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Continuous Time Learning Algorithms in Optimization and Game Theory

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Abstract
The purpose of this work is the comparison of learning algorithms in continuous time used in optimization and game theory. The first three are issued from no-regret dynamics and cover in particular "Replicator dynamics" and "Local projection dynamics". Then we study "Conditional gradient" versus "Global projection" dynamics and finally "Frank-Wolfe" versus "Best reply" dynamics. Important similarities occur when considering potential or dissipative games.

Keywords Learning algorithms · Continuous time · Optimization · Game theory

1 Presentation

We will underline several links between first order convex optimization algorithms and game learning dynamics.

The first processes aim at minimizing a function by using information on its gradient while the second class describe trajectories generated by the joint choices of the players. However we will see that, often under different names, similar ideas appear and analogous properties hold.

A first group contains three variants of the extension to continuous time of the "projected gradient dynamics" used with different regularization functions in optimization (Polyak [48]; Nemirovski and Yudin [43]; Nesterov [44]) under the names "Mirror descent" and "Dual averaging".

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We will describe the connection with replicator dynamics (Taylor and Jonker [65]) and local projection dynamics in games (Dupuis and Nagurney [18]; Lakhar and Sandholm [31]) and further properties (see e.g. Mertikopoulos and Sandholm [34]). Note that all these dynamics satisfy the so-called "no-regret property".

The next example of dynamics corresponds to "conditional gradient dynamics" in optimization (Antipin [3]; Bolte [11]) and "global projection dynamics" in game theory (Friesz et al. [22]; Tsakas and Voorneveld [66]).

In the last case we compare the famous "Frank-Wolfe algorithm" [21] and a version of the "Best-reply dynamics" (Gilboa and Matsui [23]) applied to the linearized game.

2 Optimum and Equilibria

We will work under the following framework:

V is a normed vector space, finite dimensional, with dual V^* and duality map $\langle V^*|V\rangle$,
X is a compact convex subset of V (the compactness property is not necessary in optimization but almost unavoidable when dealing with games).

2.1 Convex Optimization

Let us recall the basic property of convex optimization under constraints.

Given f convex and $\mathcal{C}^1$, the elements $\hat{x}$ achieving:

$$\min_X f(x)$$

are given by the solutions of:

$$\langle \nabla f(\hat{x}) | \hat{x} - y \rangle \leq 0, \qquad \forall y \in X. \tag{1}$$

2.2 Variational Inequalities

We generalize the previous characterization to a definition for vector fields as follows:

Definition 1 Given g a continuous vector field from V to V^* (that will play the rôle of $-\nabla f$) we introduce S_{int} (int is for internal) as the set of solutions, $\hat{x} \in X$, of the variational inequality:

$$\langle g(\hat{x}) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X. \tag{2}$$

Remark Note that in an Hilbertian framework, the solutions of (2) are equivalently the solutions of:

$$\Pi_X(\hat{x} + g(\hat{x})) = \hat{x} \tag{3}$$

where Π_C denotes the projection operator on a closed convex set C; or the solutions of:

$$\Pi_{TX(\hat{x})}(g(\hat{x})) = 0 \tag{4}$$

where $TC(x)$ is the tangent cone to a closed convex set C at $x \in C$.

Recall also that:

$$\Pi_{TX(x)}(y) = \lim_{h \to 0} \frac{\Pi_X(x + hy) - x}{h}.$$

2.3 Product Case

The product case, which is the natural framework for games, is as follows:
V and V^* are product sets: $V = \prod_i V^i$, $V^* = \prod_i V^{i*}$, $i \in I$ (finite).
For each $i \in I$, the vector field g^i maps the product $X = \prod_i X^i$ to the dual space V^{i*} and one has:

$$\langle g(x)|y\rangle = \sum_i \langle g^i(x)|y^i\rangle$$

so that $\hat{x} \in S_{\text{int}}$ if and only if:

$$\langle g^i(\hat{x})|\hat{x}^i - y^i\rangle \geq 0, \qquad \forall y^i \in X^i, \forall i \in I, \tag{5}$$

which corresponds to the representation in games, see below.

2.4 Games, Equilibria and Variational Inequalities

We will consider games where equilibria are solutions of variational inequalities.
Three basic classes where this is the case are as follows, see e.g. Sorin and Wang [63]:

(A) *Finite games*

I is the finite set of players.
A^i is the finite set of actions of player i and $X^i = \Delta(A^i)$ is the simplex of mixed strategies.
Player i's payoff G^i is a map from $A = \prod_{j \in I} A^j \to \mathbb{R}$, extended by multilinearity to $X = \prod_{j \in I} X^j$.
VG^i denotes the associated *vector payoff function* from X^{-i} to $\mathbb{R}^{A^i}$, $VG^{i,p}(x^{-i}) = G^i(p, x^{-i})$, for all $p \in A^i$, $i \in I$, so that $G^i(x) = \langle x^i, VG^i(x^{-i})\rangle$.

An *equilibrium*, Nash [41], is given by:

$$G^i(x) \geq G^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \ \forall i \in I, \tag{6}$$

thus is a solution of :

$$\langle VG(x), x - y\rangle = \sum_{i \in I} \langle VG^i(x^{-i}), x^i - y^i\rangle \geq 0, \quad \forall y \in X. \tag{7}$$

This corresponds to the set S_{int} for the vector field $g(x) = (g^i(x) = VG^i(x^{-i}), i \in I)$.

(B) *Concave* $\mathcal{C}^1$ *games*

I is the finite set of players with action sets $\{X^i, i \in I\}$ and payoff functions $\{H^i, i \in I\}$.

Assume that each X^i is convex compact and that each $H^i : X = \prod_{j \in I} X^j \to \mathbb{R}$ is of class $\mathcal{C}^1$ and concave with respect to x^i.
An *equilibrium* is as above a profile $x \in X$ satisfying:

$$H^i(x) \geq H^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \ \forall i \in I, \tag{8}$$

which under our hypotheses is equivalent to:

$$\langle \nabla^i H^i(x), \ x^i - y^i\rangle \geq 0, \quad \forall y^i \in X^i, \ \forall i \in I, \tag{9}$$

where ∇^i is the gradient w.r.t. x^i.

