Regularity and stability in shape optimization under geometric constraint

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PhD Thesis Defense,
22nd June, 2023

SORBONNE UNIVERSITÉ

IMJ-PRG

Isoperimetric problem

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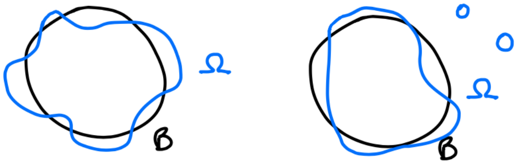

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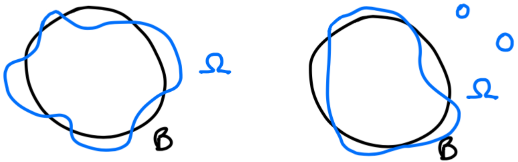

Balls minimize the perimeter at fixed volume: for any set Ω of volume 1

$$P(\Omega) \geq P(B)$$

where B is a ball of volume 1.

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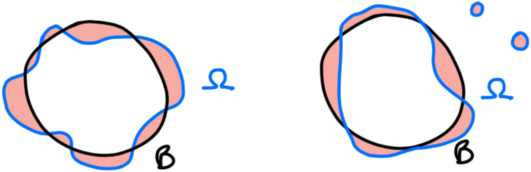

Outline of the talk

1. Stability of the ball in certain classes of shapes
   - Stability
   - Strategy

2. Stability for isoperimetric problems
   - Stability for $P - \lambda_1$
   - Stronger stability for $P - \lambda_1$

3. Stability for Faber-Krahn problems

- $\mathcal{S}_{\text{ad}}$: class of subsets of $\mathbb{R}^n$ of volume 1 ($n \geq 2$).
- $R : \mathcal{S}_{\text{ad}} \to \mathbb{R}$.
- $B \subset \mathbb{R}^n$ a ball of volume 1.
- $\lambda_1(\Omega)$: 1st Dirichlet eigenvalue of the open set $\Omega \subset \mathbb{R}^n$.

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Definition (Stability of the ball)

Let $J = P$ or $J = \lambda_1$. We say that B is stable for $J + R$ in $\mathcal{S}_{ad}$ provided that for sufficiently small ε

$$B \text{ is a minimizer of } J + \varepsilon R \text{ in } \mathcal{S}_{ad}.$$

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Example 2: Payne-Weinberger inequality.

$$\forall \Omega \subset \mathbb{R}^2 \text{ open and simply connected with } |\Omega| = 1, \text{ it holds (locally)}$$
$$P(\Omega) - P(B) \gtrsim \lambda_1(\Omega) - \lambda_1(B)$$

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Step 2: Stability in $\mathcal{S}_{\text{ad}}$: regularity theory inside $\mathcal{S}_{\text{ad}}$, *i.e.*

$$\text{any minimizer } \Omega \text{ of } \min\{(J + \varepsilon R)(\Omega),\ \Omega \in \mathcal{S}_{\text{ad}}\}$$

can be written $\Omega = B_h$ for some $h \in X$.

Stability under convexity constraint

We study $\mathcal{S}_{\text{ad}} = \{\Omega \subset \mathbb{R}^n \text{ open and convex}, |\Omega| = 1\}$.

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Theorem (P., '23)

The ball is locally stable for $P - \lambda_1$ in $\mathcal{K}_1^n$: for $\varepsilon \ll 1$

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The proof adapts to other energies ($\lambda_1 \rightsquigarrow$ capacity), which generalizes [Goldman, Novaga, Ruffini, '18] to dimension $n \geq 3$.

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Key idea: use an appropriate test function in

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$$P(B_h) - P(B) \gtrsim \|h\|_{H^1}^2$$

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$\Longrightarrow (P - \varepsilon\lambda_1)(B_h) \geq (P - \varepsilon\lambda_1)(B)$ for Lipschitz perturbations.

Step 2.

Any convex set is Lipschitz $\rightsquigarrow$ Step 1 is enough to conclude. $\square$

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Main result: strong stability for $P - \lambda_1$

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Theorem (P., '23)

There exists $c^* > 0$ s.t.

– For $c \in (0, c^*)$, the ball is a local minimizer of $P - c\lambda_1$ in $\mathcal{K}_1^n$:

$$\forall K \in \mathcal{K}_1^n \text{ with } |K \Delta B| \ll 1, (P - c\lambda_1)(K) \geq (P - c\lambda_1)(B)$$

with equality iff (up to translation) $K = B$.

