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présentée par

**Quang-Duc DAO**

# Local-global principle for integral points on certain algebraic surfaces

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## Colophon

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**Mots clés :** points entiers, points rationnels, obstruction de brauer–manin, principe local-global (hasse), approximation forte, surfaces log k3, surfaces k3 (de wehler), surfaces cubiques de type markoff, surfaces k3 de type markoff, théorie de la réduction, descente.

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*Je dédie ce travail . . .*

*À mon directeur bien-aimé !*

*À mes parents et ma petite sœur*

*À ma femme et ma fille*

Science sans conscience n'est que ruine de l'âme.

François Rabelais

What a sad era when it is easier to smash an atom than a prejudice.

Albert Einstein

Whenever you find yourself on the side of the majority, it is time to pause and reflect.

Mark Twain

**Local-global principle for integral points on certain algebraic surfaces**

**Abstract**

In this thesis, we study the problem of existence and local approximation of integral points on certain algebraic surfaces defined over number fields, particularly the field of rational numbers. In the first chapter, we introduce the history of the problem and some recent progress in the subject of our study, especially the recent work of Ghosh–Sarnak, Loughran–Mitankin, and Colliot-Thélène–Wei–Xu. In Chapter 2, we study the Brauer–Manin obstruction for Markoff-type cubic surfaces. We first provide some background on character varieties and the natural origin of the Markoff-type cubic surfaces, then we explicitly calculate the Brauer group of the smooth compactifications and the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer group to prove the failure of strong approximation which can be explained by the Brauer–Manin obstruction in an infinite family of surfaces, and then give some counting results for the frequency of the obstructions. Furthermore, we apply the reduction theory, similar to that of Markoff surfaces, in recent work by Whang to give an explicit counterexample to the integral Hasse principle for our Markoff-type cubic surfaces. We also give some analogous results to those on Markoff surfaces about the Brauer–Manin obstruction in some special cases of Markoff-type cubic surfaces. In Chapter 3, we study the Brauer–Manin obstruction for Wehler K3 surfaces of Markoff type and follow the same structure as the previous chapter. We first provide some background on Wehler K3 surfaces and a recent study of Fuchs et al. on Markoff-type K3 (MK3) surfaces, as well as introduce the *three* explicit families of MK3 surfaces that interest us. Next, we explicitly calculate the algebraic Brauer group of the projective closures for one smooth family, and then the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer groups to prove the failure of the integral Hasse principle which can be explained by the Brauer–Manin obstruction for three families of MK3 surfaces, and then give some counting results for the Hasse failures. In addition, we study some cases when the Brauer–Manin obstruction to the existence of integral points and rational points can vanish, then give some counterexamples to strong approximation which can be explained by the Brauer–Manin obstruction. Furthermore, we provide some explicit examples which show that rational points do exist on affine MK3 surfaces. To complete the thesis, in Appendix A, we give a brief introduction to the descent obstructions associated with Artin–Schreier torsors and their relation to the Brauer–Manin obstruction for integral points on affine varieties over global function fields, as studied by Harari and Voloch. Finally, we study some counterexamples to the integral Hasse principle on conics and Markoff surfaces.

**Keywords:** integral points, rational points, brauer-manin obstruction, local-global (hasse) principle, strong approximation, log k3 surfaces, (wehler) k3 surfaces, markoff-type cubic surfaces, markoff-type k3 surfaces, reduction theory, descent.

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**Résumé**

Dans cette thèse, nous étudions le problème d'existence et d'approximation locale de points entiers sur certaines surfaces algébriques définies sur des corps de nombres, en particulier le corps des nombres rationnels. Dans le premier chapitre, nous introduisons l'historique du problème et quelques progrès récents dans le sujet de notre étude, en particulier les travaux récents de Ghosh–Sarnak, Loughran–Mitankin, et Colliot-Thélène– Wei–Xu. Dans le chapitre 2, nous étudions l'obstruction de Brauer–Manin pour les surfaces cubiques de type Markoff. Nous fournissons d'abord quelques informations sur les variétés de caractères et l'origine naturelle des surfaces cubiques de type Markoff, puis nous calculons explicitement le groupe de Brauer des compactifications lisses et le groupe de Brauer algébrique des surfaces affines. Ensuite, nous utilisons le groupe de Brauer pour prouver l'échec de l'approximation forte qui peut s'expliquer par l'obstruction de Brauer–Manin dans une famille infinie de surfaces, puis donnons des estimations asymptotiques pour la fréquence des obstructions. De plus, nous appliquons la théorie de la réduction, similaire à celle des surfaces de Markoff, dans les travaux récents de Whang pour donner un contre-exemple explicite au principe de Hasse entier pour nos surfaces cubiques de type Markoff. Nous donnons aussi des résultats analogues à ceux sur les surfaces de Markoff à propos de l'obstruction de Brauer–Manin dans quelques cas particuliers de surfaces cubiques de type Markoff. Dans le chapitre 3, nous étudions l'obstruction de Brauer–Manin pour les surfaces de Wehler K3 de type Markoff et suivons la même structure que le chapitre précédent. Nous fournissons d'abord quelques informations sur les surfaces K3 de Wehler et une étude récente de Fuchs et al. sur les surfaces K3 de type Markoff (MK3), ainsi que les *trois* familles explicites de surfaces MK3 qui nous intéressent. Puis, nous calculons explicitement le groupe de Brauer algébrique des clôtures projectives pour une famille lisse, puis le groupe de Brauer algébrique des surfaces affines. Ensuite, nous utilisons les groupes de Brauer pour prouver l'échec du principe de Hasse entier qui peut être expliqué par l'obstruction de Brauer–Manin pour trois familles de surfaces MK3, puis donnons quelques estimations asymptotiques pour les échecs de Hasse. Par ailleurs, nous étudions quelques cas où l'obstruction de Brauer–Manin à l'existence de points entiers et de points rationnels peut disparaître, puis donnons quelques contre-exemples à l'approximation forte qui peut s'expliquer par l'obstruction de Brauer–Manin. De plus, nous donnons quelques exemples explicites qui montrent que des points rationnels existent sur des surfaces MK3 affines. Pour compléter la thèse, dans l'annexe A, nous donnons une brève introduction aux obstructions de descente associées aux torseurs d'Artin–Schreier et à leur relation avec l'obstruction de Brauer–Manin pour les points entiers sur les variétés affines sur un corps de fonctions d'une courbe algébrique sur un corps fini, comme étudiées par Harari et Voloch. Enfin, nous étudions quelques contre-exemples au principe de Hasse entier sur des coniques et des surfaces de Markoff.

**Mots clés :** points entiers, points rationnels, obstruction de brauer–manin, principe local-global (hasse), approximation forte, surfaces log k3, surfaces k3 (de wehler), surfaces cubiques de type markoff, surfaces k3 de type markoff, théorie de la réduction, descente.

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# Contents

# Notation

Let $k$ be a field and $\overline{k}$ a separable closure of $k$. We let $G_k := \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group. A $k$-variety is a separated $k$-scheme of finite type. If $X$ is a $k$-variety, we write $\overline{X} = X \times_k \overline{k}$. Let $k[X] = \mathrm{H}^0(X, \mathcal{O}_X)$ and $\overline{k}[X] = \mathrm{H}^0(\overline{X}, \mathcal{O}_{\overline{X}})$. If $X$ is an integral $k$-variety, let $k(X)$ denote the function field of $X$. If $X$ is a geometrically integral $k$-variety, let $\overline{k}(X)$ denote the function field of $\overline{X}$.

Let $\mathrm{Pic}\, X = \mathrm{H}^1_{\mathrm{Zar}}(X, \mathbb{G}_m) = \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m)$ denote the Picard group of a scheme $X$. Let $\mathrm{Br}\, X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ denote the Brauer group of $X$. Let

$$\mathrm{Br}_1 X := \mathrm{Ker}[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}]$$

denote the **algebraic Brauer group** of a $k$-variety $X$ and let $\mathrm{Br}_0 X \subset \mathrm{Br}_1 X$ denote the image of $\mathrm{Br}\, k \to \mathrm{Br}\, X$. The image of $\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}$ is called the **transcendental Brauer group** of $X$.

Given a field $F$ of characteristic zero containing a primitive $n$-th root of unity $\zeta = \zeta_n$, we have $\mathrm{H}^2(F, \mu_n^{\otimes 2}) = \mathrm{H}^2(F, \mu_n) \otimes \mu_n$. The choice of $\zeta_n$ then defines an isomorphism $\mathrm{Br}(F)[n] = \mathrm{H}^2(F, \mu_n) \cong \mathrm{H}^2(F, \mu_n^{\otimes 2})$. Given two elements $f, g \in F^\times$, we have their classes $(f)$ and $(g)$ in $F^\times/F^{\times n} = \mathrm{H}^1(F, \mu_n)$. We denote by $(f, g)_\zeta \in \mathrm{Br}(F)[n] = \mathrm{H}^2(F, \mu_n)$ the class corresponding to the cup-product $(f) \cup (g) \in \mathrm{H}^2(F, \mu_n^{\otimes 2})$. Suppose $F/E$ is a finite Galois extension with Galois group $G$. Given $\sigma \in G$ and $f, g \in F^\times$, we have $\sigma((f,g)_{\zeta_n}) = (\sigma(f), \sigma(g))_{\sigma(\zeta_n)} \in \mathrm{Br}(F)$. In particular, if $\zeta_n \in E$, then $\sigma((f,g)_{\zeta_n}) = (\sigma(f), \sigma(g))_{\zeta_n}$. For all the details, see [GS17, Sections 4.6, 4.7].

Let $R$ be a discrete valuation ring with fraction field $F$ and residue field $\kappa$. Let $v$ denote the valuation $F^\times \to \mathbb{Z}$. Let $n > 1$ be an integer invertible in $R$. Assume that $F$ contains a primitive $n$-th root of unity $\zeta$. For $f, g \in F^\times$, we have the residue map

$$\partial_R : \mathrm{H}^2(F, \mu_n) \to \mathrm{H}^1(\kappa, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{H}^1(\kappa, \mu_n) = \kappa^\times/\kappa^{\times n},$$

where $\mathrm{H}^1(\kappa, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{H}^1(\kappa, \mu_n)$ is induced by the isomorphism $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$ sending 1 to $\zeta$. This map sends the class of $(f, g)_\zeta \in \mathrm{Br}(F)[n] = \mathrm{H}^2(F, \mu_n)$ to

$$(-1)^{v(f)v(g)} \mathrm{class}(g^{v(f)}/f^{v(g)}) \in \kappa/\kappa^{\times n}.$$

For a proof of these facts, see [GS17]. Here we recall some precise references. Residues in Galois cohomology with finite coefficients are defined in [GS17, Construction 6.8.5]. Comparison of residues in Milnor K-Theory and Galois cohomology is given in [GS17, Proposition 7.5.1]. The explicit formula for the residue in Milnor's group K2 of a discretely valued field is given in [GS17, Example 7.1.5].

# Chapter 1

# Introduction

The problem of the existence of rational and integral points on a variety has been a classic problem for a long time. From the famous Hilbert's tenth problem posed by D. Hilbert since 1990, we have seen that there can be no algorithm determining whether a given Diophantine equation has an integral solution or not. This question has been proven to have a negative answer by the combined work of M. Davies, H. Putnam, J. Robinson, Y. Matiyasevich and G. Chudnovsky. More precisely, there exists a polynomial $f(t; x_1, \ldots, x_n) \in \mathbb{Z}[T, X_1, \ldots, X_n]$ such that there is no algorithm telling us whether for any integer $t$ the equation $f(t; x_1, \ldots, x_n) = 0$ is solvable over the integers $\mathbb{Z}$ or not.

It is natural to extend the problem of solvability to a larger domain, such as the field of rational numbers $\mathbb{Q}$ and other number fields. More generally, if $k$ is a global field, one would like to know the answer to the question that whether a variety over $k$ has a $k$-point or not. The analogue of Hilbert's tenth problem is still open in this case. When $k$ is a global function field, it has been proved that no such algorithm exists by T. Pheidas, A. Shlapentokh, Carlos R. Videla and K. Eisenträger. On the other hand, when $k$ is a number field, it is still unknown whether such an algorithm exists; even for a simple equation such as $x^3 + y^3 = a$, nothing has been found to determine whether it has a solution over $\mathbb{Q}$ or not.

However, there are certain classes of varieties that an algorithm of solvability over $\mathbb{Q}$ can be found: they are the classes of projective varieties over $\mathbb{Q}$ which satisfy *the Hasse principle*. This principle is about the requirement that the solvability over the field of $p$-adic numbers $\mathbb{Q}_p$ and the solvability over the field of real numbers $\mathbb{R}$ imply the solvability over $\mathbb{Q}$. One can also consider its integral version, namely *the integral Hasse principle*, where $\mathbb{Q}_p$ is replaced by $\mathbb{Z}_p$ to study the solvability over the ring of $p$-adic integers and over $\mathbb{Z}$. In fact, this will be of our main interest in this thesis.

The first nontrivial example when the Hasse principle holds is the case of a variety defined by a quadratic equation: this is the content of the famous Hasse–Minkowski theorem. There are several other classes of varieties that satisfy the Hasse principle; see [Sko01] for some families of examples. However, there are *counterexamples to the Hasse principle*, i.e. equations which are solvable over $\mathbb{Q}_p$ for all primes $p$ and over $\mathbb{R}$, but not over $\mathbb{Q}$: they include rather simple equations such as the homogeneous cubic equations $3x^3 + 4y^3 + 5z^3 = 0$ (Selmer) and $5x^3 + 9y^3 + 10z^3 + 12t^3 = 0$ (Cassels and Guy), where one excludes the all-zero solution. Similarly, the integral Hasse principle does not hold in general; even the analogue of the Hasse–Minkowski theorem is not true in general for quadrics defined over $\mathbb{Z}$.

In the middle of the twentieth century, following Hasse–Minkowski, many mathematicians such as Mordell, Selmer, Châtelet, and others have studied other cases of the Hasse principle as well as similar local-global principles, and analyzed the cases when the principle fails. During the course of this research, they discovered significant concepts, especially the obstruction using the Brauer group of a variety, namely the Brauer–Manin obstruction to the Hasse principle. Manin was the mathematician who first found a general obstruction to the Hasse principle, as described in his first talk at the ICM in 1970. There is a statement that *the Brauer–Manin obstruction to the Hasse principle is the only one* as a good substitute to the Hasse principle when it does not hold, which means that as long as a collection of a real solution and $p$-adic solutions for all primes $p$ satisfies certain conditions then the given equation also has a solution in $\mathbb{Q}$. These conditions, provided by the Brauer–Grothendieck group of the variety, are based on the global reciprocity law. Nearly four decades later, in a foundational paper [CX09], Colliot-Thélène and Xu defined and studied the so-called *integral Brauer–Manin obstruction* to the existence of integral points on varieties. Some failures of the integral Hasse principle can be explained by the integral Brauer-Manin obstruction; see for example [CX09], [KT08] and [JS17].

The Brauer–Manin obstruction to the Hasse principle is known to be the only obstruction for many types of homogeneous spaces of linear algebraic groups (Sansuc, Borovoi), so this gives one possible generalization of the Hasse–Minkowski theorem for quadrics. Furthermore, by computer calculations along with some partial or conditional results, there is ample evidence that the Brauer–Manin obstruction to the Hasse principle is the only one for varieties defined by one cubic or two quadratic equations over $\mathbb{Q}$. Most of these results were obtained thanks to the works of Colliot-Thélène, Sansuc, and Swinnnerton-Dyer. However, they do not generalize to their integral models: we will now look into this further in the context of the thesis. Before that, let us emphasize that there are counterexamples to the Hasse principle which cannot be explained by the Brauer–Manin obstruction, although we do not study them in this thesis; see [Sko01] and [Poo10] for some explicit examples as well as more general obstructions using the theory of torsors, namely *the descent obstruction* to the Hasse principle.

Let $X$ be an affine variety over $\mathbb{Q}$, and $\mathcal{X}$ an integral model of $X$ over $\mathbb{Z}$, i.e. an affine scheme of finite type over $\mathbb{Z}$ whose generic fiber is isomorphic to $X$. Define the set of adelic points $X(\mathbf{A}_{\mathbb{Q}}) := \prod'_p X(\mathbb{Q}_p)$, where $p$ is a prime number or $p = \infty$ (with $\mathbb{Q}_\infty = \mathbb{R}$). Similarly, define $\mathcal{X}(\mathbf{A}_{\mathbb{Z}}) := \prod_p \mathcal{X}(\mathbb{Z}_p)$ (with $\mathbb{Z}_\infty = \mathbb{R}$). We say that $X$ *fails the Hasse principle* if

$$X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.$$

We say that $\mathcal{X}$ *fails the integral Hasse principle* if

$$\mathcal{X}(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset \quad \text{but} \quad \mathcal{X}(\mathbb{Z}) = \emptyset.$$

We say that $X$ *satisfies weak approximation* if the image of $X(\mathbb{Q})$ in $\prod_v X(\mathbb{Q}_v)$ is dense, where the product is taken over all places of $\mathbb{Q}$. And we say that $\mathcal{X}$ *satisfies strong approximation* if $\mathcal{X}(\mathbb{Z})$ is dense in $\mathcal{X}(\mathbf{A}_{\mathbb{Z}})_\bullet := \prod_p \mathcal{X}(\mathbb{Z}_p) \times \pi_0(X(\mathbb{R}))$, where $\pi_0(X(\mathbb{R}))$ denotes the set of connected components of $X(\mathbb{R})$. Note that we work with $\pi_0(X(\mathbb{R}))$ since $\mathcal{X}(\mathbb{Z})$ is never dense in $X(\mathbb{R})$ for topological reasons (see [Con12, Example 2.2]).

In general, the Hasse principle for varieties does not hold. In his 1970 ICM address [Man71], Manin introduced a natural cohomological obstruction to the Hasse principle, namely the **Brauer–Manin obstruction**, which has been extended to its integral version in [CX09]. If $\mathrm{Br}\,X$ denotes the cohomological Brauer group of $X$, i.e. $\mathrm{Br}\,X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, one has a natural pairing from class field theory:

$$X(\mathbf{A}_{\mathbb{Q}}) \times \mathrm{Br}\,X \to \mathbb{Q}/\mathbb{Z}.$$

Let $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$ denote the left kernel of this pairing, then the exact sequence of Albert–Brauer–Hasse–Noether implies the relation:

$$X(\mathbb{Q}) \subseteq X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \subseteq X(\mathbf{A}_{\mathbb{Q}}).$$

We say that *the Brauer–Manin obstruction to the Hasse principle is the only one* if

$$X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset \iff \mathcal{X}(\mathbb{Q}) \neq \emptyset.$$

By defining similarly the Brauer–Manin set $\mathcal{X}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}}_{\bullet}$, one also has that

$$\mathcal{X}(\mathbb{Z}) \subseteq \mathcal{X}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}}_{\bullet} \subseteq \mathcal{X}(\mathbf{A}_{\mathbb{Z}})_{\bullet}.$$

This gives the so-called *integral* Brauer–Manin obstruction. We say that *the Brauer–Manin obstruction to the integral Hasse principle is the only one* if

$$\mathcal{X}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}}_{\bullet} \neq \emptyset \iff \mathcal{X}(\mathbb{Z}) \neq \emptyset.$$

If there is no confusion, we can omit the symbol $\bullet$ for the set of local integral points and the corresponding Brauer–Manin set.

We are particularly interested in the case where $X$ is a hypersurface, defined by a polynomial equation of degree $d$ in an affine space. The case $d = 1$ is easy and elementary. The case $d = 2$ considers the arithmetic of quadratic forms: for rational points, the Hasse principle is always satisfied by the Hasse–Minkowski theorem, and for integral points, the Brauer–Manin obstruction to the integral Hasse principle is the only one (up to an isotropy assumption) due to work of Colliot-Thélène, Xu [CX09] and Harari [Har08]. However, the case $d = 3$ (of cubic hypersurfaces) is still largely open, especially for integral points. Overall, the arithmetic of integral points on the affine cubic surfaces over number fields is still little understood. For example, the classical question to determine which integers can be written as sums of three cubes of integers is still open. In this first problem, for the affine variety defined by the equation

$$x^3 + y^3 + z^3 = a,$$

where $a$ is a fixed integer, Colliot-Thélène and Wittenberg in [CW12] proved that there is no Brauer–Manin obstruction to the integral Hasse principle (if $a$ is not of the form $9n{\pm}4$). However, the existence of such an integer $a$ remains unknown in general, with the smallest positive number at present being $a = 114$. On the other hand, in a related problem, there is no Brauer–Manin obstruction to the existence of an integral point on the cubic surface defined by

$$x^3 + y^3 + 2z^3 = a,$$

for any $a \in \mathbb{Z}$, which was also proven in [CW12].

Another interesting example of affine cubic surfaces that we consider is given by **Markoff**

**surfaces** $U_m \subset \mathbb{A}^3$ which are defined over $\mathbb{Q}$ by

$$x^2 + y^2 + z^2 - xyz = m, \tag{1.1}$$

where $m$ is an integer parameter. The origin of this family of surfaces traces back to Markoff from the year 1879. For references, we suggest reading the interesting survey "The Geometry of Markoff Numbers" by Caroline Series [Ser85] and the literature cited there, as well as the website `Markov numbers`. Here we give a brief introduction to the history of Markoff numbers.

In the study of Diophantine approximation, it is well known that any irrational number $\theta$ can be approximated by a sequence of rationals $p_n/q_n$ which are "good approximations" in the sense that there exists a constant $c$ so that $|\theta - p_n/q_n| < c/q^2$. The rationals $p_n/q_n$ are the *convergents*, or $n$th step truncations, of the continued fraction expansion

$$n_0 + \frac{1}{n_1 + \frac{1}{n_2} + \cdots} = [n_0, n_1, n_2, \dots] \text{ of } \theta.$$

It is natural to ask for the *least possible value* of $c$, in other words, for given $\theta$, one would like to find

$$v(\theta) = \inf\{c : |\theta - p/q| < c/q^2 \text{ for infinitely many } q\}.$$

It turns out that $v(\theta) \leqslant 1/\sqrt{5}$ with equality only if $\theta$ is a "noble number" whose continued fraction expansion ends in a *string of ones*. In 1879, Markoff improved this result by showing that there is a discrete set of values $v_i$ decreasing to $1/3$ so that if $v(\theta) > 1/3$ then $v(\theta) = v_i$ for some $i$. The numbers $v_i$ are called the *Markoff spectrum* and the corresponding $\theta$'s, *Markoff irrationalities.* Markoff irrationalities have continued fraction expansions whose tails satisfy a very special set of rules, often called the Dickson rules. The tail $1, 1, 1, \dots$ is the simplest example. Markoff gave a prescription for determining all of these irrationalities starting from the solutions of a certain Diophantine equation and linked his results to the minima of associated binary quadratic forms.

After nearly 100 years, since the 1970s-80s, there has been a revival of interest in this topic, starting from the realization that each $v_i$ together with its corresponding class of Markoff irrationalities is associated to a simple (non-self-intersecting) loop on the punctured torus. The details have been worked out most fully by A. Haas, based on earlier work of Cohn, and Schmidt. Lehner, Scheingorn and Beardon tackle the same problem but base their analysis on a sphere with four punctures. We will also apply these geometric viewpoints to study the arithmetic of Markoff-type cubic surfaces in this thesis.

The Markoff spectrum is frequently calculated by introducing *Markoff triples.* These are integer triples $(x, y, z)$ which are solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1.2}$$

Associated to such a triple is a pair of real quadratic numbers $\psi, \psi' = 1/2+y/xz+1/2(9-4/z^2)^{1/2}$. The numbers $\psi, \psi'$ are Markoff irrationalities with $v(\psi) = v(\psi') = (9-4/z^2)^{1/2} > 1/3$. Establishing that Markoff irrationalities are exactly endpoints of simple geodesics (loops) on the punctured torus contributed greatly to Markoff's original Diophantine approximation problem as well as another well known aspect of Markoff's theory, the minima of binary quadratic forms; see [Ser85] and the references cited there for more details. In the scope of this thesis, we only focus on the arithmetic aspect of Markoff-like numbers, in particular the local-global principle and strong approximation.

First of all, the original family of Markoff surfaces given by the equation (1.2), or (1.1) with $m = 0$ (with the same Markoff triples up to a factor of 3), appeared in a series of papers [BGS16a], [BGS16b], and recently [Che22]. They studied *strong approximation mod p* for $U_0$ for any prime $p$ and presented a Strong Approximation Conjecture ([BGS16b, Conjecture 1]):

**Conjecture 1.0.1.** *For any prime $p$, $U_0(\mathbb{Z}/p\mathbb{Z})$ consists of two $\Gamma$ orbits, namely $\{(0,0,0)\}$ and $U_0^*(\mathbb{Z}/p\mathbb{Z}) = U_0(\mathbb{Z}/p\mathbb{Z})\backslash\{(0,0,0)\}$. Here $\Gamma$ is a group of affine integral morphisms of $\mathbb{A}^3$ generated by the permutations of the coordinates and the Vieta involutions.*

With the above three papers by Bourgain, Gamburd, Sarnak, and Chen combined, the Conjecture is established for all but finitely many primes ([Che22, Theorem 5.5.5]), which also implies that $U_0$ satisfies strong approximation mod $p$ for all by finitely many primes.

On the other hand, in [GS22], Ghosh and Sarnak studied the integral points on those affine Markoff surfaces $U_m$ with general $m$, both from a theoretical point of view and by numerical evidence. They proved that for almost all $m$, the integral Hasse principle holds, and that there are infinitely many $m$'s for which it fails (Hasse failures). Furthermore, their numerical experiments suggested particularly a proportion of integers $m$ satisfying $|m| \leqslant M$ of the power $M^{0,8875\cdots+o(1)}$ for which the integral Hasse principle is not satisfied.

Subsequently, Loughran and Mitankin [LM20] proved that asymptotically only a proportion of $M^{1/2}/(\log M)^{1/2}$ of integers $m$ such that $-M \leqslant m \leqslant M$ presents an integral Brauer–Manin obstruction to the Hasse principle. They also obtained a lower bound, asymptotically $M^{1/2}/\log M$, for the number of Hasse failures which cannot be explained by the Brauer–Manin obstruction. After Colliot-Thélène, Wei, and Xu [CWX20] obtained a slightly stronger lower bound than the one given in [LM20], no better result than their number $M^{1/2}/(\log M)^{1/2}$ has been known until now. In other words, with all the current results, one does not have a satisfying comparison between the numbers of Hasse failures which can be explained by the Brauer–Manin obstruction and which cannot be explained by this obstruction. Meanwhile, for strong approximation, it has been proven to almost never hold for Markoff surfaces in [LM20] and then to always fail in [CWX20].

Let us now recall an important conjecture given by Ghosh and Sarnak.

**Conjecture 1.0.2** ([GS22, Conjecture 10.2])**.** *The number of Hasse failures satisfies that*

$$\#\{m \in \mathbb{Z} : 0 \leqslant m \leqslant M,\ \mathcal{U}_m(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}_m(\mathbb{Z}) = \emptyset\} \approx C_0 M^\theta,$$

*for some $C_0 > 0$ and some $\frac{1}{2} < \theta < 1$.*

This conjecture also means that *almost all* counterexamples to the integral Hasse principle for Markoff surfaces cannot be explained by the Brauer–Manin obstruction, thanks to the result obtained by [LM20]. While the question of counting all counterexamples to the integral Hasse principle for Markoff surfaces remains largely open, we would like to work on other related families of surfaces in this thesis. More precisely, also using the Brauer–Manin obstruction, as in the paper [Dao24], we study the set of integral points of a different family of Markoff-type cubic surfaces whose origin is similar to that of the original Markoff surfaces $U_m$, namely the *relative character varieties* as will be introduced later. While a Markoff surface comes from a relative character variety of *the one-holed torus*, a Markoff-type cubic surface that we study comes from a relative character variety of *the four-holed sphere*, given by the affine equation:

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d,$$

where $a, b, c, d \in \mathbb{Z}$ are parameters that satisfy some specific relations to be discussed later. Due to the similar appearance to that of the original Markoff surfaces, one may expect to find some similarities in their arithmetic as well. One of the main results in our paper is the following, saying that a positive proportion of these relative character varieties have no (algebraic) Brauer–Manin obstruction to the integral Hasse principle as well as fail *strong approximation*, and those failures can be explained by the Brauer–Manin obstruction.

**Theorem 1.0.3.** *Let $\mathcal{U}$ be the affine scheme over $\mathbb{Z}$ defined by*

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d,$$

*where*

$$\begin{cases} a = k_1k_2 + k_3k_4 \\ b = k_1k_4 + k_2k_3 \\ c = k_1k_3 + k_2k_4 \end{cases} \quad \text{and} \quad d = 4 - \sum_{i=1}^{4} k_i^2 - \prod_{i=1}^{4} k_i,$$

*such that the projective closure $X \subset \mathbb{P}^3_{\mathbb{Q}}$ of $U = \mathcal{U} \times_{\mathbb{Z}} \mathbb{Q}$ is smooth. Then we have*

$$\#\{k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4, |k_i| \leqslant M \; \forall\, 1 \leqslant i \leqslant 4 : \emptyset \neq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}_1} \neq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})\} \asymp M^4$$

*as $M \to +\infty$.*

For higher degrees, we study certain analogous varieties in the world of K3 surfaces instead of cubic surfaces, as in the paper [Dao23]. Let $K$ be a number field. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a (not necessarily smooth) surface over $K$, given by a $(2,2,2)$ form

$$F(X_1, X_2; Y_1, Y_2; Z_1, Z_2) \in K[X_1, X_2; Y_1, Y_2; Z_1, Z_2].$$

Then $X$ is called a *Wehler surface*. If $X$ is smooth, $X$ is an elliptic K3 surface whose projections $p_i : X \to \mathbb{P}^1$ ($i \in \{1,2,3\}$) have fibers as curves of (arithmetic) genus 1.

A **Markoff-type K3 surface** $W$ is a Wehler surface whose $(2,2,2)$-form $F$ is invariant under the action of the group $\mathcal{G} \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ generated by $(x,y,z) \mapsto (-x,-y,z)$ and permutations of $(x,y,z)$. By [Fuc+22], there exist $a,b,c,d,e \in k$ so that the $(2,2,2)$-form $F$ that defines $W$ has the affine form:

$$ax^2y^2z^2 + b(x^2y^2 + y^2z^2 + z^2x^2) + cxyz + d(x^2 + y^2 + z^2) + e = 0.$$

Our main results show the Brauer–Manin obstructions with respect to explicit elements of the algebraic Brauer groups for the existence of integral points on **three** concrete families of Markoff-type K3 surfaces (MK3 surfaces). One of them, as the most *general* one, is the following.

**Theorem 1.0.4.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surfaces defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{1.3}$$

*Let $U_k$ be the affine open subscheme defined by $W_k \setminus \{rst = 0\}$ and let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same affine equation. If $k$ satisfies the conditions:*

1. $k = -\frac{1}{4}(1 + 27\ell^2)$ *where $\ell \in \mathbb{Z}$ such that* $\ell \equiv \pm 1 \bmod 8$, $\ell \equiv 1 \bmod 5$, $\ell \equiv 3 \bmod 7$, *and* $\ell \not\equiv \pm 10 \bmod 37$*;*

2. $p \equiv \pm 1 \bmod 24$ *for any prime divisor $p$ of $\ell$,*

*then there is an algebraic Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}_k$ with respect to the subgroup $A \subset \mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$ generated by the elements $\mathcal{A}_1 = (4x^2+1, -2(4k+1))$ and $\mathcal{A}_2 = (4y^2+1, -2(4k+1))$, i.e., $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})^A = \emptyset$.*

Note that while the affine Markoff surfaces are log K3, the above affine MK3 surfaces are log general type, so the behavior of integral points may differ in general. However, since their projective closures are smooth K3 surfaces, then it is expected that in some sense, the behavior of integral points on the smooth affine Markoff surfaces may be similar to that of rational points on the smooth Wehler surfaces of Markoff type. It is well known that Skorobogatov's Conjecture states that rational points on smooth projective K3 surfaces are dense in the Brauer–Manin set, while Vojta's Conjecture states that integral points on log general type varieties are not Zariski dense. By the results in [GS22] and [CWX20], there are infinitely many Markoff surfaces (hence log K3 surfaces) where integral points are Zariski dense but are not dense in the integral Brauer–Manin sets, and so we may expect the same phenomenon in the case of our Markoff-type cubic surfaces. Therefore, it is reasonable to predict that, in some sense, there are *fewer* integral points on the affine Markoff-type K3 surfaces than on the cubic ones, so there could be *more* counterexamples to the integral Hasse principle for the former which hopefully can be explained by the Brauer–Manin obstruction.

Our next result deals with the counting problem on the number of counterexamples to the integral Hasse principle for Markoff-type K3 surfaces. Recall that for Markoff surfaces, Loughran and Mitankin [LM20] proved that asymptotically only a proportion of $M^{1/2}/(\log M)^{1/2}$ of integers $m$ such that $|m| \leqslant M$ presents an integral Brauer–Manin obstruction.

**Theorem 1.0.5.** *For the above three families of MK3 surfaces, we have*

$$\#\{k \in \mathbb{Z} : |k| \leqslant M,\ \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset,\ \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} = \emptyset\} \gg \frac{M^{1/2}}{\log M},$$

*as $M \to +\infty$.*

Finally, we study the descent and Brauer–Manin obstructions for affine varieties in positive characteristic, following a paper by Harari and Voloch [HV13]. In this paper, it was proven that the Brauer–Manin obstruction is the only one obstruction to the integral Hasse principle for affine varieties over global function fields, which is not true in general for affine varieties, such as Markoff surfaces, over number fields. One of the main results in [HV13] is the following, which says that the so-called Artin–Schreier torsors that exist naturally in positive characteristic (to be defined later) give *descent obstructions* (with relation to the Brauer–Manin obstruction) to the existence of integral points on affine varieties over global function fields.

**Theorem 1.0.6.** *Let $\mathcal{X}$ be an affine $\mathcal{O}_S$-scheme of finite type with generic fiber $X$. Let $(x_v) \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_S) \times \prod_{v \in S} X(K_v)$. Assume that $(x_v)$ is* unobstructed *by every Artin–Schreier torsor $Y \to X$. Then $(x_v) \in \mathcal{X}(\mathcal{O}_S)$.*

More precisely, we study the Markoff surfaces over global function fields of characteristic $p > 0$ (i.e., function fields of a geometrically integral curve over a finite field) and prove that integral points always exist in the case $p > 3$. For $p = 3$, we then find some similar results such as the reduction theory for the set of integral points and families of counterexamples to the integral Hasse principle as studied in the case of number fields by [GS22], [LM20] and [CWX20]. Our main analogous result is the following.

