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No-regret algorithms in on-line learning, games and convex optimization

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Abstract
The purpose of this article is to underline the links between some no-regret algorithms used in on-line learning, games and convex optimization and to compare the continuous and discrete time versions.

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Contents

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1 Introduction

General learning algorithms associate to an observation a decision and evaluate the result through a specific criteria. On-line procedures involve time hence a process of observations but require that successive decisions are taken at each instant. We will study here an example of such instance where the criteria is a "regret function".

The specificity of the current presentation lies mainly in the two following aspects:

1. We will establish properties for the general observation process and then study the so-called closed model where the observation is a function of the actual decision. A typical example will be the joint process of players choosing actions and observing each their induced private payoff. Another basic application is a first order method in convex optimization. While these three frameworks seem very different : no hypothesis on the observation in the first case, collection of unilateral procedures in parallel in the second, gradient based dynamics in the third, we will see that general optimal properties are shared and similar main concepts and tools apply.
2. We will consider both the continuous and discrete time versions and compare the basic notions, tools, ideas of proofs and results in both cases. Here also some common properties will emerge.

1.1 The model: no-regret condition in continuous and discrete time

The general framework is as follows:

V is a normed vector space, finite dimensional, with dual V^* and duality map $\langle V^*|V\rangle$,

X is a non-empty convex compact subset of V (while this last hypothesis is not needed in convex optimization - but $argminf$ non empty is often required - it is almost necesssary in the analysis for games).

The aim is to study properties of algorithms that associate to a continuous time process $\{u_t \in V^*, t \geq 0\}$, a similar procedure $\{x_t \in X, t \geq 0\}$, where x_t is function of the past $\{(x_s, u_s), 0 \leq s < t\}$.

The process corresponds to the observation, the procedure to the induced trajectory of choices.

The adequation of $\{x_t\}$ to $\{u_t\}$ is measured by a *regret function* defined on X by:

$$R_t(x) = \int_0^t \langle u_s | x - x_s \rangle ds \tag{1}$$

which compares the integral of the instantaneous score $\langle u_s | x_s \rangle$ to the one induced by a fixed action $\langle u_s | x \rangle$. (Notice that the regret can be positive or negative for some x.)

One will study procedures satisfying the *no-regret condition*:

$$\sup_{x \in X} R_t(x) \leq o(t) \tag{2}$$

which means that the (positive part of the) time average regret $\frac{R_t(x)}{t}$ vanishes.

Similarly in discrete time, $m \in \mathbb{N}$, given $\{u_m\}$ and $\{x_m\}$, with $\{x_m\}$ depending on $\{x_1, u_1, ..., x_{m-1}, u_{m-1}\}$, one defines:

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle$$

and requires:

$$\sup_{x \in X} R_n(x) \leq o(n). \tag{3}$$

1.2 The analysis

(A) We compare the performance of the algorithms in terms of evaluation of the regret under three (increasing) assumptions:

(I) *general case*: the information $\{u_t\}$ is a bounded measurable process in V^*,
(II) *closed form*: the information is determined by the choice, thus $u_t = g(x_t)$ for a continuous vector field $g : X \to V^*$,
(III) *convex gradient*: $u_t = -\nabla f(x_t)$, where f is a $\mathcal{C}^1$ convex function : $X \to \mathbb{R}$,

(with similar properties in discrete time).

(B) We consider three different procedures:

- Projected dynamics (PD),
- Mirror descent (MD),
- Dual averaging (DA)

that will satisfy the no-regret property and we will discuss their performances.

(C) We analyze the relations between the continuous and discrete time processes (in particular in terms of speed of convergence to 0 of the average regret). Recall that in continuous time u_t is not observed at time t and the information process is only measurable; moreover u_t may depend on x_t.
(D) In classes (II) and (III) it makes sense to consider the trajectories of $\{x_t\}$ or $\{x_n\}$ and we will study their convergence.

1.3 No-regret in on-line learning, games and convex optimization

1.3.1 Framework (I) corresponds to the usual model of on-line learning where at each time t the agent observes the process $\{u_s, s < t\}$ and chooses the action x_t.

Note that, for K a finite set with cardinal $|K|$, if $V = \mathbb{R}^{|K|}$ and the agent selects a component $k_t \in K$ at random, then X is the simplex on K, denoted $\Delta(K)$, x_t stands for the law of k_t and the regret is expressed in terms of conditional expectation.

Recall that since no hypothesis is made on the process u_t, no prediction on its future values makes sense, but to satisfy the no-regret criteria expresses a desirable a-posteriori property in the spirit of dynamic best-reply.

The litterature on this topic is very large and almost impossible to cover. Let us mention few aspects.

The notion of regret appears in Hannan [41], Blackwell [15, 16] in a game theoretical set-up. Algorithms and properties are studied in this spirit in Foster and Vohra [32], Fudenberg and Levine [37], Foster and Vohra [35], Hart and Mas-Colell [43], Lehrer [61], Benaim et al. [14], Cesa-Bianchi and Lugosi [26], Viossat and Zapechelnyuk [103], ... among others.

Similar tools and properties occur in statistics and in the learning community: Vvok [104], Cover [29], Littlestone and Warmuth [64], Freund and Shapire [36], Auer et al. [7], Cesa-Bianchi and Lugosi [25], Stoltz and Lugosi [97], Kalai and Vempala [54], Blum and Mansour [17], ...

This topic is analyzed in the following books:

Fudenberg and Levine [38] *The Theory of Learning in Games*, MIT Press,

Cesa- Bianchi and Lugosi [26] *Prediction, Learning and Games*, Cambridge University Press,

Hart and Mas- Colell [46] *Simple Adaptive Strategies: From Regret-Matching to Uncoupled Dynamics*, World Scientific Publishing, and the connection with related notions of approachability and consistency is well presented in the survey by Perchet [80], see also Abernethy et al. [1] .

The next two paragraphs describe more specific cases where the observation u_t is a function of the action x_t.

1.3.2 In particular, framework (II) *closed form* is relevant for game dynamics and variational inequalities as explained now.

Consider a strategic game with a finite set of players I, where $X^i \subset V^i$ is the strategy set of player i and $X = \prod_i X^i$. Assume that the equilibrium set E is obtained as the set of solutions $x = (x^i) \in X$ of the following variational inequalities:

$$\langle g^i(x)|x^i - y^i\rangle \geq 0, \qquad \forall y^i \in X^i, \forall i \in I.$$

where $g^i : X \to V^{i*}$ is the "evaluation" function of player i.

Examples include (mixed extension of) finite games where g^i is the "vector payoff" function of player i, games with smooth concave payoffs (the payoff function $F^i : X \to \mathbb{R}$ of player i is concave w.r.t. x^i and $g^i = \nabla_{x_i} F^i$), but also nonatomic population games and Wardrop equilibria (I is then the set of populations), see Sorin and Wan [96].

Notice that an equivalent formulation of the equilibrium condition is through the single variational inequality, with $g = \{g^i\}$:

$$\langle g(x)|x - y\rangle = \sum_i \langle g^i(x)|x^i - y^i\rangle \geq 0, \qquad \forall y \in X. \tag{4}$$

We will denote by $\Gamma(g)$ a game for which equilibria are solution of (4).

For each player i, the reference/observation process is specific, namely $u^i_t = g^i(x_t)$ which, as a function of x_t, is determined by the behavior of all players. In particular, in a concave smooth game, the observation of player i is the gradient w.r.t. $x_i : u^i_t = \nabla_{x_i} F^i(x_t)$.

Note that the overall global dynamics of the profile of actions $\{x_t\}$ is generated by a family of unilateral procedures since for each i, x^i_t depends on (u^i, x^i) only. In particular for each player i, the knowledge of g^j, $j \neq i$ is not assumed. Thus for each participant individually the situation is similar to (I) *general case* since the observation is not controlled, while these private observations of the participants are linked via x_t.

We will analyze the consequences on the process $\{x_t\}$ assuming only that each player uses a procedure satisfying the no-regret condition (2) or (3). Obviously the (global) algorithm associated to $g = \{g^i\}$ as a vector field on a product space associated to (4) will also share the no-regret property since:

$$\int_0^t \langle g^i(x_s)|x^i - x^i_s\rangle ds \leq o(t), \qquad \forall x^i \in X^i$$

implies:

$$\int_0^t \langle g(x_s)|x - x_s\rangle ds \leq o(t), \qquad \forall x \in X$$

but in addition it is “decentralized” in the sense that x^i depends upon g^i only.

1.3.3 Framework (III) covers the case of convex optimization where the observation, given the choice x_t, is the gradient of the (unknown) convex function, explicitly one has $u_t = -\nabla f(x_t)$.

The recent research in this area is very wide and links basic optimisation algorithms Polyak [82], Nemirovski and Yudin [74], Nesterov [76], to on-line procedures, see Zinkevich [107].

Let us mention the recent books:

Bubeck [23] Convex optimization: Algorithms and complexity, *Fondations and Trends in Machine Learning*, **8**, 231–357.

Hazan [47] The convex optimization approach to regret minimization, *Optimization for machine learning*, S. Sra, S. Nowozin, S. Wright eds, MIT Press, 287–303.

HAZAN [48] Introduction to Online Convex Optimization, Fondations and Trends in Optimization, **2**, 157–325.

HAZAN [49] Optimization for Machine Learning , arXiv:1909.03550.pdf.

SHALEV- SHWARTZ [91] Online Learning and Online Convex Optimization, Foundations and Trends in Machine Learning, **4** , 107–194.

1.3.4 One should add that related algorithms have been developped in Operations Research (transportation, networks), see e.g. Harker and Pang [40], Dupuis and Nagurney [30], Nagurney and Zhang [72].

Note that each community (learning, game theory, optimization) has its own terminology and point of view. As a consequence specific properties may inherit different names and will be rediscovered at several occasions. One of the aim of the current work is to clarify the relations between these approaches and results and unify the analysis.

In particular we will show that few basic principles are in use and we will underline the analogy between continuous and discrete time.

However there are subtle differences between the different algorithms and their properties that we will discuss (see 3.6, 4.4 and 5.4).

1.4 Summary

Section 2 is devoted to the *closed form* framework (II) and explores the links between no-regret, solutions of variational inequalities and convex optimization.
Section 3 deals with continuous time dynamics. After introducing "level functions" and "positive correlation" we describe the three algorithms (PD, MD, DA), prove that they satisfy the no-regret property and compare their performances.
Section 4 is the discrete time analog of Sect. 3.
Section 5 considers basically framework (III) under an additional regularity hypothesis on the convex function f. Section 5.4 on "Mirror prox" recalls related results using similar tools. Concluding comments are in Sect. 6.