In this framework the vector field is given by : $g(x) = \{g^i(x) = \nabla^i H^i(x), i \in I\}$.

(C) *Population games*

Consider a non atomic population with a finite set T of types. A configuration is a vector $x \in X = \Delta(T)$ specifying the proportion of each type in the population.

The payoff is defined by a continuous function ϕ from $T \times X \to \mathbb{R}$ where $\phi(p, x)$ is the outcome of a member of the population, being of type $p \in T$, given the configuration $x \in X$.

A *Wardrop equilibrium*, Wardrop [68], is a profile $x \in X$ satisfying:

$$x^p > 0 \Rightarrow \phi(p, x) \geq \phi(q, x), \quad \forall p, q \in T, \tag{10}$$

meaning that if p is used by a positive fraction of the population, it is a best choice at x.

An equivalent characterization of (10) is through the solutions of the variational inequality:

$$\sum_{p \in T} \phi(p, x)(x^p - y^p) = \langle \phi(., x), x - y \rangle \geq 0, \quad \forall y \in X,$$

so that the corresponding vector field is $g(x) = \phi(., x)$.

The extension to a finite set I of populations, each of which having a finite set of types T^i is standard.

We denote by $\Gamma(g)$ a game where the equilibrium set is defined through the vector field g.

Note that this representation of equilibria via variational inequalities is usual in transportation and congestion models, e.g. Dafermos [17], Dupuis and Nagurney [18], Smith [58] and in evolutionary game theory, e.g. Sandholm [54].

Recall that the minimization of a $\mathcal{C}^1$ convex function f on X corresponds to a variational inequality with $g = -\nabla f$. This implies two properties:

(i) g is dissipative,
(ii) g is a gradient.

These properties define two classes of vector fields that we consider now.

2.5 Dissipative Case

Let us first consider an alternative variational inequality associated to a vector field g.

Definition 2 Given a continuous vector field g, introduce the set S_{ext} (*ext* is for external) of solutions $\hat{x} \in X$ of

$$\langle g(y) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X. \tag{11}$$

Observe that S_{ext} is convex but can be empty. If g is continuous then $S_{\text{ext}} \subset S_{\text{int}}$ and if g is *dissipative* ($-g$ is monotone) in the sense that:

$$\langle g(y) - g(x) | x - y \rangle \geq 0, \qquad \forall y \in X,$$

then $S_{\text{int}} \subset S_{\text{ext}}$, see Kinderlehrer and Stampacchia [29], Facchinei and Pang [19].

In particular if g is continuous and dissipative (like $-\nabla f$), $S_{\text{int}} = S_{\text{ext}}$ and we will write simply S.

A game $\Gamma(g)$ is dissipative if g is dissipative.

This notion is related to the monotonicity requirement in Rosen [51]. The terminology is “stable” in Hofbauer and Sandholm [25] and “contractive” in Sandholm [55].

Fundamental example

If $F : X = X^1 \times X^2 \to \mathbb{R}$ is $\mathcal{C}^1$ and concave/convex, the vector field $g = (\nabla^1 F, -\nabla^2 F)$ is dissipative, Rockafellar [50]. The elements of $S_{ext} = S_{int} = S$ are the optimal strategies for the corresponding 0-sum game.

The proof of the non emptiness of S_{int} is equivalent to the fixed point theorem. On the other hand, if g is dissipative, the proof that S_{ext} is non-empty follows from the min-max theorem, Minty [37], or equivalently from a separation argument (Hahn-Banach), see Appendix.

2.6 Potential

We now consider the gradient property.

2.6.1 Potential Fields and Games

We first define a potential for a vector field, see e.g. Sorin and Wang [63].

Definition 3 A real function W of class $\mathcal{C}^1$ on $X = \prod_{i \in I} X^i$ is a *potential* for g if there exist strictly positive functions μ^i on X, $i \in I$, such that:

$$\left\langle \nabla^i W(x) - \mu^i(x) g^i(x), y^i - x^i \right\rangle = 0, \quad \forall x \in X, \forall y^i \in X^i, \forall i \in I. \tag{12}$$

A simple requirement would be $g^i = \nabla^i W$, $\forall i \in I$, (g is a gradient), but it is enough to have positive proportionality and this only on the tangent space.

The game $\Gamma(g)$ corresponding to such g is a *potential game*. Alternative previous definitions include: Monderer and Shapley [38] for finite games, Sandholm [53] for population games.

The following result is classical, see e.g. Sandholm [54].

Proposition 1 *Let $\Gamma(g)$ be a game with potential W.*

1. *Every local maximum of W is an equilibrium of $\Gamma(g)$.*
2. *If W is concave on X, then any equilibrium of $\Gamma(g)$ is a global maximum of W on X.*

Proof Since a local maximum x of W on the convex set X satisfies:

$$\langle \nabla W(x), x - y \rangle \geq 0, \quad \forall y \in X, \tag{13}$$

it follows from (12) that $\langle \mu^i(x) g^i(x), x^i - y^i \rangle \geq 0$ for all $i \in I$ and all $y \in X$.

On the other hand, if W is concave on X, a solution x of (13) is a global maximum of W on X. □

2.6.2 Positive Correlation

Given a first order dynamics x_t, f decreases on trajectories if:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x_t) = \langle \nabla f(x_t) | \dot{x}_t \rangle \leq 0.$$

The analogous property for a vector field g is:

$$\langle g(x_t) | \dot{x}_t \rangle \geq 0.$$

In the framework of games, a similar condition was described in discrete time as Myopic Adjustment Dynamics (Swinkels [64]) and writes as follows : if $x^i_{n+1} \neq x^i_n$ then

$H^i(x_{n+1}^i, x_n^{-i}) > H^i(x_n^i, x_n^{-i})$. Thus assuming that the other players do not change their move, player i modifies her action only to increase strictly her payoff.