**Proposition 1.0.7.** *Let $X$ be the Markoff surface defined by*

$$x^2 + y^2 + z^2 - xyz = M,$$

*where $M \in \mathbb{F}_3[t]$. Let $\mathcal{X}$ be the integral model of $X$ defined over $\mathbb{F}_3[t]$ by the same equation. If $M \in \mathbb{F}_3[t]$ is a* ***monic*** *polynomial* of odd degree*, denoting by* $\mathrm{lc}(P)$ *the leading coefficient of an arbitrary polynomial $P$, we consider the compact set*

$$\Delta_M := \{(a,b,c) \in \mathbb{F}_3[t] : 1 \leqslant \deg a \leqslant \deg b \leqslant \deg c,\ \mathrm{lc}(a) = \mathrm{lc}(b) = \mathrm{lc}(c) = 1,\\ a^2 + b^2 + c^2 + abc = M,\ \deg abc = \deg M\}.$$

*Then any point $(x,y,z) \in \mathcal{X}(\mathbb{F}_3[t])$ is $\Gamma$-equivalent to a point $(a,b,c)$ in $\Delta_M$.*

*On the other hand, if we also consider $M$ of even degree, then the condition* $\deg abc = \deg M$ *can be replaced by* $\deg abc \leqslant \deg M$*.*

Furthermore, by computer computations, we have found a family of Markoff surfaces which is expected to be counterexamples to the integral Hasse principle over $\mathbb{F}_3[t]$ that can be explained by the Brauer–Manin obstruction or, equivalently, descent obstructions associated with Artin–Schreier torsors.

In summary, in this thesis, we study the problem of the existence of integral points on certain algebraic surfaces defined over number fields, particularly the field of rational numbers. The structure of the thesis is as follows. In this first chapter, we introduce the history of the problem and some recent progress in the subject of our study, especially the recent work of Ghosh–Sarnak, Loughran–Mitankin, and Colliot-Thélène–Wei–Xu.

In Chapter 2, we study the Brauer–Manin obstruction for Markoff-type cubic surfaces. We first provide some background on character varieties and the natural origin of the Markoff-type cubic surfaces, then we explicitly calculate the Brauer group of the smooth compactifications and the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer group to prove the failure of strong approximation which can be explained by the Brauer–Manin obstruction in an explicit family, and then give some counting results for the frequency of the obstructions. Furthermore, we apply the reduction theory, similar to that of Markoff surfaces, in recent work by Whang to give an explicit counterexample to the integral Hasse principle for our Markoff-type cubic surfaces. We also give some analogous results to those on Markoff surfaces about the Brauer–Manin obstruction in some special cases of Markoff-type cubic surfaces.

In Chapter 3, we study the Brauer–Manin obstruction for Wehler K3 surfaces of Markoff type and follow the same structure as the previous chapter. We first provide some background on Wehler K3 surfaces and a recent study of the Markoff-type K3 (MK3) surfaces, as well as introduce the *three* explicit families of MK3 surfaces that interest us. Next, we explicitly calculate the algebraic Brauer group of the smooth projective closures, and then the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer group to prove the failure of the integral Hasse principle which can be explained by the Brauer–Manin obstruction for three families of MK3 surfaces, and then give some counting results for the Hasse failures. Afterward, we study some cases when the Brauer–Manin obstruction to the existence of integral points and rational points can vanish, then give some counterexamples to strong approximation which can be explained by the Brauer–Manin obstruction. Furthermore, we provide some explicit examples which show that rational points do exist on affine MK3 surfaces.

To complete the thesis, in Appendix A, we give a brief introduction to the descent obstructions associated with Artin–Schreier torsors and their relation to the Brauer–Manin obstruction for integral points on affine varieties over global function fields, as studied by Harari and Voloch. Finally, we study some counterexamples to the integral Hasse principle for conics and Markoff surfaces.

# Chapter 2

# Brauer–Manin obstruction for Markoff-type cubic surfaces

## 2.1 Background

The main reference to look up notations that we use here is [Wha20], mostly Chapter 2.

### 2.1.1 Character varieties

First, we introduce an important origin of the Markoff-type cubic surfaces which comes from *character varieties*, as studied in [Wha20]. Throughout this section, an *algebraic variety* is a scheme of finite type over a field. Given an affine variety $X$ over a field $k$, we denote by $k[X]$ its coordinate ring over $k$. If moreover $X$ is integral, then $k(X)$ denotes its function field over $k$. Given a commutative ring $A$ with unity, the elements of $A$ will be referred to as regular functions on the affine scheme $\operatorname{Spec} A$.

**Definition 2.1.1.** Let $\pi$ be a finitely generated group. Its ($\mathrm{SL}_2$) *representation variety* $\mathrm{Rep}(\pi)$ is the affine scheme defined by the functor

$$A \mapsto \mathrm{Hom}(\pi, \mathrm{SL}_2(A))$$

for every commutative ring $A$. Assume that $\pi$ has a sequence of generators of $m$ elements, then we have a presentation of $\mathrm{Rep}(\pi)$ as a closed subscheme of $\mathrm{SL}_2^m$ defined by equations coming from relations among the generators. For each $a \in \pi$, let $\mathrm{tr}_a$ be the regular function on $\mathrm{Rep}(\pi)$ given by $\rho \mapsto \mathrm{tr}\,\rho(a)$.

The ($\mathrm{SL}_2$) *character variety* of $\pi$ over $\mathbb{C}$ is then defined to be the affine invariant theoretic quotient

$$X(\pi) := \mathrm{Rep}(\pi) /\!\!/ \mathrm{SL}_2 = \mathrm{Spec}\left(\mathbb{C}[\mathrm{Rep}(\pi)]^{\mathrm{SL}_2(\mathbb{C})}\right)$$

under the conjugation action of $\mathrm{SL}_2$.

The regular function $\mathrm{tr}_a$ for each $a \in \pi$ clearly descends to a regular function on $X(\pi)$. Furthermore, from the fact that $\mathrm{tr}(I_2) = 2$ and $\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$, for $I_2 \in \mathrm{SL}_2(\mathbb{C})$ being the identity matrix and for any $A, B \in \mathrm{SL}_2(\mathbb{C})$, we can deduce a natural model of $X(\pi)$

over $\mathbb{Z}$ as the spectrum of

$$R(\pi) := \mathbb{Z}[\mathrm{tr}_a : a \in \pi]/(\mathrm{tr}_1 - 2, \mathrm{tr}_a \mathrm{tr}_b - \mathrm{tr}_{ab} - \mathrm{tr}_{ab^{-1}}).$$

Given any integral domain $A$ with fraction field $F$ of characteristic zero, the $A$-points of $X(\pi)$ parametrize the Jordan equivalence classes of $\mathrm{SL}_2(F)$-representations of $\pi$ having character valued in $A$.

**Example 2.1.2.** Denote by $F_m$ the free group on $m \geqslant 1$ generators $a_1, \dots, a_m$. By Goldman's results used in [Wha20], we have the following important examples:

(1) $\mathrm{tr}_{a_1} : X(F_1) \simeq \mathbb{A}^1$.

(2) $(\mathrm{tr}_{a_1}, \mathrm{tr}_{a_2}, \mathrm{tr}_{a_3}) : X(F_2) \simeq \mathbb{A}^3$.

(3) The coordinate ring $\mathbb{Q}[X(F_3)]$ is the quotient of the polynomial ring

$$\mathbb{Q}[\mathrm{tr}_{a_1}, \mathrm{tr}_{a_2}, \mathrm{tr}_{a_3}, \mathrm{tr}_{a_1a_2}, \mathrm{tr}_{a_2a_3}, \mathrm{tr}_{a_1a_3}, \mathrm{tr}_{a_1a_2a_3}, \mathrm{tr}_{a_1a_3a_2}]$$

by the ideal generated by two elements

$$\mathrm{tr}_{a_1a_2a_3} + \mathrm{tr}_{a_1a_3a_2} - (\mathrm{tr}_{a_1a_2}\mathrm{tr}_{a_3} + \mathrm{tr}_{a_1a_3}\mathrm{tr}_{a_2} + \mathrm{tr}_{a_2a_3}\mathrm{tr}_{a_1} - \mathrm{tr}_{a_1}\mathrm{tr}_{a_2}\mathrm{tr}_{a_3})$$

and

$$\begin{aligned} \mathrm{tr}_{a_1a_2a_3}\mathrm{tr}_{a_1a_3a_2} - \{&(\mathrm{tr}_{a_1}^2 + \mathrm{tr}_{a_2}^2 + \mathrm{tr}_{a_3}^2) + (\mathrm{tr}_{a_1a_2}^2 + \mathrm{tr}_{a_2a_3}^2 + \mathrm{tr}_{a_1a_3}^2) \\ &- (\mathrm{tr}_{a_1}\mathrm{tr}_{a_2}\mathrm{tr}_{a_1a_2} + \mathrm{tr}_{a_2}\mathrm{tr}_{a_3}\mathrm{tr}_{a_2a_3} + \mathrm{tr}_{a_1}\mathrm{tr}_{a_3}\mathrm{tr}_{a_1a_3}) \\ &+ \mathrm{tr}_{a_1a_2}\mathrm{tr}_{a_2a_3}\mathrm{tr}_{a_1a_3} - 4\}. \end{aligned}$$

Now given a connected smooth compact manifold $M$, we consider the *moduli of local systems* on $M$ which is the character variety $X(M) := X(\pi_1(M))$ of its fundamental group. More generally, given a smooth manifold $M = M_1 \sqcup \cdots \sqcup M_m$ with finitely many connected components $M_i$ for $1 \leqslant i \leqslant m$, define

$$X(M) := X(M_1) \times \cdots \times X(M_m).$$

The construction of the moduli space $X(M)$ is functorial in the manifold $M$. More precisely, any smooth map $f : M \to N$ of manifolds induces a morphism $f^* : X(N) \to X(M)$, depending only on the homotopy class of $f$, given by pull-back of local systems.

Let $\Sigma$ be a surface. For each curve $a \in \Sigma$, there is a well-defined regular function $\mathrm{tr}_a : X(\Sigma) \to X(a) \simeq \mathbb{A}^1$, which agrees with $\mathrm{tr}_\alpha$ for any $\alpha \in \pi_1(\Sigma)$ freely homotopic to a parametrization of $a$. The boundary curves $\partial\Sigma$ of $\Sigma$ induce a natural morphism

$$\mathrm{tr}_{\partial\Sigma} = (-)|_{\partial\Sigma} : X(\Sigma) \to X(\partial\Sigma).$$

Now since we can write $\partial\Sigma = c_1 \sqcup \cdots \sqcup c_n$, we have an identification

$$X(\partial\Sigma) = X(c_1) \times \cdots \times X(c_n) \simeq \mathbb{A}^n$$

given by taking a local system on the disjoint union $\partial\Sigma$ of $n$ circles to its sequence of traces along the curves. The morphism $\mathrm{tr}_{\partial\Sigma}$ above may be viewed as an assignment to each $\rho \in X(\Sigma)$ its sequence of traces $\mathrm{tr}\,\rho(c_1), \dots, \mathrm{tr}\,\rho(c_n)$. The fibers of $\mathrm{tr}_{\partial\Sigma}$ for $k \in \mathbb{A}^n$ will be denoted $X_k = X_k(\Sigma)$. Each $X_k$ is often called a *relative character variety* in the literature. If $\Sigma$ is a surface of type $(g, n)$ satisfying $3g + n - 3 > 0$, then the relative character variety $X_k(\Sigma)$ is an irreducible algebraic variety of dimension $6g + 2n - 6$.

Given a fixed surface $\Sigma$, a subset $K \subseteq X(\partial\Sigma, \mathbb{C})$, and a subset $A \subseteq \mathbb{C}$, we shall denote by

$$X_K(A) = X_K(\Sigma, A) := X_K(\Sigma)(A)$$

the set of all $\rho \in X(\Sigma, \mathbb{C})$ such that $\operatorname{tr}_{\partial\Sigma}(\rho) \in K$ and $\operatorname{tr}_a(\rho) \in A$ for every essential curve $a \subset \Sigma$. By [Wha20, Lemma 2.5], there is no risk of ambiguity with this notation, i.e., $X_k$ has a model over $A$ and $X_k(A)$ recovers the set of $A$-valued points of $X_k$ in the sense of algebraic geometry.

### 2.1.2 Markoff-type cubic surfaces

Now we give a description of the moduli spaces $X_k(\Sigma)$ for $(g, n) = (1, 1)$ and $(0, 4)$. These cases are special since each $X_k$ is an affine cubic algebraic surface with an explicit equation.

(1) Let $\Sigma$ be a surface of type $(g, n) = (1, 1)$, i.e. a one holed torus. Let $(\alpha, \beta, \gamma)$ be an *optimal sequence of generators* for $\pi_1(\Sigma)$, as given in [Wha20, Definition 2.1]. By Example 2.1, we have an identification $X(\Sigma) \simeq \mathbb{A}^3$. From the trace relations in $\mathrm{SL}_2$, we obtain that

$$\begin{aligned}\operatorname{tr}_\gamma &= \operatorname{tr}_{\alpha\beta\alpha^{-1}\beta^{-1}} = \operatorname{tr}_{\alpha\beta\alpha^{-1}}\operatorname{tr}_{\beta^{-1}} - \operatorname{tr}_{\alpha\beta_\alpha^{-1}\beta} \\ &= \operatorname{tr}_\beta^2 - \operatorname{tr}_{\alpha\beta}\operatorname{tr}_{\alpha^{-1}\beta} + \operatorname{tr}_{\alpha\alpha} = \operatorname{tr}_\beta^2 - \operatorname{tr}_{\alpha\beta}(\operatorname{tr}_{\alpha^{-1}}\operatorname{tr}_\beta - \operatorname{tr}_{\alpha\beta}) + \operatorname{tr}_\alpha^2 - \operatorname{tr}_1 \\ &= \operatorname{tr}_\alpha^2 + \operatorname{tr}_\beta^2 + \operatorname{tr}_{\alpha\beta}^2 - \operatorname{tr}_\alpha\operatorname{tr}_\beta\operatorname{tr}_{\alpha\beta} - 2.\end{aligned}$$

Writing $(x, y, z) = (\operatorname{tr}_\alpha, \operatorname{tr}_\beta, \operatorname{tr}_{\alpha\beta})$ so that each of the variables $x$, $y$, and $z$ corresponds to an essential curve on $\Sigma$ as depicted in [Wha20, Figure 2], the moduli space $X_k \subset X$ has an explicit presentation as an affine cubic algebraic surface in $\mathbb{A}^3_{x,y,z}$ with the equation

$$x^2 + y^2 + z^2 - xyz - 2 = k.$$

These are exactly the Markoff surfaces as studied in the two papers [LM20] and [CWX20] with $m = k + 2$.

(2) Let $\Sigma$ be a surface of type $(g, n) = (0, 4)$, i.e. a four holed sphere. Let $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be an optimal sequence of generators for $\pi_1(\Sigma)$. Set

$$(x, y, z) = (\operatorname{tr}_{\gamma_1\gamma_2}, \operatorname{tr}_{\gamma_2\gamma_3}, \operatorname{tr}_{\gamma_1\gamma_3})$$

so that each of the variables corresponds to an essential curve on $\Sigma$. By the above Example, for $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(\mathbb{C})$ the relative character variety $X_k = X_k(\Sigma)$ is an affine cubic algebraic surface in $\mathbb{A}^3_{x,y,z}$ given by the equation

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d, \tag{2.1}$$

where

$$\begin{cases} a = k_1k_2 + k_3k_4 \\ b = k_1k_4 + k_2k_3 \\ c = k_1k_3 + k_2k_4 \end{cases} \quad \text{and} \quad d = 4 - \sum_{i=1}^4 k_i^2 - \prod_{i=1}^4 k_i. \tag{2.2}$$

These are the Markoff-type cubic surfaces that we are going to study in this chapter.

## 2.2 The Brauer group of Markoff-type cubic surfaces

Our main interest is in the **second** Markoff-type cubic surfaces defined by (2.1). We are now going to give some explicit computations on the Brauer group of these surfaces. First of all, let us recall some basic definitions and results on the Brauer group of varieties over a field (see [CS21, Section 5.4]).

Let $k$ be an arbitrary field. Recall that for a variety $X$ over $k$ there is a natural filtration on the Brauer group

$$\mathrm{Br}_0 X \subset \mathrm{Br}_1 X \subset \mathrm{Br}\, X$$

which is defined as follows.

**Definition 2.2.1.** Let

$$\mathrm{Br}_0 X = \mathrm{Im}[\mathrm{Br}\, k \to \mathrm{Br}\, X], \qquad \mathrm{Br}_1 X = \mathrm{Ker}[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}].$$

The subgroup $\mathrm{Br}_1 X \subset \mathrm{Br}\, X$ is the **algebraic** Brauer group of $X$ and the quotient $\mathrm{Br}\, X/\mathrm{Br}_1 X$ is the **transcendental** Brauer group of $X$.

From the Hochschild–Serre spectral sequence, we have the following spectral sequence:

$$E_2^{pq} = \mathrm{H}^p_{\text{ét}}(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m),$$

which is contravariantly functorial in the $k$-variety $X$. It gives rise to the functorial exact sequence of terms of low degree:

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(k, \overline{k}[X]^\times) \longrightarrow \mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, \overline{X}^{G_k} \longrightarrow \mathrm{H}^2(k, \overline{k}[X]^\times) \longrightarrow \mathrm{Br}_1 X \\ \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, \overline{k}[X]^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned}$$

Let $X$ be a variety over a field $k$ such that $\overline{k}[X]^\times = \overline{k}^\times$. By Hilbert's Theorem 90 we have $\mathrm{H}^1(k, \overline{k}^\times) = 0$, then by the above sequence there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, \overline{X}^{G_k} \longrightarrow \mathrm{Br}\, k \longrightarrow \mathrm{Br}_1 X \\ \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, \overline{k}^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned}$$

This sequence is also contravariantly functorial in $X$.

**Remark 2.2.2.** Let $X$ be a variety over a field $k$ such that $\overline{k}[X]^\times = \overline{k}^\times$. This assumption $\overline{k}[X]^\times = \overline{k}^\times$ holds for any proper, geometrically connected and geometrically reduced $k$-variety $X$.

(1) If $X$ has a $k$-point, which defined a section of the structure morphism $X \to \mathrm{Spec}\, k$, then each of the maps $\mathrm{Br}\, k \longrightarrow \mathrm{Br}_1 X$ and $\mathrm{H}^3(k, \overline{k}^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ has a retraction, hence is injective. (Then $\mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, \overline{X}^{G_k}$ is an isomorphism.) Therefore, we have an isomorphism

$$\mathrm{Br}_1 X/\mathrm{Br}\, k \cong \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}).$$

(2) If $k$ is a number field, then $\mathrm{H}^3(k, \overline{k}^\times) = 0$ (see [CF67, Chapter VII, Section 11.4, p. 199]). Thus for a variety $X$ over a number field $k$ such that $\overline{k}[X]^\times = \overline{k}^\times$, we have an isomorphism

$$\mathrm{Br}_1 X/\mathrm{Br}_0 X \cong \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}).$$

### 2.2.1 Geometry of affine cubic surfaces

In this chapter, we study the geometry of affine cubic surfaces with special regards to the Brauer group. By an *affine cubic surface*, we mean an affine surface of the form

$$U : f(u_1, u_2, u_3) = 0$$

where $f$ is a polynomial of degree of 3. The closure of $U$ in $\mathbb{P}^3$ is a cubic surface $X$. The complement $H = X \setminus U$ is a hyperplane section on $S$. Much of the geometry of $U$ can be understood in terms of the geometry of $X$ and $H$, especially in the case of Markoff-type cubic surfaces. There has been already much work on the Brauer groups of affine cubic surfaces when the hyperplane section $H$ is *smooth*, for example see [CW12]. Here we shall be interested in the case where the hyperplane section $H$ is *singular*; in particular, we focus on the case where $H$ is given by 3 coplanar lines. All results here are proven in either [CWX20] or [LM20].

We begin with an important result for cubic surfaces over an algebraically closed field.

**Proposition 2.2.1** ([CWX20, Proposition 2.2]). *Let $X \subset \mathbb{P}^3_k$ be a smooth projective cubic surface over a field $k$ of characteristic zero. Suppose a plane $\mathbb{P}^2_k \subset \mathbb{P}^3_k$ cuts out on $\overline{X}$ three distinct lines $L_1, L_2, L_3$ over $\overline{k}$. Let $U \subset X$ be the complement of this plane. Then the natural map $\overline{k}^\times \to \overline{k}[U]^\times$ is an isomorphism of Galois modules and the natural sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z}L_i \longrightarrow \operatorname{Pic} \overline{X} \longrightarrow \operatorname{Pic} \overline{U} \longrightarrow 0$$

*is an exact sequence of Galois lattices.*

As $\operatorname{Pic} \overline{U}$ is torsion free, we have the following result for the algebraic Brauer group, using the computation by Magma.

**Proposition 2.2.2** ([LM20, Proposition 2.5]). *Let $X$ be a smooth projective cubic surface over a field $k$ of characteristic 0. Let $H \subset S$ be a hyperplane section which is the union of 3 distinct lines $L_1, L_2, L_3$ and let $U = X \setminus H$. Then $\operatorname{Pic} \overline{U}$ is torsion free and $\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong \mathrm{H}^1(k, \operatorname{Pic} \overline{U})$ is isomorphic to one of the following groups:*

$$0, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^r \qquad (r = 1, 2, 3, 4).$$

For the transcendental Brauer group, from the discussion in [CS21, page 140], note that $\operatorname{Br}_1 U = \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} \overline{U}^{G_k})$ so we have $\operatorname{Br} U/\operatorname{Br}_1 U \subset \operatorname{Br} \overline{U}^{G_k}$.

**Proposition 2.2.3** ([CWX20, Proposition 2.1], [LM20, Proposition 2.4]). *Let $X$ be a smooth projective cubic surface over a field $k$ of characteristic 0. Suppose that $U$ is an open subset of $X$ such that $X \setminus U$ is the union of three distinct $k$-lines, by which we mean a smooth projective curve isomorphic to $\mathbb{P}^1_k$. Suppose any two lines intersect each another transversely in one point, and that the three intersection points are distinct. Let $L$ be one of the three lines and $V \subset L$ be the complement of the 2 intersection points of $L$ with the other two lines. Then the residue map*

$$\partial_L : \operatorname{Br} \overline{k}(X) \to \mathrm{H}^1(\overline{k}(L), \mathbb{Q}/\mathbb{Z})$$

*induces a $G_k$-isomorphism*

$$\operatorname{Br} \overline{U} \simeq \mathrm{H}^1(\overline{V}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(\overline{\mathbb{G}}_m, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}(-1).$$

*In particular, if $k$ contains no non-trivial roots of unity then*

$$\operatorname{Br}\overline{U}^{G_k} = \mathbb{Z}/2\mathbb{Z}.$$

Note that $\operatorname{Br}_1 U = \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br}\overline{U}^{G_k})$ so we have $\operatorname{Br} U/\operatorname{Br}_1 U \subset \operatorname{Br}\overline{U}^{G_k}$.

**Lemma 2.2.4** ([CWX20, Lemma 2.4])**.** *Let $k$ be a field of characteristic 0. Let $G_k = \operatorname{Gal}(\overline{k}/k)$. Then $\mathbb{Q}/\mathbb{Z}(-1)^{G_k}$ is (noncanonically) isomorphic to $\mu_\infty(k)$, the group of roots of unity in $k$.*

We end this section by the following result which applies to number fields and more generally to function fields of varieties over number fields.

**Corollary 2.2.5** ([CWX20, Corollary 2.3])**.** *Let $k$ be a field of characteristic 0 such that in any finite field extension there are only finitely many roots of unity. Let $X \subset \mathbb{P}^3_k$ be a smooth projective cubic surface over $k$. Suppose that a plane cuts out on $X$ three nonconcurrent lines. Let $U \subset X$ be the complement of the plane section. Then the quotient $\operatorname{Br} U/\operatorname{Br}_0 U$ is finite.*

### 2.2.2 The geometric Picard group and algebraic Brauer group

Using the equations, we can compute explicitly the algebraic Brauer group of the Markoff-type cubic surfaces in question. First, we have the following important result.

**Lemma 2.2.6.** *Let $K$ be a number field and let $X \subset \mathbb{P}^3_K$ be a cubic surface defined by the equation*

$$t(x^2+y^2+z^2) + xyz = t^2(ax+by+cz) + dt^3,$$

*where $a, b, c, d$ are defined by (2.2) for some $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(K)$. Then $X$ is singular if and only if we are in one of the following cases:*

- *$\Delta(k) = 0$ where $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(K)$ and*

$$\Delta(k) = (2(k_1^2+k_2^2+k_3^2+k_4^2) - k_1k_2k_3k_4 - 16)^2 - (4-k_1^2)(4-k_2^2)(4-k_3^2)(4-k_4^2),$$

- *at least one of the parameters $k_1, k_2, k_3, k_4$ equals $\pm 2$.*

*If $k$ satisfies none of those two conditions and $[E : K] = 16$ where*

$$E := K(\sqrt{k_1^2-4}, \sqrt{k_2^2-4}, \sqrt{k_3^2-4}, \sqrt{k_4^2-4}),$$

*then the 27 lines on the smooth cubic surface $\overline{X}$ are defined over $E$ by the following equations*

$$L_1 : x = t = 0; \qquad L_2 : y = t = 0; \qquad L_3 : z = t = 0$$

*and*

1. $\ell_1(\epsilon,\delta) : x = \dfrac{(k_1k_2 + \epsilon\delta\sqrt{(k_1^2-4)(k_2^2-4)})}{2}t,$

$$y = -\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_2+\delta\sqrt{k_2^2-4})}{4}z - \frac{c - b\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_2+\delta\sqrt{k_2^2-4})}{4}}{\frac{\delta k_1\sqrt{k_2^2-4}+\epsilon k_2\sqrt{k_1^2-4}}{2}}t;$$

*2.* $\ell_2(\epsilon,\delta): y = \dfrac{(k_1k_4 + \epsilon\delta\sqrt{(k_1^2-4)(k_4^2-4)})}{2}t,$

$$z = -\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}x - \frac{a - c\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}}{\frac{\delta k_1\sqrt{k_4^2-4}+\epsilon k_4\sqrt{k_1^2-4}}{2}}t;$$

*3.* $\ell_3(\epsilon,\delta): z = \dfrac{(k_1k_3 + \epsilon\delta\sqrt{(k_1^2-4)(k_3^2-4)})}{2}t,$

$$y = -\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_3+\delta\sqrt{k_3^2-4})}{4}x - \frac{a - b\frac{(k_1+\epsilon\sqrt{k_1^2-4})(k_3+\delta\sqrt{k_3^2-4})}{4}}{\frac{\delta k_1\sqrt{k_3^2-4}+\epsilon k_3\sqrt{k_1^2-4}}{2}}t;$$

*4.* $\ell_4(\epsilon,\delta): x = \dfrac{(k_3k_4 + \epsilon\delta\sqrt{(k_3^2-4)(k_4^2-4)})}{2}t,$

$$y = -\frac{(k_3+\epsilon\sqrt{k_3^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}z - \frac{c - b\frac{(k_3+\epsilon\sqrt{k_3^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}}{\frac{\delta k_3\sqrt{k_4^2-4}+\epsilon k_4\sqrt{k_3^2-4}}{2}}t;$$

*5.* $\ell_5(\epsilon,\delta): y = \dfrac{(k_2k_3 + \epsilon\delta\sqrt{(k_2^2-4)(k_3^2-4)})}{2}t,$

$$z = -\frac{(k_2+\epsilon\sqrt{k_2^2-4})(k_3+\delta\sqrt{k_3^2-4})}{4}x - \frac{a - c\frac{(k_2+\epsilon\sqrt{k_2^2-4})(k_3+\delta\sqrt{k_3^2-4})}{4}}{\frac{\delta k_2\sqrt{k_3^2-4}+\epsilon k_3\sqrt{k_2^2-4}}{2}}t;$$

*6.* $\ell_6(\epsilon,\delta): z = \dfrac{(k_2k_4 + \epsilon\delta\sqrt{(k_2^2-4)(k_4^2-4)})}{2}t,$

$$y = -\frac{(k_2+\epsilon\sqrt{k_2^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}x - \frac{a - b\frac{(k_2+\epsilon\sqrt{k_2^2-4})(k_4+\delta\sqrt{k_4^2-4})}{4}}{\frac{\delta k_2\sqrt{k_4^2-4}+\epsilon k_4\sqrt{k_2^2-4}}{2}}t$$

*with $\epsilon = \pm 1$ and $\delta = \pm 1$. Furthermore, we have the intersection numbers*

$$\ell_i(\epsilon,\delta).\ell_j(\epsilon,\delta) = 0$$

*for any pair $(\epsilon,\delta)$, for all $1 \leqslant i \neq j \leqslant 6$.*

*Proof.* The necessary and sufficient condition for the affine open surface $U = X \setminus \{t = 0\}$ to be singular is proved in [CL09, Theorem 3.7]. It is easy to verify that there is no singular point at infinity on the projective surface $X$.

Now without loss of generality, we consider the system of equations

$$\begin{cases} y = \alpha_1 x + \alpha_2 t \\ z = \beta_1 x + \beta_2 t \end{cases}$$

and put them in the original equation of the cubic surfaces to solve $\alpha_i, \beta_i$ for $i = 1, 2$. We can work similarly for $(z, x)$ and $(x, y)$ to find the all the given equations of the 27 lines. $\square$

Now given the data of the lines, we can compute directly the algebraic Brauer group of the Markoff-type cubic surfaces in question.

**Proposition 2.2.7.** *Let $K$ be a number field. Let $X \subset \mathbb{P}^3_K$ be a cubic surface defined by the equation*

$$t(x^2 + y^2 + z^2) + xyz = t^2(ax + by + cz) + dt^3, \tag{2.3}$$

*where $a, b, c, d$ are defined by* (2.2) *for some* $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(K)$. *Assume that* $X$ *is smooth over* $K$ *and* $[E : K] = 16$, *then*

$$\operatorname{Br} X/\operatorname{Br}_0 X = \operatorname{Br}_1 X/\operatorname{Br}_0 X \cong \mathbb{Z}/2\mathbb{Z}.$$

*Proof.* Since $X$ is geometrically rational, one has $\operatorname{Br} X = \operatorname{Br}_1 X$. By taking $x = t = 0$ for instance, one clearly has $X(K) \neq \emptyset$, so $\operatorname{Br}_0 X = \operatorname{Br} K$. Since $K$ is a number field, by the Hochschild–Serre spectral sequence, we have an isomorphism

$$\operatorname{Br}_1 X/\operatorname{Br}_0 X \simeq \mathrm{H}^1(K, \operatorname{Pic} \overline{X}).$$

By the above lemma, we can easily verify that the six lines $\ell_1(1,1)$, $\ell_1(1,-1)$, $\ell_3(-1,1)$, $\ell_4(-1,-1)$, $\ell_4(-1,1)$, and $L_2$ on the cubic surface $\overline{X}$ are skew to each other, hence they may be simultaneously blown down to $\mathbb{P}^2$ by [Har77, Chapter V, Proposition 4.10]. For the sake of simplicity, here we shall write these six lines respectively as $\ell_i$ for $1 \leqslant i \leqslant 6$. The class $\omega$ of the canonical divisor on $\overline{X}$ is equal to $-3\ell + \Sigma_{i=1}^6 \ell_i$, where $\ell$ is the inverse image of the class of lines in $\mathbb{P}^2$. By [Har77, Chapter V, Proposition 4.8], the classes $\ell, \ell_i, i = \overline{1,6}$ form a basis of $\operatorname{Pic} \overline{X}$, and we have the following intersection properties: $(\ell.\ell) = 1, (\ell.\ell_i) = 0$ for $1 \leqslant i \leqslant 6$.

Since $(L_1.\ell_3) = 0, (L_1.\ell_i) = 1, i \neq 3$; $(L_3.\ell_3) = (L_3.\ell_6) = 1, (L_3.\ell_i) = 0, i \neq 3, 6$; and $L_2 = \ell_6$, one concludes that

$$L_1 = 2\ell - \Sigma_{i \neq 3} \ell_i, \qquad L_3 = \ell - \ell_3 - \ell_6 \tag{2.4}$$

in $\operatorname{Pic} \overline{X}$ by [Har77, Chapter V, Proposition 4.8 (e)].