2 Basic properties of the closed form

We describe first some relations with variational inequalities, when the observation process has a *closed form* : $u = g(x)$. Then we exhibit general properties of no-regret procedures.

2.1 Definition and notations

2.1.1 iS: *set of internal solutions*

Notation: iS is the set of solutions $x \in X$, (internal since it involves the value of g at $x \in X$), of the variational inequality:

$$\langle g(x)|y - x\rangle \leq 0, \qquad \forall y \in X. \tag{5}$$

Note that if g is associated to a game $\Gamma(g)$, see 1.3, (4) shows that iS corresponds to the set of equilibria.

Recall also that in an Hilbertian framework property (5) is equivalent to:

$$\Pi_X(x + \lambda g(x)) = x, \qquad \lambda > 0 \tag{6}$$

where Π_C denotes the (Hilbertian) projection operator on a closed convex set C, and (5) defines also the solutions $x \in X$ of:

$$\Pi_{TX(x)}(g(x)) = 0 \tag{7}$$

where $TX(x)$ is the tangent cône to X at x, see e.g. Kinderlehrer and Stampacchia [55], Facchinei and Pang [31].

The minimization of a $\mathcal{C}^1$ convex function f on X corresponds to the variational inequality (5) with $g = -\nabla f$.

This case presents two specific properties: (i) g is a gradient, (ii) g is dissipative.

We will consider now the analogous general definitions for vector fields.

2.1.2 *Fields with potential*

Definition: $g : X \to V^*$ is a vector field with *potential* G, if $G : X' \to \mathbb{R}$, where X' is an open neighborhood of X, satisfies:

$$\langle \nabla G(x) - g(x) | y - x \rangle = 0, \forall x, y \in X$$

or more generally if there exist a strictly positive function μ on X, such that:

$$\left\langle \nabla G(x) - \mu(x) g(x), y - x \right\rangle = 0, \quad \forall x \in X, \forall y \in X, \tag{8}$$

see Sorin and Wan [96].

Then local maxima of G on X belong to iS and if G is concave its maxima on X coincide with iS.

If g is defined through a game $\Gamma(g)$, see 1.3, the game is a *potential game*, Monderer and Shapley [70], Sandholm [86].

2.1.3 *Dissipative fields*

Definition: g is *dissipative* ($-g$ is *monotone*) if it satisfies:

$$\langle g(x) - g(y) | x - y \rangle \leq 0, \qquad \forall x, y \in X. \tag{9}$$

A fundamental example is the following: let F, defined on $X = X_1 \times X_2$, be a $\mathcal{C}^1$ concave/convex function, (like the payoff function of a two-person 0-sum game, $F = F^1 = -F^2$), then $g = \{\nabla_{x_1} F, -\nabla_{x_2} F\}$ is dissipative, Rockafellar [84].

If g is dissipative and defined through a game $\Gamma(g)$, see 1.3, the game is dissipative, as introduced by Rosen [85]. The terminology is "stable" in Hofbauer and Sandholm [51], "contractive" in Sandholm [88] and "dissipative" in Sorin and Wan [96].

It will be usefull to consider the following set.

2.1.4 eS: *set of external solutions*

Notation: eS is the set of solutions $x \in X$ (external since it involves the values of g at alternative points $y \in X$), of the variational inequality:

$$\langle g(y)|y - x\rangle \leq 0, \qquad \forall y \in X. \tag{10}$$

Observe that eS is either convex or empty, independently on the properties of g.

If g is dissipative, $eS \neq \emptyset$ follows from the—finite version of the—minmax theorem, Minty [69].

2.1.5 *Internal versus external solutions*

Let us recall the relations between internal and external solutions:
If g is dissipative, the following (weak) inclusion holds :

$$iS \subset eS$$

and if g is continuous (or simply $t \mapsto g(x + t(y - x))$ continuous in t, $\forall x, y \in X$), the reverse (weak) inclusion is satisfied:

$$eS \subset iS$$

see Kinderlehrer and Stampacchia [55], Facchinei and Pang [31].

Thus, if g defined through a game $\Gamma(g)$, see 1.3, is dissipative and continuous then $iS = eS = E$; in particular for a smooth zero-sum game this corresponds to the set of optimal strategies.

If $iS = eS$ we will also use the notation S for this set.

2.2 Results

Assume in this subsection that the procedure x_t or x_n satisfies the no-regret property (2) *or* (3).

2.2.1 *iS and convergence*

A first property deals with convergent trajectories $\{x_t\}$.

Proposition 1 *If g is continuous and x_s converges to x, then $x \in iS$.*

Proof Since $R_t(y) = \int_0^t \langle g(x_s)|y - x_s\rangle ds$:

$$\frac{R_t(y)}{t} \to \langle g(x)|y - x\rangle, \qquad \forall y \in X. \tag{11}$$

and $R_t(y) \leq o(t)$ implies $x \in iS$. □

In particular, if x is a *stationary point* for the discrete or continuous time procedure, i.e. $x_t = x$ if $x_0 = x$, then $x \in iS$.

Concerning the set eS, clearly one has:

Lemma 1 *If $x^* \in eS$, then $R_t(x^*) \geq 0$ for all $t \geq 0$.*

2.2.2 *Time average*

Define the time average trajectories as follows:

$$\bar{x}_t = \frac{1}{t}\int_0^t x_s ds \qquad \text{and} \qquad \bar{x}_n = \frac{1}{n}\sum_1^n x_m.$$

Proposition 2 *If g is dissipative, the accumulation points of $\{\bar{x}_t\}$ or $\{\bar{x}_n\}$ are in eS.*

Proof

$$\frac{R_t(y)}{t} = \frac{1}{t}\int_0^t \langle g(x_s)|y - x_s\rangle \geq \frac{1}{t}\int_0^t \langle g(y)|y - x_s\rangle = \langle g(y)|y - \bar{x}_t\rangle.$$

Hence under (2) or (3) any accumulation point $\hat{x}$ of $\{\bar{x}_t\}$ will satisfy $\langle g(y)|y - \hat{x}\rangle \leq 0$. □

Note that this result implies the non-emptiness of eS for dissipative g.

In particular the minmax theorem (in the $\mathcal{C}^1$ case) follows from the existence of no-regret procedures. This is in the spirit of the previous proofs based on dynamics: e.g. Brown and von Neumann [21], Hofbauer and Sorin [53], Lehrer and Sorin [62], Hofbauer [50].

2.2.3 *Class (III)*

Since $u_t = g(x_t) = -\nabla f(x_t)$ with f being $\mathcal{C}^1$ and convex, this corresponds to a specific case of dissipative and continuous vector field g, hence $eS = iS = S = \text{argmin}_X f$.

Use the basic convexity inequality:

$$\langle \nabla f(x_t)|y - x_t\rangle \leq f(y) - f(x_t)$$

to deduce with $u_t = -\nabla f(x_t)$ in (1):

$$\int_0^t [f(x_s) - f(y)]\,ds \leq \int_0^t \langle -\nabla f(x_s)|y - x_s\rangle ds = R_t(y)$$

which implies by Jensen's inequality:

$$f(\bar{x}_t) - f(y) \leq \frac{1}{t}\int_0^t [f(x_s) - f(y)]\,ds \leq \frac{R_t(y)}{t}. \tag{12}$$

Similarly in discrete time with $u_m = -\nabla f(x_m)$:

$$n[f(\bar{x}_n) - f(y)] \leq \sum_{m=1}^n f(x_m) - f(y) \leq \sum_{m=1}^n \langle \nabla f(x_m)|x_m - y\rangle = R_n(y).$$

In particular one obtains:

Proposition 3 (i) *The accumulation points of* $\{\bar{x}_t\}$ *or* $\{\bar{x}_n\}$ *belong to* S.

(ii) *If* $t \mapsto f(x_t)$ *(resp.* $n \mapsto f(x_n)$*) is decreasing, the accumulation points of* $\{x_t\}$ *or* $\{x_n\}$ *belong to* S.

Note that (i) is a particular instance of Proposition 2.

One can also deal with the case $u_m = -\lambda_m \nabla f(x_m)$ with $\lambda_m \geq 0$, $\sigma_m = \sum_{k\leq m} \lambda_k$ and $\sigma_n \hat{x}_n = \sum_{m\leq n} \lambda_m x_m$ to obtain:

$$\sigma_n[f(\hat{x}_n) - f(y)] \leq \sum_{m\leq n} \lambda_m [f(x_m) - f(y)] \leq \sum_{m\leq n} \langle \lambda_m \nabla f(x_m) | x_m - y\rangle \leq R_n(y).$$

This allows to compare the regret for discrete and continuous time trajectories, see Kwon and Mertikopoulos [59], Sect. 6.2.

3 Continuous time

We describe in this section three procedures in continuous time that satisfy the no-regret property. Their discrete time counterparts will be analyzed in the next section.

As usual, discrete time dynamics are easier to describe but their mathematical properties are more difficult to establish. This explains why we choose to start with the continuous time versions.

In addition a very useful tool in the form of a "level function" is available in this set-up and we start by analyzing it.

3.1 Level functions and their properties

Definition: $P : \mathbb{R}^+ \times X \to \mathbb{R}^+$ is a *level function* (for $\{u_t, x_t\}$) if $P(t; y)$ is bounded and satisfies:

$$\langle u_t, x_t - y\rangle \geq \frac{d}{dt} P(t; y), \quad \forall t \in \mathbb{R}^+, \forall y \in X. \tag{13}$$

Note that P is not defined on the trajectory $\{x_t\}$ alone, but is a function of the joint processes $\{u_t, x_t\}$.

Proposition 4 *If there exists a level function,* R_t *is upper bounded; hence* $R_t(y)/t \leq O(1/t)$.

Proof

$$R_t(y) = \int_0^t \langle u_s | y - x_s\rangle ds \leq P(0; y) - P(t; y) \leq P(0; y).$$

□

Hence the existence of a level function implies the no-regret property.

Proposition 5 *Consider class (II).*

Assume $y^* \in eS$*, then* $t \mapsto P(t; y^*)$ *is decreasing.*

Proof

$$\frac{d}{dt}P(t; y^*) \le \langle g(x_t)|x_t - y^*\rangle \le 0.$$

□

Thus a level function evaluated at an external solution in eS is a weak Lyapounov function.

Lemma 2 *Consider class (III).*
If $\{x_t\}$ is a descent procedure (meaning that $\frac{d}{dt} f(x_t) \le 0$), then:

$$E(t; y) = t(f(x_t) - f(y)) + P(t; y)$$

is decreasing, for all $y \in X$.