The corresponding property in continuous time corresponds to *positive correlation* (Sandholm [54]) defined next.

Definition 4 Given a vector field g, a dynamics satisfies positive correlation if:

$$\dot{x}_t^i \neq 0 \Longrightarrow \langle g^i(x_t), \dot{x}_t^i \rangle > 0. \tag{14}$$

The use of this notion in potential games is as follows:

Proposition 2 *Consider a game $\Gamma(g)$ with potential function W.*

If the dynamics satisfies positive correlation, then W is a strict Lyapunov function.

All ω-limit points are rest points.

Proof Let $V_t = W(x_t)$ for $t \geq 0$. Then:

$$\dot{V}_t = \langle \nabla W(x_t) | \dot{x}_t \rangle = \sum_{i \in I} \langle \nabla^i W(x_t) | \dot{x}_t^i \rangle = \sum_{i \in I} \mu^i(x) \langle g^i(x_t) | \dot{x}_t^i \rangle \geq 0.$$

Moreover, $\langle g^i(x_t) | \dot{x}_t^i \rangle = 0$ holds for all i if and only if $\dot{x}_t = 0$.

One concludes by using Lyapunov's theorem (e.g. [26, Theorem 2.6.1]). □

This result is proved by Sandholm [53] for his version of a potential population game, see extensions in Benaim, Hofbauer and Sorin [8]. A similar property for the fictitious play process in discrete time is established in Monderer and Shapley [40].

We will show that this "positive correlation" property holds for all the dynamics considered in this paper.

2.7 Comparison: Optimization/Games

1. To be eligible in a game framework, an algorithm for a vector field g on a product space $X = \prod_i X^i$ has to be decentralized: explicitly, the (first order) dynamics for the component $x^i \in X^i$ is only a function of the values of $g(x)$ on V^{i*}, namely $g^i(x)$, i.e. it is uncoupled in the sense of Hart and Mas-Colell [24], hence of the form:

$$\dot{x}_t^i = T(x_t^i, g^i(x_t)), \forall i \in I.$$

This is the way the impact of the actions of the other players on player i's payoff is modelled and this also corresponds to her information.

One could consider two extensions:

- one with less information which corresponds to the "bandit framework" in the one person case, where statistical tools are used to handle the information,
- the other where more information leads to "coordination" and various extended notions of equilibria (sunspot or correlated equilibria, common noise for mean field games).

Notice that a similar requirement (uncoupled) in discrete time makes the use of prox-like procedures problematic.

2. In optimization one considers both criteria:

- convergence of $f(x_t)$ to $\min_X f$, and
- convergence of the trajectory $\{x_t\}$ to $S_{\text{int}} = S_{\text{ext}} = S = \text{argmin}_X f$ with two arguments:

  (i) the distance $d(x_t, S)$ goes to 0, or

(ii) x_t converges to a point $x^* \in S$.

In games one considers usually only trajectories.

An alternative quantitative criteria could be defined, in the spirit of the Nikaido function [45] as follows:

$$E_g(x) = \sup_{y \in X} \sum_i \langle g^i(x) | y^i - x^i \rangle$$

hence $S_{\text{int}} = \{x \in X : E_g(x) = 0\}$.

For zero-sum games this evaluation is related to the duality gap.

We now start the presentation and analysis of the dynamics.

3 No-Regret Dynamics

The next three dynamics satisfy the *no-regret criteria* defined, in general, as follows.

We associate to a process $\{u_t \in V^*, t \geq 0\}$, a procedure $\{x_t \in X, t \geq 0\}$, where x_t is a function of the past $\{(x_s, u_s), 0 \leq s < t\}$. The adequation of $\{x_t\}$ to $\{u_t\}$ is measured by a *regret function* defined by:

$$R_t(x) = \int_0^t \langle u_s | x - x_s \rangle \mathrm{d}s, \qquad t \geq 0, x \in X \tag{15}$$

and one will deal with procedures satisfying:

$$\sup_{x \in X} R_t(x) \leq o(t) \tag{16}$$

meaning that the *time average regret* vanishes asymptotically.

The algorithm is defined for a general bounded process $\{u_t\} \in V^*$ and we study here its performance for two closed forms (where the process $\{u_t\}$ is actually produced by the procedure $\{x_t\}$ itself):

(I) equilibria or variational inequalities where $u_t = g(x_t)$ for a continuous vector field $g : X \to V^*$,

(II) convex optimization where $u_t = -\nabla f(x_t)$, for a convex, $\mathcal{C}^1$ function f on X.

3.1 Basic properties

We establish here properties under the only assumption that the procedure satisfies the no-regret criteria (16).

3.1.1 Class (I): General Vector Field

The first result identifies the set of possible limit points if the trajectory converges.

Lemma 1 *If g is continuous and x_s converges to x, then $x \in S_{\text{int}}$.*

Proof

$$\frac{R_t(y)}{t} = \frac{1}{t} \int_0^t \langle g(x_s) | y - x_s \rangle \mathrm{d}s \to \langle g(x) | y - x \rangle, \qquad \forall y \in X, \tag{17}$$

and the quantity on the left vanishes, hence $x \in S_{\text{int}}$. □

In particular if x is a stationary point for the dynamics, then $x \in S_{\text{int}}$.

Consider now the time average process defined as follows:

$$\bar{x}_t = \frac{1}{t}\int_0^t x_s \mathrm{d}s.$$

The next result characterizes the accumulation points of this process under a condition on g.

Lemma 2 *If g is dissipative, the accumulation points of $\{\bar{x}_t\}$ are in S_{ext}.*

Proof

$$\frac{R_t(y)}{t} = \frac{1}{t}\int_0^t \langle g(x_s)|y - x_s\rangle \mathrm{d}s \geq \frac{1}{t}\int_0^t \langle g(y)|y - x_s\rangle \mathrm{d}s = \langle g(y)|y - \bar{x}_t\rangle.$$

Again the quantity on the left vanishes, hence any accumulation point x^* of $\{\bar{x}_t\}$ satisfies $\langle g(y)|y - x^*\rangle \leq 0, \forall y \in X$. □

Note that this shows that the existence of no-regret dynamics implies that S_{ext} is non empty for a dissipative vector field g. In particular this corresponds to dynamical proofs of the minmax theorem.