Now we consider the action of the Galois group $G := \operatorname{Gal}(E/K)$ on $\operatorname{Pic} \overline{X}$. One clearly has $G \cong \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle \times \langle \sigma_3 \rangle \times \langle \sigma_4 \rangle$, where

$$\sigma_i(\sqrt{k_i^2 - 4}) = -\sqrt{k_i^2 - 4} \quad \text{and} \quad \sigma_i(\sqrt{k_j^2 - 4}) = \sqrt{k_j^2 - 4}, 1 \leqslant i \neq j \leqslant 4.$$

We have the following intersection numbers, noting that $\sigma_2(\ell_1) = \ell_2$, $\sigma_2(\ell_2) = \ell_1$, $\sigma_4(\ell_4) = \ell_5$, and $\sigma_4(\ell_5) = \ell_4$:

$$\begin{cases} (\sigma_1(\ell_1).\ell_1) = (\ell_1(-1,1).\ell_1(1,1)) = 0 \\ (\sigma_1(\ell_1).\ell_2) = (\ell_1(-1,1).\ell_1(1,-1)) = 1 \\ (\sigma_1(\ell_1).\ell_3) = (\ell_1(-1,1).\ell_3(-1,1)) = 1 \\ (\sigma_1(\ell_1).\ell_4) = (\ell_1(-1,1).\ell_4(-1,-1)) = 0 \\ (\sigma_1(\ell_1).\ell_5) = (\ell_1(-1,1).\ell_4(-1,1)) = 0 \\ (\sigma_1(\ell_1).\ell_6) = (\ell_1(-1,1).L_2) = 0, \end{cases} \tag{2.5}$$

$$\begin{cases} (\sigma_1(\ell_2).\ell_1) = (\ell_1(-1,-1).\ell_1(1,1)) = 1 \\ (\sigma_1(\ell_2).\ell_2) = (\ell_1(-1,-1).\ell_1(1,-1)) = 0 \\ (\sigma_1(\ell_2).\ell_3) = (\ell_1(-1,-1).\ell_3(-1,1)) = 1 \\ (\sigma_1(\ell_2).\ell_4) = (\ell_1(-1,-1).\ell_4(-1,-1)) = 0 \\ (\sigma_1(\ell_2).\ell_5) = (\ell_1(-1,-1).\ell_4(-1,1)) = 0 \\ (\sigma_1(\ell_2).\ell_6) = (\ell_1(-1,-1).L_2) = 0, \end{cases} \tag{2.6}$$

$$\begin{cases}(\sigma_1(\ell_3).\ell_1) = (\ell_3(1,1).\ell_1(1,1)) = 1\\ (\sigma_1(\ell_3).\ell_2) = (\ell_3(1,1).\ell_1(1,-1)) = 1\\ (\sigma_1(\ell_3).\ell_3) = (\ell_3(1,1).\ell_3(-1,1)) = 0\\ (\sigma_1(\ell_3).\ell_4) = (\ell_3(1,1).\ell_4(-1,-1)) = 0\\ (\sigma_1(\ell_3).\ell_5) = (\ell_3(1,1).\ell_4(-1,1)) = 0\\ (\sigma_1(\ell_3).\ell_6) = (\ell_3(1,1).L_2) = 0;\end{cases} \tag{2.7}$$

and

$$\begin{cases}(\sigma_3(\ell_3).\ell_1) = (\ell_3(-1,-1).\ell_1(1,1)) = 0\\ (\sigma_3(\ell_3).\ell_2) = (\ell_3(-1,-1).\ell_1(1,-1)) = 0\\ (\sigma_3(\ell_3).\ell_3) = (\ell_3(-1,-1).\ell_3(-1,1)) = 0\\ (\sigma_3(\ell_3).\ell_4) = (\ell_3(-1,-1).\ell_4(-1,-1)) = 1\\ (\sigma_3(\ell_3).\ell_5) = (\ell_3(-1,-1).\ell_4(-1,1)) = 1\\ (\sigma_3(\ell_3).\ell_6) = (\ell_3(-1,-1).L_2) = 0,\end{cases} \tag{2.8}$$

$$\begin{cases}(\sigma_3(\ell_4).\ell_1) = (\ell_4(1,-1).\ell_1(1,1)) = 0\\ (\sigma_3(\ell_4).\ell_2) = (\ell_4(1,-1).\ell_1(1,-1)) = 0\\ (\sigma_3(\ell_4).\ell_3) = (\ell_4(1,-1).\ell_3(-1,1)) = 1\\ (\sigma_3(\ell_4).\ell_4) = (\ell_4(1,-1).\ell_4(-1,-1)) = 0\\ (\sigma_3(\ell_4).\ell_5) = (\ell_4(1,-1).\ell_4(-1,1)) = 1\\ (\sigma_3(\ell_4).\ell_6) = (\ell_4(1,-1).L_2) = 0,\end{cases} \tag{2.9}$$

$$\begin{cases}(\sigma_3(\ell_5).\ell_1) = (\ell_4(1,1).\ell_1(1,1)) = 0\\ (\sigma_3(\ell_5).\ell_2) = (\ell_4(1,1).\ell_1(1,-1)) = 0\\ (\sigma_3(\ell_5).\ell_3) = (\ell_4(1,1).\ell_3(-1,1)) = 1\\ (\sigma_3(\ell_5).\ell_4) = (\ell_4(1,1).\ell_4(-1,-1)) = 1\\ (\sigma_3(\ell_5).\ell_5) = (\ell_4(1,1).\ell_4(-1,1)) = 0\\ (\sigma_3(\ell_5).\ell_6) = (\ell_4(1,1).L_2) = 0.\end{cases} \tag{2.10}$$

Hence, we obtain

$$\begin{cases}\sigma_1(\ell_1) = \ell - \ell_2 - \ell_3\\ \sigma_1(\ell_2) = \ell - \ell_1 - \ell_3\\ \sigma_1(\ell_3) = \ell - \ell_1 - \ell_2\\ \sigma_3(\ell_3) = \ell - \ell_4 - \ell_5\\ \sigma_3(\ell_4) = \ell - \ell_3 - \ell_5\\ \sigma_3(\ell_5) = \ell - \ell_3 - \ell_4\end{cases} \tag{2.11}$$

in $\operatorname{Pic} \overline{X}$ by [Har77, Chapter V, Proposition 4.9]. As a result, we have

$$\begin{cases}\sigma_1(\ell) = 2\ell - \ell_1 - \ell_2 - \ell_3\\ \sigma_3(\ell) = 2\ell - \ell_3 - \ell_4 - \ell_5,\end{cases} \tag{2.12}$$

and clearly $\sigma_2(\ell) = \sigma_4(\ell) = \ell$. Then

$$\begin{cases} \mathrm{Ker}(1+\sigma_1) = \langle \ell - \ell_1 - \ell_2 - \ell_3 \rangle \\ \mathrm{Ker}(1+\sigma_2) = \langle \ell_1 - \ell_2 \rangle \\ \mathrm{Ker}(1+\sigma_3) = \langle \ell - \ell_3 - \ell_4 - \ell_5 \rangle \\ \mathrm{Ker}(1+\sigma_4) = \langle \ell_4 - \ell_5 \rangle, \end{cases} \tag{2.13}$$

$$\begin{cases} \mathrm{Ker}(1-\sigma_1) = \langle \ell - \ell_1, \ell - \ell_2, \ell - \ell_3, \ell_4, \ell_5, \ell_6 \rangle \\ \mathrm{Ker}(1-\sigma_2) = \langle \ell, \ell_1 + \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \rangle \\ \mathrm{Ker}(1-\sigma_3) = \langle \ell - \ell_3, \ell - \ell_4, \ell - \ell_5, \ell_1, \ell_2, \ell_6 \rangle \\ \mathrm{Ker}(1-\sigma_4) = \langle \ell, \ell_1, \ell_2, \ell_3, \ell_4 + \ell_5, \ell_6 \rangle, \end{cases} \tag{2.14}$$

and

$$\begin{cases} (1-\sigma_1)\mathrm{Pic}\,\overline{X} = \langle \ell - \ell_1 - \ell_2 - \ell_3 \rangle \\ (1-\sigma_2)\mathrm{Pic}\,\overline{X} = \langle \ell_1 - \ell_2 \rangle \\ (1-\sigma_3)\mathrm{Pic}\,\overline{X} = \langle \ell - \ell_3 - \ell_4 - \ell_5 \rangle \\ (1-\sigma_4)\mathrm{Pic}\,\overline{X} = \langle \ell_4 - \ell_5 \rangle. \end{cases} \tag{2.15}$$

Given a finite cyclic group $G = \langle \sigma \rangle$ and a $G$-module $M$, by [NSW15, Proposition 1.7.1], recall that we have isomorphisms $\mathrm{H}^1(G, M) \cong \hat{\mathrm{H}}^{-1}(G, M)$, where the latter group is the quotient of ${}_{N_G}M$, the set of elements of $M$ of norm 0, by its subgroup $(1-\sigma)M$.

By [NSW15, Proposition 1.6.7], we have

$$\mathrm{H}^1(K, \mathrm{Pic}\,\overline{X}) = \mathrm{H}^1(G, \mathrm{Pic}\,\overline{X}),$$

where $G = \langle \sigma_i, 1 \leqslant i \leqslant 4 \rangle$. Then one has the following (inflation-restriction) exact sequence

$$0 \to \mathrm{H}^1(\langle \sigma_1, \sigma_2, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_4 \rangle}) \to \mathrm{H}^1(G, \mathrm{Pic}\,\overline{X}) \to \mathrm{H}^1(\langle \sigma_4 \rangle, \mathrm{Pic}\,\overline{X}) = 0,$$

hence $\mathrm{H}^1(G, \mathrm{Pic}\,\overline{X}) \cong \mathrm{H}^1(\langle \sigma_1, \sigma_2, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_4 \rangle})$. Continuing as above, we have

$$0 \to \mathrm{H}^1(\langle \sigma_1, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_4 \rangle}) \to \mathrm{H}^1(\langle \sigma_1, \sigma_2, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_4 \rangle}) \to \mathrm{H}^1(\langle \sigma_2 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_4 \rangle}) = 0,$$

hence $\mathrm{H}^1(\langle \sigma_1, \sigma_2, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_4 \rangle}) \cong \mathrm{H}^1(\langle \sigma_1, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_4 \rangle})$. Now we are left with

$$0 \to \mathrm{H}^1(\langle \sigma_1 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle}) \to \mathrm{H}^1(\langle \sigma_1, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_4 \rangle}) \to \mathrm{H}^1(\langle \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_4 \rangle}) = 0,$$

hence $\mathrm{H}^1(\langle \sigma_1, \sigma_3 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_4 \rangle}) \cong \mathrm{H}^1(\langle \sigma_1 \rangle, \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle}) = \mathbb{Z}/2\mathbb{Z}$. Indeed, the last group can be computed as follows. We have

$$\mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle} = \langle \ell_1 + \ell_2, \ell - \ell_3, 2\ell - \ell_4 - \ell_5, \ell_6 \rangle.$$

Considering the action of $\sigma_1$ on this invariant group, we have

$${}_{N_{\sigma_1}}\mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle} = \mathrm{Ker}(1+\sigma_1) \cap \mathrm{Pic}\,\overline{X}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle} = \langle \ell - \ell_1 - \ell_2 - \ell_3 \rangle.$$

On the other hand,

$$(1-\sigma_1)\operatorname{Pic}\overline{X}^{\langle\sigma_2,\sigma_3,\sigma_4\rangle} = [(1-\sigma_1)\operatorname{Pic}\overline{X}] \cap \operatorname{Pic}\overline{X}^{\langle\sigma_2,\sigma_3,\sigma_4\rangle} = \langle 2(\ell-\ell_1-\ell_2-\ell_3)\rangle.$$

Given these results, we conclude that

$$\mathrm{H}^1(K,\operatorname{Pic}\overline{X}) = \mathrm{H}^1(G,\operatorname{Pic}\overline{X}) \cong \mathbb{Z}/2\mathbb{Z}.$$

□

**Theorem 2.2.8.** *Let $K$ be a number field. With the same notations as before, let $k \in \mathbb{A}^4(K)$ such that $[E:K]=16$ where $E = K(\sqrt{k_i^2-4}, 1 \leqslant i \leqslant 4)$ and $X$ is smooth over $K$. Let $U$ be the affine cubic surface defined by the equation*

$$x^2+y^2+z^2+xyz = ax+by+cz+d,$$

*where $a,b,c,d$ are defined by (2.2) for some $k=(k_1,k_2,k_3,k_4) \in \mathbb{A}^4(K)$. Then we have*

$$\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong \mathbb{Z}/2\mathbb{Z}$$

*with a generator*

$$\begin{aligned}
\mathcal{A} &= \operatorname{Cor}_K^{F_1}\left(x - \frac{k_1k_2+\sqrt{(k_1^2-4)(k_2^2-4)}}{2}, (k_1\sqrt{k_2^2-4}+k_2\sqrt{k_1^2-4})^2\right)\\
&= \operatorname{Cor}_K^{F_3}\left(y - \frac{k_1k_4+\sqrt{(k_1^2-4)(k_4^2-4)}}{2}, (k_1\sqrt{k_4^2-4}+k_4\sqrt{k_1^2-4})^2\right)\\
&= \operatorname{Cor}_K^{F_2}\left(z - \frac{k_1k_3+\sqrt{(k_1^2-4)(k_3^2-4)}}{2}, (k_1\sqrt{k_3^2-4}+k_3\sqrt{k_1^2-4})^2\right),
\end{aligned}$$

*where $F_i = K(\sqrt{(k_1^2-4)(k_{i+1}^2-4)})$ for $1 \leqslant i \leqslant 3$. Furthermore, we also have*

$$\operatorname{Br} X/\operatorname{Br}_0 X = \operatorname{Br}_1 X/\operatorname{Br}_0 X \cong \operatorname{Br}_1 U/\operatorname{Br}_0 U$$

*with a generator*

$$\begin{aligned}
\mathcal{A}_0 &= \operatorname{Cor}_K^{F_1}\left(\frac{x}{t} - \frac{k_1k_2+\sqrt{(k_1^2-4)(k_2^2-4)}}{2}, (k_1\sqrt{k_2^2-4}+k_2\sqrt{k_1^2-4})^2\right)\\
&= \operatorname{Cor}_K^{F_3}\left(\frac{y}{t} - \frac{k_1k_4+\sqrt{(k_1^2-4)(k_4^2-4)}}{2}, (k_1\sqrt{k_4^2-4}+k_4\sqrt{k_1^2-4})^2\right)\\
&= \operatorname{Cor}_K^{F_2}\left(\frac{z}{t} - \frac{k_1k_3+\sqrt{(k_1^2-4)(k_3^2-4)}}{2}, (k_1\sqrt{k_3^2-4}+k_3\sqrt{k_1^2-4})^2\right)
\end{aligned}$$

*over $t \neq 0$.*

*Proof.* We keep the notation as in the previous proposition. Then $\operatorname{Pic}\overline{U}$ is given by the following quotient group

$$\operatorname{Pic}\overline{U} \cong \operatorname{Pic}\overline{X}/(\oplus_{i=1}^3 \mathbb{Z}L_i) \cong (\oplus_{i=1}^6 \mathbb{Z}\ell_i \oplus \mathbb{Z}\ell)/\langle 2\ell - \Sigma_{i\neq 3}\ell_i, \ell-\ell_3-\ell_6, \ell_6)\rangle \cong \oplus_{i=1}^4 \mathbb{Z}[\ell_i]$$

by Proposition 2.2.1 and (2.4). Here for any divisor $D \in \operatorname{Pic} \overline{X}$, denote by $[D]$ its image in $\operatorname{Pic} \overline{U}$. By Proposition 2.2.1, we also have $\overline{K}^\times = \overline{K}[U]^\times$. By the Hochschild–Serre spectral sequence, we have the following injective homomorphism

$$\operatorname{Br}_1 U / \operatorname{Br}_0 U \hookrightarrow \mathrm{H}^1(K, \operatorname{Pic} \overline{U}),$$

and in fact it is an **isomorphism** because over a number field $K$, we have $\mathrm{H}^3(K, \mathbb{G}_m) = 0$ from class field theory. Furthermore, the smooth compactification $X$ of $U$ has rational points, hence so does $U$, which comes from the fact that any smooth cubic surface over an infinite field $k$ is unirational over $k$ as soon as it has a $k$-point (see [Kol02]), so we also have $\operatorname{Br}_0 U = \operatorname{Br} K$.

Since $\operatorname{Pic} \overline{U}$ is free and $\operatorname{Gal}(\overline{K}/E)$ acts on $\operatorname{Pic} \overline{U}$ trivially, we obtain that $\mathrm{H}^1(K, \operatorname{Pic} \overline{U}) \cong \mathrm{H}^1(G, \operatorname{Pic} \overline{U})$ by [NSW15, Proposition 1.6.7]. Now in $\operatorname{Pic} \overline{U}$, as $[\ell_6] = 0$, $[\ell] = [\ell_3]$ and $2[\ell_3] = [\ell_1] + [\ell_2] + [\ell_4] + [\ell_5]$, we have the following equalities

$$\begin{cases} \sigma_1([\ell_1]) = -[\ell_2], \sigma_1([\ell_2]) = -[\ell_1], \sigma_1([\ell_3]) = [\ell_3] - [\ell_1] - [\ell_2], \sigma_1([\ell_4]) = [\ell_4]; \\ \sigma_2([\ell_1]) = [\ell_2], \sigma_2([\ell_2]) = [\ell_1], \sigma_2([\ell_3]) = [\ell_3], \sigma_2([\ell_4]) = [\ell_4]; \\ \sigma_3([\ell_1]) = [\ell_1], \sigma_3([\ell_2]) = [\ell_2], \sigma_3([\ell_3]) = [\ell_1] + [\ell_2] - [\ell_3], \sigma_3([\ell_4]) = [\ell_1] + [\ell_2] + [\ell_4] - 2[\ell_3]; \\ \sigma_4([\ell_1]) = [\ell_1], \sigma_4([\ell_2]) = [\ell_2], \sigma_4([\ell_3]) = [\ell_3], \sigma_4([\ell_4]) = 2[\ell_3] - [\ell_1] - [\ell_2] - [\ell_4]. \end{cases} \tag{2.16}$$

Using the inflation-restriction sequences similarly as in the previous proposition, with $\operatorname{Pic} \overline{X}$ replaced by $\operatorname{Pic} \overline{U}$, we can compute that

$$\mathrm{H}^1(G, \operatorname{Pic} \overline{U}) \cong \mathrm{H}^1(\langle \sigma_1 \rangle, \operatorname{Pic} \overline{U}^{\langle \sigma_2, \sigma_3, \sigma_4 \rangle}) = \mathrm{H}^1(\langle \sigma_1 \rangle, \langle [\ell_1] + [\ell_2] \rangle) = \frac{\langle [\ell_1] + [\ell_2] \rangle}{\langle 2([\ell_1] + [\ell_2]) \rangle} \cong \mathbb{Z}/2\mathbb{Z}.$$

Now we produce concrete generators in $\operatorname{Br}_1 U$ for $\operatorname{Br}_1 U / \operatorname{Br}_0 U$. Then $U$ is defined by the equation

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d.$$

We will show that the following quaternion algebras in $\operatorname{Br} K(U)$ are non-constant elements of $\operatorname{Br}_1 U$, and hence they are equal in $\operatorname{Br}_1 U / \operatorname{Br}_0 U$.

$$\begin{cases} \mathcal{A}_1 = \operatorname{Cor}_K^{F_1} \left( x - \frac{k_1 k_2 + \sqrt{(k_1^2-4)(k_2^2-4)}}{2}, (k_1 \sqrt{k_2^2 - 4} + k_2 \sqrt{k_1^2 - 4})^2 \right), \\ \mathcal{A}_2 = \operatorname{Cor}_K^{F_3} \left( y - \frac{k_1 k_4 + \sqrt{(k_1^2-4)(k_4^2-4)}}{2}, (k_1 \sqrt{k_4^2 - 4} + k_4 \sqrt{k_1^2 - 4})^2 \right), \\ \mathcal{A}_3 = \operatorname{Cor}_K^{F_2} \left( z - \frac{k_1 k_3 + \sqrt{(k_1^2-4)(k_3^2-4)}}{2}, (k_1 \sqrt{k_3^2 - 4} + k_3 \sqrt{k_1^2 - 4})^2 \right). \end{cases} \tag{2.17}$$

Indeed, it suffices to prove the claim for $\mathcal{A}_1$ and we only need to show that $\mathcal{A}_1 \in \operatorname{Br} U$, since its formula implies that $\mathcal{A}_1$ becomes zero under the field extension $K \subset K(\sqrt{k_1^2 - 4}, \sqrt{k_2^2 - 4})$, i.e., it is algebraic. By Grothendieck's purity theorem ([Poo17, Theorem 6.8.3]), for any smooth integral variety $Y$ over a field $L$ of characteristic 0, we have the exact sequence

$$0 \to \operatorname{Br} Y \to \operatorname{Br} L(Y) \to \oplus_{D \in Y^{(1)}} \mathrm{H}^1(L(D), \mathbb{Q}/\mathbb{Z}),$$

where the last map is given by the residue along the codimension-one point $D$. Therefore, to prove that our algebras come from a class in $\operatorname{Br} U$, it suffices to show that all their residues are trivial. We will show that

$$\mathcal{A}_1' = \left( x - \frac{k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)}}{2}, (k_1\sqrt{k_2^2-4} + k_2\sqrt{k_1^2-4})^2 \right) \in \operatorname{Br} U_{F_1}$$

so that its corestriction is a well-defined element over $K$. From the data of the 27 lines in Lemma 2.2.6 and the formula of $\mathcal{A}_1'$, any non-trivial residue of $\mathcal{A}_1'$ must occur along an irreducible component of the following divisor:

$$D_1 : x = \frac{(k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)})}{2},$$

$$\left(y + \frac{k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)}}{4}z - \frac{b}{2}\right)^2 - \frac{(k_1\sqrt{k_2^2-4} + k_2\sqrt{k_1^2-4})^2}{16}\left(z - \frac{2b(k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)}) - 8c}{(k_1\sqrt{k_2^2-4} + k_2\sqrt{k_1^2-4})^2}\right)^2 = 0.$$

However, clearly in the function field of any such irreducible component, $(k_1\sqrt{k_2^2-4}+k_2\sqrt{k_1^2-4})^2$ is a square; standard formulae for residues in terms of the tame symbol [GS17, Example 7.1.5, Proposition 7.5.1] therefore show that $\mathcal{A}_1'$ is unramified, and hence $\mathcal{A}_1 \in \operatorname{Br} U$. The residues of $\mathcal{A}_1$ at the lines $L_1, L_2, L_3$ which form the complement of $U$ in $X$ are easily seen to be trivial. One thus also has $\mathcal{A}_1 \in \operatorname{Br} X$.

This element is non-constant by the Faddeev exact sequence (see [CS21, Theorem 1.5.2]), since via the conic fibration $\pi : U \to \mathbb{A}^1, (x, y, z) \mapsto x$, the element

$$\operatorname{Cor}_K^{F_1}\left( x - \frac{(k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)})}{2}, (k_1\sqrt{k_2^2-4} + k_2\sqrt{k_1^2-4})^2 \right)$$

of $\operatorname{Br} K(\mathbb{A}^1) = \operatorname{Br} K(x)$ gives the trivial residue at the closed point $((2x-k_3k_4)^2-(k_3^2-4)(k_4^2-4))$ of $\mathbb{P}_K^1$, while the generator $(x^2-4, B^2+C^2+d+(a+BC)x-x^2)$ of $\operatorname{Ker}(\pi^*)$ (apply [GS17, Theorem 5.4.1] for the pull-back map $\pi^* : \operatorname{Br} K(\mathbb{A}^1) \to \operatorname{Br} K(U)$), where $B + C = \frac{b+c}{2+x}$ and $B - C = \frac{b-c}{2-x}$, gives a nontrivial residue at that point. Alternatively, we can use [CS21, Corollary 11.3.5] on the Brauer group of conic bundles to show that the element $\mathcal{A}_1$ is indeed non-constant. Furthermore, this element will contribute to the Brauer–Manin obstruction to strong approximation for $\mathcal{U}$ (the integral model of $U$ defined over $\mathbb{Z}$ by the same equation) in the next section.

Finally, the fact that $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}$ seen as an element of $\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong \mathbb{Z}/2\mathbb{Z}$ (by abuse of notation) and that $U$ is the open subset of $X$ defined by $t \neq 0$ give us the desired generator $\mathcal{A}_0$ of $\operatorname{Br} X/\operatorname{Br}_0 X \cong \mathbb{Z}/2\mathbb{Z}$. □

### 2.2.3 The transcendental Brauer group

We begin with a specific assumption.

**Assumption 2.2.3.** Let $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(\mathbb{Z})$ and $E := \mathbb{Q}(\sqrt{k_i^2-4}, 1 \leqslant i \leqslant 4)$ such that $[E : \mathbb{Q}] = 16$. For all $1 \leqslant i \neq j \leqslant 4$, assume that $|k_i| \geqslant 3$ such that $(k_i + \sqrt{k_i^2-4})(k_j + \sqrt{k_j^2-4})$ is not a square in $E$.

Now we compute the transcendental Brauer group in our particular case.

**Proposition 2.2.9.** *Let $k = (k_1, k_2, k_3, k_4) \in \mathbb{A}^4(\mathbb{Z})$ satisfy Assumption 2.2.3. Let $U$ be the affine cubic surface over $\mathbb{Q}$ defined by*

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d,$$

*where $a, b, c, d$ are defined by* (2.2)*. Assume that its natural compactification $X \subset \mathbb{P}^3$ is smooth over $\mathbb{Q}$. Set $X_E := X \times_{\mathbb{Q}} E$ and $U_E := U \times_{\mathbb{Q}} E$. Then the natural map $\operatorname{Br} X_E \to \operatorname{Br} U_E$ is an isomorphism. Moreover, $U$ has trivial transcendental Brauer group over $\mathbb{Q}$.*

*Proof.* The proof is inspired by that of [LM20, Proposition 4.1]. Let $\mathcal{B} \in \operatorname{Br} U_E$ be a non-constant Brauer element. Multiplying $\mathcal{B}$ by a constant algebra if necessary, by Proposition 2.2.3 and Theorem 2.2.8, we may assume that $\mathcal{B}$ has order dividing 4 (note that under our assumption of $k$, the field extension $E$ is totally real and thus contains no nontrivial roots of unity). In order to show that $\mathcal{B} \in \operatorname{Br} X_E$, we only need to show that $\mathcal{B}$ is unramified along the three lines $L_i$ on $X$ by Grothendieck's purity theorem ([CS21, Theorem 3.7.2]).

Let $L = L_1$ and $C = L_2 \cup L_3$. Let $L' = L \setminus C$. Note that $L$ meets $C$ at two rational points, so $L'$ is non-canonically isomorphic to $\mathbb{G}_m$. Let the point $(x : y : z : t) = (0 : 1 : 1 : 0) \in L'$ be the identity element of the group law. Then an isomorphism with $\mathbb{G}_m$ is realized via the following homomorphism:

$$\mathbb{G}_m \to X, \qquad u \mapsto (0 : u : 1 : 0). \tag{2.18}$$

The residue of $\mathcal{B}$ along $L$ lies inside $\mathrm{H}^1(L', \mathbb{Z}/2\mathbb{Z})$. Assume by contradiction that the residue is nontrivial. Since the order of $\mathcal{B}$ is a power of 2 dividing 4, then we can assume that the residue has order 2 (up to replacing $\mathcal{B}$ by $2\mathcal{B}$). This means that the residue corresponds to some irreducible degree 2 finite étale cover $f : L'' \to L'$.

Over the field $E$, the conic fiber $C$ over the coordinate $\left( x = \dfrac{k_1k_2 + \sqrt{(k_1^2-4)(k_2^2-4)}}{2} t \right)$ is split, i.e. a union of two lines over $E$. These lines meet $L$ at the points

$$Q_{\pm} = (0 : \frac{(k_1 \pm \sqrt{k_1^2-4})(k_2 \pm \sqrt{k_2^2-4})}{4} : 1 : 0).$$

Let $C_+$ be the irreducible component of $C$ containing $Q_+$, i.e., $C_+ = \ell_1(1,1)$. Consider the restriction of $\mathcal{B}$ to $C_+$. This is well-defined outside of $Q_+$, and since $C_+ \setminus Q_+ \simeq \mathbb{A}^1$ has constant Brauer group, $\mathcal{B}$ actually extends to all of $C_+$. As $C_+$ meets $L$ transversely, by the functoriality of residues ([CS21, Section 3.7]) we deduce that the residue of $\mathcal{B}$ at $Q_+$ is also trivial, so the fiber $f^{-1}(Q_+)$ consists of exactly two rational points. This implies that $L''$ is geometrically irreducible, hence $L'' \cong \mathbb{G}_m$ non-canonically.

Now by choosing a rational point over $Q_+$ and using the above group homomorphism, we may therefore identify the degree 2 cover $L'' \to L'$ with the map

$$\mathbb{G}_m \to X, \qquad u \mapsto (0 : u^2 : 1 : 0). \tag{2.19}$$

However, our assumptions on $k$ imply that $\dfrac{(k_1 + \sqrt{k_1^2-4})(k_2 + \sqrt{k_2^2-4})}{4}$ is not a square in $E^\times$, which gives a contradiction. Thus the residue of $\mathcal{B}$ along $L$ is trivial, and the same holds for the other lines. We conclude that $\mathcal{B}$ is everywhere unramified, hence $\mathcal{B} \in \operatorname{Br} X_E$.

Now let $B \in \operatorname{Br} U$ be a non-constant element. Then over the field extension $E$, the corresponding image of $B$ comes from $\operatorname{Br} X_E$ by the above argument. As $\operatorname{Br} \overline{X} = 0$, it is clear that $B$ is algebraic. The result follows. $\square$

**Remark 2.2.4.** Note that $(\operatorname{Br} U_{\overline{\mathbb{Q}}})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = \mathbb{Z}/2\mathbb{Z}$ by Proposition 2.2.3. However, in the above proposition, the Galois invariant element of order 2 does not descend to a Brauer group element over $\mathbb{Q}$, which is also the case in [LM20, Proposition 4.1].

## 2.3 The Brauer–Manin obstruction

### 2.3.1 Review of the Brauer–Manin obstruction

Here we briefly recall how the Brauer–Manin obstruction works in our setting, following [Poo17, Section 8.2] and [CX09, Section 1]. For each place $v$ of $\mathbb{Q}$ there is a pairing

$$U(\mathbb{Q}_v) \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}$$

coming from the local invariant map

$$\operatorname{inv}_v : \operatorname{Br} \mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}$$

from local class field theory (this is an isomorphism if $v$ is a prime number). This pairing is locally constant on the left by [Poo17, Proposition 8.2.9]. Any element $\alpha \in \operatorname{Br} X$ pairs trivially on $U(\mathbb{Q}_v)$ for almost all $v$, thus taking the sum of the local pairings gives a pairing

$$\prod_p U(\mathbb{Q}_v) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z}.$$

This factors through the group $\operatorname{Br} X/\operatorname{Br} \mathbb{Q}$ and pairs trivially with the elements of $U(\mathbb{Q})$. For $B \subseteq \operatorname{Br} X$, let $(\prod_v U(\mathbb{Q}_v))^B$ be the left kernel of this pairing with respect to $B$. By Theorem 3.8, the group $\operatorname{Br} X/\operatorname{Br} \mathbb{Q}$ is generated by the algebra $\mathcal{A}$. Thus in our case, it suffices to consider the sequence of inclusions $U(\mathbb{Q}) \subseteq (\prod_v U(\mathbb{Q}_v))^{\mathcal{A}} \subseteq \prod_v U(\mathbb{Q}_v)$. In particular, if $(\prod_v U(\mathbb{Q}_v))^{\mathcal{A}} = \emptyset$ then $\mathcal{A}$ obstructs the Hasse principle on $U$, and if the latter inclusion is strict, then $\mathcal{A}$ gives an obstruction to *weak approximation* on $U$. However, we do not study the (rational) Hasse principle or weak approximation for these Markoff-type cubic surfaces; instead, we focus on their *integral* models.

For **integral points**, any element $\alpha \in \operatorname{Br} U$ pairs trivially on $\mathcal{U}(\mathbb{Z}_p)$ for almost all primes $p$, so we obtain a pairing $U(\mathbf{A}_{\mathbb{Q}}) \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}$. As the local pairings are locally constant, we obtain a well-defined pairing

$$\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_\bullet \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}.$$

For $B \subseteq \operatorname{Br} U$, let $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_\bullet^B$ be the left kernel with respect to $B$, and let $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_\bullet^{\operatorname{Br}} = \mathcal{U}(\mathbf{A}_{\mathbb{Z}})_\bullet^{\operatorname{Br} U}$. By abuse of notation, from now on we write the *reduced* Brauer–Manin set $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_\bullet^B$ in the standard way as $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})^B$. The set $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})^B$ depends only on the image of $B$ in the quotient $\operatorname{Br} U/\operatorname{Br}_0 U$. By Theorem 2.2.8, the map $\langle \mathcal{A} \rangle \to \operatorname{Br}_1 U/\operatorname{Br} \mathbb{Q}$ is an isomorphism, hence $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\operatorname{Br}_1} = \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\operatorname{Br}_1 U} = \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathcal{A}}$. We have the inclusions $\mathcal{U}(\mathbb{Z}) \subseteq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathcal{A}} \subseteq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})$, so that $\mathcal{A}$ can obstruct the *integral Hasse principle* or *strong approximation* on $\mathcal{U}$.

Let $V$ be dense Zariski open in $U$. As $U$ is smooth, the set $V(\mathbb{Q}_v)$ is dense in $U(\mathbb{Q}_v)$ for all places $v$. Moreover, $\mathcal{U}(\mathbb{Z}_p)$ is open in $U(\mathbb{Q}_p)$, hence $V(\mathbb{Q}_p) \cap \mathcal{U}(\mathbb{Z}_p)$ is dense in $\mathcal{U}(\mathbb{Z}_p)$. As the local pairings are locally constant, we may restrict our attention to $V$ to calculate the local invariants of a given element in $\operatorname{Br} U$. In particular, here we take the open subset $V$ given by $((k_1^2-4)(k_2^2-4)-(2x-k_1k_2)^2)((k_1^2-4)(k_4^2-4)-(2y-k_1k_4)^2)((k_1^2-4)(k_3^2-4)-(2z-k_1k_3)^2) \neq 0$.

### 2.3.2 Brauer–Manin obstruction from a quaternion algebra

Now we consider the family of *smooth* Markoff-type cubic surfaces $U$ defined over $\mathbb{Q}$ and their integral models $\mathcal{U}$ defined over $\mathbb{Z}$ by the equation (2.1):

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d,$$

which satisfy the hypotheses of Theorem 2.2.8 and Lemma 2.2.6 with $K = \mathbb{Q}$. Set $f := x^2 + y^2 + z^2 + xyz - ax - by - cz - d \in \mathbb{Z}[x, y, z]$. First of all, we study the existence of local integral points on those affine cubic surfaces given by $f = 0$. Note that we always have $U(\mathbb{Q}) \neq \emptyset$, so $U(\mathbb{R}) \neq \emptyset$.

**Proposition 2.3.1** (Assumption A)**.** *If $k = (k_1, k_2, k_3, k_4) \in (\mathbb{Z} \setminus [-2, 2])^4$ satisfies $k_1 \equiv -1 \mod 16$, $k_i \equiv 5 \mod 16$ and $k_1 \equiv 1 \mod 9$, $k_i \equiv 5 \mod 9$ for $2 \leqslant i \leqslant 4$, such that $(k_i, k_j) = 1$, $(k_i^2 - 4, k_j^2 - 4) = 3$ for $1 \leqslant i \neq j \leqslant 4$, and $(k_1^2 - 2, k_2^2 - 2, k_3^2 - 2, k_4^2 - 2) = 1$, then $\mathcal{U}(\mathbf{A}_\mathbb{Z}) \neq \emptyset$.*

*Proof.* Our argument here makes use of Hensel's lemma (see [Conb]). With our specific choice of $k$ in the assumption, we obtain:

(1) Prime powers of $p = 2$: The only solution modulo 2 is the singular $(1, 0, 0)$ (up to permutation). However, we find a solution $(1, 0, 0)$ modulo 8 with $2x + yz \equiv 2 \mod 8$, so twice the valuation at 2 of the partial derivative at $x$ is less than the valuation at 2 of $f(x, y, z)$. This solution then lifts to a 2-adic integer solution by Hensel's lemma (fixing the variables $y, z$).

(2) Prime powers of $p = 3$: We find the non-singular solution $(1, 0, 0)$, which lifts to a 3-adic integer solution by Hensel's lemma (fixing the variables $, y, z$).

(3) Prime powers of $p \geqslant 5$: We would like to find a non-singular solution modulo $p$ of the equations $f = 0$ which does not satisfy simultaneously

$$df = 0 : 2x + yz = a, 2y + xz = b, 2z + xy = c.$$

For simplicity, we will find a sufficient condition for the existence of a non-singular solution whose at least one coordinate is zero. First, it is clear that the equation $f = 0$ always has a solution whose one coordinate is 0: indeed, take $z = 0$, then $f = 0$ is equivalent to $(2x - a)^2 + (2y - b)^2 = (k_1^2 + k_3^2 - 4)(k_2^2 + k_4^2 - 4)$, and every element in $\mathbb{F}_p$ can be expressed as a sum of two squares. Now assume that all such points in $U(\mathbb{F}_p)$ with one coordinate equal to 0 are singular. Then modulo $p$, if $z = 0$ the required equations become $f = 0, (2x = a, 2y = b, xy = c)$, plus all the permutations for $x = 0$ and $y = 0$. For convenience, we drop the phrase "modulo $p$". From these equations, we get

$$(k_1^2 + k_3^2 - 4)(k_2^2 + k_4^2 - 4) = 0, ab = 4c \, (= 4xy) \tag{2.20}$$

plus all the permutations, respectively

$$(k_1^2 + k_2^2 - 4)(k_3^2 + k_4^2 - 4) = 0, bc = 4a \tag{2.21}$$

and

$$(k_1^2 + k_4^2 - 4)(k_2^2 + k_3^2 - 4) = 0, ac = 4b. \tag{2.22}$$

We will choose $k$ which does not satisfy all of these equations simultaneously to get a contradiction to our assumption at the beginning.