Proof

$$\begin{aligned}\frac{d}{dt}E(t; y) &= f(x_t) - f(y) + t\frac{d}{dt}f(x_t) + \frac{d}{dt}P(t; y)\\ &\le f(x_t) - f(y) + \langle\nabla f(x_t)|y - x_t\rangle \le 0\end{aligned}$$

□

We recover the fact that the accumulation points of $\{x_t\}$ are in $S = \text{argmin}_X\ f$ and that the speed of convergence of $f(x_t)$ to min f is $O(1/t)$.

3.2 Positive correlation

Given a first order dynamics of the form $\dot{x}_t = A(x_t)$, f decreases on trajectories if:

$$\frac{d}{dt} f(x_t) = \langle\nabla f(x_t)|\dot{x}_t\rangle \le 0.$$

The analogous property for a vector field g is:

$$\langle g(x_t)|\dot{x}_t\rangle \ge 0. \tag{14}$$

In the framework of games, a similar condition was described in discrete time as Myopic Adjustment Dynamics, Swinkels [100] : if $x^i_{n+1} \neq x^i_n$ then $G^i(x^i_{n+1}, x^{-i}_n) > G^i(x^i_n, x^{-i}_n)$, $i \in I$.

The corresponding property in continuous time corresponds to *positive correlation*, (between the dynamics and the vector field), Sandholm [87]:

$$\dot{x}^i_t \neq 0 \Longrightarrow \langle g^i(x_t)|\dot{x}^i_t\rangle > 0, i \in I. \tag{15}$$

The use of this notion for vector fields with potential is a follows:

Proposition 6 *Consider a vector field g with potential G.*

If the dynamics satisfies positive correlation, then G is a strict (increasing) Lyapunov function.

All ω-limit points are rest points.

Proof Let $V_t = G(x_t)$ for $t \geq 0$. Then:

$$\dot{V}_t = \langle \nabla G(x_t) | \dot{x}_t \rangle = \sum_{i \in I} \langle \nabla_{x_i} G(x_t) | \dot{x}_t^i \rangle = \sum_{i \in I} \mu^i(x) \langle g^i(x_t) | \dot{x}_t^i \rangle \geq 0.$$

Moreover $\langle g^i(x_t) | \dot{x}_t^i \rangle = 0, \forall i \in I$, holds if and only if $\dot{x_t} = 0$.

One concludes by using Lyapunov's theorem (e.g. Theorem 2.6.1 in [52]). □

This result is proved by Sandholm [86] for his version of potential population game, see extensions in Benaim et al. [14].

A similar property for fictitious play in discrete time is established in Monderer and Shapley [70].

We will show that this property (15) *holds for the three dynamics defined below.*

We now introduce and study three dynamics:

- Projected dynamics (PD),
- Mirror descent (MD),
- Dual averaging (DA).

In each case we first define the dynamics, then control the evaluation of the regret by exhibiting a level function and finally study the trajectories for class (II) and (III).

3.3 Projected dynamics

We assume in this subsection that $V = V^*$ is an Euclidean space with scalar product denoted by $\langle , \rangle$.

3.3.1 *Dynamics*

The continuous time analog of the generalization of the *Projected Gradient Descent*, Levitin and Polyak [63], Polyak [82], see also Sect. 4.1, is defined by $x_t \in X$ satisfying:

$$\boxed{\langle u_t - \dot{x}_t, y - x_t \rangle \leq 0, \forall y \in X.} \tag{16}$$

which is:

$$\dot{x}_t = \Pi_{T_X(x_t)}(u_t) \tag{17}$$

since $T_X(x_t)$ is a cône.

3.3.2 *Values*

Let:

$$V(t; y) = \frac{1}{2}\|x_t - y\|^2, \quad y \in X. \tag{18}$$

Proposition 7 *V is a level function.*

Proof One has:

$$\frac{d}{dt}V(t;y) = \langle \dot{x}_t, x_t - y\rangle \leq \langle u_t, x_t - y\rangle$$

by (16). □

Thus the properties of Sect. 3.1 hold.

3.3.3 *Trajectories*

• Consider class (II) : $u_t = g(x_t)$.

One has the following convergence result:

Proposition 8 *Assume g dissipative.*

Then $\{\bar{x}_t\}$ *converges to a point in* eS.

Proof - The limit points of $\{\bar{x}_t\}$ are in eS by Proposition 2.

- $\|x_t - y^*\|$ converges when $y^* \in eS$ by Propositions 5 and 7.

Hence using Opial's Lemma [79], which states:

$$In\ an\ Hilbert\ space,\ if\ for\ any\ weak\ accumulation\ point\ \hat{x}\ of\ \{x_t\}(resp. \{\bar{x}_t\}),$$
$$\|x_t - \hat{x}\|\ has\ a\ limit\ as\ t \to \infty,\ then\ \{x_t\}(resp. \{\bar{x}_t\})\ weakly\ converges. \tag{19}$$

it follows that $\bar{x}_t$ converges to a point in eS. □

Proposition 9 *Positive correlation holds.*

Proof

$$\langle g(x_t), \dot{x}_t\rangle = \|\dot{x}_t\|^2$$

since $u_t = g(x_t)$ and $\langle u_t - \dot{x}_t, \dot{x}_t\rangle = 0$, by Moreau's decomposition, Moreau [71]. □

• Consider class (III) : $u_t = -\nabla f(x_t)$.

Proposition 10 (i) $\{x_t\}$ *converges to a point in* S.

(ii) $f(x_t)$ *decreases to* min f *with speed* $O(1/t)$.

Proof (i) Both Proposition 3 or Lemma 2, and Proposition 9 imply that the accumulation points of $\{x_t\}$ are in S. Then using Proposition 5, Opial's Lemma (19) applies.

(ii) Follows from Lemma 2.

□

3.3.4 *Hilbert case*

We assume in this subsection that V is an Hilbert space with scalar product $\langle , \rangle$ and that $X \subset V$ is non-empty, convex and closed.

The results of Sects. 2 and 3.1 will extend, while considering weak accumulation points and assuming eS non empty.

• Consider class (II).

Lemma 3 *Assume $eS \neq \emptyset$. Then the trajectory $\{x_t\}$ is bounded.*

Proof By Proposition 5, for $y^* \in eS$, $V(t; y^*)$ is decreasing. □

In particular this implies the following convergence result:

Proposition 11 *Assume $eS \neq \emptyset$ and g dissipative.*
Then $\{\bar{x}_t\}$ converges weakly to a point in eS.

Proof - $\{\bar{x}_t\}$ is bounded by Lemma 3 hence has weak accumulation points.
Then follow the proof of Proposition 8. □

Recall that if g is dissipative and X is bounded, $eS \neq \emptyset$ by Lemma 2.
• Consider class (III).
With a proof as above, Proposition 12 extends to:

Proposition 12 *Assume $S \neq \emptyset$.*

(i) *$\{x_t\}$ weakly converges to a point in S.*
(ii) *$f(x_t)$ decreases to* min f *with speed $O(1/t)$.*

• Maximal monotone operators.
The preceeding results show strong links with properties of maximal monotone operators, see e.g. Brezis [20].
Recall that A is a monotone operator on V if $Ax \subset V$ and

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x \in V, u \in Ax, y \in V, v \in Ay.$$

A is a maximal monotone operator if in addition its graph $G_A = \{(x, u); x \in V, u \in Ax\}$ is maximal for the inclusion among monotone operators.
If A is a maximal monotone operator with domain $\{x \in V; Ax \neq \emptyset\} = \text{dom } A$, the set of external solutions associated to $X = \text{dom } A$ and $-A$, that will play the role of eS, is the set T with:

$$T = A^{-1}(0) = \{x \in V; 0 \in Ax\} = \{x \in V; \langle Ay, y - x \rangle \geq 0, \forall y \in \text{dom } A\} \quad (20)$$

where as usual $\langle Ay, z \rangle \geq 0$ means $\langle u, z \rangle \geq 0, \ \forall u \in Ay$.
Let $w_t \in \text{dom } A$ be an absolutely continuous trajectory that satisfies a.e. (for the existence, see Brezis [20]):

$$\dot{w}_t \in -Aw_t, t \in \mathbb{R}^+; w_0 \in \text{dom } A. \quad (21)$$

The following result is due to Baillon and Brezis [10] and is the exact analog of Proposition 11 in terms of statements and proof.

Proposition 13 *Assume $T \neq \emptyset$ and A maximal monotone. Then $\{\bar{w}_t\}$ converges weakly to a point in T.*

Proof Let $z \in \operatorname{dom} A$. Then :

$$\frac{d}{dt}\|w_t - z\|^2 = \langle \dot{w}_t, w_t - z\rangle \in \langle -Aw_t, w_t - z\rangle \tag{22}$$

Hence if $z^* \in T$:

$$\frac{d}{dt}\|w_t - z^*\|^2 \leq 0 \tag{23}$$

and $\|w_t - z^*\|^2$ is decreasing, thus $\{w_t\}$ is bounded hence has weak accumulation points.

From monotonicity:

$$\frac{d}{dt}\|w_t - z\|^2 \in \langle -Aw_t, w_t - z\rangle \leq \langle Az, z - w_t\rangle \tag{24}$$

one deduces:

$$\langle -Az, z - \bar{w}_t\rangle = \frac{1}{t}\int_0^t \langle -Az, z - w_t\rangle \leq \frac{1}{t}\|w_0 - z\|^2 \tag{25}$$

so that any weak accumulation point $\hat{w}$ of $\{\bar{w}_t\}$ satisfies :

$$\langle -Az, z - \hat{w}\rangle \leq 0, \quad \forall z \in \operatorname{dom} A$$

hence belongs to T.

Thus by Opial's Lemma (19), $\bar{w}_t$ converges weakly to a point in T. □

Similarly the following result, due to Bruck [22], and its proof, corresponds to Proposition 12. Let $f : V \to \mathbb{R} \cup \{+\infty\}$ be convex, l.s.c. and proper. A is the subdifferential ∂f and we consider the dynamics (21).

Proposition 14 (i) *$f(w_t)$ decreases to $\inf_V f$ with speed $O(1/t)$.*
(ii) *If $T = \operatorname{argmin}_V f \neq \emptyset$, $\{w_t\}$ weakly converges to a point in T.*

Proof The subdifferential inequality writes :

$$f(w_s) - f(w_t) \geq \langle -\dot{w}_t, w_s - w_t\rangle$$

so that :

$$\limsup_{s \to t^-} \frac{f(w_t) - f(w_s)}{t - s} \leq -\|\dot{w}_t\|^2$$

hence $f(w_t)$ is decreasing.

It follows that for any $y \in V$ and $s \in [0, t]$:

$$f(y) \geq f(w_s) + \langle \dot{w}_s, w_s - y\rangle \geq f(w_t) + \frac{1}{2}\frac{d}{ds}\|w_s - y\|^2$$

hence by integration :

$$tf(y) \geq tf(w_t) + \frac{1}{2}\|w_t - y\|^2 - \frac{1}{2}\|w_0 - y\|^2$$

so that :

$$f(w_t) \leq f(y) + \frac{1}{2t}\|w_0 - y\|^2, \quad \forall y \in V.$$

This gives (i).