3.1.2 Class (II): Convex Optimization

We use the basic convexity property:

$$\langle \nabla f(x_t)|y - x_t\rangle \leq f(y) - f(x_t)$$

to get with $u_t = -\nabla f(x_t)$ in (15):

$$\int_0^t (f(x_s) - f(y))\mathrm{d}s \leq \int_0^t \langle -\nabla f(x_s)|y - x_s\rangle \mathrm{d}s = R_t(y)$$

which implies by Jensen's inequality:

$$f(\bar{x}_t) - f(y) \leq \frac{1}{t}\int_0^t [f(x_s) - f(y)]\mathrm{d}s \leq \frac{R_t(y)}{t}. \tag{18}$$

In particular one obtains:

Lemma 3 (i) *The accumulation points of $\{\bar{x}_t\}$ belong to $S = argmin_X f$.*
(ii) *If $f(x_t)$ is decreasing, the accumulation points of $\{x_t\}$ belong to $S = argmin_X f$.*

3.1.3 Level Functions

We introduce here a basic tool to check the no-regret property (16).

Definition 5 $P : \mathbb{R}^+ \times X \to \mathbb{R}^+$ is a *level function* for the dynamics (u_t, x_t) if it satisfies:

$$\langle u_t, x_t - y\rangle \geq \frac{\mathrm{d}}{\mathrm{d}t}P(t; y). \tag{19}$$

The existence of a level function allows to control the regret as follows:

Lemma 4 *If there exists a level function, $R_t(y)/t$ converges to 0 at a rate $1/t$.*
In particular the "no-regret" property holds.

Proof Integrating (19) gives:

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle \mathrm{d}s \leq P(0; y) - P(t; y) \leq P(0; y).$$

□

Moreover in this framework, the use of points in S_{ext} allow to obtain Lyapounov monotonicity, in the following sense:

Lemma 5 *Assume $\hat{y} \in S_{\text{ext}}$, then $t \mapsto P(t; \hat{y})$ is decreasing.*

Proof

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t; \hat{y})) \leq \langle g(x_t), x_t - \hat{y}) \rangle \leq 0.$$

□

3.2 Projection Dynamics: Euclidean Framework

This dynamics is defined in the following set-up: V is an Euclidean space with scalar product $\langle , \rangle$.

3.2.1 Dynamics

Recall the *projected gradient descent*, Polyak [48], defined in discrete time by:

$$x_{m+1} = \text{argmax}_X \left[\langle u_m, x \rangle - \frac{1}{2\eta_m} \| x - x_m \|^2 \right] \tag{20}$$

with $u_m = -\nabla f(x_m)$ and decreasing step size η_m. The objective function is the linearization of f and the penalization is the squared distance, both at x_m.

Alternatively:

$$\begin{aligned} x_{m+1} &= \text{argmin}_X \left[\langle -u_m, x \rangle + \frac{1}{2\eta_m} \| x - x_m \|^2 \right] \\ &= \text{argmin}_X \| x - (x_m + \eta_m u_m) \|^2 \end{aligned} \tag{21}$$

which corresponds to the Euler algorithm (recall that Π_X is the projection):

$$x_{m+1} = \Pi_X [x_m + \eta_m u_m] \tag{22}$$

thus with variational characterization:

$$\langle x_m + \eta_m u_m - x_{m+1}, y - x_{m+1} \rangle \leq 0, \quad \forall y \in X. \tag{23}$$

The continuous time analog is given by:

$$\boxed{\langle u_t - \dot{x}_t, y - x_t \rangle \leq 0, \quad \forall y \in X} \tag{24}$$

which is also:

$$\dot{x}_t = \Pi_{TX(x_t)}(u_t) \tag{25}$$

since $TX(x_t)$ is a cone.

When $u_t = g(x_t)$ this corresponds to the *local projection dynamics*, Dupuis and Nagurney [18], Lahkar and Sandholm [31].

Note that the decomposition property writes:

$$\dot{x}_t^i = \Pi_{TX^i(x_t^i)}(g^i(x_t)), \quad \forall i \in I.$$

3.2.2 Level Function

A first series of properties of the dynamics follows from the existence of a level function. In fact let:

$$V(t; y) = \frac{1}{2}\|x_t - y\|^2, \quad y \in X. \tag{26}$$

Proposition 3 *V is a level function.*

Proof One has:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t; y) = \langle \dot{x}_t, x_t - y\rangle \le \langle u_t, x_t - y\rangle$$

by (24). □

Hence the results of 3.1.3. apply and the properties of 3.1.1. and 3.1.2 hold. In addition by Lemma 5, the points in S_{ext} are Lyapounov stable.

3.2.3 Trajectories

Compared to 3.1.3 one has the following stronger convergence result for the trajectory:

Proposition 4 *Assume g dissipative.*

Then $\{\bar{x}_t\}$ converges to a point in S_{ext}.

Proof The limit points of $\{\bar{x}_t\}$ are in S_{ext} by Lemma 2.

$\|x_t - \hat{y}\|$ converges when $\hat{y} \in S_{\text{ext}}$ by Lemma 5 and Proposition 3.

Hence by Opial's lemma [46] which states: "In an Hilbert space, if $\|x_t - y\|$ converges for any y weak accumulation point of $\{x_t\}$ (resp.$\{\bar{x}_t\}$), then x_t (resp. $\bar{x}_t$) weakly converges", $\bar{x}_t$ converges to a point in S_{ext}. □

Lemma 6 *Positive correlation holds.*

Proof One has:

$$\langle g(x_t), \dot{x}_t\rangle = \|\dot{x}_t\|^2$$

since $\langle u_t - \dot{x}_t, \dot{x}_t\rangle = 0$ by (25) and Moreau's decomposition, Moreau [40]. □

Consider now class (II), convex optimization.