Now if $k$ satisfies all the above equations, we first require that $(k_i^2 - 4, k_j^2 - 4) = 3$ for any $1 \leqslant i \neq j \leqslant 4$. Next, without loss of generality (WLOG), assume in (2.22) that $k_1^2 + k_4^2 - 4 = 0$ with $ac = 4b = (k_1^2 + k_4^2)b$, then putting the formulae of $a, b, c$ in $ac - (k_1^2 + k_4^2)b = 0$ gives us $k_1 k_4 (k_2^2 + k_3^2 - k_1^2 - k_4^2) = 0$, hence either $k_2^2 + k_3^2 - 4 = 0$ or $k_1 k_4 = 0$. (We also have similar equations for all the other cases.)

If $k_2^2+k_3^2-4=0$, then $\sum_{i=1}^4 k_i^2-8=0$, and so from (2.20), (2.21) we obtain $k_i^2+k_j^2-4=0$ for any $i\neq j$, hence $k_i^2-2=0$ for all $1\leqslant i\leqslant 4$. Otherwise, if $k_2^2+k_3^2-4\neq 0$ but $k_1k_4=0$, WLOG assume in (2.22) that $k_1=0$, then $k_4^2-4=0$ and so from (2.20), (2.21) we have the following possibilities: $k_2^2-4=k_3^2-4=0$, or $k_2=k_3=0$. Therefore, we immediately deduce a sufficient condition for the nonexistence of singular solutions modulo $p\geqslant 5$: $(k_i,k_j)=1$, $(k_i^2-4,k_j^2-4)=3$ for any $1\leqslant i\neq j\leqslant 4$, and $(k_1^2-2,k_2^2-2,k_3^2-2,k_4^2-2)=1$.

As a result, assuming the hypothesis of the proposition, it is clear that $U(\mathbb{F}_p)$ has a smooth point, which then lifts to a $p$-adic integral point by Hensel's lemma (with respect to the variable at which the partial derivative is nonzero modulo $p$, fixing other variables).

□

We keep Assumption A to ensure that $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})\neq\emptyset$ and study the Brauer–Manin obstruction to the existence of integral points. Note that our specific choice of $k$ implies that $[E:\mathbb{Q}]=16$. Indeed, it is clear that with our assumption, $k_i^2-4>0$ is not a square in $\mathbb{Q}$ for $1\leqslant i\leqslant 4$. Then $[E:\mathbb{Q}]<16$ if and only if there are some $i\neq j$ such that $(k_i^2-4)(k_j^2-4)$ is a square in $\mathbb{Q}$; however, since $(k_i^2-4,k_j^2-4)=3$ and $k_i^2-4=3l^2$ does not have any solution modulo 8 with $k_i$ odd, that cannot happen.

Now let us calculate the local invariants of the following quaternion algebras as elements of the algebraic Brauer group $\operatorname{Br}_1 U$:

$$\begin{cases}\mathcal{A}_1=\operatorname{Cor}_{\mathbb{Q}}^{F_1}\left(x-\frac{k_1k_2+\sqrt{(k_1^2-4)(k_2^2-4)}}{2},(k_1\sqrt{k_2^2-4}+k_2\sqrt{k_1^2-4})^2\right),\\ \mathcal{A}_2=\operatorname{Cor}_{\mathbb{Q}}^{F_3}\left(y-\frac{k_1k_4+\sqrt{(k_1^2-4)(k_4^2-4)}}{2},(k_1\sqrt{k_4^2-4}+k_4\sqrt{k_1^2-4})^2\right),\\ \mathcal{A}_3=\operatorname{Cor}_{\mathbb{Q}}^{F_2}\left(z-\frac{k_1k_3+\sqrt{(k_1^2-4)(k_3^2-4)}}{2},(k_1\sqrt{k_3^2-4}+k_3\sqrt{k_1^2-4})^2\right),\end{cases}$$

where $F_i=\mathbb{Q}(\sqrt{(k_1^2-4)(k_{i+1}^2-4)})$ for $1\leqslant i\leqslant 3$. Now for each $i$, we have the local invariant map

$$\operatorname{inv}_p\mathcal{A}_i:U(\mathbb{Q}_p)\to\mathbb{Z}/2\mathbb{Z},\qquad x\mapsto\operatorname{inv}_p\mathcal{A}_i(x).$$

**Lemma 2.3.2** (Assumption B). *Let $k=(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4$ and $p\geqslant 11$ be a prime such that $(2,p)_p=1/2$ and $k_1-2\equiv p \bmod p^3$. Then there exist $k_2,k_3,k_4$ such that $k_2^2-4$ is a quadratic nonresidue* mod $p$, *$k_3\equiv k_4\not\equiv 0,2$ mod $p$, $k_2-2\not\equiv 2k_3$ mod $p$, and $k_2,k_3,k_4$ satisfy the cubic equation defining an affine Markoff surface $S$:*

$$k_2^2+k_3^2+k_4^2-k_2k_3k_4\equiv(p+2)^2 \mod p^2.$$

*Furthermore, for any $k$ satisfying all the above conditions (Assumption B) and Assumption A, the local invariant map of the quaternion algebra $\mathcal{A}_1$ at $p$ is surjective.*

*Proof.* For convenience (and by abuse of notation), we will also write $\mathcal{A}_1$ (resp. $F_1$) as $\mathcal{A}$ (resp. $F$). We denote $(k_1\sqrt{k_2^2-4}+k_2\sqrt{k_1^2-4})$ by $\alpha_1$ and write it simply as $\alpha$. Set $D:=(k_1^2-4)(k_2^2-4)$.

First of all, over $F = \mathbb{Q}(\sqrt{D})$ the affine cubic equation of $\mathcal{U}$ can be rewritten equivalently as

$$\begin{aligned} f(x,y,z,k_1,k_2,k_3,k_4) = \\ \left(x - \frac{k_1k_2+\sqrt{D}}{2}\right)\left(x - \frac{k_1k_2-\sqrt{D}}{2} - k_3k_4 + yz\right) + \left(y + \frac{k_1k_2+\sqrt{D}}{4}z - \frac{b}{2}\right)^2 \\ - \frac{\alpha^2}{16}\left(z - \frac{2b(k_1k_2+\sqrt{D})-8c}{\alpha^2}\right)^2 = 0, \end{aligned} \tag{2.23}$$

for all $(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4$ satisfying our hypothesis. Since $\mathrm{v}_p(D) = 1$, $p$ ramifies over $F = \mathbb{Q}(\sqrt{D})$, i.e., $p\mathcal{O}_F = \mathfrak{p}^2$ where $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_F$. Therefore, we have $\mathrm{N}\mathfrak{p} = p$ and

$$\mathcal{O}_F/\mathfrak{p} \cong \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}.$$

Following the proof of [LM20, Proposition 5.5], for all $u_2, u_3, u_4 \in \mathbb{F}_p^*$, there exists an $\mathbb{F}_p$-point on the variety

$$(2k_2 - k_3k_4)^2 = (k_3^2-4)(k_4^2-4), k_2 - 2 = u_2v_2^2, k_3 - 2 = u_3v_3^2, k_4 - 2 = u_4v_4^2$$

which satisfy $v_2v_3v_4 \neq 0$. As $(k_2-2)(k_3-2)(k_4-2) \neq 0$, we find from [LM20, Lemma 5.3] that this gives rise to a smooth $\mathbb{F}_p$-point of $S$, hence a $\mathbb{Z}/p^2\mathbb{Z}$-point with the same residue modulo $p$ by Hensel's lemma. Now to construct the given $\mathbb{F}_p$-point, we restrict our attention to the subvariety given by $k_3 = k_4 \bmod p$ then assume that $u_3 = u_4 \bmod p$ and $u_2$ is a quadratic nonresidue mod $p$. The above equations then become

$$(2k_2 - k_3k_4)^2 = (k_3^2-4)^2, k_2 - 2 = u_2v_2^2, k_3 - 2 = u_3v_3^2.$$

Factoring the left hand side, it suffices to solve the equations

$$k_2 = k_3^2 - 2, k_2 - 2 = u_2v_2^2, k_3 - 2 = u_3v_3^2.$$

This then gives the equation of an affine curve

$$u_2v_2^2 = u_3^2v_3^4 + 4u_3v_3^2.$$

By the argument in the proof of [LM20, Proposition 5.5], this affine curve has a unique singular point $(v_2, v_3) = (0,0)$ and has $p-2$ many $\mathbb{F}_p$-points. Of these points at most 3 satisfy $v_2v_3 = 0$, and $k_3 = 0$ gives only at most 4 points, hence providing $p - 2 - 3 - 4 = p - 9 > 0$, there exists an $\mathbb{F}_p$-point $(k_2, k_3, k_4)$ with the properties:

$$(k_2-2)(k_3-2)(k_4-2) \neq 0, k_3 = k_4 \neq 0.$$

Since $k_3 \neq 0$, we have $k_2 = k_3^2 - 2 \neq -2$, so $k_2^2 - 4 \neq 0$ in $\mathbb{F}_p$. Moreover, as $u_2$ is a quadratic nonresidue modulo $p$, so is $k_2 - 2 = k_3^2 - 4$. Since $k_2 + 2 = k_3^2$ is a nonzero square in $\mathbb{F}_p$, we deduce that $k_2^2 - 4$ is a quadratic nonresidue modulo $p$, as required.

Now after lifting from the smooth $\mathbb{F}_p$-point to a $\mathbb{Z}/p^2\mathbb{Z}$-point (with the same residue modulo $p$), if $k_2 - 2 \equiv 2k_3 \bmod p$ then we can take $k_2' = k_2$, $k_3' = -k_3$ and $k_4' = -k_4$ (which still satisfy the cubic equation mod $p^2$) to have $k_2' - 2 \not\equiv 2k_3' \bmod p$ since $k_3 \not\equiv 0 \bmod p$ and $p$ is odd. Therefore, there exist integers $k_2, k_3, k_4$ satisfying the required properties by the Chinese Remainder Theorem.

Now assume that $k = (k_1, k_2, k_3, k_4)$ satisfies all the above congruence conditions and Assumption A. Since $p\mathcal{O}_F = \mathfrak{p}^2$ and $\mathcal{O}_F/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$, by Hensel's lemma we deduce that when $k_2^2 - 4$ is a quadratic nonresidue modulo $p$ over $\mathbb{Z}$, then $k_2^2 - 4$ is a quadratic nonresidue modulo $\mathfrak{p}$ over $\mathcal{O}_F$. Recalling that $k_1 - 2 \equiv p \bmod p^3$, we can choose a $\mathbb{Z}/p^3\mathbb{Z}$-point $(x, y, z)$ of $\mathcal{U}$ such that

$$x - k_2 = p, y - k_4 = p, z - k_3 = p,$$

from which we have (in $\mathbb{Z}/p^3\mathbb{Z}$):

$$\begin{aligned}
&f(x, y, z, k_1, k_2, k_3, k_4) = f(x, y, z, 2, k_2, k_3, k_4) - \sum_{i\in\{2,3,4\}} pk_i(k_i + p) + 4p + p^2 + pk_2k_3k_4 \\
&= (x - k_2)(x - k_2 + yz - k_3k_4) + \left(y + \frac{k_2z}{2} - \frac{b}{2}\right)^2 - \frac{k_2^2 - 4}{4}(z - k_3)^2 \\
&- \sum_{i\in\{2,3,4\}} pk_i(k_i + p) + 4p + p^2 + pk_2k_3k_4 \\
&= p(p + p^2 + p(k_3 + k_4)) + p^2\left(1 + \frac{k_2}{2}\right)^2 - p^2\left(\frac{k_2^2 - 4}{4}\right) - \sum_{i\in\{2,3,4\}} pk_i(k_i + p) + 4p + p^2 + pk_2k_3k_4 \\
&= p(p + p^2 + p(k_3 + k_4)) + p(2p + pk_2) - \sum_{i\in\{2,3,4\}} p(k_i^2 + pk_i) + p(4 + p + k_2k_3k_4) \\
&= p(4 + 4p + p^2 - (k_2^2 + k_3^2 + k_4^2) + k_2k_3k_4) \\
&= p((p + 2)^2 - (k_2^2 + k_3^2 + k_4^2 - k_2k_3k_4)) = 0,
\end{aligned}$$

and

$$f'_x(x, y, z) = 2x + yz - k_1k_2 - k_3k_4 = 2(x - k_2) - pk_2 + p^2 + p(k_3 + k_4) = p(p + 2 - k_2 + k_3 + k_4).$$

Therefore $\mathrm{v}_p(f) \geqslant 3 > 2 = 2\mathrm{v}_p(f'_x)$, by Hensel's lemma (fixing the variables $, y, z)$, this point lifts to a $\mathbb{Z}_p$-point $(x, y, z)$ (abuse of notation) with the same residue modulo $p^2$. Then since $\mathrm{v}_\mathfrak{p}(\sqrt{D}) = \mathrm{v}_p(D) = 1$, the local invariant at $p$ of $\mathcal{A}$ at this point is equal to

$$\left(x - \frac{k_1k_2 + \sqrt{D}}{2}, \alpha^2\right)_\mathfrak{p} = \left(x - \frac{k_1k_2 + \sqrt{D}}{2}, k_2^2 - 4\right)_\mathfrak{p} = 1/2$$

by the formulae in [Neu13, Proposition II.1.4 and Proposition III.3.3].

It is also clear that there exists a $\mathbb{Z}_p$-point such that $x - k_2$ is not divisible by $p$, which gives the local invariant of $\mathcal{A}$ at $p$ the value 0. Indeed, if every $\mathbb{Z}_p$-point satisfies that $x - k_2$ is divisible by $p$, then as $k_1 - 2 \equiv 0 \bmod p$, the affine equation of $\mathcal{U}$ over $\mathbb{F}_p$ becomes

$$(x - k_2)(x - k_2 + yz - k_3k_4) + \left(y + \frac{k_2z}{2} - \frac{b}{2}\right)^2 - \frac{k_2^2 - 4}{4}(z - k_3)^2 = 0.$$

From the fact that $k_2^2 - 4$ is a quadratic nonresidue mod $p$, one must obtain that (in $\mathbb{F}_p$):

$$y + \frac{k_2z}{2} - \frac{b}{2} = 0, z - k_3 = 0,$$

which gives $y = k_4, z = k_3$. Therefore, in $\mathbb{F}_p$ we have

$$2x + yz = k_1k_2 + k_3k_4, 2y + xz = k_1k_4 + k_2k_3, 2z + xy = k_1k_3 + k_2k_4,$$

which implies that $f'_x = f'_y = f'_z = 0$. This cannot be true for every $\mathbb{Z}_p$-point $(x, y, z)$ of $\mathcal{U}$, since by Assumption A we have proved in Proposition 2.3.1 that there always exists a non-singular solution modulo $p$ of the affine Markoff-type cubic equation (with at least one coordinate zero) which lifts to a $\mathbb{Z}_p$-point by Hensel's lemma.

In conclusion, the local invariant map of $\mathcal{A}$ at the prime $p$ satisfying our hypothesis is indeed surjective. $\square$

**Theorem 2.3.3.** *Let $k \in (\mathbb{Z} \setminus [-2, 2])^4$ satisfy Assumptions A and B. Then we have a Brauer–Manin obstruction to strong approximation for $\mathcal{U}$ given by the class of $\mathcal{A} = \mathcal{A}_1$ in $\mathrm{Br}_1 U/\mathrm{Br}\,\mathbb{Q}$ (i.e. $\mathcal{U}(\mathbf{A}_\mathbb{Z}) \neq \mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathcal{A}} = \mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathrm{Br}_1}$) and no* algebraic *Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}$ (i.e. $\mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathrm{Br}_1} \neq \emptyset$).*

*Proof.* By abuse of notation, we also denote the class of $\mathcal{A}$ by the Brauer group element itself. For any point $\mathbf{u} = (x, y, z) \in \mathcal{U}(\mathbf{A}_\mathbb{Z})$, from the above Lemma we can find a local point $\mathbf{u}_p = (x_p, y_p, z_p) \in \mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\, \mathcal{A}(\mathbf{u}_p) = -\sum_{q \neq p} \mathrm{inv}_q \mathcal{A}(\mathbf{u})$ and $\mathbf{u}'_p = (x'_p, y'_p, z'_p) \in \mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\, \mathcal{A}(\mathbf{u}'_p) = 1/2 - \sum_{q \neq p} \mathrm{inv}_q\, \mathcal{A}(\mathbf{u})$. Then we obtain a point in $\mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathcal{A}}$ by replacing the $p$-part of $\mathbf{u}$ by $\mathbf{u}_p$, and a point in $\mathcal{U}(\mathbf{A}_\mathbb{Z})$ but not in $\mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathcal{A}}$ by replacing the $p$-part of $\mathbf{u}$ by $\mathbf{u}'_p$.

Therefore, we have a Brauer–Manin obstruction to strong approximation and no algebraic Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}$. $\square$

**Example 2.3.1.** For $p = 11$: Take $k_1 = 11^3.102 + 13 = 135775, k_2 = 11^2.86 + 111 = 10517, k_3 = 11^2.(119 - 144) + 6 = -3019, k_4 = 11^2.(119 + 144) + 6 = 31829$.

**Remark 2.3.2.** In addition, if $k$ satisfies Assumption 2.2.3 from the previous section, then we can drop the term "algebraic" from the statement of Theorem 2.3.3, and we will have no Brauer–Manin obstruction to the integral Hasse principle.

### 2.3.3 Some counting results

In this part, we compute the number of examples of existence for local integral points as well as the number of counterexamples to strong approximation for the Markoff-type cubic surfaces in question which can be explained by the Brauer–Manin obstruction. More precisely, we consider the natural density of $k \in (\mathbb{Z} \setminus [-2, 2])^4$ satisfying $k_1 \equiv -1 \bmod 16$, $k_i \equiv 5 \bmod 16$ and $k_1 \equiv 1 \bmod 9$, $k_i \equiv 5 \bmod 9$ for $2 \leqslant i \leqslant 4$, such that $(k_i, k_j) = 1$, $(k_i^2 - 4, k_j^2 - 4) = 3$ for $1 \leqslant i \neq j \leqslant 4$, and $(k_1^2 - 2, k_2^2 - 2, k_3^2 - 2, k_4^2 - 2) = 1$ (then satisfying the additional hypothesis in Lemma 2.3.2). Note that the finite number of $k \in [-2, 2]^4$ is negligible here.

In fact, for now we can only give an asymptotic lower bound. To get a lower bound, it is enough to count the number of examples satisfying stronger conditions, namely the congruence conditions for the $k_i$ and the common divisor condition

$$\gcd(k_i(k_i^2 - 2)(k_i^2 - 4), k_j(k_j^2 - 2)(k_j^2 - 4)) = 3$$

for $1 \leqslant i \neq j \leqslant 4$. To do this, we make use of a natural generalization of Ekedahl–Poonen's formula in [Poo03, Theorem 3.8] as follows.

**Proposition 2.3.4.** *Let $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, $1 \leqslant i \leqslant s$ for some $s \in \mathbb{Z}_{>1}$, be $s$ polynomials that are mutually relatively prime as elements of $\mathbb{Q}[x_1, \ldots, x_n]$. For each $1 \leqslant i \neq j \leqslant s$, let*

$$\mathcal{R}_{i,j} := \{a \in \mathbb{Z}^n : \gcd(f_i(a), f_j(a)) = 1\}.$$

*Then* $\mu(\bigcap_{i\neq j} \mathcal{R}_{i,j}) = \prod_p (1 - c_p/p^n)$, *where* $p$ *ranges over all primes of* $\mathbb{Z}$, *and* $c_p$ *is the number of* $x \in (\mathbb{Z}/p\mathbb{Z})^n$ *satisfying at least one of* $f_i(x) = f_j(x) = 0$ *in* $\mathbb{Z}/p\mathbb{Z}$ *for* $1 \leqslant i \neq j \leqslant s$.

*Proof.* We also have a generalization of [Poo03, Lemma 5.1] as follows.

**Lemma 2.3.5.** *Let* $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, $1 \leqslant i \leqslant s$ *for some* $s \in \mathbb{Z}_{>1}$, *be* $s$ *polynomials that are mutually relatively prime as elements of* $\mathbb{Q}[x_1, \ldots, x_n]$. *For each* $1 \leqslant i \neq j \leqslant s$, *let*

$$\mathcal{Q}_{i,j,M} := \{a \in \mathbb{Z}^n : \exists\, p \text{ such that } p \geqslant M \text{ and } p \mid f_i(a), f_j(a)\}.$$

*Then* $\lim_{M\to\infty} \overline{\mu}(\bigcup_{i\neq j} \mathcal{Q}_{i,j,M}) = 0$.

*Proof.* The result immediately follows from the inequality $\overline{\mu}(\bigcup_{i\neq j} \mathcal{Q}_{i,j,M}) \leqslant \sum_{i\neq j} \overline{\mu}(\mathcal{Q}_{i,j,M})$ and the original result of [Poo03, Lemma 5.1]. $\square$

Next, we proceed similarly as in the proof of [Poo03, Theorem 3.1]. Let $P_M$ denote the set of prime numbers of $\mathbb{Z}$ such that $p < M$. Approximate $\mathcal{R}_{i,j}$ by

$$\mathcal{R}_{i,j,M} := \{a \in \mathbb{Z}^n : f_i(a) \text{ and } f_j(a) \text{ are not both divisible by any prime } p \in P_M\}.$$

Define the ideal $I$ as the product of all $(p)$ for $p \in P_M$. Then $\mathcal{R}_{i,j,M}$ is a union of cosets of the subgroup $I^n \subset \mathbb{Z}^n$. Hence $\mu(\bigcap_{i\neq j} \mathcal{R}_{i,j,M})$ is the fraction of residue classes in $(\mathbb{Z}/I)^n$ in which for all $p \in P_M$, for all $1 \leqslant i \neq j \leqslant s$, at least one of $f_i(a)$ and $f_j(a)$ is nonzero modulo $p$. Applying the Chinese Remainder Theorem, we obtain that $\mu(\bigcap_{i\neq j} \mathcal{R}_{i,j,M}) = \prod_{p\in P_M}(1 - c_p/p^n)$. By the above lemma,

$$\mu(\bigcap_{i\neq j} \mathcal{R}_{i,j}) = \lim_{M\to\infty} \mu(\bigcap_{i\neq j} \mathcal{R}_{i,j,M}) = \prod_p (1 - c_p/p^n). \tag{2.24}$$

Since $f_i$ and $f_j$ are relatively prime as elements of $\mathbb{Q}[x_1, \ldots, x_n]$ for any $i \neq j$, there exists a nonzero $u \in \mathbb{Z}$ such that every $f_i = f_j = 0$ defines a subscheme of $\mathbb{A}^n_{\mathbb{Z}[1/u]}$ of codimension at least 2. Thus $c_p = O(p^{n-2})$ as $p \to \infty$, and the product converges. $\square$

**Theorem 2.3.6.** *Let* $\mathcal{U}$ *be the affine scheme over* $\mathbb{Z}$ *defined by*

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d,$$

*where*

$$\begin{cases} a = k_1k_2 + k_3k_4 \\ b = k_1k_4 + k_2k_3 \\ c = k_1k_3 + k_2k_4 \end{cases} \quad \text{and} \quad d = 4 - \sum_{i=1}^{4} k_i^2 - \prod_{i=1}^{4} k_i,$$

*such that the projective closure* $X \subset \mathbb{P}^3_{\mathbb{Q}}$ *of* $U = \mathcal{U} \times_{\mathbb{Z}} \mathbb{Q}$ *is smooth. Then we have*

$$\#\{k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4, |k_i| \leqslant M \ \forall\, 1 \leqslant i \leqslant 4 : \mathcal{U}(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset\} \asymp M^4 \tag{2.25}$$

*and also*

$$\#\{k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4, |k_i| \leqslant M \ \forall\, 1 \leqslant i \leqslant 4 : \mathcal{U}(\mathbf{A}_{\mathbb{Z}}) \neq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}_1} \neq \emptyset\} \asymp M^4 \tag{2.26}$$

*as* $M \to +\infty$.

*Proof.* Denote by $\mathcal{S}$ the set in question and $\mathcal{M}$ is the set of $k$ such that $|k_i| \leqslant M$ for $1 \leqslant i \leqslant 4$. Hence, using the same notation as in Proposition 2.3.4, the density we are concerned with is given by

$$\mu(\mathcal{S}) = \lim_{M\to\infty} \frac{\#(\mathcal{S}\cap\mathcal{M})}{\#\mathcal{M}} = \lim_{M\to\infty} \frac{\#(\mathcal{S}\cap\mathcal{M})}{M^4}.$$

We apply Proposition 2.3.4 with the polynomials $f_i = x_i(x_i^2-2)(x_i^2-4)/3 \in \mathbb{Z}[x_1', x_2', x_3', x_4']$ for $1 \leqslant i \leqslant 4$ in which we write $x_1 = 144x_1' + 127$, $x_j = 144x_j' + 5$ for $2 \leqslant j \leqslant 4$ (resp. $x_i = 144.p_0^3 x_i' + r_i$ where the $x_i$ is actually the $k_i$ and $p_0 \geqslant 11$ is the chosen prime in Lemma 2.3.2 and the residues $r_i$ satisfy the hypotheses of the lemma), and using the inclusion-exclusion principle we can compute that $c_2 = 0$, $c_3 = 0$,

$$c_{p,1} = C_4^2.3^2.p^2 - (3.C_4^3.3^3.p + 3.3^4) + (4.3^3.p + 16.3^4) - C_6^4.3^4 + C_6^5.3^4 - 3^4 = 27(2p^2 - 8p + 9)$$

if $p > 3$ and $p \equiv \pm 3 \pmod 8$ and

$$c_{p,2} = C_4^2.5^2.p^2 - (3.C_4^3.5^3.p + 3.5^4) + (4.5^3.p + 16.5^4) - C_6^4.5^4 + C_6^5.5^4 - 5^4 = 25(6p^2 - 40p + 75)$$

if $p > 3$ and $p \equiv \pm 1 \pmod 8$ $\big($resp. except for $p_0$ with $p_0 \equiv \pm 3 \bmod 8$, since $k_2 \equiv k_3^2 - 2 \equiv k_4^2 - 2$, $k_3 \equiv k_4 \not\equiv 0$ and $k_2^2 - 4 \not\equiv 0 \pmod{p_0}$, we have $k_i(k_i^2-2)(k_i^2-4) \not\equiv 0 \bmod p_0$ for all $2 \leqslant i \leqslant 4$, and so $c_{p_0} = 0\big)$. Applying (2.24) gives us a positive density in (2.25):

$$\mu\Big(\bigcap_{i\neq j} \mathcal{R}_{i,j}\Big) = \frac{1}{144^4} \prod_{\substack{p>3\\ p\equiv\pm3 \pmod 8}} \left(1 - \frac{c_{p,1}}{p^4}\right) \prod_{\substack{p>3\\ p\equiv\pm1 \pmod 8}} \left(1 - \frac{c_{p,2}}{p^4}\right),$$

and also in (2.26):

$$\mu\Big(\bigcap_{i\neq j} \mathcal{R}_{i,j}\Big) = \frac{1}{144^4.p_0^{12}} \left(1 - \frac{c_{p_0}}{p_0^4}\right) \prod_{\substack{p>3, p\neq p_0\\ p\equiv\pm3 \pmod 8}} \left(1 - \frac{c_{p,1}}{p^4}\right) \prod_{\substack{p>3\\ p\equiv\pm1 \pmod 8}} \left(1 - \frac{c_{p,2}}{p^4}\right).$$

Finally, we need to consider the number of surfaces which are singular (see necessary and sufficient conditions given in Lemma 2.2.6). By Lemma 2.2.6, the total number of $k$ with $|k_i| \leqslant M$ for $1 \leqslant i \leqslant 4$ such that the surfaces are singular is just $O(M^3)$ as $M \to \infty$, hence it is negligible. Therefore, we obtain that $\mu(\mathcal{S}) \geqslant \mu(\bigcap_{i\neq j} \mathcal{R}_{i,j}) > 0$. □

**Remark 2.3.3.** Continuing from the previous remark, it would be interesting if one can find a way to include Assumption 2.2.3 into the counting result, which would help us consider the Brauer–Manin set with respect to the whole Brauer group instead of only its algebraic part.

## 2.4 Further remarks

In this section, we compare the results that we obtain in this chapter with those in the previous papers studying Markoff surfaces, namely [GS22], [LM20] and [CWX20].

First of all, for Markoff surfaces, we see from [LM20] that given $|m| \leqslant M$ as $M \to +\infty$, the number of counterexamples to the integral Hasse principle which can be explained by the Brauer–Manin obstruction is $M^{1/2}/(\log M)^{1/2}$ asymptotically. This implies that *almost all* Markoff surfaces with a nonempty set of local integral points have a nonempty Brauer–Manin set, and for the surfaces (relative character varieties) that we study in this chapter, a similar phenomenon is expected to occur (looking at the order of magnitude in the main counting result above). However, at present, we are not able to compute the number of these surfaces for which the Brauer–Manin obstruction can or cannot explain the integral Hasse principle in a similar way as in [LM20] and [CWX20]. If we can solve (partly) this problem, it will be really significant and then we may see a bigger picture of the arithmetic of Markoff-type cubic surfaces.

### 2.4.1 Markoff descent and reduction theory

Recall that, in order to show that the integral Hasse principle fails in [CWX20], the authors also make use of the fundamental set, or *box*, in [GS22] as a very useful tool to bound the set of integral points significantly and then use the Brauer group elements to finish the proofs. This mixed method has been the most effective way to prove counterexamples to the integral Hasse principle for Markoff surfaces which cannot be explained by the Brauer–Manin obstruction until now. However, in the case of Markoff-type cubic surfaces which we consider in this chapter, the bounds do not prove themselves to be so effective when considered in a similar way. Indeed, let us recall the **Markoff descent** below for convenience; see [GS22] and [Wha20] for more details on the notation. Given $k = (k_1, \dots, k_n) \in \mathbb{A}^n(\mathbb{C})$, we write

$$\mathrm{H}(k) = \max\{1, |k_1|, \dots, |k_n|\}.$$

*Surfaces of type* $(1,1)$. Let $\Sigma$ be a surface of type $(1,1)$. By Section 2, we have an identification of the moduli space $X_k = X_k(\Sigma)$ with the affine cubic algebraic surface in $\mathbb{A}^3_{x,y,z}$ given by the equation

$$x^2 + y^2 + z^2 - xyz - 2 = k.$$

The mapping class group $\Gamma = \Gamma(\Sigma)$ acts on $X_k$ via polynomial transformations. Up to finite index, it coincides with the group $\Sigma'$ of automorphisms of $X_k$ generated by the transpositions and even sign changes of coordinates as well as the Vieta involutions of the form $(x, y, z) \mapsto (x, y, xy - z)$.

The mapping class group dynamics on $X_k(\mathbb{R})$ was analyzed in detail by Goldman as discussed in [Wha20], and the work of Ghosh–Sarnak [GS22] (Theorem 1.1) establishes a remarkable exact fundamental set for the action of $\Sigma'$ on the integral points $X_k(\mathbb{Z})$ for admissible $k$. More generally, the results below establish Markoff descent for complex points.

**Lemma 2.4.1** ([Wha20, Lemma 4.2])**.** *Let $\Sigma$ be a surface of type $(1,1)$. There is a constant $C > 0$ independent of $k \in \mathbb{C}$ such that, given any $\rho \in X_k(\Sigma, \mathbb{C})$, there exists some $\gamma \in \Gamma$ such that $\gamma^*\rho = (x, y, z)$ satisfies*

$$\min\{|x|, |y|, |z|\} \leqslant C \cdot \mathrm{H}(k)^{1/3}.$$

*Surfaces of type* $(0,4)$. Let $\Sigma$ be a surface of type $(0,4)$. By Section 2, we have an identification of the moduli space $X_k = X_k(\Sigma)$ with the affine cubic algebraic surface in $\mathbb{A}^3_{x,y,z}$ given by the equation

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d$$

with $a, b, c, d$ appropriately determined by $k$. The mapping class group $\Gamma = \Gamma(\Sigma)$ acts on $X_k$ via polynomial transformations. Let $\Sigma'$ be the group of automorphisms of $X_k$ generated by the Vieta involutions

$$\begin{aligned} \tau_x^* &: (x, y, z) \mapsto (a - yz - x, y, z), \\ \tau_y^* &: (x, y, z) \mapsto (x, b - xz - y, z), \\ \tau_z^* &: (x, y, z) \mapsto (x, y, c - xy - z). \end{aligned}$$

Two points $\rho, \rho' \in X_k(\mathbb{C})$ are $\Sigma'$-equivalent if and only if they are $\Sigma$-equivalent or $\rho$ is $\Sigma$-equivalent to all of $\tau_x^*\rho', \tau_y^*\rho'$, and $\tau_z^*\rho'$.

**Lemma 2.4.2** ([Wha20, Lemma 4.4])**.** *Let $\Sigma$ be a surface of type $(0,4)$. There is a constant*

*$C > 0$ independent of $k \in \mathbb{C}^4$ such that, given any $\rho \in X_k(\mathbb{C})$, there exists some $\gamma \in k$ such that $\gamma^* \rho = (x, y, z)$ satisfies one of the following conditions:*

*(1)* $\min\{|x|, |y|, |z|\} \leqslant C$,

*(2)* $|yz| \leqslant C \cdot \mathrm{H}(a)$,

*(3)* $|xz| \leqslant C \cdot \mathrm{H}(b)$,

*(4)* $|xy| \leqslant C \cdot \mathrm{H}(c)$,

*(5)* $|xyz| \leqslant C \cdot \mathrm{H}(d)$.

Clearly, the restrictions in this lemma are *weaker* than those in the previous lemma, and so seems their effect. It would be interesting if one can find a way to apply these fundamental sets to produce some family of counterexamples to the integral Hasse principle which cannot be explained by the Brauer–Manin obstruction. A natural continuation from our work would be to find some sufficient hypothesis for $k$ such that the Brauer–Manin set of the general Markoff-type surface is nonempty, as inspired by [LM20, Corollary 5.11], which we will discuss in some particular cases in the next part. Ultimately, similar to the case of Markoff surfaces, it is still reasonable for us to expect that the number of counterexamples which cannot be explained by the Brauer–Manin obstruction for these relative character varieties is asymptotically *greater* than the number of those which can be explained by this obstruction.

**Example 2.4.1.** Take $k_1 = 127, k_2 = 5, k_3 = 5 + 144.5 = 725, k_4 = 5 + 144.10 = 1445$, then $k = (k_1, k_2, k_3, k_4)$ satisfies Assumption A and we now have an explicit example of a Markoff-type cubic surface (where local integral points exist) given by the equation

$$x^2 + y^2 + z^2 + xyz = 1048260x + 187140y + 99300z - 667871675.$$

By using the box as discussed above, we will prove that this example gives no integral solution, i.e., it is a counterexample to the integral Hasse principle.

Indeed, from the proof of [Wha20, Lemma 4.4], we can find that the constant $C$ independent of $k \in \mathbb{C}^4$ may take the value $C = \mathbf{48}$ in the condition (1) and $C = \mathbf{24}$ in all the other conditions $(2), (3), (4), (5)$ (thus we may choose $\mathbf{48}$ to be the desired constant in the statement of the Lemma). With this information, we can run a program on SageMath [SJ05] to find integral points satisfying one of the restrictions $(2), (3), (4), (5)$, along with the help of Dario Alpern's website (Alpertron) [Alp] to find integral points whose one coordinate satisfies the restriction (1). More precisely, SageMath shows that there is no integral point in any of the boxes defined by $(2), (3), (4), (5)$, while Alpertron deals with conic equations after fixing one variable bounded in (1) by transforming them into homogeneous quadratic equations and showing that some corresponding modular equations do not have solutions. As a result, we are able to prove that there is clearly no integral point in all those five cases, hence no integral point on the corresponding Markoff-type cubic surface.