– For $c \in (c^*, \infty)$, the ball is not a local minimizer of $P - c\lambda_1$ in $\mathcal{K}_1^n$.

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Definition (q.m.p.c.c.)

$K \in \mathcal{K}^n$ is a quasi-minimizer of the perimeter under convexity constraint if there exists $\Lambda, \eta > 0$ s.t.

$$\forall\left(\widetilde{K} \in \mathcal{K}^n,\ |K\Delta\widetilde{K}| \leq \eta\right),\ P(K) \leq P(\widetilde{K}) + \Lambda|K\Delta\widetilde{K}|$$

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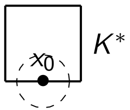
- 

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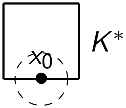
- 

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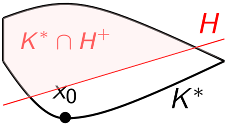
- Cutting procedure :



Use $K^* \cap H^+$ as a competitor.

Competitors:

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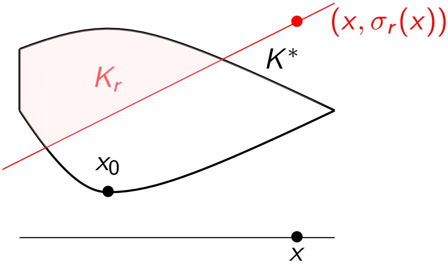

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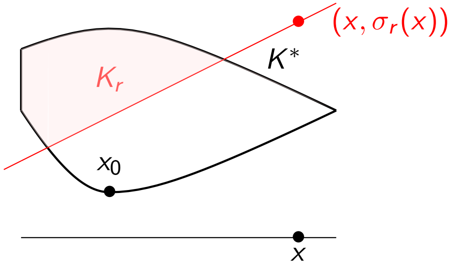

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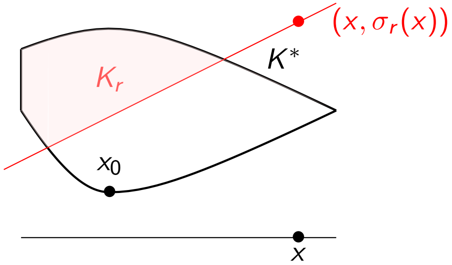

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$\rightsquigarrow$ Interpret $P(K^*) - P(K_r)$ and $|K \setminus K_r|$ in a calculus of var. framework.
□

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Deny minimality of B: by contradiction, there exists $(K_j) \in \mathcal{K}_1^n$ s.t.

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Replace K_j by $\tilde{K}_j$ in (1), where $\tilde{K}_j$ is a minimizer of $P - c\lambda_1$ (+ some vol. term involving $|K_j \Delta B|$, enforcing $|\tilde{K}_j \Delta B| \to 0$).

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$\overset{Reg.Theorem}{\Longrightarrow} \tilde{K}_j = B_{h_j}$ for some $h_j \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|h_j\|_{\mathcal{C}^{1,\alpha}} \to 0$ for any $\alpha \in (0, 1)$.

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$\rightsquigarrow$ Goal: prove stability for $X = \mathcal{C}^{1,\alpha}$ perturbations of the ball.

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Theorem 2 (P., '23)

There exists $\alpha \in (0,1)$ *s.t. for all* $h \in \mathcal{C}^{1,\alpha}(\partial B)$ *with* $\|h\|_{\mathcal{C}^{1,\alpha}} \ll 1$*:*

$$\lambda_1(B_h) = \lambda_1(B) + \lambda_1'(B) \cdot h + \frac{1}{2}\lambda_1''(B) \cdot (h,h) + \omega\left(\|h\|_{\mathcal{C}^{1,\alpha}}\right) \|h\|_{H^1}^2$$

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$\rightsquigarrow$ Concludes Step 1 hence the proof. □

1. Stability of the ball in certain classes of shapes
   - Stability
   - Strategy

2. Stability for isoperimetric problems
   - Stability for $P - \lambda_1$
   - Stronger stability for $P - \lambda_1$

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Several works with non-sharp exponents:
– $\gamma = \frac{1}{80n}$ in [Bertrand, Colbois, '06],
– $\gamma = \frac{1}{12}$ in [Mazzoleni, Pratelli, '19].

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$\rightsquigarrow \gamma = 1/2$ if $\lambda_k(B)$ is multiple, $\gamma = 1$ if $\lambda_k(B)$ is simple.

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Thank you for your attention !