Lemma 6 and Proposition 2 imply:

Lemma 7 *$f(x_t)$ is decreasing.*

Lemma 8 (i) *$\{x_t\}$ converges to a point in S.*
(ii) *$f(x_t)$ decreases to* min f *with speed $O(1/t)$.*

Proof (i) Lemmas 3 and 7 imply that the accumulation points of x_t are in S. Then using Lemma 5, Opial's lemma applies.
(ii) Follows from Lemma 4.

□

3.2.4 Hilbert Case

All the previous results extend to the case where V is a Hilbert space, X is convex and closed in V, if one assumes either $S_{\text{ext}} \neq \emptyset$ or X is bounded, and the convergence is weak.

3.3 Mirror Descent: Differential/Incremental Approach

The algorithm uses a regularization function H which is strictly convex, $\mathcal{C}^1$ and $X \subset dom\ H$.

The Bregman distance associated to H is:

$$D_H(x, y) = H(x) - H(y) - \langle \nabla H(y) | x - y \rangle \ (\geq 0). \tag{27}$$

3.3.1 Dynamics

The discrete version corresponds to the *mirror descent algorithm*, Nemirovski and Yudin [43], Beck and Teboulle [7] defined in convex optimization by:

$$x_{m+1} = \operatorname{argmax}_X \{\langle u_m | x \rangle - (1/\eta_m) D_H(x, x_m)\} \tag{28}$$

where $u_m = -\nabla f(x_m)$ and η_m is the step size, which gives the first order condition:

$$\langle \nabla H(x_m) + \eta_m u_m - \nabla H(x_{m+1}) | x - x_{m+1} \rangle \leq 0, \forall x \in X. \tag{29}$$

This is, like (23), an incremental property.

The continuous time procedure satisfies: $x_t \in X$ and:

$$\boxed{\langle u_t - \frac{\mathrm{d}}{\mathrm{d}t} \nabla H(x_t) | x - x_t \rangle \leq 0, \quad \forall x \in X} \tag{30}$$

which is, like (24), a differential characterization, with $u_t = g(x_t)$.

The previous analysis of Sect. 3.2 corresponds to the Euclidean case with regularization function:

$$H(x) = \frac{1}{2} \|x\|^2.$$

3.3.2 Level Function

The regularization function allows to construct a level function as follows:

Proposition 5 *$P(t; y) = D_H(y, x_t)$ is a level function.*

Proof Note the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t} D_H(y, x_t) = -\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \nabla H(x_t) | y - x_t \right\rangle \tag{31}$$

so that (30) implies:

$$\frac{\mathrm{d}}{\mathrm{d}t} D_H(y, x_t) \leq \langle u_t | x_t - y \rangle. \tag{32}$$

□

3.3.3 Interior Trajectory

The use of a specific function H adapted to X, with $\|\nabla H(x)\| \to +\infty$ as $x \to \partial X$ allows to produce a trajectory that remains in $intX$ and has been much analyzed, Attouch and Teboulle [4], Bolte and Teboulle [12].

In this case (30) leads to an equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla H(x_t) = u_t \tag{33}$$

thus:

$$\nabla H(x_t) = \int_0^t u_s \mathrm{d}s \tag{34}$$

and then:

$$\dot{x}_t = \nabla^2 H(x_t)^{-1} u_t. \tag{35}$$

$\nabla^2 H(x)$ induces a Hessian Riemannian metric as analyzed in Alvarez, Bolte and Brahic [2] and in Mertikopoulos and Sandholm [35] for games.

Lemma 9 *Positive correlation holds.*

Proof One has:

$$\langle g(x_t)|\dot{x}_t\rangle = \langle g(x_t)|\nabla^2 H(x_t)^{-1} g(x_t)\rangle \geq 0.$$

□

To prove convergence in the convex optimization case, one uses the following properties:

$[H1]$ if $z^k \to y^* \in S$ then $D_H(y^*, z^k) \to 0$.

For example H L-smooth and then:

$$0 \leq D_H(x, y) \leq \frac{L}{2}\|x - y\|^2.$$

$[H2]$ if $D_H(y^*, z^k) \to 0$, $y^* \in S$ then $z^k \to y^*$.

For example H β-strongly convex and then:

$$D_H(x, y) \geq \frac{\beta}{2}\|x - y\|^2.$$

Proposition 6 *If H is smooth and strongly convex, $\{x_t\}$ converges to some $x^* \in S$.*

Proof Consider an accumulation point x^* of $\{x_t\}$. Then $x^* \in S$ by Proposition 2, Lemmas 3 and 9. Thus $D_H(x^*, x_t)$ is decreasing by Lemma 5. Since this sequence is decreasing to 0 on a subsequence $x_{t_k} \to x^*$ by $[H1]$, it is decreasing to 0, hence by $[H2]$ $x_t \to x^*$. □

3.4 Dual Averaging: Integral/Cumulative Approach

A third alternative algorithm uses again a regularization function h with the following assumptions: h is a bounded strictly convex s.c.i. function with $dom\, h = X$.

3.4.1 Dynamics

The dynamics corresponds to the continuous time version of *dual averaging*, introduced in optimization by Nesterov [44].

We follow the analysis in Kwon and Mertikopoulos [30] .

Notice that the approach is cumulative and relies on the quantity:

$$U_t = \int_0^t u_s \mathrm{d}s$$

where again $u_s = g(x_s)$.

Then x_t is the argmax (on V or X) of:

$$\langle U_t | x \rangle - h(x).$$

Let $h^*(w) = \sup_{x \in V} \langle w|x \rangle - h(x)$ be the Fenchel conjugate of h. h being strictly convex, h^* is differentiable, Rockafellar [49].

The dynamics can be written as:

$$\boxed{x_t = \nabla h^*(U_t) \in X.} \tag{36}$$

Note the integral formulation, compared to (24) and (30).

3.4.2 Level Function

Here again we can exhibit a level function, constructed through the regularization function. Note that it is defined via the dual space and using the cumulative process U_t.