### 2.4.2 Some special cases of Markoff-type cubic surfaces

Since $a = b = c = 0$ in (2) implies that either three of the $k_i$ are equal to 0 or $k_1 = k_2 = k_3 = -k_4$ or any other permutation of $k_i$ (with $1 \leqslant i \leqslant 4$), the only cases that our Markoff-type cubic surfaces recover the original Markoff surfaces are given by equations of the form

$$x^2 + y^2 + z^2 + xyz = 4 - k_0^2,$$

or
$$x^2 + y^2 + z^2 + xyz = (2 - k_0^2)^2$$
for some $k_0 \in \mathbb{Z}$. For the former equation, it is an interesting Markoff surface that can be studied similarly as in previous work, with a remark that for any odd $k_0$ the equation will not be everywhere locally solvable as the right-hand side is congruent to 3 mod 4 (see [GS22]). For the latter equation, it always has the integral solutions $(\pm(2 - k_0^2), 0, 0)$. Although these special cases are only of magnitude $M^{1/4}$ compared to the total number of cases that we consider, it is clear that our family of examples does not recover these particular surfaces. We will discuss it here in a more general situation.

Naturally, one can find different hypotheses from that of Assumption A to work for more general cases, since Assumption A exists for technical reasons and specific counting results. More precisely, we will now consider the special case when $k_1 \neq k_2 = k_3 = k_4$ are integers such that the total field extension $[E : \mathbb{Q}] = 4$ and the set of local integral points on the Markoff-type cubic surface is nonempty. Interestingly, we are under the same framework as that of the general case in [CWX20, Section 3], to compute the algebraic Brauer group of the affine surface, which gives us
$$\mathrm{Br}_1 U/\mathrm{Br}_0 U \cong (\mathbb{Z}/2\mathbb{Z})^3$$
with three generators
$$\{(x - 2, k_2^2 - 4), (y - 2, k_2^2 - 4), (z - 2, k_2^2 - 4)\}.$$
Following [LM20, Proposition 5.7 and Lemma 5.8] or [CWX20, Lemmas 5.4 and 5.5]), we have
$$\{(x - 2, k_2^2 - 4)_2, (y - 2, k_2^2 - 4)_2, (z - 2, k_2^2 - 4)_2\} = \{0, 1/2, 1/2\}$$
and
$$\{(x - 2, k_2^2 - 4)_3, (y - 2, k_2^2 - 4)_3, (z - 2, k_2^2 - 4)_3\} = \{0, 0, -\}$$
as multisets. As $k_2$ are odd, so $k_2^2 - 4 \equiv 5 \bmod 8$, hence there is no way to describe this number by the form $3v^2$ for $v \in \mathbb{Z}$ which is favorable to give Hasse failures as in previous work. Due to the complexity of the integral values of the polynomial $X^2 - 4$ and their prime divisors, for now we can only deduce the vanishing of Brauer–Manin obstructions to the integral Hasse principle using a similar method as in [LM20].

**Proposition 2.4.3.** *Let $k_1 \neq k_2 = k_3 = k_4$ be integers satisfying the congruence conditions $k_1 \equiv -k_2 \equiv -5$ mod $16$ (resp. mod $9$), and the divisibility conditions: $(k_1, k_2) = 1$, $(k_1, k_2^2 - 4) = (k_1^2 - 4, k_2) = 1$, and $(k_1^2 - 2, k_2^2 - 2) = 1$, such that $[E : \mathbb{Q}] = 4$. Moreover, assume that there is a prime $p > 3$ such that $p$ divides $k_2^2 - 4$ to an odd order and $k_1 \equiv -k_2$ mod $p$. Then there is no algebraic Brauer–Manin obstruction to the integral Hasse principle, but there is a Brauer–Manin obstruction to strong approximation for $\mathcal{U}$.*

*Proof.* Following similar arguments as in the proof of Proposition 2.3.1, we can prove that the set of local integral points is indeed nonempty. Note that under our assumption, $a = b = c \equiv 0$ mod $p$ and $d = (2 - k_2^2)^2 \equiv 4$ mod $p$, so $ax + by + cz + d \equiv 4$ mod $p$ in the defining equation (1) of $\mathcal{U}$. Now for $p > 5$, we let $\mathcal{B} = \langle (x - 2, k_2^2 - 4), (y - 2, k_2^2 - 4), (z - 2, k_2^2 - 4) \rangle$ and follow the same arguments in the proof of [LM20, Proposition 5.5] to prove that the map
$$\mathcal{U}(\mathbb{Z}_p) \to \mathrm{Hom}(\mathcal{B}, \mathbb{Q}/\mathbb{Z}), \qquad \mathbf{u} \mapsto (\beta \mapsto \mathrm{inv}_p\, \beta(\mathbf{u})),$$
induced by the Brauer–Manin pairing, is surjective. Now we need to show that $\emptyset \neq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}_1} \neq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})$. Let $\mathbf{u} \in \mathcal{U}(\mathbf{A}_{\mathbb{Z}})$. Then by surjectivity, there exist $\mathbf{u}_p \in \mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\, \beta(\mathbf{u}_p) =$

$-\sum_{v\neq p}\mathrm{inv}_v\beta(\mathbf{u})$ for all $\beta\in\mathcal{B}$ and $\mathbf{u}'_p\in\mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\,\beta(\mathbf{u}'_p)=1/2-\sum_{v\neq p}\mathrm{inv}_v\,\beta(\mathbf{u})$ for some $\beta\in\mathcal{B}$. Then we obtain a point in $\mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathrm{Br}_1}$ by replacing the $p$-part of $\mathbf{u}$ by $\mathbf{u}_p$, and a point in $\mathcal{U}(\mathbf{A}_\mathbb{Z})$ but not in $\mathcal{U}(\mathbf{A}_\mathbb{Z})^{\mathrm{Br}_1}$ by replacing the $p$-part of $\mathbf{u}$ by $\mathbf{u}'_p$, as required.

We are left with the case $p=5$. Using [LM20, Proposition 5.7] and [CWX20, Lemma 5.5], we know that the image of the above map induced by the Brauer–Manin pairing contains all the nontrivial elements. Let $\mathbf{u}\in\mathcal{U}(\mathbf{A}_\mathbb{Z})$. In fact, there always exists $\mathbf{u}'_p\in\mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\,\beta(\mathbf{u}'_p)=1/2-\sum_{v\neq p}\mathrm{inv}_v\,\beta(\mathbf{u})$ for some $\beta\in\mathcal{B}$. Next, if $\sum_{v\neq p}\mathrm{inv}_v\beta'(\mathbf{u})\neq 0$ for some $\beta'\in\mathcal{B}$, there still exists $\mathbf{u}_p\in\mathcal{U}(\mathbb{Z}_p)$ such that $\mathrm{inv}_p\,\beta(\mathbf{u}_p)=-\sum_{v\neq p}\mathrm{inv}_v\beta(\mathbf{u})$ for all $\beta\in\mathcal{B}$. Now if $\sum_{v\neq p}\mathrm{inv}_v\beta(\mathbf{u})=0$ for all $\beta\in\mathcal{B}$, we can consider another local point $\mathbf{u}''$ whose $p$-parts ($p\neq 2$) are the same as those of $\mathbf{u}$ and 2-part $(x''_2,y''_2,z''_2)$ is a permutation of $\mathbf{u}_2=(x_2,y_2,z_2)$ (note that the Markoff-type cubic equation here is symmetric in $x,y,z$) such that their images under the local invariant map at 2 are *different* permutations of $(0,1/2,1/2)$, hence we will get $\sum_{v\neq p}\mathrm{inv}_v\beta(\mathbf{u})\neq 0$ for some $\beta\in\mathcal{B}$. The proof is now complete. □

Again, the hypothesis of Proposition 2.4.3 can be modified or generalized to give the same results, which we hope to achieve in possible future work. For now, let us give some concrete examples from Proposition 2.4.3 using the help of SageMath [SJ05] and Alpertron [Alp] which have been mentioned in Example 2.4.1.

**Example 2.4.2.** Consider $k_1=144.7.2-5=2011$ and $k_2=k_3=k_4=5$. Then there is no integral point on the corresponding Markoff-type cubic surface. By Proposition 2.4.3 (with $p=7$), we obtain a counterexample to the integral Hasse principle which cannot be explained by the *algebraic* Brauer–Manin obstruction.

**Example 2.4.3.** Consider $k_1=144.7.3-5=3019$ and $k_2=k_3=k_4=5$. Then we find an integral point $(x,y,z)=(24,409,672)$ on the corresponding Markoff-type cubic surface. By Proposition 2.4.3 (with $p=7$), we get a counterexample to strong approximation which can be explained by the Brauer–Manin obstruction.

Finally, we end with a counterexample to the integral Hasse principle for which it is unclear whether the (algebraic) Brauer–Manin obstruction exists or not.

**Example 2.4.4.** Consider $k_1=127$ and $k_2=k_3=k_4=5$. Since $(127^2-4,5^2-4)=3$, this example does not completely satisfy any of our previous assumptions. However, by using the programs on SageMath and Alpertron as discussed above, we find that there is indeed no integral point on the corresponding Markoff-type cubic surface.

# Chapter 3

# Brauer–Manin obstruction for Wehler K3 surfaces of Markoff type

## 3.1 Background

We give some notations and results about Wehler K3 surfaces and the so-called *Markoff-type* K3 surfaces that we study in this chapter.

### 3.1.1 Wehler K3 surfaces

Consider the variety $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi_1$, $\pi_2$, and $\pi_3$ be the projections on the first, second, and third factor: $\pi_i(z_1, z_2, z_3) = z_i$. Denote by $L_i$ the line bundle $\pi_i^*(\mathcal{O}(1))$ and set

$$L = L_1^2 \otimes L_2^2 \otimes L_3^2 = \pi_1^*(\mathcal{O}(2)) \otimes \pi_2^*(\mathcal{O}(2)) \otimes \pi_3^*(\mathcal{O}(2)).$$

Since $K_{\mathbb{P}^1} = \mathcal{O}(-2)$, this line bundle $L$ is the dual of the canonical bundle $K_M$. By definition, $|L| \simeq \mathbb{P}(\mathrm{H}^0(M, L))$ is the linear system of surfaces $W \subset M$ given by the zeroes of global sections $P \in \mathrm{H}^0(M, L)$. Using affine coordinates $(x_1, x_2, x_3)$ on $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, such a surface is defined by a polynomial equation $F(x_1, x_2, x_3) = 0$ whose degree with respect to each variable is $\leqslant 2$. These surfaces will be referred to as **Wehler surfaces**; modulo $\mathrm{Aut}(M)$, they form a family of dimension 17.

Fix $k \in \{1, 2, 3\}$ and denote by $i < j$ the other indices. If we project $W$ to $\mathbb{P}^1 \times \mathbb{P}^1$ by $\pi_{ij} = (\pi_i, \pi_j)$, we get a 2 to 1 cover (the generic fiber is made of two points, but some fibers may be rational curves). As soon as $W$ is *smooth*, the involution $\sigma_k$ that permutes the two points in each (general) fiber of $\pi_{ij}$ is an involutive automorphism of $W$; indeed $W$ is a K3 surface and any birational self-map of such a surface is an automorphism (see [Bil97, Lemma 1.2]). By [CD22, Proposition 3.1], we have the following general result.

**Proposition 3.1.1.** *There is a countable union of proper Zariski closed subsets $(S_i)_{i \geqslant 0}$ in $|L|$ such that:*

*(1) If $W$ is an element of $|L| \setminus S_0$, then $W$ is a smooth K3 surface and $W$ does not contain any fiber of the projections $\pi_{ij}$, i.e., each of the three projections $(\pi_{ij})_{|W} : W \to \mathbb{P}^1 \times \mathbb{P}^1$ is a finite map;*

*(2) If $W$ is an element of $|L| \setminus (\cup_{i \geqslant 0} S_i)$, the restriction morphism $\mathrm{Pic}\, M \to \mathrm{Pic}\, W$ is surjective. In particular, the Picard number of $W$ is equal to 3.*

From the second assertion, we deduce that for a *very general* $W$, $\operatorname{Pic} W$ is isomorphic to $\operatorname{Pic} M$: it is the free Abelian group of rank 3, generated by the classes

$$D_i := [(L_i)_{|W}].$$

The elements of $|(L_i)_{|W}|$ are the curves of $W$ given by the equations $z_i = \alpha$ for some $\alpha \in \mathbb{P}^1$. The arithmetic genus of these curves is equal to 1: in other words, the projection $(\pi_i)_{|W} : W \to \mathbb{P}^1$ is a genus 1 fibration (see [Bil97, Lemma 1.1]). Moreover, for a general choice of $W$ in $|L|$, $(\pi_i)_{|W}$ has 24 singular fibers of type $\mathrm{I}_1$, i.e. isomorphic to a rational curve with exactly one simple double point. The intersection form is given by $D_i^2 = 0$ and $(D_i.D_j) = 2$ if $i \neq j$, so that its matrix is given by

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Note that if $W$ is a smooth K3 surface, then $\operatorname{Pic} W \simeq \operatorname{NS} W$. By [Bil97, Proposition 1.5] or [CD22, Lemma 3.2], we have the following result about the actions of the subgroup of $\operatorname{Aut}(W)$ generated by $\sigma_1, \sigma_2, \sigma_3$ on the geometry of $W$.

**Proposition 3.1.2.** *Assume that $W$ does not contain any fiber of the projection $\pi_{ij}$. Then the involution $\sigma_k^*$ preserves the subspace $\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \mathbb{Z}D_3$ of $\operatorname{NS} W$ and*

$$\sigma_k^*(D_i) = D_i, \quad \sigma_k^*(D_j) = D_j, \quad \sigma_k^*(D_k) = -D_k + 2D_i + 2D_j.$$

*In other words, the matrices of the $\sigma_i^*$ in the basis $(D_1, D_2, D_3)$ are:*

$$\sigma_1^* = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \ \sigma_2^* = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \ \sigma_3^* = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Combining these two propositions, we have the following (see [Bil97, Proposition 1.3] and [CD22, Proposition 3.3]):

**Proposition 3.1.3.** *If $W$ is a very general Wehler surface then:*

*(1) $W$ is a smooth K3 surface with Picard number 3;*

*(2) $\operatorname{Aut}(W) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, which is a free product of three copies of $\mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Aut}(W)^*$ is a finite index subgroup in the group of integral isometries of $\operatorname{Pic} W = \operatorname{NS} W$. Here $\operatorname{Aut}(W)^*$ denotes the image of $\operatorname{Aut}(W)$ in $\mathsf{GL}(\mathrm{H}^2(W, \mathbb{Z}))$* (see [CD22, Section 2.1]).

Besides the three involutions $\sigma_1, \sigma_2, \sigma_3$, depending on the symmetries of the defining polynomial $F$, the automorphism group of a Wehler surface $W$ may contain additional automorphisms. Typical examples include symmetry in $x, y, z$ that allows permutation of the coordinates, and power symmetry that allows the signs of two of $x, y, z$ to be reversed. For example, the original Markoff equation permits these extra automorphisms; and hereafter we consider analogous Markoff-type surfaces. Note that all the above results are true for *very general* Wehler surfaces; as we will see, our examples of surfaces to study in this chapter are in fact very far from being general, which leads to many different results in the end.

### 3.1.2 Markoff-type K3 surfaces

Now let $K$ be a field. A Wehler surface $W$ over $K$ is then a surface

$$W = \{\overline{F} = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

defined by a $(2,2,2)$-form

$$\overline{F}(x,r;y,s;z,t) \in K[x,r;y,s;z,t].$$

Using the affine coordinates $(x,y,z)$, we let

$$F(x,y,z) = \overline{F}(x,1;y,1;z,1),$$

and then $W$ is the closure in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the affine surface, which by abuse of notation we also denote by

$$W : F(x,y,z) = 0.$$

We say that $W$ is *non-degenerate* if it satisfies the following two conditions:

(i) The projection maps $\pi_{12}, \pi_{13}, \pi_{23}$ are finite.

(ii) The generic fibers of the projection maps $\pi_1, \pi_2, \pi_3$ are smooth curves, in which case the smooth fibers are necessarily curves of genus 1, since they are $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

By analogy with the classical Markoff equation, we say that $W$ is of *Markoff type* (MK3) if it is symmetric in its three coordinates and invariant under double sign changes. An MK3 surface admits a group of automorphisms $\Gamma$ generated by the three involutions, coordinate permutations, and sign changes. Following the notations in [Fuc+22], we define:

**Definition 3.1.1.** We let $\mathfrak{S}_3$, the symmetric group on 3 letters, act on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by permuting the coordinates, and we let the group

$$(\mu_2^3)_1 := \{(\alpha,\beta,\gamma) : \alpha,\beta,\gamma \in \mu_2 \,\text{and}\, \alpha\beta\gamma = 1\}$$

act on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ via sign changes,

$$(\alpha,\beta,\gamma)(x,y,z) = (\alpha x, \beta y, \gamma z).$$

In this way, we obtain an embedding

$$\mathcal{G} := (\mu_2^3)_1 \rtimes \mathfrak{S}_3 \hookrightarrow \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1).$$

**Definition 3.1.2.** A *Markoff-type K3* (MK3) surface $W$ is a Wehler surface whose $(2,2,2)$-form $F(x,y,z)$ is invariant under the action of $\mathcal{G}$, i.e., the $(2,2,2)$-form $F$ defining $W$ satisfies

$$\begin{aligned} F(x,y,z) &= F(-x,-y,z) = F(-x,y,-z) = F(x,-y,-z), \\ F(x,y,z) &= F(z,x,y) = F(y,z,x) = F(x,z,y) = F(y,x,z) = F(z,y,x). \end{aligned}$$

By [Fuc+22, Proposition 7.5], we have the following key result about the defining form of MK3 surfaces.

**Proposition 3.1.4.** *Let $W/K$ be a (possibly degenerate) MK3 surface.*

*(a) There exist $a,b,c,d,e \in K$ so that the $(2,2,2)$-form $F$ that defines $W$ has the form*

$$F(x,y,z) = ax^2y^2z^2 + b(x^2y^2 + x^2z^2 + y^2z^2) + cxyz + d(x^2+y^2+z^2) + e = 0. \quad (3.1)$$

(b) *Let $F$ be as in (a). Then $W$ is a non-degenerate, i.e., the projections $\pi_{ij} : W \to \mathbb{P}^1 \times \mathbb{P}^1$ are* quasi-finite, *if and only if*

$$c \neq 0, \quad be \neq d^2, \quad \text{and} \quad ad \neq b^2.$$

**Remark 3.1.3.** We can recover the original Markoff equation for a surface $S_k$ as a special case of a form $F$ with $a = b = 0, c = -1, d = 1, e = -k$. More precisely, $S_k$ is given by the affine equation

$$F(x, y, z) = x^2 + y^2 + z^2 - xyz - k = 0.$$

We note, however, that the Markoff equation is degenerate, despite the involutions being well-defined on the affine Markoff surface $S_k$. This occurs because the involutions are not well-defined at some of the points at infinity in the closure of $S_k$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; for example, the inverse image $\pi_{12}^{-1}([1:0],[1:0])$ in $X_k$ is a line isomorphic to $\mathbb{P}^1$.

Now we are ready to introduce the **three** families of MK3 surfaces that we study in this chapter. For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by one of the following $(2, 2, 2)$-forms:

$$F_1(x, y, z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 - k = 0; \tag{3.2}$$

$$F_2(x, y, z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - k = 0; \tag{3.3}$$

$$F_3(x, y, z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.4}$$

It is important to note that all these families of Markoff-type K3 surfaces are *degenerate* in the sense that every member of each family contains a fiber (a line isomorphic to $\mathbb{P}^1$) of the projection $\pi_{ij}$. Furthermore, there exist $\mathbb{Q}$-rational points at infinity on every member of each family of Markoff-type K3 surfaces considered above:

$$\begin{cases} ([1:0],[1:1],[1:2]) \in \{\overline{F_1} = 0\}; \\ ([1:0],[1:0],[1:2]) \in \{\overline{F_2} = 0\}; \\ ([1:0],[1:0],[1:2]) \in \{\overline{F_3} = 0\}. \end{cases}$$

In this chapter, we study some explicit cases when there are however no integral points due to the Brauer–Manin obstruction.

## 3.2 The Brauer group of Markoff-type K3 surfaces

We are particularly interested in the geometry of the **third** family of Markoff-type K3 surfaces defined by (3.4), as they are more complicated and general than the other two. In addition, under our specific conditions, the first and second surfaces are always *singular* at infinity (for example, at the points $([1:0],[0:1],[1:0])$ and $([1:0],[1:2],[1:2])$, respectively), but the third ones are *smooth.* Before studying the arithmetic problem of integral points, we will give some explicit computations on the (geometric) Picard group and the (algebraic) Brauer group of these surfaces. Recall that by [Bil97, Proposition 1.3] or [CD20, Proposition 3.3], for a *very general* $W$, $\operatorname{Pic} \overline{W}$ is isomorphic to $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, i.e. $\operatorname{Pic} \overline{W}$ is generated by the classes $D_i$ so the Picard number of $\overline{W}$ equals 3. However, as previously discussed, we will see in this section that our example of MK3 surfaces is very *special.*

### 3.2.1 Geometry of K3 surfaces

Let $K$ be a number field with a fixed algebraic (separable) closure $\overline{K}$. If $X$ is a K3 surface over $K$, or more generally, $X$ is a smooth, projective and geometrically integral $K$-variety such that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then the Picard group $\operatorname{Pic} \overline{X}$ and the Néron–Severi group $\operatorname{NS} \overline{X}$ are equal (see [CS21, Corollary 5.1.3]).

Now let $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth surface over $K$ defined by a $(2, 2, 2)$-form $F = 0$ (so $W$ is a Wehler K3 surface). For distinct $i, j \in \{1, 2, 3\}$, we keep the notations $\pi_i : W \to \mathbb{P}^1$ and $\pi_{ij} : W \to \mathbb{P}^1 \times \mathbb{P}^1$ of the various projections of $W$ onto one or two copies of $\mathbb{P}^1$. Let $D_i$ denote the divisor class represented by a fiber of $\pi_i$. We find that $(D_i.D_j) = 2$ for $i \neq j$ and since any two different fibers of $\pi_i$ are disjoint, we have $D_i^2 = 0$. It follows that the intersection matrix $((D_i.D_j))_{i,j}$ has rank 3, so the $D_i$ generates a subgroup of rank 3 of the Néron–Severi group $\operatorname{NS} \overline{W}$.

We have the following result for the geometric Picard group of Wehler surfaces.

**Proposition 3.2.1.** *Let $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth, projective, geometrically integral Wehler surface over $K$. Suppose that the three planes at infinity $\{rst = 0\}$ cut out on $\overline{W}$ three distinct irreducible fibers $D_1, D_2, D_3$ over $\overline{K}$. Let $U \subset W$ be the complement of these fibers. Then $\overline{K}^\times = \overline{K}[U]^\times$ and the natural sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z} D_i \longrightarrow \operatorname{Pic} \overline{W} \longrightarrow \operatorname{Pic} \overline{U} \longrightarrow 0$$

*is exact.*

*Proof.* Note that by [Ful98, Proposition 1.8], in order to show that the above sequence is exact, it suffices to prove that the second arrow is an injective homomorphism. Let

$$a_1 D_1 + a_2 D_2 + a_3 D_3 = 0 \in \operatorname{Pic} \overline{W}$$

with $a, b, c \in \mathbb{Z}$. By the assumption that $(D_i.D_i) = 0$ and $(D_i.D_j) = 2$ for $1 \leqslant i \neq j \leqslant 3$, one has

$$2a_2 + 2a_3 = 2a_1 + 2a_3 = 2a_1 + 2a_2 = 0,$$

so $a_1 = a_2 = a_3 = 0$. In other words, this is another proof of the fact that $D_1, D_2, D_3$ are linearly independent in $\operatorname{Pic} \overline{W}$ and it also shows that $\overline{K}^\times = \overline{K}[U]^\times$ as desired. □

Now let $k$ be an arbitrary field. Recall that for a variety $X$ over $k$ there is a natural filtration on the Brauer group

$$\operatorname{Br}_0 X \subset \operatorname{Br}_1 X \subset \operatorname{Br} X$$

which is defined as

$$\operatorname{Br}_0 X = \operatorname{Im}[\operatorname{Br} k \to \operatorname{Br} X], \qquad \operatorname{Br}_1 X = \operatorname{Ker}[\operatorname{Br} X \to \operatorname{Br} \overline{X}].$$

Let $X$ be a variety over a field $k$ such that $\overline{k}[X]^\times = \overline{k}^\times$. By Hilbert's Theorem 90 we have $\mathrm{H}^1(k, \overline{k}^\times) = 0$, then by the Hochschild–Serre spectral sequence, there is a functorial exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Pic} X \longrightarrow \operatorname{Pic} \overline{X}^{G_k} \longrightarrow \operatorname{Br} k \longrightarrow \operatorname{Br}_1 X \\ \longrightarrow \mathrm{H}^1(k, \operatorname{Pic} \overline{X}) \longrightarrow \operatorname{Ker}[\mathrm{H}^3(k, \overline{k}^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{3.5}$$

For convenience, let us recall some remarks as discussed in Section 2.2.

**Remark 3.2.1.** Let $X$ be a variety over a field $k$ such that $\overline{k}[X]^\times = \overline{k}^\times$. This assumption $\overline{k}[X]^\times = \overline{k}^\times$ holds for any proper, geometrically connected and geometrically reduced $k$-variety $X$.

(1) If $X$ has a $k$-point, then each of the maps $\operatorname{Br} k \longrightarrow \operatorname{Br}_1 X$ and $\mathrm{H}^3(k, \overline{k}^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ is injective. (Then $\operatorname{Pic} X \longrightarrow \operatorname{Pic} \overline{X}^{G_k}$ is an isomorphism.) Therefore, we have an isomorphism
$$\operatorname{Br}_1 X / \operatorname{Br} k \cong \mathrm{H}^1(k, \operatorname{Pic} \overline{X}).$$

(2) If $k$ is a number field, then $\mathrm{H}^3(k, \overline{k}^\times) = 0$ from class field theory. Therefore, we have an isomorphism
$$\operatorname{Br}_1 X / \operatorname{Br}_0 X \cong \mathrm{H}^1(k, \operatorname{Pic} \overline{X}).$$

We have the following result (see [CS21, Theorem 5.5.1]).

**Theorem 3.2.2.** *Let $X$ be a smooth, projective and geometrically integral variety over a field $k$. Assume that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ and $\operatorname{NS} \overline{X}$ is torsion-free. Then $\mathrm{H}^1(k, \operatorname{Pic} \overline{X})$ and $\operatorname{Br}_1 X / \operatorname{Br}_0 X$ are finite groups.*

The assumption of the above theorem is always true if $X$ is a K3 surface. Furthermore, by Skorobogatov and Zarhin, we have a stronger result for the Brauer group of K3 surfaces (see [CS21, Theorem 16.7.2 and Collorary 16.7.3]).

**Theorem 3.2.3.** *Let $X$ be a K3 surface over a field $k$ finitely generated over $\mathbb{Q}$. Then $(\operatorname{Br} \overline{X})^\Gamma$ is finite. Moreover, the group $\operatorname{Br} X / \operatorname{Br}_0 X$ is finite.*

Next, we will give an explicit computation of the geometric Picard group and the algebraic Brauer group for the family of Markoff-type K3 surfaces defined by (3.4).

### 3.2.2 The geometric Picard group

Using the explicit equations, we compute the geometric Picard group of the Markoff-type K3 surfaces in question. To bound the Picard number we use the method described in [Lui07b]. Let $X$ be any smooth surface over a number field $K$ and let $\mathfrak{p}$ be a prime of good reduction with residue field $k$. Let $\mathcal{X}$ be an integral model for $X$ over the localization $\mathcal{O}_\mathfrak{p}$ of the ring of integers $\mathcal{O}$ of $K$ at $\mathfrak{p}$ for which the reduction is smooth. Let $k'$ be any extension field of $k$. Then by abuse of notation, we will write $X_{k'}$ for $\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_\mathfrak{p}} \operatorname{Spec} k'$. We need the following important result which describes the behavior of the Néron–Severi group under good reduction (see [Lui07a, Proposition 6.2 and Corollary 6.4] or [BL07, Proposition 2.3]).

**Proposition 3.2.4.** *Let $X$ be a smooth surface over a number field $K$ and let $\mathfrak{p}$ be a prime of good reduction with residue field $k$. Let $l$ be a prime not dividing $q = \#k$. Let $\operatorname{Frob}_q^*$ denote the automorphism on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_l(1))$ induced by the $q$-th power Frobenius. Then there are natural injective homomorphisms*
$$\operatorname{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow \operatorname{NS}(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_l)(1)$$
*of finite-dimensional vector spaces over $\mathbb{Q}_l$, that respect the intersection pairing and the action of Frobenius respectively. The rank of $\operatorname{NS}(X_{\overline{k}})$ is at most the number of eigenvalues of $\operatorname{Frob}_q^*$ that are roots of unity, counted with multiplicity.*

Recall that if $X$ is a K3 surface, then linear, algebraic, and numerical equivalence all coincide (see [Huy16]). This means that the Picard group $\operatorname{Pic}\overline{X}$ and the Néron–Severi group $\operatorname{NS}\overline{X}$ of $\overline{X} := X_{\overline{K}}$ are naturally isomorphic, finitely generated, and free. Their rank is called the *geometric* Picard number of $X$ or the Picard number of $\overline{X}$. By the *Hodge Index Theorem*, the intersection pairing on $\operatorname{Pic}\overline{X}$ is even, non-degenerate, and of signature $(1, \operatorname{rk}\operatorname{NS}\overline{X} - 1)$.

**Proposition 3.2.5.** *Let $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0,$$

*where $k \in \mathbb{Z}$. If $k \equiv 3 \bmod 5$, then $W$ is a smooth K3 surface and the Picard number of $\overline{W} = W_{\overline{\mathbb{Q}}}$ equals* 18.

*Proof.* Since the surface $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is defined over $\mathbb{Q}$ by a $(2,2,2)$-form $F = 0$ with $k(4k+1)((4k-5)^2 - 32) \neq 0$, it is clear that $W$ is a smooth K3 surface. For $i = 1,2,3$, let $\pi_i : W \to \mathbb{P}^1$ be the projection from $W$ to the $i$-th copy of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $D_i$ denote the divisor class represented by a *smooth* fiber of $\pi_i$. By considering all the smooth fibers and the *singular* fibers, the corresponding divisor classes on $\overline{W}$ are given explicitly as follows (denote by $[x:r], [y:s], [z:t]$ the coordinates for each point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$):

$$\begin{cases} D_1 : [x:r] = [1:0], s^2t^2 + 4y^2t^2 + 4z^2s^2 - 16y^2z^2 = 0, \\ D_2 : [y:s] = [1:0], r^2t^2 + 4x^2t^2 + 4z^2r^2 - 16x^2z^2 = 0, \\ D_3 : [z:t] = [1:0], r^2s^2 + 4x^2s^2 + 4y^2r^2 - 16x^2y^2 = 0; \end{cases}$$

$$\begin{cases} A_1 : [x:r] = [\pm\sqrt{k}:1], (4k+1)y^2t^2 + (4k+1)z^2s^2 - (16k-4)y^2z^2 = 0, \\ A_2 : [y:s] = [\pm\sqrt{k}:1], (4k+1)x^2t^2 + (4k+1)z^2r^2 - (16k-4)x^2z^2 = 0, \\ A_3 : [z:t] = [\pm\sqrt{k}:1], (4k+1)x^2s^2 + (4k+1)y^2r^2 - (16k-4)x^2y^2 = 0; \end{cases}$$

$$\begin{cases} B_1 : [x:r] = [\pm\frac{1}{2}:1], y^2t^2 + z^2s^2 - \frac{4k-1}{8}s^2t^2 = 0, \\ B_2 : [y:s] = [\pm\frac{1}{2}:1], x^2t^2 + z^2r^2 - \frac{4k-1}{8}r^2t^2 = 0, \\ B_3 : [z:t] = [\pm\frac{1}{2}:1], x^2s^2 + y^2r^2 - \frac{4k-1}{8}r^2s^2 = 0; \end{cases}$$

$$\begin{cases} C_1^{\pm\pm} : [x:r] = [\pm\sqrt{\frac{-1}{4}}:1], yz = \pm\sqrt{\frac{4k+1}{32}}st, \\ C_2^{\pm\pm} : [y:s] = [\pm\sqrt{\frac{-1}{4}}:1], xz = \pm\sqrt{\frac{4k+1}{32}}rt, \\ C_3^{\pm\pm} : [z:t] = [\pm\sqrt{\frac{-1}{4}}:1], xy = \pm\sqrt{\frac{4k+1}{32}}rs; \end{cases}$$

and for $1 \leqslant i \neq j \leqslant 3$,

1. $\ell_{ij}^{\pm\pm} : [x_i : r_i] = [\pm\sqrt{\alpha}:1], [x_j : r_j] = [\pm\sqrt{\bar{\alpha}}:1]$,

2. $\overline{\ell_{ij}^{\pm\pm}} : [x_i : r_i] = [\pm\sqrt{\bar{\alpha}}:1], [x_j : r_j] = [\pm\sqrt{\alpha}:1]$,

where $[x_1 : r_1], [x_2 : r_2], [x_3 : r_3]$ denote $[x:r], [y:s], [z:t]$ respectively, while $(\pm\sqrt{\alpha}, \pm\sqrt{\bar{\alpha}})$ are the solutions of the polynomial system

$$\begin{cases} 1 + 4a^2 + 4b^2 - 16a^2b^2 = 0, \\ a^2 + b^2 + 4a^2b^2 - k = 0; \end{cases}$$

i.e., they are deduced from the solutions of the polynomial equation

$$T^4 - \frac{4k-1}{8}T^2 + \frac{4k+1}{32} = 0,$$

where $\alpha = \frac{1}{2}\left(\frac{4k-1}{8} + \sqrt{\Delta}\right), \bar{\alpha} = \frac{1}{2}\left(\frac{4k-1}{8} - \sqrt{\Delta}\right)$ and $\Delta = \frac{(4k-5)^2-32}{64}$ is the discriminant of the associated quadratic polynomial $T^2 - \frac{4k-1}{8}T + \frac{4k+1}{32}$.