The proof of the previous Proposition 13 and (i) implies that the weak accumulation points of $\{w_t\}$ are in T.

Using Opial's Lemma 19 then gives (ii). □

To compare with Proposition 12 take $f_X = f + \mathbf{1}_X$ with $int(\mathrm{dom} f \cap X) \neq \emptyset$, where $\mathbf{1}_X$ is the indicator function of X.

3.4 Mirror descent: differential/incremental approach

We study here the continuous version of the extension of the *mirror descent algorithm* studied in convex optimization, Nemirovski and Yudin [74], Beck and Teboulle [13], see also Sect. 4.2.

3.4.1 *Dynamics*

The assumptions are:

H is a strictly convex, $\mathcal{C}^1$ function from V to $\mathbb{R} \cup \{+\infty\}$.

$X \subset V$ is nonempty, compact, convex and $X \subset \mathrm{dom} H$.

The continuous time procedure satisfies $x_t \in X$ with:

$$\langle u_t - \frac{d}{dt}\nabla H(x_t)|y - x_t\rangle \leq 0, \forall y \in X. \tag{26}$$

Recall that the *Bregman distance* associated to H is:

$$D_H(y, x) = H(y) - H(x) - \langle \nabla H(x)|y - x\rangle \ (\geq 0). \tag{27}$$

3.4.2 *Values*

The use of the Bregman distance is the following:

Proposition 15 *$P(t; y) = D_H(y, x_t)$ is a level function.*

Proof Note the relation:

$$\frac{d}{dt}D_H(y, x_t) = -\langle \nabla H(x_t)|\dot{x}_t\rangle - \frac{d}{dt}\langle \nabla H(x_t)|y - x_t\rangle = \left\langle \frac{d}{dt}\nabla H(x_t)|x_t - y\right\rangle \tag{28}$$

so that (26) implies

$$\frac{d}{dt} D_H(y, x_t) \leq \langle u_t | x_t - y \rangle. \tag{29}$$

□

Hence the properties of Sect. 3.1 apply.

The previous analysis of Sect. 3.3 corresponds to the Euclidean case with the regularization function:

$$H(x) = \frac{1}{2} \|x\|^2$$

for the dynamics and the level function.

3.4.3 *Interior trajectory*

The use of a specfic function H adapted to X, with $dom H = X$, H $\mathcal{C}^2$ on $int X$ and $\|\nabla H(x)\| \to +\infty$ as $x \to \partial X$ allows to produce a trajectory that remains in $int X$.

In this case (26) leads to an equality:

$$\frac{d}{dt} \nabla H(x_t) = u_t \tag{30}$$

thus:

$$\nabla H(x_t) = \int_0^t u_s ds \tag{31}$$

or, with H^* being the Fenchel conjugate of H:

$$x_t = \nabla H^* \left(\int_0^t u_s ds \right) \tag{32}$$

and then:

$$\dot{x}_t = \nabla^2 H(x_t)^{-1} u_t. \tag{33}$$

$\nabla^2 H(x)$ induces a Riemannian metric, see Alvarez et al. [2], Mertikopoulos and Sandholm [67].

In this framework one has a monotonic algorithm for a vector field g with potential G, since (recall 3.2.):

Proposition 16 *Positive correlation holds.*

Proof

$$\langle g(x_t) | \dot{x}_t \rangle = \langle g(x_t) | \nabla^2 H(x_t)^{-1} g(x_t) \rangle \geq 0.$$

since H is convex and $\dot{x}_t = \nabla^2 H(x_t)^{-1} g(x_t) \neq 0$ implies $\langle g(x_t) | \nabla^2 H(x_t)^{-1} g(x_t) \rangle > 0$. □

Consider now class (III).

By Proposition 3, the accumulation points of $\{x_t\}$ are in S.

To prove convergence one introduces the following :

Hypothesis [H1]: if $z^k \to y^* \in S$ then $D_H(y^*, z^k) \to 0$.

For example H L-smooth (see e.g. Nesterov [76] Section 1.2.2.) and then:

$$0 \leq D_H(x, y) \leq \frac{L}{2}\|x - y\|^2.$$

Hypothesis [H2]: if $D_H(y^*, z^k) \to 0$, $y^* \in S$ then $z^k \to y^*$.

For example H β-strongly convex (see e.g. Nesterov [76] Section 2.1.3.) and then:

$$D_H(x, y) \geq \frac{\beta}{2}\|x - y\|^2.$$

Proposition 17 *Consider class (III). If H is smooth and strongly convex or more generally if [H1] and [H2] hold, $\{x_t\}$ converges to some $x^* \in S$.*

Proof Let x^* be an accumulation point of $\{x_t\}$. Then $x^* \in S$ by Proposition 3 and thus $D_H(x^*, x_t)$ is decreasing by Propositions 5 and 15. Since this sequence is decreasing to 0 on a subsequence $x_{t_k} \to x^*$ by [H1], it is decreasing to 0, hence by [H2] $x_t \to x^*$. □

3.5 Dual averaging: integral/cumulative approach

We consider here the continuous version of the extension of *dual averaging*, Nesterov [77], see also Sect. 4.3.

We follow the analysis and results in Kwon and Mertikopoulos [59]. *3.5.1 Dynamics*

The assumption is: the regularization function h is a bounded strictly convex l.s.c. function from V to $\mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} h = X$.

Let $h^*(w) = \sup_{x \in V} \langle w|x\rangle - h(x)$ be the Fenchel conjugate of h. h^* is differentiable since h is strictly convex.

Introduce the integral:

$$U_t = \int_0^t u_s ds$$

and let us define x_t by:

$$x_t = argmax\{\langle U_t|x\rangle - h(x); x \in V\} = argmax\{\langle U_t|x\rangle - h(x); x \in X\}.$$

The dynamics can be written as:

$$x_t = \nabla h^*(U_t) \in X. \tag{34}$$

3.5.2 *Values*

Define, for $y \in X$:

$$W(t; y) = h^*(U_t) - \langle U_t | y \rangle + h(y) \tag{35}$$

which corresponds to the Fenchel coupling between the cumulative input $U_t \in V^*$ and a reference point $y \in X \subset V$.

Proposition 18 *W is a level function.*

Proof $W(t; y) \geq 0$ by Fenchel inequality.

Use that:

$$\frac{d}{dt} h^*(U_t) = \langle u_t | \nabla h^*(U_t) \rangle = \langle u_t | x_t \rangle \tag{36}$$

by (34), thus:

$$\frac{d}{dt} W(t; y) = \langle u_t | x_t - y \rangle.$$

□

In particular one has:

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle ds = W(0; y) - W(t; y) \leq \left[-\inf_X h + h(y) \right] \leq r_X(h) \tag{37}$$

with $r_X(h) = \sup_X h(x) - \inf_X h(x)$.

Note that due to the integral formulation of the dynamics (34) (x_t as a function of U_t) compared to the differential formulation (26) ($\dot{x}_t$ as a function of u_t) the level function is expressed through the dual space, however properties of Sect. 3.1 applies as well.

3.5.3 *Trajectories*

Proposition 19 *Positive correlation holds.*

Proof

$$\frac{d}{dt} G(x_t) = \langle \nabla G(x_t) | \dot{x}_t \rangle = \langle g(x_t) | \nabla^2 h^*(U_t)(u_t) \rangle$$

using (34) with $u_t = g(x_t)$. □

Hence in class (III), using Proposition 3 the accumulation points of $\{x_t\}$ are in S.

3.5.4 *Remarks*

In the interior smooth case both dynamics and level functions of Sects. 3.4 and 3.5 are the same, since one has:

$$x_t = \nabla h^*(U_t), \qquad \nabla h(x_t) = U_t, \qquad h^*(U_t) + h(x_t) = \langle U_t | x_t \rangle$$

and

$$\begin{aligned} D_h(y, x_t) &= h(y) - h(x_t) - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | x_t \rangle - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | y \rangle. \end{aligned}$$

3.6 Comments on the continuous time dynamics framework

(1) One obtains the existence of a level function and same speed of convergence of the no-regret values in classes (I), (II) or (III) : $O(\frac{1}{t})$.
(2) Hence by Sect. 2 the accumulation points of the average $\{\bar{x}_t\}$ in class (II) with g dissipative are in eS.
(3) In addition one has convergence of the average $\{\bar{x}_t\}$ in class (II) with g dissipative, with (PD), via Opial's Lemma.
The linear aspect of the derivative of the level function seems crucial to obtain this property.
Note that the Hilbertian structure is used first in the definition of the dynamics, then for the level function and in the reference to Opial's Lemma.
(4) Similarly convergence of $\{x_t\}$ in case (III) holds for (PD), and (MD) with an adapted penalization function H.
(5) Positive correlation holds for the three dynamics (under conditions on H for (MD)).
(6) For vector fields g with potential G, $G(x_t)$ is strictly decreasing in (PD) and (DA), and under conditions on H for (MD).
(7) In the framework of games the entropy function:

$$h(x) = \sum_{p \in S} x^p Log x^p$$

defined on the simplex $X = \Delta(S)$ leads (via (MD) or (DA)) to the *replicator dynamics* on $int\ X$, Taylor and Jonker [101], Hofbauer and Sigmund [52], Sorin [93, 95].
(MD) gives the differential version:

$$\dot{x}_t^p = x_t^p (u_t^p - \langle x_t, u_t \rangle), \quad p \in S$$

while (DA) corresponds to the integral representation:

$$x_t^p = \frac{e^{U_t^p}}{\sum_{s \in S} e^{U_t^s}}, \quad p \in S.$$

The corresponding Riemannian metric $[a, b]_x = \sum_{s \in S} \frac{1}{x^s} a^s b^s$, $x \in int\ X$, is introduced in Shahshahani [90].

Recall also that the replicator dynamics is the continuous time version of the *multiplicative weight algorithm*, Littlestone and Warmuth [64], Vovk [104], Sorin [93, 95].

On the other hand, $h(x) = \frac{1}{2}\|x\|^2$ leads to the *local/direct projection dynamics*, for a comparison, see Lahkar and Sandholm [60], Sandholm et al. [89].

(8) There is an important literature on continuous time dynamics enjoying similar features, see e.g. :

- in convex optimization: Attouch and Teboulle [5], Attouch et al. [3], Auslender and Teboulle [8, 9], Bolte and Teboulle [18], Teboulle [102] ...

- in game theory: Hofbauer and Sandholm [51], Coucheney et al. [28], Mertikopoulos and Sandholm [66], Mertikopoulos and Sandholm [67], Mertikopoulos and Zhou [68] ...