Introduce, for $y \in X$:

$$W(t; y) = h^*(U_t) - \langle U_t | y \rangle + h(y). \tag{37}$$

Proposition 7 *$W(t; y)$ is a level function.*

Proof $W(t; y) \geq 0$ by the Fenchel inequality.

Note that:

$$\frac{\mathrm{d}}{\mathrm{d}t} h^*(U_t) = \langle u_t | \nabla h^*(U_t) \rangle = \langle u_t | x_t \rangle \tag{38}$$

by (36) thus:

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t; y) = \langle u_t | x_t - y \rangle.$$

□

Similarly one has:

Lemma 10 *Positive correlation holds.*

Proof

$$\langle g(x_t) | \dot{x}_t \rangle = \langle g(x_t) | \nabla^2 h^*(U_t)(u_t) \rangle$$

with $u_t = g(x_t)$. □

This, again by Proposition 2, implies that in convex optimization $f(x_t)$ is decreasing. Thus, using Lemma 3, the accumulation points of x_t are in S.

3.4.3 Remark

In the interior smooth case, both level functions of Sects. 3.3 and 3.4 are the same, since:

$$x_t = \nabla h^*(U_t), \qquad \nabla h(x_t) = U_t, \qquad h^*(U_t) + h(x_t) = \langle U_t | x_t \rangle,$$

so that:

$$\begin{aligned} D_h(y, x_t) &= h(y) - h(x_t) - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | x_t \rangle - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | y \rangle \\ &= W(t; y) \end{aligned}$$

For more properties see Mertikopoulos and Sandholm [34], Mertikopoulos and Zhou [36].

3.4.4 Comparison with Dynamic in Games

We describe here the connection with the *replicator dynamics*, Taylor and Jonker [65].

Given a finite set A and a $A \times A$ matrix M the replicator dynamics is defined on $X = \Delta(A)$ by:

$$\dot{x}_t^s = x_t^s (e^s M x_t - x_t M x_t) \tag{39}$$

where e^s is the s-unit vector, $s \in A$.

Consider now the *entropy function* h:

$$h(x) = \sum_{p \in A} x^p Log(x^p)$$

as a regularization function and introduce the *Logit function* L defined on $\mathbb{R}^A$ by:

$$L(V) = \text{argmax}_X (\langle V, x \rangle - h(x))$$

which takes the form:

$$L(V)^s = \frac{\exp(V^s)}{\sum_{p \in S} \exp(V^p)}, \quad s \in A.$$

The main property is that:

$$x_t^s = L\left(\int_0^t e^s M x_u \, du \right)$$

satisfies (39) on int X, see Rustichini [52], Hofbauer et al. [28].

More generally for a continuous vector field g defined on X, a finite product of simplex $X^i = \Delta(A^i), i \in I$, with value in $\mathbb{R}^A$, ($A = \prod A^i$), the replicator dynamics takes the differential form (see (30)):

$$\dot{x}_t^{ip} = x_t^{ip} [g_t^{ip}(x_t) - \langle x^i, g^i(x) \rangle], \quad p \in A^i, i \in I,$$

or the integral form (see (36)):

$$x_t^i = L\left(\int_0^t g^i(x_u) \, du \right), \quad i \in I.$$

The corresponding Riemannian metric is introduced in Shahshahani [57] and studied in Akin [1].

More properties are described in Hofbauer and Sigmund [26], Sorin [64–67].

Recall (Sect. 3.2) that the regularization $H(x) = \frac{1}{2}\|x^2\|$ leads to the *local/direct projection dynamics*, for a comparison with the replicator dynamics, see Sandholm, Dokumaci and Lahkar [56].

The next two dynamics (Sects. 4 and 5) are of the form "aiming dynamics", i.e.:

$$\dot{x} = y(x) - x \tag{40}$$

for some map (or more generally correspondence) $x \mapsto y(x)$ from X to itself. A general analysis in optimization is in Bolte and Teboulle [12].

4 Conditional Gradient and Global Projection

4.1 Definition and General Properties

The same dynamics appears as *conditional gradient* in convex optimization (Antipin [3], Bolte [11]) and as *global/target projection dynamics* in operations research and game theory (Friesz et al. [22], Tsakas and Voorneveld [66]).

The dynamics is given in an Euclidean space by (40) with $y(x) = \Pi_X(x + g(x))$, thus explicitly:

$$\boxed{\dot{x}^i = \Pi_{X^i}[x^i + g^i(x)] - x^i, \quad i \in I} \tag{41}$$

which comes from the discrete process with step size λ_n:

$$x^i_{n+1} - x^i_n = \lambda_n[\Pi_{X^i}[x^i_n + g^i(x_n)] - x^i_n].$$

Compare with (22). Rather than following the vector field on a small time interval and then projecting to X, one follows g during one time unit, then projects and this defines the 'aiming point" which is thus independent of the step size.

Obviously the rest points are still the set S_{int} (recall Sect. 2.2).

The variational expression of (41) is :

$$\langle g^i(x_t) - \dot{x}^i_t, z^i - (\dot{x}^i_t + x^i_t)\rangle \leq 0, \qquad \forall z^i \in X^i. \tag{42}$$

Lemma 11 *Positive correlation holds.*

Proof Use (42) for $z^i_t = x^i_t$:

$$\langle g^i(x_t) - \dot{x}^i_t, -\dot{x}^i_t)\rangle \leq 0$$

hence :

$$\langle g^i(x_t), \dot{x}^i_t)\rangle \geq \|\dot{x}^i_t\|^2. \tag{43}$$

□

4.2 Convex Optimization

Consider the case of a convex function f. We follow Antipin [3] and Bolte [11] to reach results similar to Sect. 3.