We will now find explicit generators for the geometric Picard group of $W$. It is clear that $W$ is a K3 surface admitting an elliptic fibration $\pi_1 : W \to \mathbb{P}^1$ with a zero section defined by $\ell_{23}^{+} \simeq \mathbb{P}^1$. The Néron–Severi group of an elliptic fibration on the K3 surface is the lattice generated by the class of a (general) fiber, the class of the zero section, the classes of the irreducible components of the reducible fibers which do not intersect the zero section, and the Mordell–Weil group (the set of the sections). Following this property, we find a set of 18 linearly independent divisor classes consisting of:

(i) $D_1$ (a smooth fiber), $\ell_{23}^{++}$ (a zero section);

(ii) $\{\ell_{12}^{++}, \ell_{12}^{+-}, \ell_{13}^{+-}, \ell_{12}^{--}, \ell_{12}^{-+}, \ell_{13}^{--}, \overline{\ell_{13}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{13}^{+-}}, \overline{\ell_{12}^{--}}, \overline{\ell_{13}^{-+}}, \overline{\ell_{13}^{--}}\}$ (the classes of irreducible components of singular fibers not intersecting the zero section);

(iii) $\{\overline{\ell_{23}^{++}}, \ell_{23}^{+-}, C_2^{+-}, C_3^{+-}\}$ (the set of some other sections).

Their Gram matrix of the intersection pairing on $\operatorname{Pic} \overline{W}$ has determinant $-192$, which is nonzero, so they are indeed linearly independent as the intersection pairing is *non-degenerate*. However, after considering other divisor classes, we are able to find and work with another lattice of 18 classes as follows, for technical reasons such as more symmetry for the fibers and a smaller absolute value of the Gram determinant (in fact, the former lattice is a sublattice of the latter). More precisely, the intersection matrix associated to the sequence of classes

$$S = \{D_1, D_2, D_3, \ell_{12}^{++}, \ell_{12}^{+-}, \ell_{13}^{++}, \ell_{23}^{++}, \ell_{12}^{-+}, \ell_{13}^{-+}, \ell_{23}^{--}, \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{13}^{++}}, \overline{\ell_{23}^{++}}, \overline{\ell_{12}^{-+}}, \overline{\ell_{13}^{-+}}, C_1^{+-}, C_2^{+-}\}$$

is

$$\begin{pmatrix}
0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
2 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0\\
2 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 1\\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2
\end{pmatrix},$$

so it has determinant $-48$, which is nonzero. Consequently, the above classes are also linearly independent, so the Picard number of $\overline{W}$ is at least 18. We will show that these classes form a *basis* of the free $\mathbb{Z}$-module $\operatorname{Pic}\overline{W}$.

Under our assumption on $k$, one can check easily that $W_5$ is smooth, so $W$ has good reduction at $p = 5$. We will now show that the Picard number of $\overline{W}_5$ equals exactly 18. Let $\overline{W}_5$ be the base change of $W_5$ to an algebraic closure of $\mathbb{F}_5$, and $\mathrm{Frob}_5 : \overline{W}_5 \to \overline{W}_5$ the geometric Frobenius morphism, defined by $([x : r], [y : s], [z : t]) \mapsto ([x^5 : r^5], [y^5 : s^5], [z^5 : t^5])$. Choose a prime $l \neq 5$ and let $\mathrm{Frob}_5^*$ be the endomorphism of $\mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l(1))$ induced by $\mathrm{Frob}_5$. By Proposition 3.2.4, the Picard rank of $\overline{W}$ is bounded above by that of $\overline{W}_5$, which in turn is at most the number of eigenvalues of $\mathrm{Frob}_5^*$ that are roots of unity. As in [Lui07a], we find the characteristic polynomial of $\mathrm{Frob}_5^*$ by counting points on $W_5$. Almost all fibers of the fibration $\pi_1$ are smooth curves of genus 1. Using Magma we can count the number of points over small fields fiber by fiber. The first three results are:

$$W_5(\mathbb{F}_5) = 42, \qquad W_5(\mathbb{F}_{5^2}) = 1032, \qquad W_5(\mathbb{F}_{5^3}) = 16122.$$

By [Lui07a, Lemma 6.1], we have $\dim \mathrm{H}^i(\overline{W}, \mathbb{Q}_l) = \dim \mathrm{H}^i(\overline{W}_5, \mathbb{Q}_l)$ for $0 \leqslant i \leqslant 4$. Since $\overline{W}$ is a K3 surface, the Betti numbers equal $\dim \mathrm{H}^i(\overline{W}_5, \mathbb{Q}_l) = 1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$, respectively. Therefore, from the Weil conjectures and the Lefschetz trace formula, we find that the trace of the $n$-th power of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l)$ equals $\#W_5(\mathbb{F}_{5^n}) - 5^{2n} - 1$; the trace on the Tate twist $\mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l(1))$ is obtained by *dividing* by $5^n$. Meanwhile, on the subspace $V \subset \mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l(1))$ generated over $\mathbb{Q}_l$ by the following set of 18 linearly independent divisor classes (of Gram determinant $-192$)

$$S' = \{D_2, D_3, \ell_{12}^{++}, \ell_{12}^{+-}, \ell_{12}^{-+}, \ell_{12}^{--}, \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{12}^{-+}}, \overline{\ell_{12}^{--}}, \ell_{13}^{++}, \overline{\ell_{13}^{++}}, \ell_{13}^{-+}, \overline{\ell_{13}^{-+}}, \ell_{23}^{++}, \overline{\ell_{23}^{++}}, C_1^{+-}, C_2^{+-}\},$$

$\mathrm{Frob}_5^*$ acts trivially on $D_2, D_3, C_1^{+-}, C_2^{+-}$, while $\mathrm{Frob}_5^*(\ell_{ij}^{\pm\pm}) = \overline{\ell_{ij}^{\pm\pm}}$ and $\mathrm{Frob}_5^*(\overline{\ell_{ij}^{\pm\pm}}) = \ell_{ij}^{\pm\pm}$, so the characteristic polynomial of the Frobenius acting on $V$ is $(t-1)^{11}(t+1)^7$. The trace $t_n$ is thus equal to 18 if $n$ is even, and equal to 4 if $n$ is odd; hence, on the 4-dimensional quotient $Q = \mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l(1))/V$, the trace equals

$$\frac{\#W_5(\mathbb{F}_{5^n})}{5^n} - 5^n - \frac{1}{5^n} - t_n.$$

These traces are sums of powers of eigenvalues, and we use the Newton identities to compute the elementary symmetric polynomials in these eigenvalues, which are the coefficients of the characteristic polynomial $f$ of the Frobenius acting on $Q$ (see [Lui07b, Lemma 2.4]). This yields the first half of the coefficients of $f$, including the middle coefficient, which turns out to be non-zero. This implies that the sign in the functional equation $t^4 f(1/t) = \pm f(t)$ is $+1$, so this functional equation determines $f$, which we calculate to be

$$f(t) = t^4 + \frac{4}{5}t^3 + \frac{6}{5}t^2 + \frac{4}{5}t + 1.$$

As a result, we find that the characteristic polynomial of the Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{W}_5, \mathbb{Q}_l(1))$ is equal to $(t-1)^{11}(t+1)^7 f$. The polynomial $5f \in \mathbb{Z}[t]$ is irreducible, primitive and not monic, so its roots are not roots of unity. Thus, we obtain an upper bound of 18 for the Picard number of $\overline{W}$.

Therefore, we deduce that $\operatorname{rk} \operatorname{Pic} \overline{W} = 18$, and the aforementioned sequence $S$ of 18 divisor classes (of Gram determinant $-48$) forms a sublattice $\Lambda \subset \mathrm{NS}\,\overline{W} = \mathrm{Pic}\,\overline{W}$ of finite index. We now verify that it is actually the whole lattice. Indeed, assume that $\Lambda$ is a proper sublattice of $\mathrm{NS}\,\overline{W}$, so their discriminants differ by a square factor. We know that $\mathrm{disc}(\Lambda) = -48 = -3.2^4$, so $\Lambda$ would be a sublattice of index 2 or 4. In other words, there would exist a divisor class of the form

$$E = \frac{1}{2}\sum_{E_i \in S} a_i E_i, \qquad a_i \in \{0,1\},$$

in $\mathrm{Pic}\,\overline{W}$. Since the intersection pairing between $E$ and each divisor class in $S$ would give an *integer* value, we find that there are only two possibilities:

(a) $E = \frac{1}{2}(D_1 + \ell_{12}^{+-} + \ell_{23}^{++} + \ell_{12}^{-+} + \ell_{23}^{--} + \overline{\ell_{12}^{++}} + \overline{\ell_{12}^{+-}} + \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}})$;

(b) $E = \frac{1}{2}(D_2 + D_3 + \ell_{13}^{++} + \ell_{23}^{++} + \ell_{13}^{-+} + \ell_{23}^{--} + \overline{\ell_{12}^{++}} + \overline{\ell_{12}^{-+}})$.

In the first case, we can check that $E^2 = -1$ is odd, which is a contradiction since the intersection pairing on $\mathrm{Pic}\,\overline{W}$ is *even*. In the second case, we have $E^2 = 2$, which is even. However, using the linear relations in $\mathrm{Pic}\,\overline{W}$:

$$D_3 = \ell_{13}^{++} + \ell_{13}^{-+} + \ell_{23}^{++} + \ell_{23}^{-+}$$

and

$$D_2 = \ell_{21}^{-+} + \ell_{21}^{--} + \ell_{23}^{-+} + \ell_{23}^{--} = \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{--}} + \ell_{23}^{-+} + \ell_{23}^{--},$$

we can rewrite

$$E = D_3 + \ell_{23}^{--} + \frac{1}{2}(\overline{\ell_{12}^{++}} + \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{-+}} + \overline{\ell_{12}^{--}}).$$

This implies that if (b) were true, then we would have $\frac{1}{2}(\overline{\ell_{12}^{++}} + \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{-+}} + \overline{\ell_{12}^{--}}) \in \mathrm{Pic}\,\overline{W}$. By contrast, using the argument in the proof of [Nik75, Lemma 3], one shows that the sum of divisor classes of *four* non-singular, non-intersecting rational curves on a K3 surface cannot be divisible by 2, since the total number of elements in such a set of classes can only be 0, 8, or 16. This is a contradiction, so the lattice generated by $S$ is indeed the whole Picard lattice $\mathrm{Pic}\,\overline{W}$, thus completing our proof. □

**Remark 3.2.2.** The above 18 divisor classes that form a basis of $\mathrm{Pic}\,\overline{W}$ are not unique, because one can find other first 16 divisors in the set of $D_i$ and $\ell_{ij}^{\pm\pm}, \overline{\ell_{ij}^{\pm\pm}}$ for $1 \leqslant i \neq j \leqslant 3$, and find the other 2 remaining divisors in the set of $C_i^{\pm\pm}$ for $1 \leqslant i \leqslant 3$ with different indexes $i$. Note that the divisors $A_1, A_2, A_3$ and $B_1, B_2, B_3$ defined by *irreducible* singular fibers have the same classes as $D_1, D_2, D_3$, respectively.

Next, we consider the geometric Picard group of the affine surface $U$ defined by the same equation.

**Corollary 3.2.6.** *Let $U = W \setminus \{rst = 0\}$ be the affine surface defined by*

$$x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 = k,$$

*where $k \in \mathbb{Z}$. If $k \equiv 3$ mod 5, then the Picard number of $\overline{U} = U_{\overline{\mathbb{Q}}}$ equals* 15.

*Proof.* By the exact sequence in Proposition 3.2.1, we obtain

$$\mathrm{Pic}\,\overline{U} \cong \mathrm{Pic}\,\overline{W}/(\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \mathbb{Z}D_3),$$

so $\mathrm{Pic}\,\overline{U}$ is *free* and the Picard number of $\overline{U}$ is equal to $18 - 3 = 15$. □

### 3.2.3 The algebraic Brauer group

Now given the geometric Picard group, we can compute directly the algebraic Brauer group of the Markoff-type cubic surfaces in question.

**Theorem 3.2.7.** *For $k \in \mathbb{Z}$, let $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.6}$$

*If $k \equiv 3 \bmod 5$, then*

$$\operatorname{Br}_1 W/\operatorname{Br}_0 W \cong (\mathbb{Z}/2\mathbb{Z})^3.$$

*Furthermore, for the affine surface $U = W \setminus \{rst = 0\}$, we even have*

$$\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

*Proof.* Since $W$ is smooth, projective, geometrically integral over $K$, we have $\overline{\mathbb{Q}}[W]^\times = \overline{\mathbb{Q}}^\times$. One already has $W(\mathbb{Q}) \neq \emptyset$, so $\operatorname{Br}_0 X = \operatorname{Br}\mathbb{Q}$. Since $\mathbb{Q}$ is a number field, by the Hochschild–Serre spectral sequence, we have an isomorphism

$$\operatorname{Br}_1 W/\operatorname{Br}_0 W \simeq \mathrm{H}^1(\mathbb{Q}, \operatorname{Pic}\overline{W}).$$

By Proposition 3.2.5, the geometric Picard number of $W$ is equal to 18 and a basis of $\operatorname{Pic}\overline{W}$ is given by

$$S = \{D_1, D_2, D_3, \ell_{12}^{++}, \ell_{12}^{+-}, \ell_{13}^{++}, \ell_{23}^{++}, \ell_{12}^{-+}, \ell_{13}^{-+}, \ell_{23}^{--}, \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{13}^{++}}, \overline{\ell_{23}^{++}}, \overline{\ell_{12}^{-+}}, \overline{\ell_{13}^{-+}}, C_1^{+-}, C_2^{+-}\}$$

along with the intersection matrix. If we denote by $(S)$ the column vector of elements of $S$, then from the intersection pairings of the classes in $S$ with the other classes in the list of Proposition 3.2.5, we find that

$$\begin{pmatrix} \overline{\ell_{12}^{--}} \\ \overline{\ell_{12}^{--}} \\ \overline{\ell_{23}^{--}} \\ \overline{\ell_{13}^{--}} \\ \overline{\ell_{13}^{--}} \\ \overline{\ell_{13}^{+-}} \\ \overline{\ell_{13}^{+-}} \\ \overline{\ell_{23}^{+-}} \\ \overline{\ell_{23}^{+-}} \\ \overline{\ell_{23}^{-+}} \\ \overline{\ell_{23}^{-+}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} (S)$$

and also

$$\begin{cases} D_1 = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{2,3\}} \ell_{1j}^{\epsilon\delta} = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{2,3\}} \overline{\ell_{1j}^{\epsilon\delta}}, \\ D_2 = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{1,3\}} \ell_{1j}^{\epsilon\delta} = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{1,3\}} \overline{\ell_{1j}^{\epsilon\delta}}, \\ D_3 = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{1,2\}} \ell_{1j}^{\epsilon\delta} = \sum_{\epsilon=\pm \text{ fixed},\delta=\pm \text{ varied}} \sum_{j\in\{1,2\}} \overline{\ell_{1j}^{\epsilon\delta}}, \\ C_1^{--} = C_1^{+-}; C_1^{++} = D_1 - C_1^{+-}, \\ C_2^{--} = C_2^{+-}; C_2^{++} = D_2 - C_2^{+-}, \\ C_3^{--} = C_3^{+-} = \ell_{12}^{++} + \ell_{13}^{++} + \ell_{23}^{++} + \overline{\ell_{12}^{++}} + \overline{\ell_{13}^{++}} + \overline{\ell_{23}^{++}} - C_1^{+-} - C_2^{+-}; C_3^{++} = D_3 - C_3^{+-}. \end{cases} \tag{3.7}$$

Consider the field extension $K := \mathbb{Q}(\sqrt{-1}, \sqrt{\alpha}, \sqrt{\bar{\alpha}})$ where $\Delta = \frac{(4k-5)^2-32}{64}$, $\alpha = \frac{1}{2}\left(\frac{4k-1}{8} + \sqrt{\Delta}\right)$ and $\bar{\alpha} = \frac{1}{2}\left(\frac{4k-1}{8} - \sqrt{\Delta}\right)$. Since $k \equiv 3 \bmod 5$, none of $\pm 2(4k+1), \pm\Delta, \pm 2(4k+1)\Delta$ is a square in $\mathbb{Q}$ so that $G := \mathrm{Gal}(K/\mathbb{Q}) \cong D_4 \times \mathbb{Z}/2\mathbb{Z}$ is the Galois group of the polynomial $(T^2+1)\left(T^4 - \frac{4k-1}{8}T^2 + \frac{4k+1}{32}\right)$ (see [Cona]). Now we study the action of the absolute Galois group on $\mathrm{Pic}\,\overline{W}$, which can be reduced to the action of $G = \mathrm{Gal}(K/\mathbb{Q})$. One clearly has $G \cong D_4 \times \mathbb{Z}/2\mathbb{Z} \cong (\langle\sigma\rangle \rtimes \langle\tau\rangle) \times \langle\rho\rangle$, where

$$\sigma(\alpha) = \bar{\alpha}, \quad \sigma(\bar{\alpha}) = -\alpha,$$

$$\tau(\alpha) = \alpha, \quad \tau(\bar{\alpha}) = -\bar{\alpha},$$

$$\rho(\sqrt{-1}) = -\sqrt{-1}.$$

Note that for $1 \leqslant i \neq j \leqslant 3$, $\sigma(\ell_{ij}^{\pm\pm}) = \overline{\ell_{ij}^{\pm\mp}}$, $\sigma(\overline{\ell_{ij}^{\pm\pm}}) = \ell_{ij}^{\mp\pm}$; $\tau(\ell_{ij}^{\pm\pm}) = \ell_{ij}^{\pm\mp}$, $\tau(\overline{\ell_{ij}^{\pm\pm}}) = \overline{\ell_{ij}^{\mp\pm}}$; $\rho(C_i^{\pm\pm}) = C_i^{\mp\pm}$ and $\sigma(C_i^{\pm\pm}) = C_i^{\pm\mp} = D_i - C_i^{\pm\pm}$. For technical reasons, we consider the following matrices of $\langle\sigma\rangle$ and $\langle\tau\rangle$ acting *stably* on the first 16 divisor classes of $\mathrm{Pic}\,\overline{W}$ in a specific permutation of the ordered basis given by $(S)$ as follows.

$$\{D_1, D_2, D_3, \ell_{12}^{++}, \ell_{12}^{+-}, \ell_{12}^{-+}, \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{12}^{-+}}, \ell_{13}^{++}, \ell_{13}^{-+}, \overline{\ell_{13}^{++}}, \overline{\ell_{13}^{-+}}, \ell_{23}^{++}, \ell_{23}^{--}, \overline{\ell_{23}^{++}}\}$$

gives the corresponding matrices for the action of $\sigma$ and $\tau$ respectively on $\mathrm{Pic}\,\overline{W}$:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0
\end{pmatrix},$$

and

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1
\end{pmatrix},$$

As a result, we obtain

$$\operatorname{Ker}(1+\rho) = \langle C_1^{+-} - C_1^{--}, C_2^{+-} - C_2^{--} \rangle,$$

and $\operatorname{Ker}(1+\sigma+\sigma^2+\sigma^3) = \langle D_1 - \overline{\ell_{12}^{-+}} - \overline{\ell_{12}^{++}} - \overline{\ell_{13}^{++}} - \overline{\ell_{13}^{-+}}, D_2 - \overline{\ell_{12}^{-+}} - \overline{\ell_{12}^{++}} - \ell_{23}^{--} - \overline{\ell_{23}^{++}}, D_3 - \overline{\ell_{13}^{++}} - \overline{\ell_{13}^{-+}} - \ell_{23}^{--} - \overline{\ell_{23}^{++}}, \ell_{12}^{++} - \overline{\ell_{12}^{-+}}, \ell_{12}^{+-} - \overline{\ell_{12}^{++}}, \ell_{12}^{-+} - \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}} - \overline{\ell_{12}^{-+}}, \ell_{13}^{++} - \overline{\ell_{13}^{-+}}, \ell_{13}^{-+} - \overline{\ell_{13}^{++}}, \ell_{23}^{++} - \ell_{23}^{--} \rangle$.
We also have

$$\operatorname{Ker}(1-\rho) = \langle D_1, D_2, D_3, \ell_{12}^{++}, \ell_{12}^{+-}, \ell_{13}^{++}, \ell_{23}^{++}, \ell_{12}^{-+}, \ell_{13}^{-+}, \ell_{23}^{--}, \overline{\ell_{12}^{++}}, \overline{\ell_{12}^{+-}}, \overline{\ell_{13}^{++}}, \overline{\ell_{23}^{++}}, \overline{\ell_{12}^{-+}}, \overline{\ell_{13}^{-+}} \rangle,$$

and $\operatorname{Ker}(1-\sigma) \cap \operatorname{Pic} \overline{W}^{\langle\rho\rangle} = \langle D_1, D_2, D_3, \ell_{12}^{+-} + \ell_{12}^{++} - \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{-+}} + 2\ell_{13}^{++} - \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} + \ell_{23}^{++} - \ell_{23}^{--}, \ell_{12}^{-+} + \ell_{12}^{++} + \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} - \ell_{23}^{++} - \ell_{23}^{--} + 2\overline{\ell_{23}^{++}}, \overline{\ell_{12}^{++}} - 2\ell_{12}^{++} - \overline{\ell_{12}^{-+}} - \ell_{13}^{++} + \ell_{13}^{-+} - 2\overline{\ell_{13}^{-+}} + \ell_{23}^{++} + \ell_{23}^{--} - 2\overline{\ell_{23}^{++}} \rangle$.

Given a finite cyclic group $G = \langle\sigma\rangle$ and a $G$-module $M$, by [NSW15, Proposition 1.7.1], recall that we have isomorphisms $\mathrm{H}^1(G, M) \cong \hat{\mathrm{H}}^{-1}(G, M)$, where the latter group is the quotient of ${}_{N_G}M$, the set of elements of $M$ of norm 0, by its subgroup $(1-\sigma)M$.

By [NSW15, Proposition 1.6.7], we have

$$\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \overline{W}) = \mathrm{H}^1(G, \operatorname{Pic} \overline{W}),$$

where $G = (\langle\sigma\rangle \rtimes \langle\tau\rangle) \times \langle\rho\rangle \cong D_4 \times \mathbb{Z}/2\mathbb{Z}$. Then one has the following (inflation-restriction) exact sequence

$$0 \to \mathrm{H}^1((\langle\tau\rangle \ltimes \langle\sigma\rangle), \operatorname{Pic} \overline{W}^{\langle\rho\rangle}) \to \mathrm{H}^1(G, \operatorname{Pic} \overline{W}) \to \mathrm{H}^1(\langle\rho\rangle, \operatorname{Pic} \overline{W}) = \frac{\operatorname{Ker}(1+\rho)}{(1-\rho)\operatorname{Pic} \overline{W}} = 0,$$

so $\mathrm{H}^1(G, \operatorname{Pic} \overline{W}) \cong \mathrm{H}^1(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic} \overline{W}^{\langle\rho\rangle})$. Now we are left with

$$\begin{aligned} 0 \to \mathrm{H}^1(\langle\tau\rangle, \operatorname{Pic} \overline{W}^{\langle\sigma,\rho\rangle}) &\to \mathrm{H}^1(G, \operatorname{Pic} \overline{W}) \\ &\to \mathrm{H}^1(\langle\sigma\rangle, \operatorname{Pic} \overline{W}^{\langle\rho\rangle}) = \frac{\operatorname{Ker}(1+\sigma+\sigma^2+\sigma^3) \cap \operatorname{Pic} \overline{W}^{\langle\rho\rangle}}{(1-\sigma)\operatorname{Pic} \overline{W}^{\langle\rho_2\rangle}} = 0, \end{aligned}$$

so $\mathrm{H}^1(G, \operatorname{Pic} \overline{W}) \cong \mathrm{H}^1(\langle\tau\rangle, \operatorname{Pic} \overline{W}^{\langle\sigma,\rho\rangle})$. The latter group can be computed as follows. We already have

$$\operatorname{Pic} \overline{W}^{\langle\sigma,\rho\rangle} = \operatorname{Ker}(1-\sigma) \cap \operatorname{Pic} \overline{W}^{\langle\rho\rangle}.$$

We find that $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \overline{W}) = \mathrm{H}^1(G, \operatorname{Pic} \overline{W})$
$\cong [\operatorname{Ker}(1+\tau) \cap \operatorname{Pic} \overline{W}^{\langle\sigma,\rho\rangle}]/(1-\tau)\operatorname{Pic} \overline{W}^{\langle\sigma,\rho\rangle}$
$= \langle \ell_{12}^{+-} + \ell_{12}^{++} - \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{-+}} + 2\ell_{13}^{++} - \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} + \ell_{23}^{++} - \ell_{23}^{--} - D_1, \ell_{12}^{-+} + \ell_{12}^{++} + \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} - \ell_{23}^{++} - \ell_{23}^{--} + 2\overline{\ell_{23}^{++}} - D_1, \overline{\ell_{12}^{++}} - 2\ell_{12}^{++} - \overline{\ell_{12}^{-+}} - \ell_{13}^{++} + \ell_{13}^{-+} - 2\overline{\ell_{13}^{-+}} + \ell_{23}^{++} + \ell_{23}^{--} - 2\overline{\ell_{23}^{++}} + D_1 \rangle$
$/2\langle \ell_{12}^{+-} + \ell_{12}^{++} - \overline{\ell_{12}^{+-}} + \overline{\ell_{12}^{-+}} + 2\ell_{13}^{++} - \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} + \ell_{23}^{++} - \ell_{23}^{--} - D_1, \ell_{12}^{-+} + \ell_{12}^{++} + \overline{\ell_{13}^{++}} + \overline{\ell_{13}^{-+}} - \ell_{23}^{++} - \ell_{23}^{--} + 2\overline{\ell_{23}^{++}} - D_1, \overline{\ell_{12}^{++}} - 2\ell_{12}^{++} - \overline{\ell_{12}^{-+}} - \ell_{13}^{++} + \ell_{13}^{-+} - 2\overline{\ell_{13}^{-+}} + \ell_{23}^{++} + \ell_{23}^{--} - 2\overline{\ell_{23}^{++}} + D_1 \rangle$
$\cong (\mathbb{Z}/2\mathbb{Z})^3$.

We keep the notation as above. Now $\operatorname{Pic} \overline{U}$ is given by the following quotient group

$$\operatorname{Pic} \overline{U} \cong \operatorname{Pic} \overline{W}/(\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \mathbb{Z}D_3)$$

by Proposition 3.2.1. Here for any divisor $D \in \operatorname{Pic} \overline{X}$, denote by $[D]$ its image in $\operatorname{Pic} \overline{U}$. By Proposition 3.2.1, we also have $\overline{\mathbb{Q}}^\times = \overline{\mathbb{Q}}[U]^\times$. By the Hochschild–Serre spectral sequence, we have the following isomorphism

$$\mathrm{Br}_1 U/\mathrm{Br}_0 U \cong \mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \overline{U})$$

as $\mathbb{Q}$ is a number field. Since $\operatorname{Pic} \overline{U}$ is free and $\operatorname{Gal}(\overline{\mathbb{Q}}/K)$ acts on $\operatorname{Pic} \overline{U}$ trivially, we obtain that $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \overline{U}) \cong \mathrm{H}^1(G, \operatorname{Pic} \overline{U})$. With the action of $G$, we can compute in the quotient group $\operatorname{Pic} \overline{U}$:

$$\begin{cases} \operatorname{Ker}(1+\rho) = \langle [C_1^{+-}] - [C_1^{--}], [C_2^{+-}] - [C_2^{--}] \rangle, \\ \operatorname{Ker}(1-\rho) = \langle [\ell_{12}^{++}], [\ell_{12}^{+-}], [\ell_{13}^{++}], [\ell_{23}^{++}], [\ell_{12}^{-+}], [\ell_{13}^{-+}], [\ell_{23}^{--}], [\overline{\ell_{12}^{++}}], [\overline{\ell_{12}^{+-}}], [\overline{\ell_{13}^{++}}], [\overline{\ell_{23}^{++}}], [\overline{\ell_{12}^{-+}}], [\overline{\ell_{13}^{-+}}] \rangle, \end{cases}$$

$\operatorname{Ker}(1-\sigma) \cap \operatorname{Pic} \overline{U}^{\langle\rho\rangle} = \langle [\ell_{12}^{+-}] + [\ell_{12}^{++}] - [\overline{\ell_{12}^{+-}}] + [\overline{\ell_{12}^{-+}}] + 2[\ell_{13}^{++}] - [\overline{\ell_{13}^{++}}] + [\overline{\ell_{13}^{-+}}] + [\ell_{23}^{++}] - [\ell_{23}^{--}], [\ell_{12}^{-+}] + [\ell_{12}^{++}] + [\overline{\ell_{13}^{++}}] + [\overline{\ell_{13}^{-+}}] - [\ell_{23}^{++}] - [\ell_{23}^{--}] + 2[\overline{\ell_{23}^{++}}], [\overline{\ell_{12}^{++}}] - 2[\ell_{12}^{++}] - [\overline{\ell_{12}^{-+}}] - [\ell_{13}^{++}] + [\ell_{13}^{-+}] - 2[\overline{\ell_{13}^{-+}}] + [\ell_{23}^{++}] + [\ell_{23}^{--}] - 2[\overline{\ell_{23}^{++}}] \rangle$,
and
$\operatorname{Ker}(1+\sigma+\sigma^2+\sigma^3) = \langle [\ell_{12}^{++}] - [\overline{\ell_{12}^{-+}}], [\ell_{12}^{+-}] + [\overline{\ell_{12}^{-+}}], [\ell_{12}^{-+}] + [\overline{\ell_{12}^{-+}}], [\overline{\ell_{12}^{++}}] + [\overline{\ell_{12}^{-+}}], [\overline{\ell_{12}^{+-}}] - [\overline{\ell_{12}^{-+}}], [\ell_{13}^{++}] - [\overline{\ell_{13}^{-+}}], [\ell_{13}^{-+}] + [\overline{\ell_{13}^{-+}}], [\overline{\ell_{13}^{++}}] + [\overline{\ell_{13}^{-+}}], [\ell_{23}^{++}] + [\overline{\ell_{23}^{++}}], [\ell_{23}^{--}] + [\overline{\ell_{23}^{++}}] \rangle$. Then

$$\mathrm{H}^1(\langle\rho\rangle, \operatorname{Pic} \overline{U}) = \frac{\operatorname{Ker}(1+\rho)}{(1-\rho)\operatorname{Pic} \overline{U}} = 0,$$

$$\mathrm{H}^1(\langle\sigma\rangle, \operatorname{Pic} \overline{U}^{\langle\rho\rangle})^{\langle\tau\rangle} = \left( \frac{\operatorname{Ker}(1+\sigma+\sigma^2+\sigma^3) \cap \operatorname{Pic} \overline{U}^{\langle\rho\rangle}}{(1-\sigma)\operatorname{Pic} \overline{U}^{\langle\rho\rangle}} \right)^{\langle\tau\rangle} = \left( \frac{\langle [\overline{\ell_{12}^{++}}] + [\overline{\ell_{12}^{-+}}] \rangle}{2\langle [\overline{\ell_{12}^{++}}] + [\overline{\ell_{12}^{-+}}] \rangle} \right)^{\langle\tau\rangle} \cong \mathbb{Z}/2\mathbb{Z},$$

and

$$\mathrm{H}^1(\langle\tau\rangle, \operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}) \cong \frac{\operatorname{Ker}(1+\tau)\cap \operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}}{(1-\tau)\operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}} = \frac{\operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}}{2\operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}} \cong (\mathbb{Z}/2\mathbb{Z})^3.$$

Since $\operatorname{Pic}\overline{U}^G = 0$, which gives $\mathrm{H}^0(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}) = \mathrm{H}^2(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}) = 0$, and moreover $\mathrm{H}^0(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}/2) \cong (\mathbb{Z}/2\mathbb{Z})^4$, then from the short exact sequence

$$0 \longrightarrow \operatorname{Pic}\overline{U}^{\langle\rho\rangle} \xrightarrow{\times 2} \operatorname{Pic}\overline{U}^{\langle\rho\rangle} \longrightarrow \operatorname{Pic}\overline{U}^{\langle\rho\rangle}/2 \longrightarrow 0$$

along with all the (similar as above) inflation-restriction exact sequences, we obtain the following exact sequences:

$$0 \to (\mathbb{Z}/2\mathbb{Z})^3 \cong \mathrm{H}^1(\langle\tau\rangle, \operatorname{Pic}\overline{U}^{\langle\sigma,\rho\rangle}) \to \mathrm{H}^1(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}) \to \mathrm{H}^1(\langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle})^{\langle\tau\rangle} \cong \mathbb{Z}/2\mathbb{Z} \to 0,$$

and

$$0 \to \mathrm{H}^0(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}/2) \cong (\mathbb{Z}/2\mathbb{Z})^4 \to \mathrm{H}^1(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}) \xrightarrow{\times 2} \mathrm{H}^1(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}),$$

we conclude that

$$\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong \mathrm{H}^1(G, \operatorname{Pic}\overline{U}) \cong \mathrm{H}^1(\langle\tau\rangle \ltimes \langle\sigma\rangle, \operatorname{Pic}\overline{U}^{\langle\rho\rangle}) \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

Now we produce some concrete generators in $\operatorname{Br}_1 U$ for $\operatorname{Br}_1 U/\operatorname{Br}_0 U$. The affine scheme $U \subset \mathbb{A}^3$ is defined over $\mathbb{Q}$ by the equation

$$x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 = k. \tag{3.8}$$

This affine equation is equivalent to

$$(4x^2+1)(4y^2+1)(4z^2+1) = (4k+1) + 128x^2y^2z^2, \tag{3.9}$$

$$(4x^2+1)(1+4y^2+4z^2-16y^2z^2) = (4k+1) - 32y^2z^2, \tag{3.10}$$

and also implies the following equation over $\{xyz \neq 0\}$:

$$(16x^2y^2 - 4x^2 - 4y^2 - 1)(16x^2z^2 - 4x^2 - 4z^2 - 1) = 2\left(\left(4x^2 - \frac{4k-1}{4}\right)^2 - \frac{(4k-5)^2 - 32}{16}\right), \tag{3.11}$$

as well as similar ones obtained by permutation of coordinates in all the above equations. Here we note that

$$\{4x^2+1=0\} \cap \{(4y^2+1)(4z^2+1)=0\},$$
$$\{4x^2+1=0\} \cap \{16y^2z^2 - 4y^2 - 4z^2 - 1 = 0\},$$
$$\{16x^2y^2 - 4x^2 - 4y^2 - 1 = 0\} \cap \{16x^2z^2 - 4x^2 - 4z^2 - 1 = 0\}$$

are closed subsets of codimension $\geqslant 2$ on $U$. By Grothendieck's purity theorem ([Poo17, Theorem 6.8.3]), we have the exact sequence

$$0 \to \operatorname{Br} U \to \operatorname{Br}\mathbb{Q}(U) \to \oplus_{D\in U^{(1)}}\mathrm{H}^1(\mathbb{Q}(D), \mathbb{Q}/\mathbb{Z}),$$

where the last map is given by the residue along the codimension-one point $D$. We consider the following quaternion algebras:

$$\mathcal{A}_1 = (4x^2+1, -2(4k+1)),$$

$$\mathcal{A}_2 = (4y^2+1, -2(4k+1)),$$
$$\mathcal{B} = (16x^2y^2 - 4x^2 - 4y^2 - 1, (4k-5)^2 - 32).$$

In order to prove that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ come from a class in $\operatorname{Br} U$, it suffices to show that their residues along the irreducible components of the divisors that belong to $\{4x^2+1=0\}$, $\{4y^2+1=0\}$, and $\{16x^2y^2-4x^2-4y^2-1=0\}$ are all trivial. Indeed, in the function field of each one of these irreducible divisors, $-2(4k+1)$ or $(4k-5)^2-32$ is clearly a square; standard formulae for residues in terms of the tame symbol [GS17, Example 7.1.5, Proposition 7.5.1] therefore show that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ are unramified, so they are elements of $\operatorname{Br} U$ and moreover they are clearly algebraic. We also obtain that

$$(4x^2+1, -2(4k+1)) = (2(4y^2+1)(4z^2+1), -2(4k+1)) = (2(16y^2z^2-4y^2-4z^2-1), 2(4k+1))$$

and

$$(16x^2y^2-4x^2-4y^2-1, (4k-5)^2-32) = (2(16x^2z^2-4x^2-4z^2-1), (4k-5)^2-32)$$

in $\operatorname{Br}_1 U$, as well as similar ones given by permutation of coordinates. The residues of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ at the irreducible divisors $D_1, D_2, D_3$ given by $rst = 0$ which form the complement of $U$ in $W$ are easily seen to be trivial. One thus has $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B} \in \operatorname{Br}_1 W$. Furthermore, the elements $\mathcal{A}_1, \mathcal{A}_2$ will contribute to the Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}$ (the integral model of $U$ defined over $\mathbb{Z}$ by the same equation) in the next section.

