists x \in X, \forall y \in Y, \quad f(x,y) \in W \tag{4}$$

which is the alternative for half spaces W.
When applied to a repeated game it leads, in the case of a convex domain D, to the duality property: approachability vs excludability. In fact if player 2 cannot force some set W disjoint from D, player 1 can inductively mimick a viable trajectory that generates a sequence of payoffs such that the average converges to D.
In the more general case with signals on the outcomes, necessary and sufficient conditions of this kind for approachability are available, Perchet (2011) [64]

3.2.2. *Operator approach and recursive formula.*
This domain analyzes the extension of the recursive structure for discounted stochastic games, introduced by Shapley (1953).

For a general repeated game, the natural state space is given by the consistent probabilities on the universal belief space, Ω, Mertens and Zamir (1985) [57], Mertens, Sorin and Zamir (2015) [56]. A profile of one-stage strategies defines an information structure and the signals allow to compute the transition kernel on this state space Ω.

When the minmax theorem holds, the repeated game G has the same value as an auxiliary game G' where the one-stage strategies are announced. However in G' a basic recursive formula on the value function as a function on Ω is available, [56] Th. IV.3.2.

This tool is fundamental to study asymptotic properties of the value, but also regularity aspects and the link between discrete and continuous time approach (Hamilton-Jacobi equation).

3.2.3. *Uniform and asymptotic criteria.*
The analysis of zero-sum repeated games can be divided into :
- an *asymptotic approach* which is concerned by the limit properties of the value function as the (expected) duration of the game goes to infinity, like existence of $\lim v_n$ (n-stage game) or $\lim v_\lambda$ (λ-discounted game),
- a *uniform approach* based on robustness properties of strategies in any game with long duration, leading to the notion of uniform maxmin and minmax (where the existence has to be proved), and of uniform value v_∞ when they coincide.
In particular the existence of a uniform value implies the existence of an asymptotic value and their equality.

3.3. **Comments.**

Note that the topics under 3.1 are trivial or vacuous when applied to the value of zero-sum games.

On the other hand there are no properties equivalent to 3.2 for equilibria of non-zero sum games.
The viewpoint described in 3.2.1 is in the spirit of the basic approach of Blackwell and Girshik (1954) [13] and applies as well to calibration procedures (see e.g. Cesa-Bianchi and Lugosi [17] and 4.2.) or to the link between α and β characteristic functions.
The minmax theorem is one example of a duality result: expressing a property from two perspectives (achieve or defend an amount).

Property 3.2.2 has no counter-part in non zero-sum games. Strategies have to be specified on (private) histories, not on Ω, and this prevents the emergence of a recursive structure.
In addition the "value operator" exhibits monotonic aspects (with respect to the payoff or the strategy spaces) that are crucial to deduce regularity aspects and analyze limit behavior.
One cannot establish similar properties for equilibria in non-zero sum games.

Concerning 3.2.3 there are no analogous results in the non-zero sum case, see for incomplete information games, Aumann and Maschler (1995) [4], and for stochastic games, Sorin (1986) [74].

4. Some links

We describe here two important connections between equilibrium and value.

4.1. **Variational inequalities.**
The equilibrium condition (3) for smooth concave payoff functions takes the form:

$$\langle \nabla_i g^i(s), s^i - t^i \rangle \geq 0 \qquad \forall t^i \in S^i, \forall i \in I, \tag{5}$$

where $\nabla_i g^i$ denotes the gradient of g^i w.r.t. s^i.
More generally for a vector field F on a compact convex set Z, one introduces the sets of internal solutions:

$$\langle F(z), z - z' \rangle \geq 0 \quad \forall z' \in Z$$

and external solutions:

$$\langle F(z'), z - z' \rangle \geq 0 \quad \forall z' \in Z$$

and for continuous fields the existence of an internal solution is equivalent to the fixed point theorem, while external solutions may no exist.
The field is dissipative if:

$$\langle F(z') - F(z), z - z' \rangle \geq 0 \quad \forall z, z' \in Z.$$

For dissipative continuous fields, there is equivalence between internal and external solutions of variational inequalities and an existence proof is available via the minmax theorem, Minty (1967) [58].

A basic example corresponds to $F = (\nabla_1 f, -\nabla_2 f)$ for smooth concave/convex zero-sum game, Rockafellar (1970) [66].
This property has fundamental applications in the study of no-regret dynamics, Sorin (2021) [76].

A complementary perspective on the difference between fixed point and min max approaches for variational inequalities is discussed in the recent article by Foster and Hart (2021) [21], Sections III.D and VII.

4.2. **Correlated equilibria.**
Correlated equilibria corresponds to equilibria of a game extended by an information structure, Aumann (1974) [2]. Hart and Schmeidler (1989) [35] proved existence by identifying the set of correlated equilibrium distributions as the set of optimal strategies in an associated two-person zero-sum game.

In repeated games several learning procedures based on the no-regret property, Hannan (1957) [32], converge to the set of optimal strategies in zero-sum games.

A refinement asking for internal vs external consistency, Hart and Mas Colell (2013) [34], Foster and Vohra (1999) [22] allows to prove that if every player uses such a procedure then: all accumulation points of the empirical joint distribution of moves are correlated equilibria.
Note that one obtains convergence of the *average behavior* to the set of *correlated equilibria* under a profile of algorithms satisfying a *unilateral property* (equivalent to calibration).

5. RESEARCH DIRECTIONS

5.1. **General theory of two-person zero-sum games.**
In the spirit of sections 3.2.2.and 3.2.3. the aim is to build a comprehensive theory including stochastic aspects, incomplete information and signals, Mertens, Sorin and Zamir (2015) [56] and developing the link with differential games and games in continuous time, Laraki and Sorin (2015) [47].

5.2. **Equilibria as solutions of variational inequalities.**
The objective here is to extend and generalize the results of section 3.1.2. to the framework of section 4.1. . Games will appear there as special vector fields that decompose in the sense that:

$$\langle \phi(z), v \rangle = \sum_{i \in I} \langle \phi^i(z), v^i \rangle, \quad z, v \in Z \tag{6}$$

and the manifold of solutions is defined on the vector space $\{\phi(.) + v, v \in Z\}$.

5.3. **Nash fields and dynamic stability.**
The analysis of the dynamical system $\dot{s}_t = \Phi(G, s_t)$, Φ being a Nash field, is of first importance, especially concerning its asymptotic properties. In particular one can observe that attractors may not consist of subsets of fixed points, Shapley [72]. The link with discrete time related dynamics is also of interest, in the spirit of Benaim, Hofbauer and Sorin [8, 9].

6. SUMMARY

To summarize it appears that value and equilibria are quite different concepts with specific properties and fields of applications.
They correspond to two approaches that are complementary and cannot be reduced to each other.
A typical example is the uniform approach for zero-sum games with lack of information on one side: an optimal strategy of the informed player is the celebrated splitting strategy, while an optimal strategy of the uninformed player relies on the approachability of an orthant. These are unilateral guarantees rather than equilibrium properties.

On the other hand, there are similar aspects between zero-sum games and certain classes of non-zero sum games, mainly potential games, Monderer and Shapley (1996) [59] or dissipative games, in particular in terms of learning dynamics.

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