Proposition 8 (1) *$f(x_t)$ converges to the minimum of f with speed $\frac{1}{t}$,*
(2) *the trajectory $\{x_t\}$ converges to a point in S.*

Proof From convexity:

$$f(z) - f(x_t) \geq \langle \nabla f(x_t), z - x_t \rangle$$

one obtains:

$$f(x_t) - f(z) \leq \langle \nabla f(x_t), -\dot{x}_t \rangle + \langle \nabla f(x_t), \dot{x}_t + x_t - z \rangle$$

thus using (42) one deduces:

$$f(x_t) - f(z) \leq \langle \nabla f(x_t), -\dot{x}_t \rangle - \langle \dot{x}_t, \dot{x}_t + x_t - z \rangle$$

which implies:

$$\frac{d}{dt}\left[\frac{1}{2}\|x_t - z\|^2 + f(x_t)\right] \leq f(z) - f(x_t). \tag{44}$$

Integrating and using $f(x_t)$ decreasing (from Lemma 11) gives:

$$\frac{1}{2}\|x_t - z\|^2 + f(x_t) + t[f(x_t - f(z)] \leq \frac{1}{2}\|x_0 - z\|^2 + f(x_0)$$

hence the convergence of $f(x_t)$ to the minimum of f with speed $\frac{1}{t}$.

For $\hat{z} \in S$, (44) implies that $\frac{1}{2}\|x_t - \hat{z}\|^2 + f(x_t)$ is decreasing. Using that $f(x_t)$ is decreasing, $\|x_t - \hat{z}\|^2$ converges.

Let now z^* be an accumulation point of $\{x_t\}$. Thus $z^* \in S$ and $\|x_t - z^*\|$ converges, hence by Opial's lemma, $\{x_t\}$ converges. □

4.3 Vector Field

Let $M(x, y) = \frac{1}{2}\|(x + g(x)) - y)\|^2$, $L(x, y) = M(x, x) - M(x, y)$ so that $L(x, y) = \langle y - x, g(x)\rangle - \frac{1}{2}\|y - x\|^2$ and finally $H(x) = \sup_{y \in X} L(x, y)$, for $x, y \in X$.

Proposition 9 *Let g be a smooth dissipative vector field. Then H is a Lyapunov function for S.*

Proof Note that $H(x) = L(x, y(x)) \geq 0$. By definition of the projection Π_X, equality holds if and only if $x = y(x)$.

Using the Enveloppe theorem, one obtains:

$$\nabla H(x) = \nabla_x L(x, y(x)) = -g(x) + (y(x) - x) + (y(x) - x) J_g(x)$$

where J_g is the Jacobian of g so that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H(x_t) &= \langle \nabla H(x_t), \dot{x}_t \rangle \\ &= \langle -g(x_t) + (y(x_t) - x_t), y(x_t) - x_t \rangle + (y(x_t) - x_t) J_g(x_t)(y(x_t) - x_t) \\ &\leq 0.\end{aligned}$$

The first term is negative as a property of Π_X. The second term is negative because g is dissipative.

Thus H is a Lyapunov function.

Note that H is a strict Lyapunov function when g is strictly dissipative. □

This result is proved by Pappalardo and Passacantando [47] in the one-population game setting.

All the results in this section extend to the Hilbertian framework, convergence being understood as weak convergence.

5 Frank-Wolfe and Best Reply

5.1 Definition

Recall that the best reply dynamics is usually defined trough the best reply correspondence. In the framework of a strategic game with payoff function $H^i : X = \prod_j X^j \to \mathbb{R}$ for player i, the definition is :

$$BR^i(x) = \{y^i \in X^i; H^i(y^i, x^{-i}) \geq H^i(z^i, x^{-i}), \forall z^i \in X^i\}$$

Note that it is independent of x^i.

In our framework we will use the linearization of the payoff and the first order optimality condition, thus introduce:

$$br^i(x) = \{y^i \in X^i; \langle y^i - z^i, g^i(x)\rangle \geq 0, \forall z^i \in X^i\}$$

where g^i is the vector field used to define equilibria.

(Remark that in the case of a finite game - with multilinear extension - both definitions agree).

The *best reply dynamics* (Gilboa and Matsui [23]) is defined by the differential inclusion:

$$\dot{x}^i \in BR^i(x) - x^i, \quad i \in I.$$

We consider here the version:

$$\boxed{\dot{x}^i \in br^i(x) - x^i, \quad i \in I} \tag{45}$$

which can also be written as (40):

$$\dot{x} = y(x) - x$$

with $y(x) \in br(x)$.

Note that in the framework of a vector field $g = -\nabla f$ this corresponds precisely to the Frank-Wolfe algorithm [21]. In fact recall that the discrete time version is:

$$x_{n+1} - x_n = \lambda_n[y(x_n) - x_n]$$

where $y(x) \in argmin\{\langle \nabla f(x), z\rangle, z \in X\}$ and λ_n is the step size.
Notice again that the aiming point $y(x)$ which minimizes the evaluation $\langle \nabla f(x), z\rangle$ is independent of the step size. Thus the continuous time analog, corresponding to $\lambda_n = O(\frac{1}{n})$ is of the form:

$$\dot{x}_t = \frac{1}{t}[y(x_t) - x_t] \tag{46}$$

which is (40) up to a time change.

A first property is:

Lemma 12 *Positive correlation holds.*

Proof

$$\begin{aligned}\langle g(x_t), \dot{x}_t\rangle &= \langle g(x_t), y(x_t) - x_t\rangle \\ &\geq 0\end{aligned}$$

and the inequality is strict if $0 \notin br(x_t) - x_t$. □

5.2 Convex Optimization

The convergence result concerns the evaluation $f(x_t)$.

Proposition 10 *There is exponential convergence of $f(x_t)$ to its minimum.*

Proof Letting $\delta_t = f(x_t) - f(\hat{x})$ with $\hat{x} \in S$ one has $\delta_t \geq 0$ and:

$$\begin{aligned}\dot{\delta}_t &= \frac{\mathrm{d}}{\mathrm{d}t} f(x_t) \\ &= \langle \nabla f(x_t), y(x_t) - x_t\rangle \\ &\leq \langle \nabla f(x_t), \hat{x} - x_t\rangle \\ &\leq f(\hat{x}) - f(x_t)\end{aligned}$$

hence :

$$\dot{\delta}_t \leq -\delta_t$$

and $\delta_t \leq \delta_0 e^{-t}$, thus convergence of the order $O(\frac{1}{t})$ before the time change in (46). □

5.3 Vector Field

The analysis is similar to the one in Sect. 5.3.