In conclusion, we have $\operatorname{Br}_1 W/\operatorname{Br}_0 W \cong (\mathbb{Z}/2\mathbb{Z})^3$, which can be seen as a subgroup of $\operatorname{Br}_1 U/\operatorname{Br}_0 U \cong (\mathbb{Z}/2\mathbb{Z})^4$ on using the vanishing of $\mathrm{H}^1(G, \oplus_{i=1}^3 \mathbb{Z}D_i)$ (and we also have $\operatorname{Br}_0 W = \operatorname{Br}_0 U = \operatorname{Br}\mathbb{Q}$ since the natural composite map $\operatorname{Br}\mathbb{Q} \hookrightarrow \operatorname{Br}_1 W \hookrightarrow \operatorname{Br}_1 U$ is injective). $\square$

**Remark 3.2.3.** One can hope to find more explicit generators for the quotient group $\operatorname{Br}_1 U/\operatorname{Br}_0 U$ by studying further the equation and its geometric nature. Furthermore, it would be more interesting if one can compute the transcendental part of the Brauer group for this family of Markoff-type K3 surfaces like what the authors in [LM20] and [CWX20] did for Markoff surfaces, which in general should be difficult.

## 3.3 The Brauer–Manin obstruction

### 3.3.1 Review of the Brauer–Manin obstruction

Let us again briefly recall how the Brauer–Manin obstruction works in our setting, following Section 2.3.1. For each place $v$ of $\mathbb{Q}$ there is a pairing

$$U(\mathbb{Q}_v) \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}$$

coming from the local invariant map

$$\operatorname{inv}_v : \operatorname{Br}\mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}$$

from local class field theory (this is an isomorphism if $v$ is a prime number). This pairing is locally constant on the left by [Poo17, Proposition 8.2.9]. For **integral points**, any element $\alpha \in \operatorname{Br} U$

pairs trivially on $\mathcal{U}(\mathbb{Z}_p)$ for almost all primes $p$, so we obtain a pairing $U(\mathbf{A}_{\mathbb{Q}}) \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}$. As the local pairings are locally constant, we obtain a well-defined pairing

$$\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_{\bullet} \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}.$$

For $B \subseteq \operatorname{Br} U$, let $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_{\bullet}^{B}$ be the left kernel with respect to $B$, and let $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_{\bullet}^{\operatorname{Br}} = \mathcal{U}(\mathbf{A}_{\mathbb{Z}})_{\bullet}^{\operatorname{Br} U}$. By abuse of notation, from now on we write the reduced Brauer–Manin set $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})_{\bullet}^{B}$ in the standard way as $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{B}$. We have the inclusions $\mathcal{U}(\mathbb{Z}) \subseteq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{B} \subseteq \mathcal{U}(\mathbf{A}_{\mathbb{Z}})$, so that $B$ can obstruct the *integral Hasse principle* or *strong approximation* on $\mathcal{U}$.

Let $V$ be dense Zariski open in $U$. As $U$ is smooth, the set $V(\mathbb{Q}_p)$ is dense in $U(\mathbb{Q}_p)$ for all places $p$. Moreover, $\mathcal{U}(\mathbb{Z}_p)$ is open in $U(\mathbb{Q}_p)$, hence $V(\mathbb{Q}_p) \cap \mathcal{U}(\mathbb{Z}_p)$ is dense in $\mathcal{U}(\mathbb{Z}_p)$. As the local pairings are locally constant, we may restrict our attention to $V$ to calculate the local invariants of a given element in $\operatorname{Br} U$.

### 3.3.2 Brauer–Manin obstruction from quaternion algebras

Now we consider the **three** explicit families of Markoff-type K3 (MK3) surfaces over $\mathbb{Q}$ as introduced before. From now on, we always denote by $W_k$ the projective MK3 surfaces, $U_k$ the affine open subscheme defined by $W_k \setminus \{rst = 0\}$ and $\mathcal{U}_k$ the integral model of $U_k$ defined by the same equation.

#### Existence of local points

First of all, we study the existence of local integral points on the affine MK3 surfaces. It is interesting to note that there always exist $\mathbb{Q}$-points at infinity (when $rst = 0$) on these surfaces.

**Proposition 3.3.1** (Assumption I). *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 - k = 0. \tag{3.12}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. *$k \equiv -1 \bmod 8$;*

2. *$k \not\equiv 0 \bmod 3, 5, 7$,*

*then $\mathcal{U}_k(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset$.*

*Proof.* For the place at infinity, it is clear that there exist real solutions: If $k \geqslant 0$ then take $(x,y,z) = (\sqrt{k}, 0, 0)$; if $k \leqslant -1$ then take $x = y = z \geqslant 1$ which satisfies $3x^2 - 4x^6 = k$ as the left hand side is a strictly decreasing continuous function of value $\leqslant -1$ on $[1, +\infty)$. For solutions at finite places $p$, with our specific conditions for $k$ in the assumption, we have:

(i) Prime powers of $p = 2$: It is clear that every solution modulo 2 is singular. Thanks to the condition (1), we find the solution $(1,1,1)$ modulo 8 with $(F_1)'_x(1,1,1) \equiv 2 \bmod 4$, which then lifts to a 2-adic integer solution by Hensel's lemma.

(ii) Prime powers of $p \geqslant 3$: We would like to find a non-singular solution modulo $p$ of the equations $F_1 = 0$ which does **not** satisfy simultaneously

$$dF_1 = 0 : 2x(1 - 4y^2z^2) = 0, 2y(1 - 4z^2x^2) = 0, 2z(1 - 4x^2y^2) = 0.$$

For simplicity, we fix $z = 0$ over $\mathbb{Z}_p$. First, it is clear that the equation $F_1 = 0$ always has a solution mod $p$ when $z = 0$: indeed, take $z = 0$, then $F_1 = 0$ becomes $x^2 + y^2 = k$, and every element in $\mathbb{F}_p$ can be expressed as a sum of two squares. Note that such a solution is singular if and only if $x = y = 0$, which means that $p$ divides $k$. Hence, if $p$ does not divide $k$, then we can find a non-singular solution mod $p$ which lifts to a $p$-adic integer solution by Hensel's lemma. In particular, this is true for $p = 3, 5, 7$ thanks to the condition (2).

Next, consider the case when $p \geqslant 11$ and $p$ divides $k$. We will fix instead $z = 1$ over $\mathbb{Z}_p$. The equation becomes

$$F_1(x, y, 1) = x^2 + y^2 - 4x^2y^2 + (1 - k) = 0$$

which defines an affine curve $C \subset \mathbb{A}^2_{(x,y)}$ over $\mathbb{F}_p$. If we consider its projective closure in $\mathbb{P}^2_{[x:y:t]}$ defined by

$$t^2(x^2 + y^2) - 4x^2y^2 + (1 - k)t^4 = 0,$$

then we can see that it has only two singularities which are *ordinary* of multiplicity 2, namely $[1 : 0 : 0]$ and $[0 : 1 : 0]$. By the genus–degree formula and the fact that the geometric genus is a birational invariant, we obtain

$$g(C) = \frac{(\deg C - 1)(\deg C - 2)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i - 1)}{2},$$

where $n$ is the number of ordinary singularities and $r_i$ is the multiplicity of each singularity for $i = 1, \dots, n$; in particular, $g(C) = 3 - 2 = 1$.

Now we consider the original projective closure $C^1 \subset \mathbb{P}^1_{[x:r]} \times \mathbb{P}^1_{[y:s]}$ defined by

$$x^2s^2 + y^2r^2 - 4x^2y^2 + (1 - k)r^2s^2 = 0.$$

The projective curve $C^1$ is **smooth** over $\mathbb{F}_p$ under our assumption on $k$. Then by the Hasse–Weil bound for smooth, projective and geometrically integral curves of genus 1, we have

$$|C^1(\mathbb{F}_p)| \geqslant p + 1 - 2\sqrt{p} = (\sqrt{p} - 1)^2 > (3 - 1)^2 = 4$$

since $p \geqslant 11$, so $|C^1(\mathbb{F}_p)| \geqslant 5$. As $C^1$ has exactly 4 points at infinity (when $rs = 0$), the affine curve $C$ has at least one *smooth* $\mathbb{F}_p$-point which then lifts to a $p$-adic integral point by Hensel's lemma.

□

**Proposition 3.3.2** (Assumption II). *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_2(x, y, z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - k = 0. \tag{3.13}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. $k \equiv 2 \bmod 8$, $k \equiv -9 \bmod 27$, $k \equiv -2 \bmod 5$, *and* $k \equiv 2 \bmod 7$;
2. $p \equiv \pm 1 \bmod 8$ *for any odd prime divisor $p$ of $k$,*

*then* $\mathcal{U}_k(\mathbf{A}_\mathbb{Z}) \neq \emptyset$.

*Proof.* For the place at infinity, it is clear that there exist real solutions: If $k \geqslant 0$ then take $(x, y, z) = (\sqrt{k}, 0, 0)$; if $k \leqslant -1$ then take $y = 1, z = 0$ and $x = \sqrt{\frac{1-k}{3}}$. For solutions at finite places $p$, with our specific conditions for $k$ in the assumption, we have:

(i) Prime powers of $p = 2$: It is clear that every solution modulo 2 is singular. Thanks to the condition (1), we find the solution $(1, 1, 2)$ modulo 8 with $(F_2)'_x(1, 1, 2) \equiv 2 \bmod 4$, which then lifts to a 2-adic integer solution by Hensel's lemma.

(ii) Prime powers of $p = 3, 5$: Thanks to the condition (1), we find the solutions $(3, 3, 0)$ modulo 27 ($(F_2)'_x(3, 3, 0) \equiv 6 \bmod 27$) and $(1, 1, 0)$ modulo 5 ($(F_2)'_x(1, 1, 0) \not\equiv 0 \bmod 5$), which respectively lift to a 3-adic and 5-adic integer solution by Hensel's lemma.

(iii) Prime powers of $p \geqslant 7$: We would like to find a non-singular solution modulo $p$ of the equations $F_2 = 0$ which does **not** satisfy simultaneously

$$dF_2 = 0 : 2x(1 - 4y^2)(1 - 4z^2) = 0, 2y(1 - 4z^2)(1 - 4x^2) = 0, 2z(1 - 4x^2)(1 - 4y^2) = 0.$$

First, note that the equation $F_2 = 0$ is equivalent to

$$(4x^2 - 1)(4y^2 - 1)(4z^2 - 1) = 4k - 1.$$

We observe that there are two special cases: if $p$ divides $k$ then there exists a non-singular solution $(a, b, 0)$ where $2a^2 \equiv 2b^2 \equiv 1 \bmod p$ thanks to the condition (2); if $p$ divides $4k - 1$ then there clearly exists a non-singular solution $(\frac{1}{2}, 0, 0)$. In particular, this is true for $p = 7$ thanks to the condition (1).

Next, consider the case when $p \geqslant 11$ and $p$ does not divide either $k$ or $4k - 1$. For simplicity, we fix $z = 0$ over $\mathbb{Z}_p$. The equation becomes

$$F_2(x, y, 0) = x^2 + y^2 - 4x^2y^2 - k = 0$$

which defines an affine curve $C \subset \mathbb{A}^2_{(x,y)}$ over $\mathbb{F}_p$. If we consider its projective closure in $\mathbb{P}^2_{[x:y:t]}$ defined by

$$t^2(x^2 + y^2) - 4x^2y^2 - kt^4 = 0,$$

then we can see that it has only two singularities which are *ordinary* of multiplicity 2, namely $[1 : 0 : 0]$ and $[0 : 1 : 0]$. By the genus–degree formula and the fact that the geometric genus is a birational invariant, we obtain

$$g(C) = \frac{(\deg C - 1)(\deg C - 2)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i - 1)}{2},$$

where $n$ is the number of ordinary singularities and $r_i$ is the multiplicity of each singularity for $i = 1, \ldots, n$; in particular, $g(C) = 3 - 2 = 1$.

Now we consider the original projective closure $C^1 \subset \mathbb{P}^1_{[x:r]} \times \mathbb{P}^1_{[y:s]}$ defined by

$$x^2s^2 + y^2r^2 - 4x^2y^2 - kr^2s^2 = 0.$$

The projective curve $C^1$ is **smooth** over $\mathbb{F}_p$ under our assumption on $k$. Then by the Hasse–Weil bound for smooth, projective and geometrically integral curves of genus 1, we have

$$|C^1(\mathbb{F}_p)| \geqslant p + 1 - 2\sqrt{p} = (\sqrt{p} - 1)^2 > (3-1)^2 = 4$$

since $p \geqslant 11$, so $|C^1(\mathbb{F}_p)| \geqslant 5$. As $C^1$ has exactly 4 points at infinity (when $rs = 0$), the affine curve $C$ has at least one *smooth* $\mathbb{F}_p$-point which then lifts to a $p$-adic integral point by Hensel's lemma.

□

**Proposition 3.3.3** (Assumption III). *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.14}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. *$k \equiv 1 \bmod 4$, $k \equiv 2 \bmod 3$, $k \equiv 3 \bmod 5$;*
2. *$k \not\equiv 0, -2 \bmod 7$ and $k \not\equiv 0 \bmod 37$,*

*then $\mathcal{U}_k(\mathbf{A}_\mathbb{Z}) \neq \emptyset$.*

*Proof.* For the place at infinity, it is clear that there exist real solutions: If $k \geqslant 0$ then take $(x,y,z) = (\sqrt{k}, 0, 0)$; if $k \leqslant -1$ then take $x = y = z \geqslant 1$ which satisfies $3x^2 + 12x^4 - 16x^6 = k$ as the left hand side is a strictly decreasing continuous function of value $\leqslant -1$ on $[1, +\infty)$. For solutions at finite places $p$, with our specific conditions for $k$ in the assumption, we have:

(i) Prime powers of $p = 2$: It is clear that every solution modulo 2 is singular. Thanks to the condition (1), we find the solutions $(1,0,0) \bmod 8$ if $k \equiv 1 \bmod 8$ and $(1,2,0) \bmod 8$ if $k \equiv 5 \bmod 8$ (with $(F_3)'_x \equiv 2 \bmod 8$), each of which then lifts to a 2-adic integer solution by Hensel's lemma.

(ii) Prime powers of $p = 3, 5$: Thanks to the condition (1), we find the non-singular solutions $(1,1,1)$ for $p = 3$ if $k \equiv 2 \bmod 3$ and $(1,2,1)$ for $p = 5$ if $k \equiv 3 \bmod 5$, which then respectively lift to a 3-adic and 5-adic integer solution by Hensel's lemma (with respect to $x$, fixing $y, z$).

(iii) Prime powers of $p \geqslant 7$: We would like to find a non-singular solution modulo $p$ of the equations $F_3 = 0$ which does **not** satisfy simultaneously

$$dF_3 = 0 : 2x(1+4y^2+4z^2-16y^2z^2) = 2y(1+4z^2+4x^2-16z^2x^2) = 2z(1+4x^2+4y^2-16x^2y^2) = 0.$$

For simplicity, we fix $z = 0$ over $\mathbb{Z}_p$. The equation becomes

$$F_3(x,y,0) = x^2 + y^2 + 4x^2y^2 - k = 0$$

which defines an affine curve $C \subset \mathbb{A}^2_{(x,y)}$ over $\mathbb{F}_p$. First we consider its projective closure in $\mathbb{P}^2_{[x:y:t]}$ defined by

$$t^2(x^2 + y^2) + 4x^2y^2 - kt^4 = 0.$$

If $p$ does not divide either $k$ or $4k+1$, then we can see that it has only two singularities which are *ordinary* of multiplicity 2, namely $[1:0:0]$ and $[0:1:0]$. By the genus–degree formula and the fact that the geometric genus is a birational invariant, we obtain

$$g(C) = \frac{(\deg C - 1)(\deg C - 2)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i - 1)}{2},$$

where $n$ is the number of ordinary singularities and $r_i$ is the multiplicity of each singularity for $i = 1, \ldots, n$; in particular, $g(C) = 3 - 2 = 1$.

Now we consider the original projective closure $C^1 \subset \mathbb{P}^1_{[x:r]} \times \mathbb{P}^1_{[y:s]}$ defined by

$$x^2s^2 + y^2r^2 + 4x^2y^2 - kr^2s^2 = 0.$$

If $p$ does not divide either $k$ or $4k+1$, then the projective curve $C^1$ is **smooth** over $\mathbb{F}_p$ under our assumption on $k$. Then by the Hasse–Weil bound for smooth, projective and geometrically integral curves of genus 1, we have

$$|C^1(\mathbb{F}_p)| \geqslant p + 1 - 2\sqrt{p} = (\sqrt{p} - 1)^2,$$

so $|C^1(\mathbb{F}_p)| \geqslant 3$ if $p = 7$ and $|C^1(\mathbb{F}_p)| \geqslant 5$ if $p \geqslant 11$. As $C^1$ has exactly 0 and 4 points at infinity (when $rs = 0$) if $p \equiv 3$ and $p \equiv 1 \bmod 4$ respectively, the affine curve $C$ has at least one *smooth* $\mathbb{F}_p$-point which then lifts to a $p$-adic integral point by Hensel's lemma. In particular, this is true for $p = 7$ thanks to the condition (2).

Next, consider the case when $p \geqslant 11$ and $p$ divides $k$ or $4k+1$. We will fix instead $z = 1$ over $\mathbb{Z}_p$. The equation becomes

$$F_3(x, y, 1) = 5x^2 + 5y^2 - 12x^2y^2 + (1 - k) = 0$$

which defines an affine curve $D \subset \mathbb{A}^2_{(x,y)}$ over $\mathbb{F}_p$. If we consider its projective closure in $\mathbb{P}^2_{[x:y:t]}$ defined by

$$5t^2(x^2 + y^2) - 12x^2y^2 + (1 - k)t^4 = 0,$$

then we can see that it also has only two singularities which are *ordinary* of multiplicity 2, namely $[1 : 0 : 0]$ and $[0 : 1 : 0]$. By the genus–degree formula and the fact that the geometric genus is a birational invariant, we obtain

$$g(D) = \frac{(\deg D - 1)(\deg D - 2)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i - 1)}{2},$$

where $n$ is the number of ordinary singularities and $r_i$ is the multiplicity of each singularity for $i = 1, \ldots, n$; in particular, $g(D) = 3 - 2 = 1$.

Now we consider the original projective closure $D^1 \in \mathbb{P}^1_{[x:r]} \times \mathbb{P}^1_{[y:s]}$ defined by

$$5x^2s^2 + 5y^2r^2 - 12x^2y^2 + (1 - k)r^2s^2 = 0.$$

The projective curve $D^1$ is **smooth** over $\mathbb{F}_p$ under our assumption on $k$ (especially the additional hypothesis $k \not\equiv 0 \bmod 37$). Then by the Hasse–Weil bound for smooth, projective and geometrically integral curves of genus 1, we have $|D^1(\mathbb{F}_p)| \geqslant 5$ since $p \geqslant 11$. As $D^1$ has at most 4 points at infinity (when $rs = 0$), the affine curve $D$ has at least one *smooth* $\mathbb{F}_p$-point which then lifts to a $p$-adic integral point by Hensel's lemma. The proof is now complete.

□

**Integral Brauer–Manin obstructions**

It is important to recall that there always exist $\mathbb{Q}$-points (at infinity) on every member $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of each family of these Markoff-type K3 surfaces, hence they satisfy the (rational) Hasse principle. Now we prove the Brauer–Manin obstructions to the integral Hasse principle for the integral model $\mathcal{U}_k$ of the affine subscheme $U_k \subset W_k$ by calculating the local invariants for some quaternion algebra classes $\mathcal{A}$ in their Brauer groups:

$$\mathrm{inv}_p \mathcal{A} : \mathcal{U}_k(\mathbb{Z}_p) \to \mathbb{Z}/2\mathbb{Z}, \qquad \mathbf{u} = (x, y, z) \mapsto \mathrm{inv}_p \mathcal{A}(\mathbf{u}).$$

**Theorem 3.3.4.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2, 2, 2)$-form*

$$F_1(x, y, z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 - k = 0. \tag{3.15}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. *$k = -(1 + 16\ell^2)$ where $\ell \in \mathbb{Z}$ such that $\ell$ is odd and $\ell \not\equiv \pm 2 \bmod 5$;*
2. *$p \equiv 1 \bmod 4$ for any prime divisor $p$ of $\ell$,*

*then there is an algebraic Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}_k$ with respect to the element $\mathcal{A} = (4x^2y^2 - 1, k + 1) = (4y^2z^2 - 1, k + 1) = (4z^2x^2 - 1, k + 1)$ in $\mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$. In other words, $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}} = \emptyset$.*

*Proof.* For any local point in $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$, we calculate its local invariants at every prime $p \leqslant \infty$. First of all, note that $U_k$ is smooth over $\mathbb{Q}$ and the affine equation implies

$$(4x^2y^2 - 1)(4y^2z^2 - 1) = (2y^2 + 1)^2 - 4(k + 1)y^2.$$

Therefore, we obtain the equality

$$\mathcal{A} = (4x^2y^2 - 1, k + 1) = (4y^2z^2 - 1, k + 1) = (4z^2x^2 - 1, k + 1)$$

in $\mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$. Now by abuse of notation, at each place $p$ we consider a local point denoted by $(x, y, z)$.

At $p = \infty$: From the equation $z^2(4x^2y^2 - 1) = x^2 + y^2 - k > 0$ for all $x, y, z \in \mathbb{R}$ since $k \leqslant -1$ by our assumption, so $4x^2y^2 - 1 > 0$ for every point $(x, y, z) \in U_k(\mathbb{R})$. Hence we have $\mathrm{inv}_\infty \mathcal{A}(x, y, z) = 0$.

At $p = 2$: Since $k \equiv -1 \bmod 8$, all the coordinates $x, y, z$ are in $\mathbb{Z}_2^\times$, then $4x^2y^2 - 1 \equiv 3 \bmod 8$ so $\mathrm{inv}_2 \mathcal{A}(x, y, z) = (4x^2y^2 - 1, k + 1)_2 = (3, -1)_2 = \frac{1}{2}$.

At $p \geqslant 3$: Since $k + 1 = -16\ell^2$ and every odd prime divisor $p$ of it satisfies $(-1, p)_p = 0$, if $p$ divides $k + 1$ then $\mathrm{inv}_p \mathcal{A}(x, y, z) = 0$. Otherwise, if $p$ divides $4x^2y^2 - 1$ then $p$ cannot divide $y$ and so by the above equation we have $k + 1 \in \mathbb{Z}_p^{\times 2}$, which implies that $(4x^2y^2 - 1, k + 1)_p = 0$. Finally, if $4x^2y^2 - 1$ and $k + 1$ are both in $\mathbb{Z}_p^\times$ then the local invariant is trivial as well.

In conclusion, we have

$$\sum_{p \leqslant \infty} \mathrm{inv}_p \mathcal{A}(x, y, z) = \frac{1}{2} \neq 0,$$

so $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}} = \emptyset$. □

**Theorem 3.3.5.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_2(x,y,z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - k = 0. \tag{3.16}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. $k = 18\ell^2$ *where $\ell \in \mathbb{Z}$ such that $\ell \not\equiv 0 \bmod 2, 3$, $\ell \equiv 1 \bmod 5$, and $\ell \equiv 2 \bmod 7$;*
2. $p \equiv \pm 1 \bmod 8$ *for any prime divisor $p$ of $\ell$,*

*then there is an algebraic Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}_k$ with respect to the subgroup $A \subset \mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$ generated by the elements $\mathcal{A}_1 = (4x^2-1, k)$ and $\mathcal{A}_2 = (4y^2-1,k)$, i.e., $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^A = \emptyset$.*

*Proof.* For any local point in $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$, we calculate its local invariants at every prime $p \leqslant \infty$. First of all, note that $U_k$ is smooth over $\mathbb{Q}$ and the affine equation implies

$$(4x^2-1)(4y^2-1)(4z^2-1) = 4k-1.$$

Therefore, we obtain the equality

$$(4x^2-1,k) + (4y^2-1,k) + (4z^2-1,k) = 0$$

in $\mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$. Now by abuse of notation, at each place $p$ we consider a local point denoted by $(x,y,z)$.

At $p=\infty$: For any $\ell \in \mathbb{R}$, we always have $k = 18\ell^2 > 0$, hence $\mathrm{inv}_\infty \mathcal{A}_{1,2}(x,y,z) = 0$.

At $p=2$: Since $k \equiv 2 \bmod 8$, exactly two of the coordinates $x,y,z$ are in $\mathbb{Z}_2^\times$, so without loss of generality let one of them be $x$, then $4x^2-1 \equiv 3 \bmod 8$ so $\mathrm{inv}_2 \mathcal{A}_1(x,y,z) = (4x^2-1,k)_2 = (3,2)_2 = \frac{1}{2}$.

At $p=3$: Since $k = 18\ell^2$, all of the coordinates $x,y,z$ must be divisible by 3, so $\mathrm{inv}_3 \mathcal{A}_1(x,y,z) = (-1,18)_3 = 0$.

At $p \geqslant 5$: Since $k = 18\ell^2$ and every odd prime divisor $p \neq 3$ satisfies $(2,p)_p = 0$, if $p$ divides $k$ then $\mathrm{inv}_p \mathcal{A}_1(x,y,z) = 0$. Otherwise, if $p$ divides $4x^2-1$ then by the above equation we have $k \in \mathbb{Z}_p^{\times 2}$, which implies that $(4x^2-1,k)_p = 0$. Finally, if $4x^2-1$ and $k$ are both in $\mathbb{Z}_p^\times$ then the local invariant is trivial as well.

In conclusion, we have

$$\sum_{p \leqslant \infty} \mathrm{inv}_p \mathcal{A}(x,y,z) = \frac{1}{2} \neq 0,$$

so $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^A = \emptyset$. □

**Theorem 3.3.6.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.17}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. $k = -\frac{1}{4}(1+27\ell^2)$ *where $\ell \in \mathbb{Z}$ such that $\ell \equiv \pm 1 \bmod 8$, $\ell \equiv 1 \bmod 5$, $\ell \equiv 3 \bmod 7$, and $\ell \not\equiv \pm 10 \bmod 37$;*

*2.* $p \equiv \pm 1 \bmod 24$ *for any prime divisor* $p$ *of* $\ell$,

*then there is an algebraic Brauer–Manin obstruction to the integral Hasse principle for* $\mathcal{U}_k$ *with respect to the subgroup* $A \subset \mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$ *generated by the elements* $\mathcal{A}_1 = (4x^2+1, -2(4k+1))$ *and* $\mathcal{A}_2 = (4y^2+1, -2(4k+1))$*, i.e.,* $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})^A = \emptyset$.

*Proof.* For any local point in $\mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})$, we calculate its local invariants at every prime $p \leqslant \infty$. First of all, note that $U_k$ is smooth over $\mathbb{Q}$ and the affine equation implies

$$(4x^2+1)(4y^2+1)(4z^2+1) = (4k+1) + 128x^2y^2z^2.$$

Therefore, we obtain the equality

$$(4x^2+1, -2(4k+1)) + (4y^2+1, -2(4k+1)) + (4z^2+1, -2(4k+1)) = 0$$

in $\mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$. Now by abuse of notation, at each place $p$ we consider a local point denoted by $(x, y, z)$.

At $p = \infty$: For any $x \in \mathbb{R}$, we always have $4x^2 + 1 > 0$, hence $\mathrm{inv}_\infty \mathcal{A}_1(x, y, z) = 0$.

At $p = 2$: Since $k \equiv 1 \bmod 4$, exactly two of the coordinates $x, y, z$ are in $2\mathbb{Z}_2$, so without loss of generality let one of them be $x$, then $4x^2+1 \equiv 1 \bmod 8$ so $\mathrm{inv}_2 \mathcal{A}_1(x, y, z) = (4x^2+1, -2(4k+1))_2 = 0$.

At $p = 3$: Since $k \equiv 2 \bmod 3$, all of the coordinates $x, y, z$ are in $\mathbb{Z}_3^\times$, so $\mathrm{inv}_3 \mathcal{A}_1(x, y, z) = (2, 54\ell^2)_3 = (-1, 3)_3 = \frac{1}{2}$.

At $p \geqslant 5$: Since $-2(4k+1) = 54\ell^2$ and every odd prime divisor $p \neq 3$ satisfies $(6, p)_p = 0$, if $p$ divides $4k+1$ then $\mathrm{inv}_p \mathcal{A}_1(x, y, z) = 0$. Otherwise, if $p$ divides $4x^2+1$ then by the above equation we have $-2(4k+1) \in \mathbb{Z}_p^{\times 2}$, which implies that $(4x^2+1, -2(4k+1))_p = 0$. Finally, if $4x^2+1$ and $4k+1$ are both in $\mathbb{Z}_p^\times$ then the local invariant is trivial as well.

In conclusion, we have

$$\sum_{p \leqslant \infty} \mathrm{inv}_p \mathcal{A}(x, y, z) = \frac{1}{2} \neq 0,$$

so $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})^A = \emptyset$. □

**Example 3.3.1.** We give some explicit counterexamples to the integral Hasse principle for the three families of Markoff-type K3 surfaces that we have discussed. Note that in theory, there always exist **primes** $\ell$ which satisfy all the hypotheses for each family, thanks to the well-known Dirichlet's theorem on arithmetic progressions.

(1) For $\ell = 1$, we have

$$x^2 + y^2 + z^2 - 4x^2y^2z^2 = -(1 + 16.1^2) = -17.$$

(2) For $\ell = 191$, we have

$$x^2 + y^2 + z^2 - 4(x^2y^2 + z^2x^2 + x^2y^2) + 16x^2y^2z^2 = 18.191^2 = 656658.$$

(3) For $\ell = 241$, we have

$$x^2 + y^2 + z^2 + 4(x^2y^2 + z^2x^2 + x^2y^2) - 16x^2y^2z^2 = -\frac{1}{4}(1 + 27.241^2) = -392047.$$

### 3.3.3 Counting the Hasse failures

In this part, we calculate the number of examples of existence for local integral points as well as the number of counterexamples to the integral Hasse principle for our Markoff-type K3 surfaces which can be explained by the Brauer–Manin obstruction. More precisely, we compute the natural density of $k \in \mathbb{Z}$ satisfying Assumptions I, II, III and the three main Theorems about the Brauer–Manin obstruction.

**Theorem 3.3.7.** *For the above three families of MK3 surfaces, we have*

$$\#\{k \in \mathbb{Z} : |k| \leqslant M,\ \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset\} \asymp M$$

*and*

$$\#\{k \in \mathbb{Z} : |k| \leqslant M,\ \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset,\ \mathcal{U}_k(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} = \emptyset\} \gg \frac{M^{1/2}}{\log M},$$

*as $M \to +\infty$.*

*Proof.* For the first estimate, the result follows directly from the fact that Assumptions I, II, III only give finitely many congruence conditions on $k$, so the total numbers of $k$ are approximately a proportion of $M$ as $M \to +\infty$.

For the second estimate, we only give an asymptotic lower bound under the condition that $\ell$ is a **prime**. The result follows from the fact that $|k| \leqslant M$ is a linear function of $\ell^2$ and as $M \to +\infty$, the number of primes smaller than $\sqrt{N}$ (where $N$ is a proportion of $M$) satisfying finitely many congruence conditions is asymptotically equal to $\dfrac{\sqrt{M}}{\log M}$, up to a constant factor (see [Apo76, Section 7.9]). □

**Remark 3.3.2.** Continuing from a previous remark, it would be interesting if one can find a way to include the transcendental Brauer group into the counting result, which would help us consider the Brauer–Manin set with respect to the whole Brauer group instead of only its algebraic part.

## 3.4 Further remarks

In this section, we compare the results that we obtain in this chapter with those in the previous papers studying Markoff surfaces, namely [GS22], [LM20], [CWX20], and [Dao24].

### 3.4.1 Existence of the Brauer–Manin obstruction

First of all, recall that in the case of Markoff surfaces, we see from [LM20] that the number of counterexamples to the integral Hasse principle which can be explained by the Brauer–Manin obstruction is asymptotically equal to $M^{1/2}/(\log M)^{1/2}$; this number is also the asymptotic lower bound for the number of Markoff surfaces such that there is no Brauer–Manin obstruction to the integral Hasse principle, as done in [CWX20] (slightly better than the result $M^{1/2}/\log M$ in [LM20]).

We begin our study in the case of Markoff-type K3 surfaces by the following two results.

**Proposition 3.4.1.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x, y, z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.18}$$

*Denote by $\mathcal{U}_k$ the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. $k = \ell(\ell+1)$ *where* $\ell \in \mathbb{Z}$ *such that* $\ell \equiv 5 \bmod 8$, $\ell \equiv 4 \bmod 27$, $\ell \equiv 1 \bmod 35$, *and* $\ell \not\equiv 0, -1 \bmod 37$;
2. $p \equiv \pm 1, 3 \bmod 8$ *for any prime divisor* $p$ *of* $2\ell + 1$,

*then there is a Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}_k$ with respect to the subgroup $A \subset \mathrm{Br}_1\, U_k/\mathrm{Br}_0\, U_k$ generated by the elements $\mathcal{A}_1 = (4x^2+1, 2(4k+1))$ and $\mathcal{A}_2 = (4y^2+1, 2(4k+1))$, i.e., $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^A = \emptyset$.*

*Proof.* The proof is similar to above, with notice that only the local invariant at $p = 2$ is nonzero which makes the total sum of invariants nonzero, hence a contradiction. □

**Proposition 3.4.2.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x, y, z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.19}$$

*Denote by $\mathcal{U}_k$ the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $k$ satisfies the conditions:*

1. $k = \ell(\ell+1)$ *where* $\ell \in \mathbb{Z}$ *such that* $\ell \equiv 3 \bmod 8$, $\ell \equiv 4 \bmod 27$, $\ell \equiv 1 \bmod 35$, *and* $\ell \not\equiv 0, -1 \bmod 37$;
2. $p \equiv \pm 1, 3 \bmod 8$ *for any prime divisor* $p$ *of* $2\ell + 1$,

*then there is **no** Brauer–Manin obstruction to the integral Hasse principle for $\mathcal{U}_k$ with respect to the subgroup $A \subset \mathrm{Br}_1\, U_k/\mathrm{Br}_0\, U_k$ generated by the elements $\mathcal{A}_1 = (4x^2+1, 2(4k+1))$ and $\mathcal{A}_2 = (4y^2+1, 2(4k+1))$, i.e., $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^A \neq \emptyset$.*

*Proof.* The proof is similar to above, except that with $k = \ell(\ell+1) \equiv 4 \bmod 8$, the local invariants at $p = 2$ are $(0, 0)$, which makes the total sum of invariants always zero, hence the conclusion. In fact, it even shows that $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^A = \mathcal{U}_k(\mathbf{A}_\mathbb{Z})$. □

**Remark 3.4.1.** The first Proposition is only used to give a different family of Markoff-type K3 surfaces for which there is a Brauer–Manin obstruction and to make an interesting comparison with the second Proposition. In fact, one may give an elementary proof for the fact that the set of integral points is empty as follows.