4 Discrete time: general case

We consider now discrete time algorithms.

Remark that the dynamics depends on an additional parameter, the *step size*.

4.1 Projected dynamics

Recall that V is Euclidean and let $m(X)$ denote the diameter of X.

Assumption: $\|u_m\| \leq M, \forall m \in \mathbb{N}$.

4.1.1 *Dynamics*

The standard discrete dynamics (*Gradient projection method* in class (III), Levitin and Polyak [63], Polyak [82]) is given by:

$$\boxed{x_{m+1} = argmax_X\{\langle u_m, x\rangle - \frac{1}{2\eta_m}\|x - x_m\|^2\},} \tag{38}$$

with $\eta_m > 0$ decreasing, which corresponds to:

$$x_{m+1} = \Pi_X[x_m + \eta_m u_m], \tag{39}$$

or with variational characterization, $x_{m+1} \in X$ and :

$$\langle x_m + \eta_m u_m - x_{m+1}, y - x_{m+1}\rangle \leq 0, \forall y \in X, \tag{40}$$

which is:

$$\left\langle u_m - \frac{x_{m+1} - x_m}{\eta_m}, y - x_{m+1}\right\rangle \leq 0, \forall y \in X,$$

and leads to (16) as $\eta_m \to 0$.

4.1.2 *Values*

Recall that :

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m\rangle.$$

Proposition 20

$$R_n(x) \leq \frac{1}{2\eta_n} m(X)^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m.$$

Hence with $\eta_n = 1/\sqrt{n}$*:*

$$R_n(x) \leq O(\sqrt{n}).$$

Proof Let $x_{m+1} = \Pi_X(y_{m+1})$ with $y_{m+1} = x_m + \eta_m u_m$.
So that:

$$\begin{aligned} 2\eta_m \langle u_m, x - x_m \rangle &= 2\langle y_{m+1} - x_m, x - x_m \rangle \\ &= \| y_{m+1} - x_m \|^2 + \| x - x_m \|^2 - \| x - y_{m+1} \|^2. \end{aligned}$$

Note that for $x \in X$, (40) implies $\| y_{m+1} - x \|^2 \geq \| x_{m+1} - x \|^2$ hence:

$$2\eta_m \langle u_m, x - x_m \rangle \leq \eta_m^2 \| u_m \|^2 + \| x - x_m \|^2 - \| x - x_{m+1} \|^2 \tag{41}$$

which is the discrete analog of the level function property in Proposition 7.

This gives:

$$\begin{aligned} R_n(x) &= \sum_{m=1}^{n} \langle u_m, x - x_m \rangle \\ &\leq \frac{1}{2\eta_1} \| x - x_1 \|^2 - \frac{1}{2\eta_n} \| x - x_{n+1} \|^2 + \sum_{m=2}^{n} \left[\frac{1}{2\eta_m} - \frac{1}{2\eta_{m-1}} \right] \| x - x_m \|^2 \\ &\quad + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m. \end{aligned}$$

Thus, with $m(X)$ being the diameter of X:

$$R_n(x) \leq \frac{1}{2\eta_n} m(X)^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m$$

and the choice of $\eta_m = \frac{1}{\sqrt{m}}$ gives:

$$.R_n(x) \leq O(\sqrt{n}).$$

□

4.1.3 *Trajectories*

Consider class (II).

Lemma 4 *For $x^* \in eS$, $\|x_m - x^*\|$ converges if $\{\eta_n\} \in \ell^2$.*

Proof If $x^* \in eS$ then:

$$\|x^* - x_{m+1}\|^2 \leq \|x^* - (x_m + \eta_m g(x_m))\|^2 \leq \eta_m^2 \|g(x_m)\|^2 + \|x^* - x_m\|^2$$

so that $\|x_m - x^*\|$ converges if $\{\eta_n\} \in \ell^2$. □

This corresponds to Proposition 5 in this framework.

Lemma 5 *If g is dissipative and $\{\eta_n\} \in \ell^2$, $\{\bar{x}_n\}$ converges to a point in eS.*

Proof The limit points of $\{\bar{x}_n\}$ are in eS by Lemma 2, hence by Lemma 4 and Opial's Lemma (19), $\{\bar{x}_n\}$ converges to a point in eS. □

This property is the counterpart of Proposition 8.

4.2 Mirror descent

Assumptions:

(a) H is a $\mathcal{C}^1$ function from V to $\mathbb{R} \cup \{+\infty\}$, L-strongly convex for some norm $\|.\|$ on $V = \mathbb{R}^n$ and $X \subset \text{dom } H$,

(b) $\|u_n\|_* \leq M, \forall n \in \mathbb{N}$.

4.2.1 *Dynamics*

The classical *mirror descent algorithm*, introduced for class (III) in Nemirovski and Yudin [74], see also Beck and Teboulle [13], is given by (recall (27)):

$$x_{m+1} = argmax_X\{\langle u_m | x\rangle - \frac{1}{\eta_m} D_H(x, x_m)\}. \tag{42}$$

The variational expression takes the form:

$$\langle \nabla H(x_m) + \eta_m u_m - \nabla H(x_{m+1}) | x - x_{m+1}\rangle \leq 0, \forall x \in X. \tag{43}$$

which leads to the continuous formulation (26).

Note that $D_H(x, y)$ plays the rôle of $\frac{1}{2}\|x - y\|^2$ in (38).

4.2.2 *Values*

We will use the identity:

$$D_H(x, z) - D_H(x, y) - D_H(y, z) = \langle \nabla H(y) - \nabla H(z) | x - y\rangle. \tag{44}$$

which is a direct consequence of the definition of D_H.

Proposition 21 *Let the step size* $\eta_n = \frac{1}{\sqrt{n}}$, *then:*

$$R_n(x) \leq O(\sqrt{n}).$$

Proof

$$\begin{aligned}\langle \eta_n u_n | x - x_n \rangle &= \langle \eta_n u_n | x - x_{n+1} \rangle + \langle \eta_n u_n | x_{n+1} - x_n \rangle \\ &\leq \langle \nabla H(x_{n+1}) - \nabla H(x_n) | x - x_{n+1} \rangle + \langle \eta_n u_n | x_{n+1} - x_n \rangle \\ &= D_H(x, x_n) - D_H(x, x_{n+1}) - D_H(x_{n+1}, x_n) + \langle \eta_n u_n | x_{n+1} - x_n \rangle \end{aligned} \tag{45}$$

by using (44).

Now, H is L strongly convex, hence:

$$D_H(x_{n+1}, x_n) \geq \frac{L}{2} \|x_{n+1} - x_n\|^2 \tag{46}$$

and moreover :

$$\begin{aligned}\langle \eta_n u_n | x_{n+1} - x_n \rangle - \frac{L}{2} \|x_n - x_{n+1}\|^2 &\leq M \eta_n \|x_n - x_{n+1}\| - \frac{L}{2} \|x_n - x_{n+1}\|^2 \\ &\leq \frac{(\eta_n M)^2}{2L}\end{aligned}$$

so that one obtains the analogous of the level function property in Proposition 15:

$$\langle \eta_n u_n | x - x_n \rangle \leq D_H(x, x_n) - D_H(x, x_{n+1}) + \frac{(\eta_n M)^2}{2L}.$$

Summing leads to :

$$R_n(x) \leq \sum_m \left[D_H(x, x_m) \left(\frac{1}{\eta_{m+1}} - \frac{1}{\eta_m} \right) + \eta_m \frac{M^2}{2L} \right]. \tag{47}$$

Hence the bound like in Proposition 20. □

4.2.3 *Trajectories*

Consider class (II).

Lemma 6 *For* $x^* \in eS$, $D_H(x^*, x_n)$ *converges if* $\{\eta_n\} \in \ell^2$.

Proof Start with (45) for $x^* \in eS$ and use $D_H \geq 0$ to get:

$$D_H(x^*, x_{m+1}) \leq D_H(x^*, x_m) - \langle \eta_m u_m | x_m - x_{m+1} \rangle \tag{48}$$

thus it remains to control $\langle \eta_m u_m | x_m - x_{m+1} \rangle$.

H being L strongly convex implies:

$$\langle \nabla H(x_m) - \nabla H(x_{m+1}) | x_m - x_{m+1} \rangle \geq L \|x_{m+1} - x_m\|^2$$

but one has by (43):

$$\begin{aligned} &\langle \nabla H(x_m) - \nabla H(x_{m+1}) | x_m - x_{m+1} \rangle \\ &\quad \leq \langle -\eta_m u_m | x_m - x_{m+1} \rangle \leq \|\eta_m u_m\|_* \|x_m - x_{m+1}\|. \end{aligned}$$

It follows that:

$$\|x_m - x_{m+1}\| \leq \frac{1}{L} \|\eta_m u_m\|_*$$

hence:

$$\langle -\eta_m u_m | x_m - x_{m+1} \rangle \leq \frac{1}{L} \|\eta_m u_m\|_*^2$$

Altogether this implies from (48) that $D_H(x^*, x_m)$ converges if $\{\eta_n\} \in \ell^2$. □

This is the counterpart of Proposition 5.

4.3 Dual averaging

Assumptions :

(a) h is a l.s.c. function from V to $\mathbb{R} \cup \{+\infty\}$, L-strongly convex for some norm $\|.\|$ on $V = \mathbb{R}^n$, with $\operatorname{dom} h = X$.

(b) $\|u_m\|_* \leq M, \forall n \in \mathbb{N}$.

4.3.1 *Dynamics*

We extend the formulation in Nesterov [77]. The starting point is again a maximization property:

$$x_{m+1} = argmax_X \left\{ \langle U_m | x \rangle - \frac{1}{\eta_m} h(x) \right\}, \tag{49}$$

with $U_m = \sum_{k=1}^{m} u_k$ and where $\{\eta_m\}$ is decreasing.

Note that there is an explicit form without using a variational formulation. Hence the *dual averaging* algorithm is given by:

$$\boxed{x_{m+1} = \nabla h^*(\eta_m U_m).} \tag{50}$$

This cumulative representation corresponds to the integral dynamics (34) while the incremental algorithm (43) is associated to the differential dynamics (26).