Introduce $Q(x) = \sup_{y\in X} R(x, y)$ with $R(x, y) = \langle y - x, g(x)\rangle$, for $x, y \in X$.

Proposition 11 *Assume g dissipative and smooth. Then Q is a strict Lyapounov function for S.*

Proof Note that $Q(x) = R(x, y(x)) \geq 0$ with $y(x) \in br(x)$. By the Enveloppe theorem:

$$\nabla Q(x) = \nabla_x R(x, y(x)) = -g(x) + (y(x) - x) J_g(x).$$

Hence:

$$\begin{aligned}\langle \nabla Q(x_t), \dot{x}_t\rangle &= \langle -g(x_t) + (y(x_t) - x_t) J_g(x_t), y(x_t) - x_t\rangle \\ &= -Q(x_t) + [y(x_t) - x_t] J_g(x_t) [y(x_t) - x_t] \leq 0.\end{aligned}$$

The second term is negative because g is dissipative. Then equality to zero holds if and only if $Q(x) = 0$, hence $x \in S$. Therefore Q is a strict Lyapunov function. More precisely if $\alpha_t = Q(x_t)$, one has $\dot{\alpha}_t \leq \alpha_t$ and $\alpha_t \leq \alpha_0 e^{-t}$. □

This result appears for population games in Hofbauer and Sandholm [25]. One recovers the speed of convergence to 0 of the duality gap in the analysis of Fictitious Play for two person zero-sum games, Hofbauer and Sorin [27].

6 Final Comments

(A) There are strong analogies between several learning dynamics in both areas: convex optimization and equilibria in games.

Properties holding in convex optimization are obviously a test for games (one person case).

On the other hand properties for games are a proof of robustness for optimization (sensitivity analysis).

(B) Properties were obtained only for the average processes $\{\bar{x}_t\}$ in Sect. 3.

However recall that the Fictitious Play process (Brown [14]) defined on the moves in discrete time by :

$$Y^i_{n+1} \in BR^i(\overline{Y}_n), \quad i \in I$$

with $\overline{Y}_n = \frac{1}{n}\sum_{k=1}^n Y_k$, leads to:

$$\overline{Y}^i_{n+1} - \overline{Y}^i_n \in \frac{1}{n+1}[BR^i(\overline{Y}_n) - \overline{Y}^i_n]$$

so that x_t in the continuous time best reply dynamics (45) corresponds to the average $\overline{Y}_n$ in discrete time.

Explicit links for two person finite games between replicator dynamics and best reply dynamics are analyzed in Hofbauer, Sorin and Viossat [28].

(C) Among the natural extensions one would like to deal with games satisfying Nash's conditions [42] for equilibrium: each H^i is continuous in x and quasi-concave in x^i. Preliminary results in this directions are in Hofbauer and Sorin [27], Barron, Goebel and Jensen [6].

In the same spirit the link with subgradient dynamics and maximal monotone operators, Brézis [13], Bruck [16] should be studied.

(D) Also we did not mention analysis on learning with stochastic perturbation in the spirit of Foster and Young [20], see e.g. Avrachenkov and Borkar [5] for interesting connections.

(E) As a general comment one could summarize that positive results hold:

– for potential games when the dynamics satisfies the quite natural "positive correlation" condition. The framework looks then similar to (pseudo) gradient dynamics.
– for dissipative games: this covers zero-sum games and appears as an extension of the initial dynamics due to Brown and von Neumann [15] and based on a Lyapounov function. Then convergence occurs to a set defined via "elementary properties" in particular convex.

Recall that one cannot expect in general convergence of dynamics in games to equilibria, see Hart and Mas-Colell [24], even to correlated equilibria, Viossat [67]; for learning algorithms see also Mazumdar, Ratliff and Sastry [33].

(F) Further study of learning dynamics in games could follow alternative approaches like:

– introduce a new notion of "selected profiles" in a game, that would play the role of "equilibrium strategies", and study the associated classes of dynamics and games.
  Examples include: ESS and replicator dynamics, Maynard Smith [32], Hofbauer and Sigmund [26], or in the same spirit: Mertikopoulos and Zhou [36].
– alternatively define "natural" dynamics associated to a game and study the induced attractors.

In this direction, analysis of Internally Chain Transitive sets, Benaim, Hofbauer and Sorin [8–10] show that stable components may differ from subsets of rest points.

7 Appendix

Assume g continuous and dissipative and recall that X is convex and compact. Let us prove that S_{ext} is non-empty. Define :

$$S^y_{\text{ext}} = \{x \in X; \langle g(y)|x - y\rangle \geq 0\}$$

so that $S_{\text{ext}} = \cap_{y \in X} S^y_{\text{ext}}$. Hence by compactness (weak-compactness in an Hilbert framework) it is enough to establish the following:
Claim
For any finite collection $y_i \in X, i \in I$, there exists $x \in co\{y_i, i \in I\}$ such that:

$$\langle g(y_i)|x - y_i\rangle \geq 0, \qquad \forall i \in I. \tag{47}$$

Consider the finite two-person zero-sum game defined by the following $I \times I$ matrix A:

$$A_{ij} = \langle g(y_j)|y_i - y_j\rangle.$$

Introduce $B = \frac{1}{2}[A + {}^tA]$ and $C = \frac{1}{2}[A - {}^tA]$.

The crucial point is that B has non negative coefficients since:

$$B_{ij} = \langle g(y_j)|y_i - y_j\rangle + \langle g(y_i)|y_j - y_i\rangle = \langle g(y_j) - g(y_i)|y_i - y_j\rangle \geq 0.$$

Hence an optimal strategy $u \in \Delta(I)$ in the game C (which has value 0) gives $uA_j \geq 0, \forall j \in I$. Letting $x = \sum_i u_i y_i$ this writes as (47).

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