4.3.2 *Values*

A direct proof, see Xiao [106] or a discrete approximation of (34), see Kwon and Mertikopoulos [59], allows to obtain:

Proposition 22

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle \leq \frac{r_X(h)}{\eta_n} + \frac{\sum_{m=1}^{n} \eta_{m-1} \|u_m\|_*^2}{2L} \tag{51}$$

Proof Fenchel inequality:

$$\langle \eta_n U_n | x \rangle \leq h^*(\eta_n U_n) + h(x)$$

implies:

$$\langle U_n | x \rangle \leq \frac{h^*(0)}{\eta_0} + \sum_{m=1}^{n} \left(\frac{h^*(\eta_m U_m)}{\eta_m} - \frac{h^*(\eta_{m-1} U_{m-1})}{\eta_{m-1}} \right) + \frac{\max_X h}{\eta_n}. \tag{52}$$

Now:

$$\begin{aligned} \frac{h^*(\eta_m U_m)}{\eta_m} &= \sup_X \left[\langle U_m | x \rangle - \frac{h(x)}{\eta_m} \right] \\ &\leq \sup_X \left[\langle U_m | x \rangle - \frac{h(x)}{\eta_{m-1}} \right] + \sup_X \left[-\frac{h(x)}{\eta_m} + \frac{h(x)}{\eta_{m-1}} \right] \\ &= \frac{h^*(\eta_{m-1} U_m)}{\eta_{m-1}} + \left(\frac{1}{\eta_{m-1}} - \frac{1}{\eta_m} \right) \min_X h, \end{aligned}$$

so that replacing in (52) gives:

$$\begin{aligned} \langle U_n | x \rangle \leq \frac{h^*(0)}{\eta_0} &+ \sum_{m=1}^{n} \frac{1}{\eta_{m-1}} [h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1})] \\ &+ \min_X h \left(\frac{1}{\eta_0} - \frac{1}{\eta_n} \right) + \frac{\max_X h}{\eta_n}. \end{aligned} \tag{53}$$

h is L strongly convex for $\|.\|$, so that h^* is $1/L$ smooth for $\|.\|_*$ hence:

$$\begin{aligned} & h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1}) - \langle \eta_{m-1} U_m - \eta_{m-1} U_{m-1} | \nabla h^*(\eta_{m-1} U_{m-1}) \rangle \\ &= h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1}) - \langle \eta_{m-1} u_m | x_m \rangle \\ &\leq \frac{\eta_{m-1}^2}{2L} \|u_m\|_*^2. \end{aligned} \tag{54}$$

This leads to a property similar to the level function property in Proposition 18 since:

$$\langle u_m | x - x_m \rangle \leq \langle u_m | x \rangle + \frac{1}{\eta_{m-1}} [h^*(\eta_{m-1} U_{m-1}) - h^*(\eta_{m-1} U_m)] + \frac{\eta_{m-1}}{2L} \|u_m\|_*^2$$

$$\leq \frac{1}{\eta_{m-1}}[h^*(\eta_{m-1}U_{m-1}) + h(x) - \langle \eta_{m-1}U_{m-1}, x\rangle]$$
$$-\frac{1}{\eta_m}[h^*(\eta_m U_m) + h(x) - \langle \eta_m U_m, x\rangle]$$
$$+\frac{\eta_{m-1}}{2L}\|u_m\|_*^2 + \left(\frac{1}{\eta_{m-1}} - \frac{1}{\eta_m}\right)\left(\min_X h - h(x)\right).$$

Now inserting (54) in (53) gives:

$$\langle U_n|x\rangle - \sum_{m=1}^{n}\langle u_m|x_m\rangle \leq \frac{h^*(0)}{\eta_0} + \sum_{m=1}^{n}\frac{\eta_{m-1}}{2L}\|u_m\|_*^2 + \left(\frac{1}{\eta_0} - \frac{1}{\eta_n}\right)\min_X h + \frac{\max_X h}{\eta_n}$$

and recall that $h^*(0) = -\min_X h$. □

Hence the convergence rate $O(\sqrt{n})$ with time varying parameters $\eta_n = 1/\sqrt{n}$.

4.4 Comments on the discrete dynamics framework

(1) The three algorithms achieve the same bound $O(1/\sqrt{n})$ for the speed of convergence of the average regret, which is optimal already in class (III), Nesterov (2004), using time varying step sizes $\eta_n = 1/\sqrt{n}$.

(2) The no-regret property holds for the three algorithms hence the results of Sect. 2.2 apply for the closed form.

(3) More precise properties concerning the trajectories are available only in the (PD) set-up. The results are similar to the ones in the continuous case, Sect. 3.2 if $\eta_n \in \ell^2$, for class (II). (Compare to the analysis in Peypouquet and Sorin [81] for dynamics induced by maximal monotone operators in discrete and continuous time, see Sect. 3.3.4).

(4) There is no discrete counterpart of "positive correlation".

(5) For vector fields g with potential one does not have the crucial property $G(x_n)$ decreasing.

5 Discrete time: regularity

This section deals mainly with class (III) *convex gradient*, where in addition f satisfies some regularity properties.
Recall that f is β smooth if:

$$|f(y) - f(x) - \langle \nabla f(x)|y - x\rangle| \leq \frac{\beta}{2}\|x - y\|^2. \quad (55)$$

Alternatively ∇f is β-Lipschitz.

A last part is devoted to the so-called mirror-prox procedure, class (II) with g β-Lipschitz.

5.1 Projected dynamics

Assumption: f is β smooth.

The algorithm is like (40) with constant step size $\eta_m = 1/\beta$.

$$x_{m+1} = argmax_X \left\{ \langle -\nabla f(x_m), x \rangle - \frac{1}{2\beta} \|x - x_m\|^2 \right\}, \tag{56}$$

which gives:

$$x_{m+1} = \Pi_X \left[x_m - \frac{1}{\beta} \nabla f(x_m) \right]. \tag{57}$$

The analysis in this section is standard, see e.g. Nesterov [76].

5.1.1 *Preliminaries*

Define $Tx = \Pi_X[x - \frac{1}{\beta}\nabla f(x)]$ and $v(x) = \beta(x - Tx)$ (which plays the role of $\nabla f(x)$, corresponding to $X = V$).

The projection property gives:

$$\left\langle x - \frac{1}{\beta}\nabla f(x) - Tx, y - Tx \right\rangle \le 0, \quad \forall y \in X \tag{58}$$

so that:

$$\langle \nabla f(x), Tx - y \rangle \le \langle v(x), Tx - y \rangle, \quad \forall y \in X. \tag{59}$$

Now one has:

$$\begin{aligned} f(Tx) - f(y) &= f(Tx) - f(x) + f(x) - f(y) \\ &\le \langle \nabla f(x), Tx - x \rangle \\ &\quad + \frac{\beta}{2}\|Tx - x\|^2 + \langle \nabla f(x), x - y \rangle \qquad f\ \beta \text{ smooth, convex} \\ &= \langle \nabla f(x), Tx - y \rangle + \frac{1}{2\beta}\|v(x)\|^2 \\ &\le \langle v(x), Tx - y \rangle + \frac{1}{2\beta}\|v(x)\|^2, \quad \forall y \in X \qquad \text{by (59)} \end{aligned}$$

hence:

$$f(Tx) - f(y) \le \langle v(x), x - y \rangle - \frac{1}{2\beta}\|v(x)\|^2, \quad \forall y \in X. \tag{60}$$

The following decreasing property is crucial and shows the difference with the general non-smooth case.

Lemma 7 *Descent lemma*

$$f(x_{m+1}) - f(x_m) \le -\frac{1}{2\beta}\|v(x_m)\|^2 = -\frac{\beta}{2}\|x_{m+1} - x_m\|^2 \tag{61}$$

Proof By the previous inequality (60) with $x_m = x = y$. □

Note that in the previous lemma the convexity of f is actually not usued.

A consequence of (61) is:

$$\frac{1}{2\beta}\sum_{m=1}^{n}\|v(x_m)\|^2 \leq f(x_1) - f(x_{n+1}) \leq f(x_1) - f^*$$

hence $\{\|v(x_n)\|\} \in \ell^2$.

5.1.2 *Values*

Proposition 23

$$f(x_n) - f^* \leq O\left(\frac{1}{n}\right).$$

Proof Consider the algorithm defined by $\{z_n\}$ and the process $\{v_n\}$ with $z_1 = x_1$, $v_n = -v(z_n)$ and $z_{n+1} = z_n + \eta v_n$ with $\eta = 1/\beta$. Clearly $z_n = x_n$.

From Sect. 3.1, Proposition 20 with η_m constant, one obtains:

$$R_n^v(y) = \sum_{m=1}^{n}\langle v_m, y - z_m\rangle \leq \frac{1}{2\eta}\|y - z_1\|^2 + \frac{\eta}{2}\sum_{m=1}^{n}\|v_m\|^2 \tag{62}$$

which is bounded since $\{\|v(x_n)\|\} \in \ell^2$. This implies, using f decreasing and (60):

$$n[f(x_{n+1}) - f(y)] \leq R_n^v(y) - \frac{1}{2\beta}\|\sum_{m=1}^{n}\|v_m\|^2 = \frac{\beta}{2}\|y - x_1\|^2.$$

□

5.1.3 *Trajectories*

Lemma 8 *For $y^* \in S$, $\|x_n - y^*\|$ is decreasing.*

Proof From (60) one has:

$$0 \leq f(x_{n+1}) - f(y^*) \leq \langle v(x_n), x_n - y^*\rangle - \frac{1}{2\beta}\|v(x_n)\|^2$$

hence :

$$\langle v(x_n), x_n - y^*\rangle \geq \frac{1}{2\beta}\|v(x_n)\|^2.$$

So that:

$$\|x_{n+1} - y^*\|^2 = \|x_{n+1} - x_n\|^2 + \|x_n - y^*\|^2 + 2\langle x_{n+1} - x_n, x_n - y^*\rangle$$

$$= \frac{1}{\beta^2}\|v(x_n)\|^2 + \|x_n - y^*\|^2 + 2\left\langle -\frac{1}{\beta}v(x_n), x_n - y^*\right\rangle$$
$$\leq \|x_n - y^*\|^2.$$

□

Proposition 24 $\{x_n\}$ *converges to a point in* S.

Proof Since $f(x_n)$ decreases, the accumulation points of $\{x_n\}$ are in S and by the previous Lemma 8, Opial's Lemma (19) applies. □

5.2 Mirror descent

We first assume only that H and f are $\mathcal{C}^1$.

The next analysis follows Bauschke et al. [12].

The main assumption in this section is the existence of a constant $L > 0$ such that (recall that D_H is the Bregman distance (27)):

$$(A) \qquad L\, D_H - D_f \geq 0$$

which is equivalent to : $LH - f$ convex.

Note that if H is strongly convex and f is smooth (not assumed convex), there exists L such that (A) holds. However f is not required to be smooth.

(A similar pre-oder on convex functions appears in Nguyen [78]).

Recall the procedure (43) with constant step size λ:

$$\boxed{\langle \lambda \nabla f(x_n) + \nabla H(x_{n+1}) - \nabla H(x_n) | x - x_{n+1}\rangle \geq 0, \forall x \in X} \qquad (63)$$

and the identity:

$$D_H(x, z) - D_H(x, y) - D_H(y, z) = \langle \nabla H(y) - \nabla H(z) | x - y\rangle. \qquad (64)$$

Let us consider a step size satisfying $2\lambda L = 1$.

5.2.1 *Values*

Lemma 9

$$\lambda[f(x_{n+1}) - f(y)] \leq D_H(y, x_n) - D_H(y, x_{n+1}) - \frac{1}{2}D_H(x_{n+1}, x_n) - \lambda D_f(y, x_n), \forall y \in X. \qquad (65)$$

Proof Since:

$$D_f(x, z) = D_f(y, z) + f(x) - f(y) - \langle \nabla f(z) | x - y\rangle, \qquad (66)$$

one has, by (A):

$$f(x) \leq f(y) + \langle \nabla f(z) | x - y \rangle + L D_H(x, z) - D_f(y, z)$$

Let $x = x_{n+1}$, $z = x_n$

$$f(x_{n+1}) - f(y) \leq \langle \nabla f(x_n) | x_{n+1} - y \rangle + L D_H(x_{n+1}, x_n) - D_f(y, x_n)$$

so that by (63):

$$\begin{aligned}\lambda[f(x_{n+1}) - f(y)] \leq \langle \nabla H(x_{n+1}) - \nabla H(x_n) | y - x_{n+1} \rangle \\ + \lambda L D_H(x_{n+1}, x_n) - \lambda D_f(y, x_n).\end{aligned}$$

Uses then (64):

$$\begin{aligned}\lambda[f(x_{n+1}) - f(y)] \leq D_H(y, x_n) - D_H(y, x_{n+1}) - D_H(x_{n+1}, x_n) \\ + \lambda L D_H(x_{n+1}, x_n) - \lambda D_f(y, x_n)\end{aligned}$$

Hence the result since $2\lambda L = 1$. □

Proposition 25 *Assume H convex . Then:*
(1) *$f(x_n)$ is decreasing.*
(2) $\sum D_H(x_{n+1}, x_n) < +\infty$.
Assume f convex. Then :
(3)

$$f(x_n) - f(y) \leq \frac{2L}{n} D_H(y, x_1), \quad \forall y \in X.$$

Proof (1) and (2) Take $y = x_n$ in (65).
(3) Use $f(x_n)$ decreasing, $D_f \geq 0$ and the telescoping sum in (65). □

5.2.2 *Trajectories*

Proposition 26 *Assume f convex.*
(1) *$y \in S$ implies: $D_H(y, x_n)$ decreases.*
(2) *Assume:*
$[H1] : x^k \to x^* \in S \Rightarrow D_H(x^*, x^k) \to 0$,
$[H2] : x^* \in S, D_H(x^*, x^k) \to 0 \Rightarrow x^k \to x^*$,
then $\{x_n\}$ converges to a point in S.

Proof (1) follows from (65).
(2) By the previous Proposition 25, the accumulation points of $\{x_n\}$ are in S. Let thus $x_{n_k} \to x^* \in S$.
By $[H1]$, $D_H(x^*, x_{n_k}) \to 0$ then by 1) $D_H(x^*, x_n) \to 0$. Now use $[H2]$. □

Compare with the proof of Proposition 17.

Note that the result is more precise than in the continuous case, Sect. 3.3 where there was no decreasing property (for general H).

Let us finally mention the very recent result due to Bui and Combettes [24] Theorem 3.9., where the use of variable metrics H_n allows to reach $f(x_n) - f^* = o(1/n)$.

5.3 Dual averaging

We follow the analysis in Lu et al. [65].

Recall that we consider class (III) : f convex and $\mathcal{C}^1$ and as in the previous Sect. 4.2 the main hypothesis is the existence of $L > 0$ with:

$$(A) \qquad L\,D_h - D_f \geq 0.$$

but we will simply assume here $L\,h - f$ convex, where $h : V \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. with dom $h = X$. Let $x_0 = argmin_X h(x)$ and assume $h(x_0) = 0$. Define :

$$G_k(x) = \sum_{i=0}^{k-1} [\langle \nabla f(x_i), x - x_i \rangle + f(x_i)] + Lh(x) \tag{67}$$

and as in (49):

$$\boxed{x_k = argmin_X G_k(x) = argmax_X \{\langle U_{k-1}, x\rangle - Lh(x)\}} \tag{68}$$

with as usual $u_k = -\nabla f(x_k)$ and $U_m = \sum_{k=1}^m u_k$.

Proposition 27

$$f(\bar{x}_k) - f(x) \leq \frac{L}{k} h(x), \tag{69}$$

$$\min_{i=0,\ldots,k} f(x_i) - f(x) \leq \frac{L}{k} h(x). \tag{70}$$

Proof By definition:

$$G_{k+1}(x_{k+1}) = G_k(x_{k+1}) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + f(x_k). \tag{71}$$

Note that (A) implies that each $G_k - f$ is convex hence:

$$G_k(x_{k+1}) - f(x_{k+1}) \geq G_k(x_k) - f(x_k) + \langle \partial G_k(x_k) - \nabla f(x_k), x_{k+1} - x_k \rangle. \tag{72}$$

Thus one has:

$$G_{k+1}(x_{k+1}) \geq f(x_{k+1}) + G_k(x_k) + \langle \partial G_k(x_k), x_{k+1} - x_k \rangle \tag{73}$$

but there exists $u_k \in \partial G_k(x_k)$ with :

$$\langle u_k, x - x_k\rangle \geq 0, \forall x \in X$$

by the choice of x_k.

Finally, with $g_{k+1} = G_{k+1}(x_{k+1})$ one obtains:

$$g_{k+1} \geq f(x_{k+1}) + g_k. \tag{74}$$

Using f convex, thus $G_k(x) \leq kf(x) + Lh(x)$ by (67), this implies that:

$$\sum_{i=0}^{k+1} f(x_i) \leq g_{k+1} \leq (k+1)f(x) + Lh(x) \tag{75}$$

hence the result. □

5.4 Comments on the regular case

(1) In the three cases (PD), (MD) and (DA) the speed of convergence of the values is $O(1/n)$ and the algorithms use a constant step parameter.

(2) Using (PD) with f smooth implies $f(x_n)$ decreasing and the convergence of $\{x_n\}$.

(3) The approach in Sect. 5.2 shows that similar results can be obtained using (MD) without assuming f with Lipschitz gradient if the regularization function H is adapted to f: condition (A).

(4) Analogous results for the values are much simpler to obtain in the (DA) framework. However the properties concern the value at the average $f(\bar{x}_n)$ and no result is available on the trajectories.

5.5 Mirror prox

We follow in this section the model introduced by Korpelevich [56], and further studied by Nemirovski [73].

We present it for two main reasons: first it gives faster convergence than in Sect. 4 for a Lipschitz vector field g; second the tools used in the proofs are very similar to the one used in the previous sections.

5.5.1 *Dynamics*

The algorithm corresponds to a two-stage procedure.

The first step produces y_{n+1} from x_n via usual (MD) (43) applied to $u_n = g(x_n)$ with constant step size λ:

$$\langle -\lambda g(x_n) + \nabla H(y_{n+1}) - \nabla H(x_n) | x - y_{n+1}\rangle \geq 0, \forall x \in X. \tag{76}$$

The second step produces x_{n+1} from x_n via (MD) (43) applied to $v_n = g(y_{n+1})$ with constant step size λ:

$$\langle -\lambda g(y_{n+1}) + \nabla H(x_{n+1}) - \nabla H(x_n) | x - x_{n+1}\rangle \geq 0, \forall x \in X. \tag{77}$$

Altogether, the decision after x_n uses a more refined information $g(y_{n+1})$ rather than the usual (in (MD)) $g(x_n)$.

5.5.2 *Values*

The result concerns the process $\{y_n\}$.

Proposition 28 *Nemirovski* [73] *Assume H α strongly convex and g β-Lipschitz with $\alpha \geq \lambda\beta$, then:*

$$R_n(x) = \sum_{m=1}^{n} \langle g(y_m)|x - y_m\rangle \leq \frac{1}{\lambda}[D_H(x, x_1) - D_H(x, x_n)].$$

Proof Decompose:

$$\begin{aligned}\langle g(y_{n+1})|x - y_{n+1}\rangle = &\langle g(y_{n+1})|x - x_{n+1}\rangle \\ &+ \langle g(x_n)|x_{n+1} - y_{n+1}\rangle \\ &+ \langle g(y_{n+1}) - g(x_n)|x_{n+1} - y_{n+1}\rangle.\end{aligned}$$

The first term gives via (77):

$$\begin{aligned}\lambda\langle g(y_{n+1})|x - x_{n+1}\rangle &\leq \langle \nabla H(x_{n+1}) - \nabla H(x_n)|x - x_{n+1}\rangle \\ &= D_H(x, x_n) - D_H(x, x_{n+1}) - D_H(x_{n+1}, x_n)\end{aligned}$$

using (44).

Invoking (76), the second term leads to:

$$\begin{aligned}\lambda\langle g(x_n)|x_{n+1} - y_{n+1}\rangle &\leq \langle \nabla H(x_n) - \nabla H(y_{n+1})|y_{n+1} - x_{n+1}\rangle \\ &= D_H(x_{n+1}, x_n) - D_H(x_{n+1}, y_{n+1}) - D_H(y_{n+1}, x_n)\end{aligned}$$

and for the last one, using g β-Lipschitz :

$$\begin{aligned}\langle g(y_{n+1}) - g(x_n)|x_{n+1} - y_{n+1}\rangle &\leq \|g(y_{n+1}) - g(x_n)\|_* \|x_{n+1} - y_{n+1}\| \\ &\leq \beta\|y_{n+1} - x_n\|\|y_{n+1} - x_{n+1}\| \\ &\leq \frac{\beta}{2}[\|y_{n+1} - x_n\|^2 + \|y_{n+1} - x_{n+1}\|^2].\end{aligned}$$

So that:

$$\begin{aligned}\lambda\langle g(y_{n+1})|x - y_{n+1}\rangle \leq\ & D_H(x, x_n) - D_H(x, x_{n+1}) - D_H(x_{n+1}, y_{n+1}) \\ & - D_H(y_{n+1}, x_n) + \frac{\lambda\beta}{2}[\|y_{n+1} - x_n\|^2 + \|y_{n+1} - x_{n+1}\|^2].\end{aligned} \tag{78}$$

Hence if H is α strongly convex and $\alpha \geq \lambda\beta$ one has a telescopic majorant leading to:

$$\lambda R_n(x) = \lambda \sum_{m=1}^{n} \langle g(y_m) | x - y_m \rangle \leq D_H(x, x_1) - D_H(x, x_n).$$

□

Comments :
If $eS \neq \emptyset$, take $x \in eS$ in (78) and $\alpha > \lambda\beta$ to get:
- $\sum_n \|y_{n+1} - x_n\|^2 + \|y_{n+1} - x_{n+1}\|^2 < \infty$
- $D_H(x, x_n)$ decreasing.
Notice that g was not assumed dissipative.
Adding g dissipative gives:

$$\langle g(x) | x - \bar{y}_n \rangle \leq R_n(x)$$

hence the accumulation points of $\{\bar{y}_n\}$ are in S.

5.5.3 *Trajectories*
The result involves the trajectory $\{x_n\}$.

Proposition 29 *Korpelevich* [56]
Assume g β-Lipschitz and dissipative.
(i) *(PD) case:*
for $\lambda < 1/\beta$, x_n converges to a point in S.
(ii) *(MD) case:*
Assume H α strongly convex with $\alpha > \lambda\beta$. $y \in S$ implies: $D_H(y, x_n)$ decreases.
Assume:
$[H1] : x^k \to x^* \in S \Rightarrow D_H(x^*, x^k) \to 0,$
$[H2] : D_H(z^k, x^k) \to 0 \Rightarrow \|z^k - x^k\| \to 0,$
then $\{x_n\}$ converges to a point in S.

Proof (i) The property of the projection gives:

$$\|z - y\|^2 \geq \|z - \Pi_X(z)\|^2 + \|\Pi_X(z) - y\|^2, \qquad \forall y \in X$$

hence with $z = x_n + \lambda g(y_{n+1})$,
$\Pi_X(z) = x_{n+1}$ by (77) and one deduces:

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \|x_n + \lambda g(y_{n+1}) - y\|^2 - \|x_n + \lambda g(y_{n+1}) - x_{n+1}\|^2 \\ &= \|x_n - y\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda \langle g(y_{n+1}), x_{n+1} - y \rangle. \end{aligned} \tag{79}$$

Taking $y = x^* \in eS = S$ and adding $2\lambda\langle g(y_{n+1}), x^* - y_{n+1}\rangle \geq 0$, we obtain:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda\langle g(y_{n+1}), x_{n+1} - y_{n+1}\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_{n+1}\|^2 - \|y_{n+1} - x_{n+1}\|^2 \end{aligned}$$

$$-2\langle x_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle + 2\lambda\langle g(y_{n+1}), x_{n+1} - y_{n+1}\rangle. \tag{80}$$

Now by (76) one has:

$$\langle x_n + \lambda g(x_n) - y_{n+1}, x_{n+1} - y_{n+1}\rangle \leq 0 \tag{81}$$

replacing in (80) and using:

$$\begin{aligned}\langle g(y_{n+1}) - g(x_n), x_{n+1} - y_{n+1}\rangle &\leq \|g(y_{n+1}) - g(x_n)\| \|x_{n+1} - y_{n+1}\| \\ &\leq \beta \|y_{n+1} - x_n\| \|x_{n+1} - y_{n+1}\|\end{aligned} \tag{82}$$

one finally achieves:

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq &\|x_n - x^*\|^2 - \|x_n - y_{n+1}\|^2 - \|y_{n+1} - x_{n+1}\|^2 \\ &+2\lambda\beta\|y_{n+1} - x_n\| \|x_{n+1} - y_{n+1}\| \\ &\leq \|x_n - x^*\|^2 - (1 - \lambda^2\beta^2)\|y_{n+1} - x_n\|^2.\end{aligned} \tag{83}$$

This implies :

$$\|y_{n+1} - x_n\|^2 \to 0. \tag{84}$$

By (83) the sequence $\{x_n\}$ is bounded. Let $\hat{x}$ be an accumulation point. Properties (76) and (84) imply that $\hat{x}$ satisfies:

$$\langle g(\hat{x}), \hat{x} - y\rangle \geq 0, \qquad \forall y \in X$$

hence $\hat{x} \in iS = S$, since g is dissipative. Taking now $x^* = \hat{x}$ in (83) implies that $\|x_n - \hat{x}\|$ is decreasing, but going to 0 on a subsequence, it converges to 0.

(II) Let us start with $x^* \in S$ and the identity:

$$D_H(x^*, x_{n+1}) = D_H(x^*, x_n) - D_H(x_{n+1}, x_n) - \langle \nabla H(x_{n+1}) - \nabla H(x_n) | x^* - x_{n+1}\rangle. \tag{85}$$

Denote $A = -\langle \nabla H(x_{n+1}) - \nabla H(x_n) | x^* - x_{n+1}\rangle$ and from (77) with $x = x^*$ one deduces:

$$A \leq \langle -\lambda g(y_{n+1}) | x^* - x_{n+1}\rangle$$

But $x^* \in S$ hence:

$$\langle -\lambda g(y_{n+1}) | x^* - y_{n+1}\rangle \leq 0$$

so that:

$$A \leq \langle -\lambda g(y_{n+1}) | y_{n+1} - x_{n+1}\rangle.$$

Adding (76) with $x = x_{n+1}$ gives:

$$A \leq \langle \lambda[g(x_n) - g(y_{n+1})] - \nabla H(y_{n+1}) + \nabla H(x_n) | y_{n+1} - x_{n+1} \rangle.$$

Use that:

$$\begin{aligned}\langle -\nabla H(y_{n+1}) + \nabla H(x_n) | y_{n+1} - x_{n+1} \rangle = D_H(x_{n+1},, x_n) \\ - D_H(x_{n+1}, y_{n+1}) - D_H(y_{n+1}, x_n)\end{aligned}$$

and coming back to (85) we obtain:

$$\begin{aligned}D_H(x^*, x_{n+1}) = D_H(x^*, x_n) - D_H(x_{n+1}, y_{n+1}) - D_H(y_{n+1}, x_n) \\ + \lambda \langle g(x_n) - g(y_{n+1}) | y_{n+1} - x_{n+1} \rangle.\end{aligned} \tag{86}$$

Recall that H is α strongly convex so that:

$$D_H(u, v) \geq \frac{\alpha}{2}]\|u - v\|^2$$

and use that g is β- Lipschitz to bound:

$$\begin{aligned}\langle g(x_n) - g(y_{n+1}) | y_{n+1} - x_{n+1} \rangle &\leq \|g(x_n) - g(y_{n+1})\|_* \|y_{n+1} - x_{n+1}\| \\ &\leq \beta \|x_n - y_{n+1}\| \|y_{n+1} - x_{n+1}\| \\ &\leq \frac{\beta}{2} \left[\rho \|x_n - y_{n+1}\|^2 + \frac{1}{\rho} \|y_{n+1} - x_{n+1}\|^2 \right].\end{aligned}$$

Replacing in (86) with $\rho = \frac{\lambda\beta}{\alpha}$ gives:

$$D_H(x^*, x_{n+1}) \leq D_H(x^*, x_n) - \left(1 - \left(\frac{\lambda\beta}{\alpha}\right)^2\right) D_H(y_{n+1}, x_n).$$

Taking $\lambda < \frac{\alpha}{\beta}$ implies $D_H(y_{n+1}, x_n) \to 0$ and $D_H(x^*, x_n)$ decreasing.

Finally let $\tilde{x}$ be an accumulation point of $\{x_n\}$ and $x_{n_k} \to \tilde{x}$. By [H2], $y_{n_k} \to \tilde{x}$ as well.

Using (76) this implies $\tilde{x} \in iS = S$ since g is dissipatif. But then $D_H(\tilde{x}, x_n)$ decreases hence goes to 0, since it is the case on $\{x_{n_k}\}$ by [H1].

Using again [H2] $x_n \to \tilde{x}$ hence the result. □

The result has the same flavour than Proposition 26.

Notice that the algorithm is well defined for any vector field g but the interpretation in the decentralized framework of games is difficult. The main point is that x^i_{n+1} is determined via $g^i(y_{n+1})$ where all the components of y_{n+1} appear, hence some coordination is required.

6 Concluding remarks

For the three dynamics PG, MD and DA (1), (2) and (3) holds:

(1) In continuous time the speed of convergence of the average regret to 0, of the order $O(1/t)$, is not better in the general gradient convex case than in the on-line learning case. This means that the specificity of the observation process cannot be exploited.

(2) Similarly in discrete time the speed of convergence of the average regret to 0, of the order $O(1/\sqrt{n})$ is not better in the general gradient convex case than in the on-line learning case.

(3) Adding a smoothness hypothesis on the convex function does not change the speed in the continuous time framework but allow a better convergence rate in discrete time from $O(1/\sqrt{n})$ to $O(1/n)$, and with a simpler algorithm : constant step size. Basically an additional property is required to get a descent lemma.

(4) A similar phenomena appears with the so-called acceleration procedures following Nesterov [75].

In the continuous time case a second order ODE leads to a speed of convergence $f(x_t) - f(x^*) \leq O(\frac{1}{t^2})$ with no further hypothesis on f, see Su et al. [98, 99], Krichene et al. [57, 58], Attouch et al. [4].

To obtain a similar property in discrete time, namely $f(x_n) - f(x^*) \leq O(\frac{1}{n^2})$ one has to assume f smooth.

The same remark apply to the (weak) convergence of the trajectory, where f smooth is needed in discrete time and not in continuous time, Chambolle and Dossal [27] and Attouch et al. [4].

(5) Concerning the link between discrete and continuous time dynamics, there are no direct results of the form: no-regret in continuous time imply no-regret in discrete time but analogy of the tools used and ad-hoc choice of the step parameter, see Sorin [93], Kwon and Mertikopoulos [59]. Similar properties hold in the closed form for Lyapounov functions in Krichene, Bayen and Bartlett [57], [58], Wibisono et al. [105], see also Bansal and Gupta [11].

(6) The Hilbert framework for (PD) allows to obtain convergence results on the trajectories. The two other algorithms ((MD and (DA)) are more flexible and can achieve better explicit speed of convergence of the values by choosing an adequate norm, see the discussion in Bauschke et al. [12]. For (MD) specific regularization functions H can also lead to convergence of the trajectories. (DA) is much simpler to implement due to its integral formulation. However no convergence properties of the trajectories are available.

(7) In the framework of games, general positive results are obtained in the class of dissipative games. Accumulation points of the average trajectory are equilibria.

(8) For potential games or more generally vector fields with potential the three continuous time algorithms imply the inclusion of the set of accumulation point in the set of rest points.

A further study in the framework of tame functions and invoving quasi-gradients, in the spirit of Bégoutet al. [19], would be very interesting.

(9) Obviously potential games with a concave potential P will share the properties of class (III) since basically one can replace for each i, $\langle g^i(x), y^i - x^i \rangle$ by $\langle \nabla_i P(x), y^i - x^i \rangle$.

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