Assume that there is an integral point $(x, y, z) \in \mathcal{U}_k(\mathbb{Z})$, then if $|x|, |y|, |z| \geqslant 1$, $F(x, y, z) < 0$ as $k = \ell(\ell+1) > 0$. Therefore, at least one of $x, y, z$ must be zero, and without loss of generality, we may assume that $z = 0$. The equation is equivalent to

$$(4x^2+1)(4y^2+1) = (2\ell+1)^2.$$

As the right hand side is divisible by 3 since $\ell \equiv 4 \bmod 27$, so is the left hand side. However, this is a contradiction as $-1$ is not a square modulo 3.

The second Proposition only gives the result with respect to a proper subgroup of the Brauer group since we are not able to determine explicitly the whole (algebraic) Brauer–Manin set to prove whether it is nonempty or not. That is also the reason why we have not yet found a similar counting result to the ones in [LM20] and [CWX20].

### 3.4.2 Failure of strong approximation

Next, we consider some cases when strong approximation, instead of the integral Hasse principle, fails, while integral points can exist.

**Proposition 3.4.3.** *For $k \equiv 2 \bmod 8$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_2(x,y,z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - k = 0. \tag{3.20}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $\mathcal{U}_k(\mathbb{Z}) \neq \emptyset$, then there is a Brauer–Manin obstruction to strong approximation for $\mathcal{U}_k$ with respect to the element $\mathcal{A}_1 = (4x^2 - 1, k) \in \mathrm{Br}_1 U_k / \mathrm{Br}_0 U_k$, i.e., $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1} \neq \mathcal{U}_k(\mathbf{A}_\mathbb{Z})$.*

To illustrate our choice of $k$, we can choose an integral point $(x,y,z) = (1,1,10) \in \mathcal{U}_k(\mathbb{Z})$ to have $k = 898$.

*Proof.* Assume that we have $(x,y,z) \in \mathcal{U}(\mathbb{Z})$, so with $k \equiv 2 \bmod 8$ we can assume further without loss of generality that $x, y$ are odd and $z = 2a$ is even. Since $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$, the set $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$ is nonempty, and so is $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$. Viewing $(x,y,z)$ as an element of $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$ via the diagonal embedding, we can find another local integral point $(x', y', z')$ with the 2-part $(x'_2, y'_2, z'_2) = (z_2, x_2, y_2)$ and the same $p$-parts as those of $(x,y,z)$ for every $p \neq 2$, so that $\mathrm{inv}_2\, \mathcal{A}_1(x', y', z') = (4.4a^2 - 1, k)_2 = 0 \neq 1/2 = (3,k))_2 = (4x_2^2 - 1, k)_2$. Consequently,

$$\sum_p \mathrm{inv}_p\, \mathcal{A}_1(x', y', z') \neq \sum_p \mathrm{inv}_p\, \mathcal{A}_1(x,y,z) = 0.$$

Therefore, $(x', y', z') \notin \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$ and the result follows. □

**Proposition 3.4.4.** *For $k \equiv 1 \bmod 4$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_3(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.21}$$

*Let $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. If $\mathcal{U}_k(\mathbb{Z}) \neq \emptyset$, then there is a Brauer–Manin obstruction to strong approximation for $\mathcal{U}_k$ with respect to the element $\mathcal{A}_1 = (4x^2 + 1, -2(4k+1)) \in \mathrm{Br}_1 U_k / \mathrm{Br}_0 U_k$, i.e., $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1} \neq \mathcal{U}_k(\mathbf{A}_\mathbb{Z})$.*

To illustrate our choice of $k$, we can choose an integral point $(x,y,z) = (1,4,4) \in \mathcal{U}_k(\mathbb{Z})$ to have $k = -2911$.

*Proof.* Assume that we have $(x,y,z) \in \mathcal{U}(\mathbb{Z})$, so with $k \equiv 1 \bmod 4$ we can assume further without loss of generality that $x$ is odd and $y = 2a, z = 2b$ are even. Since $\mathcal{U}_k(\mathbb{Z}) \subset \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$, the set $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$ is nonempty, and so is $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$. Viewing $(x,y,z)$ as an element of $\mathcal{U}_k(\mathbf{A}_\mathbb{Z})$ via the diagonal embedding, we can find another local integral point $(x', y', z')$ with the 2-part $(x'_2, y'_2, z'_2) = (y_2, x_2, z_2)$ and the same $p$-parts as those of $(x,y,z)$ for every $p \neq 2$, so that $\mathrm{inv}_2\, \mathcal{A}_1(x', y', z') = (4.4a^2+1, -2(4k+1))_2 = 0 \neq 1/2 = (5, -2(4k+1))_2 = (4x_2^2+1, -2(4k+1))_2$. Consequently,

$$\sum_p \mathrm{inv}_p\, \mathcal{A}_1(x', y', z') \neq \sum_p \mathrm{inv}_p\, \mathcal{A}_1(x,y,z) = 0.$$

Therefore, $(x', y', z') \notin \mathcal{U}_k(\mathbf{A}_\mathbb{Z})^{\mathcal{A}_1}$ and the result follows. □

**Remark 3.4.2.** In the case of $F_1$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 - k = 0 \tag{3.22}$$

and $\mathcal{U}_k$ be the integral model of $U_k$ defined over $\mathbb{Z}$ by the same equation. When $\mathcal{U}_k(\mathbb{Z}) \neq \emptyset$, it seems likely that the local invariant at $p = 2$ of the Brauer element $\mathcal{A} = (4x^2y^2 - 1, k+1) = (4y^2z^2 - 1, k+1) = (4z^2x^2 - 1, k+1)$ is constant for various choices of 2-adic integral points. In other words, we may need to work with other primes to (possibly) find a Brauer–Manin obstruction to strong approximation.

**Remark 3.4.3.** For the third family of MK3 surfaces $W_k$ defined by $F_3 = 0$, from the previous section we can see that the divisor $K + D$ is *big*, where $K$ is the (trivial) canonical divisor on $W_k$ and $D := D_1 + D_2 + D_3$ is an *ample* divisor. Therefore, $U_k = W_k \setminus D$ is of *log general type*, and Vojta's Conjecture asserts that integral points on $\mathcal{U}_k$ are *not Zariski-dense*.

### 3.4.3 Rational points on affine surfaces

Finally, we study the existence of rational points on affine Markoff-type K3 surfaces. For Markoff surfaces, we know from [Kol02], [LM20] and [CWX20] that there are always rational points on smooth affine Markoff sufaces; this comes from the fact that any smooth cubic surface over an infinite field $k$ is $k$-unirational as soon as it has a $k$-rational point. However, such a phenomenon does not happen for smooth affine MK3 surfaces, since their projective closures are elliptic surfaces and lie in $(\mathbb{P}^1)^3$ instead of $\mathbb{P}^3$. We know that there are always rational points at infinity for our families of MK3 surfaces, but we are not certain whether there are also rational points on the affine open subschemes or not. As a modest contribution to the existence problem of rational points, we have the following results.

**Proposition 3.4.5.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 - k = 0. \tag{3.23}$$

*Let $U_k \subset W_k$ be the affine open subscheme defined by $\{rst \neq 0\}$ over $\mathbb{Q}$ by the same equation. If $k$ satisfies the conditions:*

1. *$k = -(1 + 16\ell^2)$ where $\ell \in \mathbb{Z}$ such that $\ell$ is odd and $\ell \not\equiv \pm 2 \bmod 5$;*

2. *$p \equiv 1 \bmod 4$ for any prime divisor $p$ of $\ell$,*

*then there is no Brauer–Manin obstruction to the (rational) Hasse principle for $U_k$ with respect to the element $\mathcal{A} = (4x^2y^2 - 1, k+1) = (4y^2z^2 - 1, k+1) = (4z^2x^2 - 1, k+1)$ in $\mathrm{Br}_1\, U_k / \mathrm{Br}_0\, U_k$. In other words, $U_k(\mathbf{A}_{\mathbb{Q}})^{\mathcal{A}} \neq \emptyset$.*

*Proof.* The proof proceeds similarly as before, with notice that for $p = 2$, besides the local integral point lifted from $(1,1,1) \in \mathcal{U}_k(\mathbb{Z}/8\mathbb{Z})$ which gives the local invariants $(1/2, 1/2)$, there exists another local point $(x_2, y_2, z_2) \in U_k(\mathbb{Q}_2)$ with $\mathrm{v}_2(x_2) = -1, \mathrm{v}_2(y_2) = -3$, and $\mathrm{v}_2(z_2) = 0$ which gives the local invariants $(0,0)$. $\square$

**Proposition 3.4.6.** *For $k \in \mathbb{Z}$, let $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surface defined over $\mathbb{Q}$ by the $(2,2,2)$-form*

$$F_2(x,y,z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - k = 0. \tag{3.24}$$

*Let $U_k \subset W_k$ be the affine open subscheme defined by $\{rst \neq 0\}$ over $\mathbb{Q}$ by the same equation. If $k$ satisfies the conditions:*

1. $k = 18\ell^2$ *where* $\ell \in \mathbb{Z}$ *such that* $\ell \not\equiv 0 \bmod 2, 3$, $\ell \equiv 1 \bmod 5$, *and* $\ell \equiv 2 \bmod 7$;
2. $p \equiv \pm 1 \bmod 8$ *for any prime divisor* $p$ *of* $\ell$,

*then there is no Brauer–Manin obstruction to the (rational) Hasse principle for* $U_k$ *with respect to the subgroup* $A \subset \mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$ *generated by the elements* $\mathcal{A}_1 = (4x^2 - 1, k)$ *and* $\mathcal{A}_2 = (4y^2 - 1, k)$, *i.e.,* $U_k(\mathbf{A}_{\mathbb{Q}})^A \neq \emptyset$.

*Proof.* The proof proceeds similarly as before, with notice that for $p = 3$, besides the local integral point lifted from $(3, 3, 0) \in \mathcal{U}_k(\mathbb{Z}/27\mathbb{Z})$ which gives the local invariants $(0, 0)$, there exists another local point $(x_3, y_3, z_3) \in U_k(\mathbb{Q}_3)$ with $\mathrm{v}_3(x_3) = \mathrm{v}_3(y_3) = 0, \mathrm{v}_3(z_3) = -1$ and $2z_3 = \frac{a}{b}$ such that $a \equiv 2, b \equiv 3$, $x_3 \equiv -2, y_3 \equiv 13 \pmod{27}$: this gives the local invariants $(1/2, 1/2)$. $\square$

**Proposition 3.4.7.** *For* $k \in \mathbb{Z}$, *let* $W_k \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *be the MK3 surface defined over* $\mathbb{Q}$ *by the* $(2, 2, 2)$*-form*

$$F_3(x, y, z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 - k = 0. \tag{3.25}$$

*Let* $U_k \subset W_k$ *be the affine open subscheme defined by* $\{rst \neq 0\}$ *over* $\mathbb{Q}$ *by the same equation. If* $k$ *satisfies the conditions:*

1. $k = -\frac{1}{4}(1 + 27\ell^2)$ *where* $\ell \in \mathbb{Z}$ *such that* $\ell \equiv \pm 1 \bmod 8$, $\ell \equiv 1 \bmod 5$, $\ell \equiv 3 \bmod 7$, *and* $\ell \not\equiv \pm 10 \bmod 37$;
2. $p \equiv \pm 1 \bmod 24$ *for any prime divisor* $p$ *of* $\ell$,

*then there is no Brauer–Manin obstruction to the (rational) Hasse principle for* $U_k$ *with respect to the subgroup* $A \subset \mathrm{Br}_1 U_k/\mathrm{Br}_0 U_k$ *generated by the elements* $\mathcal{A}_1 = (4x^2 + 1, -2(4k + 1))$ *and* $\mathcal{A}_2 = (4y^2 + 1, -2(4k + 1))$, *i.e.,* $U_k(\mathbf{A}_{\mathbb{Q}})^A \neq \emptyset$.

*Proof.* The proof proceeds similarly as before, with notice that for $p = 3$, besides the local integral point lifted from $(1, 1, 1) \in \mathcal{U}_k(\mathbb{Z}/3\mathbb{Z})$ which gives the local invariants $(1/2, 1/2)$, there exists another local point $(x_3, y_3, z_3) \in U_k(\mathbb{Q}_3)$ with $\mathrm{v}_3(x_3) < 0, \mathrm{v}_3(y_3) < 0$, and $\mathrm{v}_3(z_3) = 0$ which gives the local invariants $(0, 0)$. $\square$

**Remark 3.4.4.** Once again, we do not know whether the Brauer–Manin set with respect to the whole Brauer group is nonempty or not, but at least we know that there is no Brauer–Manin obstruction to the existence of rational points with respect to the Brauer elements that we are interested in. We believe that there should exist rational points on those families of affine MK3 surfaces, but we do not know how to prove or disprove this claim in general. Following Yang Cao's suggestion to our work [Dao23], it seems that rational points of Wehler K3 surfaces should have some similar phenomena as integral points of affine Markoff surfaces, since one side is K3 and the other side is log K3. From Ghosh and Sarnak's results in [GS22], we expect that in a similar nature, our families of affine MK3 surfaces would satisfy the (rational) Hasse principle and the Brauer–Manin obstruction would not be enough to explain *almost all* counterexamples to the Hasse principle.

**Example 3.4.5.** We consider the first surface in Example 4.3.1 where the affine MK3 surface in question contains no integral points (due to the Brauer–Manin obstruction as previously shown) but indeed contains rational points. Let $W_{-17} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surfaces defined over $\mathbb{Q}$ by the $(2, 2, 2)$-form

$$F_1(x, y, z) = x^2 + y^2 + z^2 - 4x^2y^2z^2 + 17 = 0. \tag{3.26}$$

Denote by $\mathcal{U}_{-17}$ the integral model of $U_{-17}$ defined over $\mathbb{Z}$ by the same equation. Then $\mathcal{U}_{-17}(\mathbb{Z}) = \emptyset$; however, we can find a few rational points of small height in $U_{-17}(\mathbb{Q})$ using SageMath [SJ05]: (1/2, 49/24, 13/5), (1/3, 5/2, 29/8), (1/3, 15/8, 109/18), (1/5, 13/2, 77/24), (7/32, 46/15, 23/4), (22/25, 23/16, 23/12), (27/29, 47/34, 15/8).

**Example 3.4.6.** We consider an example of the second family of surfaces in Theorem 4.3.5 where the affine MK3 surface in question contains no integral points (due to the Brauer–Manin obstruction as previously shown) but indeed contains rational points. Let $W_{18} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surfaces defined over $\mathbb{Q}$ by the $(2,2,2)$-form

$$F_2(x,y,z) = x^2 + y^2 + z^2 - 4(x^2y^2 + y^2z^2 + z^2x^2) + 16x^2y^2z^2 - 18 = 0. \tag{3.27}$$

Denote by $\mathcal{U}_{18}$ the integral model of $U_{18}$ defined over $\mathbb{Z}$ by the same equation. Note that here we choose $\ell = 1$ in Theorem 4.3.5, and although Assumption II is not satisfied when considering $k$ modulo 7, but we can find $(2,2,2) \in \mathcal{U}_{18}(\mathbb{Z}/7\mathbb{Z})$ which is non-singular and so can still lift to a point in $\mathcal{U}_{18}(\mathbb{Z}_7)$. Therefore, $\mathcal{U}_{18}(\mathbb{Z}) = \emptyset$; however, we can find a few rational points of small height in $U_{18}(\mathbb{Q})$ using SageMath [SJ05]: (1/3, 1/3, 38/5), (1/7, 38/5, 11/27), (2/7, 37/3, 5/11), (3, 3, 18/35), (3/8, 3/8, 135/14), (3/44, 35/6, 259/760), (5/6, 38/5, 13/24), (5/17, 11/49, 158/27), (6/5, 39/7, 9/17).

**Example 3.4.7.** We consider an example of the third family of surfaces in Theorem 4.3.6 where the affine MK3 surface in question contains no integral points (due to the Brauer–Manin obstruction as previously shown) but indeed contains rational points. Let $W_{-7} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the MK3 surfaces defined over $\mathbb{Q}$ by the $(2,2,2)$-form

$$F_3(x,y,z) = x^2 + y^2 + z^2 + 4(x^2y^2 + y^2z^2 + z^2x^2) - 16x^2y^2z^2 + 7 = 0. \tag{3.28}$$

Denote by $\mathcal{U}_{-7}$ the integral model of $U_{-7}$ defined over $\mathbb{Z}$ by the same equation. Note that here we choose $\ell = 1$ in Theorem 4.3.6, and although Assumption III is not satisfied when considering $k$ modulo 7, but we can find $(0,1,2) \in \mathcal{U}_{-7}(\mathbb{Z}/7\mathbb{Z})$ which is non-singular and so can still lift to a point in $\mathcal{U}_{-7}(\mathbb{Z}_7)$. Then $\mathcal{U}_{-7}(\mathbb{Z}) = \emptyset$; however, we can find a rational point of small height in $U_{-7}(\mathbb{Q})$ using SageMath [SJ05]: (29/4, 91/86, 631/988).

# Appendix A

# Integral points on Markoff surfaces over global function fields

## A.1 Integral points on affine conics

Throughout this section, let $K$ denote a global field of characteristic $p > 0$, which is the function field of a geometrically integral curve over a finite field $\mathbb{F}_q$, where $q$ is a power of $p$. We recall an example of affine scheme $\mathcal{X}$ over $\mathbb{F}_p[t]$ studied by Harari and Voloch in [HV13], using the Artin–Schreier torsors, where $K := \operatorname{Frac} \mathcal{O}_S = \mathbb{F}_p(t)$ and the finite set $S$ consists of the prime at infinity, such that $\mathcal{X}$ has points over every local completion of $\mathcal{O}_S$ but $\mathcal{X}(\mathcal{O}_S) = \emptyset$. In particular, every point of $\prod_{v \neq \infty} \mathcal{X}(\mathcal{O}_S) \times X(K_\infty)$ is *obstructed* by some torsor. First, let us recall the definition and the main result regarding the Artin–Schreier torsors in [HV13]. For more details on the descent obstructions, one should see the cited reference.

**Definition A.1.1.** Let $X$ be a $K$-variety. An *Artin–Schreier torsor* over $X$ is a torsor under the étale group scheme $\mathbf{F}_s$ (for some $s = q^e, e > 0$) given by the equation

$$z^s - z = g$$

for some $g \in K[X]$. Such a torsor corresponds to the cohomology class $u_{s,X}(g) \in \mathrm{H}^1(X, \mathbf{F}_s)$, where $g$ is viewed as an element of $K[X]/\Phi_s(K[X])$.

**Theorem A.1.1.** *Let $\mathcal{X}$ be an affine $\mathcal{O}_S$-scheme of finite type with generic fiber $X$. Let $(x_v) \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_S) \times \prod_{v \in S} X(K_v)$. Assume that $(x_v)$ is* unobstructed *by every Artin–Schreier torsor $Y \to X$. Then $(x_v) \in \mathcal{X}(\mathcal{O}_S)$.*

Let $s = q^e$ be a power of $q$. The *Cartier dual* $\mathbf{G}_s := \mathscr{H}\!om(\mathbf{F}_s, \mathbb{G}_m)$ of the étale $K$-group scheme $\mathbf{F}_s$ is a finite $K$-group scheme of order $s$; for example if $s = p$ is a prime number, then $\mathbf{G}_s$ is just the $K$-group scheme $\mu_p$ of $p$-roots of unity. For a $K$-variety $X$, the duality pairing

$$\mathbf{G}_s \times \mathbf{F}_s \longrightarrow \mu_s \hookrightarrow \mathbb{G}_m$$

induces a cup-product map in fppf cohomology

$$\mathrm{H}^1(X, \mathbf{G}_s) \times \mathrm{H}^1(X, \mathbf{F}_s) \longrightarrow \mathrm{H}^2(X, \mathbb{G}_m) = \operatorname{Br} X.$$

Hence, for any Artin–Schreier torsor $Y \to X$ under $\mathbf{F}_s$ and any element $a \in \mathrm{H}^1(X, \mathbf{G}_s)$ (fppf cohomology), there is a cup-product $(a \cup [Y]) \in \operatorname{Br} X$, where $[Y]$ is the class of $Y$ in $\mathrm{H}^1(X, \mathbf{F}_s)$ and $a$ stands for the image of $a$ in $\mathrm{H}^1(X, \mathbf{G}_s)$.

Harari and Voloch defined a subgroup of the Brauer group, on using this cup-product map, in order to study the Brauer–Manin obstruction to the integral Hasse principle.

**Definition A.1.2.** Let $X$ be a $K$-variety. Set $\operatorname{Br}_0 X := \operatorname{Im}(\operatorname{Br} K \to \operatorname{Br} X)$. For each $e > 0$ and $s = q^e$, define a subgroup $B_{AS,e}(X)$ of the Brauer group $\operatorname{Br} X$ as the subgroup generated by $\operatorname{Br}_0 X$ and the cup-products $(a \cup [Y])$, where $Y$ runs over all Artin–Schreier $X$-torsors under $\mathbf{F}_s$ and $a$ runs over all elements of $\mathrm{H}^1(K, \mathbf{G}_s)$. Then set

$$B_{AS}(X) := \bigcup_{e>0} B_{AS,e}(X) \subset \operatorname{Br} X.$$

**Theorem A.1.2.** *Let $X$ be an affine $K$-variety. Let $(a_v) \in \prod_v X(K_v)$ be an adelic point on $X$. Assume that for every element $\theta \in B_{AS}(X)$, the evaluation $\theta((a_v))$ is global, i.e, comes from an element of $\operatorname{Br} K$ by the diagonal embedding. Then $(a_v) \in X(K)$.*

Using this theorem, Harari and Voloch proved that *the Brauer–Manin obstruction is the only obstruction to the existence of integral points on affine varieties over global fields of positive characteristic.* More precisely, the obstructions come from étale covers of exponent $p$ or, alternatively, from flat covers coming from torsors under connected group schemes of exponent $p$. This result leads to an explicit family of counterexample to the integral Hasse principle for affine conics over global function fields, as also studied in [HV13].

**Proposition A.1.3.** *Let $p > 2$ and $A, B \in \mathbb{F}_p[t]$ with $\deg A = 3, \deg B = 1$. Consider the affine conic over $\mathcal{O}_S = \mathbb{F}_p[t]$*

$$\mathcal{X} : x^2 + Ay^2 = B.$$

*Assume that $A, B$ are chosen so that the generic fiber $X = \mathcal{X} \times_{\mathcal{O}_S} K$ (defined over $K = \mathbb{F}_p(t)$) has a $K$-point. Then every point in $\prod_{v \neq \infty} \mathcal{X}(\mathcal{O}_v) \times X(K_\infty)$ is* obstructed *by the torsor*

$$Y : z^p - z = y.$$

*In particular, $\mathcal{X}$ has no $\mathbb{F}_p[t]$-point.*

*Proof.* By contradiction, we assume that $\mathcal{X}$ has a $\mathbb{F}_p[t]$-point. This means that there exists a point in $\prod_{v \neq \infty} \mathcal{X}(\mathcal{O}_S) \times X(K_\infty)$ which is *unobstructed* by the Artin–Scheier torsor $z^p - z = y$. Hence as already discussed, there exists a global twist

$$z^p - z = y + c$$

with local points everywhere. From the conditions at $v \neq \infty$, we get that $c$ is a polynomial. Now we consider the place $v = \infty$ in particular. Recall that the equation for $X$ is given by

$$X : x^2 + Ay^2 = B$$

over $K$, from which we can compute the valuation of the component $y_\infty$. Indeed, since there exist $x, y \in K$ such that $Ay^2 = B - x^2$ by assumption, if $\operatorname{ord}(x_\infty) \leqslant -1$ then $\operatorname{ord}(B - x_\infty^2) = \operatorname{ord}(x_\infty^2)$ (as $\deg B = 1 < 2$) is even but $\operatorname{ord}(Ay_\infty^2) = \operatorname{ord}(A) + \operatorname{ord}(y_\infty^2)$ is odd (as $\deg A = 3$ is odd), so we obtain a contradiction. Hence we must have $\operatorname{ord}(x_\infty) \geqslant 0$, then $\operatorname{ord}(B - x_\infty^2) = \operatorname{ord}(B) = -1$ and so $\operatorname{ord}(y_\infty) = (-1+3)/2 = 1$. Now as the equation $z^p - z = y + c$ has a solution over $K = \mathbb{F}_p(t)$, then for any $y$ with $\operatorname{ord}(y_\infty) = 1$ we have $\operatorname{ord}(z_\infty) > 0$ so $y_\infty \in \Phi_p(K_\infty)$. It follows that $c \in \Phi_p(K_\infty)$ also and as a polynomial, $c$ is actually in $\Phi_p(\mathbb{F}_p[t])$ by Hensel's Lemma applied to the completion at $v = \infty$ with the uniformizing parameter $\pi = 1/t$. Therefore, without loss of generality, we may assume that $c = 0$.

Now, as $p > 2$ and $B$ is linear, there exists $\alpha \in \mathbb{F}_p$ such that $B(\alpha)$ is not a square (since the set of square residues modulo $p$ is always smaller the set of (linear) residues modulo $p$). The condition that $z^p - z = y$ has a local point at the place $v = t - \alpha$ shows that $y_v$ vanishes at $\alpha$ and so we get $x_v(\alpha)^2 = B(\alpha)$ is a square, which is a contradiction. The proof is complete. $\square$

**Remark A.1.3.** In this proposition, we can check directly that $\mathcal{X}$ has no $\mathbb{F}_p[t]$-point by degree considerations. For example, when $p = 3$, we can take $A = t^2(t+1), B = t+1$, for which the affine scheme $X$ has the rational point $(0, 1/t)$.

Theoretically, the Brauer–Manin obstruction is the only obstruction to the existence of integral points on affine conics over global fields, i.e., both number fields and global function fields. We have just seen the result for the global function field case; for the number field case, we recall the following result by Harari [Har08], saying that the *algebraic* Brauer–Manin obstruction is the only one.

**Proposition A.1.4.** *Let $\mathcal{X}$ be an affine conic over a number field $k$ defined by the equation*

$$ax^2 + by^2 = c, \qquad a, b, c \in \mathcal{O}_k, abc \neq 0.$$

*We denote by $X$ the generic fiber over $k$. If for every place $v \in \Omega_k$, there exists a solution $(x_v, y_v) \in \mathcal{O}_v \times \mathcal{O}_v$, such that the adelic point $(P_v) = (x_v, y_v)_{v \in \Omega_k}$ is orthogonal to $\mathrm{Br}_1 X/\mathrm{Br}\, k$, then there exists a global solution $(x, y) \in \mathcal{O}_k \times \mathcal{O}_k$.*

## A.2 Integral points on Markoff surfaces

In this last section, we would like to study the Markoff cubic surfaces defined by the equation

$$x^2 + y^2 + z^2 - xyz = m$$

with $m \in \mathbb{F}_p[t]$. As we have known, this cubic equation has been studied in [GS22], [CWX20] and [LM20] in the case $m \in \mathbb{Z}$. It now seems interesting for us to study the existence problem of integral points over global fields of characteristic $p$ (here $p > 2$ due to the nature of the equation). If possible, we would prefer to use the Artin–Schreier torsors to construct some explicit counterexamples to the integral Hasse principle for Markoff surfaces over the function field $\mathbb{F}_p(t)$.

**Remark A.2.1.** With this type of equation, we could start by making the following transformation: The cubic equation of the Markoff surfaces can be turned into a quadratic equation of two variables, for example, $x$ and $y$:

$$(2x - zy)^2 - (z^2 - 4)y^2 = 4(m - z^2),$$

where we can set new variables: $z := z_0 \in \mathbb{F}_p[t]$, $x := 2x - z \cdot y$ and $y := y$. Then we consider the equation over the field $K = \mathbb{F}_p(t)$ where $p > 2$, $m \in \mathbb{F}_p[t]$ and $m \neq 0, 4$. Now let $A := 4 - z_0^2$ and $B := 4(m - z_0^2)$ as elements of $\mathbb{F}_p[t]$, we get the following similar type of conic equations as in the previous section:

$$\mathcal{X} : x^2 + Ay^2 = B;$$

however, $\deg A = \deg B = 2$, which makes the problem more complicated than the previous one. On the other hand, if one variable is fixed, we can construct another explicit counterexample to the integral Hasse principle for this type of affine conics. For example, we can take $z_0 = t^2 - 2$ and $m = z_0^2 + t + 2$, and then replace them into the equation to obtain

$$x^2 + t^2(4 - t^2)y^2 = 4(t + 2).$$

By elementary arguments, one can show that this equation has a solution $(t + 2, 1/t)$ over $\mathbb{F}_p(t)$, but has no solution over $\mathbb{F}_p[t]$.

For convenience, let us recall systematically some important original results from the papers [GS22], [CWX20] and [LM20]. First of all, a key tool which Ghosh and Sarnak used to study the integral points on Markoff surfaces is the automorphism group $\Gamma$ of polynomial affine transformations generated by the following three types of elements:

(i) the Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$,

(ii) the sign change: $(x, y, z) \mapsto (-x, -y, z)$,

(iii) the permutations of $x, y, z$.

This implies that $\Gamma$ preserves $\mathcal{U}_m(\mathbb{Z})$. Their cases of interest are that of $k \geqslant 5$ or $k < 0$ with $k$ not a square, and in those cases $\mathcal{U}_m(\mathbb{Z})$ decomposes into a finite number $h(m)$ of $\Gamma$-orbits, with each orbit being infinite and even Zariski dense in $\mathcal{U}_m$ as long as $\mathcal{U}_m(\mathbb{Z}) \neq \emptyset$. The goal of their paper is to study the set of $m$'s for which $h(m) > 0$. Any admissible $m$ for which $h(m) = 0$ is called a *Hasse failure*. The number of $0 < m \leqslant M$ (or $-M \leqslant m < 0$) which are admissible is shown to be $\frac{7}{12}M + O(1)$.

Ghosh and Sarnak developed an explicit reduction theory for the action of $\Gamma$ on $\mathcal{U}_m(\mathbb{Z})$. For convenience, they removed a set of special admissible $m$'s, namely those for which there is a point in $\mathcal{U}_m(\mathbb{Z})$ with the absolute value of one coordinate being 0, 1 or 2: they are of the form $m = a^2 + b^2$ or $4(m - 1) = a^2 + 3b^2$ or $m = 4 + a^2$. These special $m$'s are called *exceptional*, and the remaining admissible $m$'s are called *generic*. Ghosh and Sarnak obtained the following elegant reduced forms of the fundamental sets (note that all negative admissible $m$'s are generic).

**Theorem A.2.1** ([GS22, Theorem 1.1]). (i) *Let $m \geqslant 5$ be generic and consider the compact set*

$$\Delta_m^+ := \{(a, b, c) \in \mathbb{R}^3 : 3 \leqslant a \leqslant b \leqslant c, a^2 + b^2 + c^2 + abc = m\}.$$

*The points in $\Delta_m^+(\mathbb{Z}) = \Delta_m^+ \cap \mathbb{Z}^3$ are $\Gamma$-equivalent, and any $(x, y, z) \in \mathcal{U}_m(\mathbb{Z})$ is $\Gamma$-equivalent to a unique point $(-a, b, c)$ with $(a, b, c) \in \Delta_m^+(\mathbb{Z})$.*

(ii) *Let $m < 0$ be admissible and consider the compact set*

$$\Delta_m^- := \{(a, b, c) \in \mathbb{R}^3 : 3 \leqslant a \leqslant b \leqslant c \leqslant \frac{1}{2}ab, a^2 + b^2 + c^2 - abc = m\}.$$

*The points in $\Delta_m^-(\mathbb{Z}) = \Delta_m^- \cap \mathbb{Z}^3$ are $\Gamma$-equivalent, and any $(x, y, z) \in \mathcal{U}_m(\mathbb{Z})$ is $\Gamma$-equivalent to a unique point $(a, b, c) \in \Delta_m^-(\mathbb{Z})$.*

By the above reduction theory, for each $m$ generic, if $\mathcal{U}_m(\mathbf{A}_\mathbb{Z}) \neq \emptyset$ but $\Delta_m^\pm(\mathbb{Z}) = \emptyset$, then $\mathcal{U}_m(\mathbb{Z}) = \emptyset$, i.e. $\mathcal{U}_m$ fails the integral Hasse principle. Here are some simple consequences of this theorem (see the discussion in the Introduction of [GS22]):

(a) $\mathcal{U}_m(\mathbb{Z}) = \emptyset$, that is $h(46) = 0$, being the first positive Hasse failure, while $m = -4$ is the first negative Hasse failure.

(b) $h(m) \ll_\epsilon |m|^{\frac{1}{3}+\epsilon}$ as $k \to \pm\infty$, due to the restrictions imposed by the fundamental sets.

(c) Let $h^{\pm}(m) = |\Delta_m^{\pm}(\mathbb{Z})|$, then the above theorem implies that for generic $m$, $h^{\pm}(m) = h(m)$ while otherwise $k(m) \leqslant h^{\pm}(m)$. By [GS22, Lemmas 7.2 and 7.3], we have

$$\sum_{m\neq 4, |m|\leqslant K} h^{\pm}(m) \sim C^{\pm} K(\log K)^2,$$

where $C^{\pm}$ is some fixed positive constant and $K \to \infty$.

Hence, on average the numbers $h(m)$ is small, and the fact that this average grows slowly is a key feature which suggests that $h(m)$ might be nonzero for many integer $m$. Next, Ghosh and Sarnak showed that although there are infinitely many Hasse failures and give a lower bound for their number, the Markoff surfaces are **almost perfect but not perfect** in the following sense.

**Theorem A.2.2** ([GS22, Theorem 1.2]). (i) *There are* infinitely many *Hasse failures. More precisely, the number of $0 < m \leqslant K$ and $-K \leqslant m < 0$ for which the Hasse principle fails is at least $K^{1/2}/(\log K)^{1/2}$ for $K$ sufficiently large.*

(ii) *The Markoff surfaces are* almost perfect*, that is*

$$\#\{|m| \leqslant K : m \textit{ is admissible}, h(m) = 0\} = o(K),$$

*as $K \to \infty$.*

Furthermore, in [GS22, Section 10], they gave some numerical evidence from computing experiments using Theorem 6.1 to find the Hasse failures among the generic $m$'s when $0 < m < 6 \times 10^8$, and it suggests that the number of Hasse failures when $0 < m \leqslant K$ is asymptotically $C_0 K^{\gamma}$, with $C_0 > 0$ and $\gamma \approx 0.8875\ldots$.

**Conjecture A.2.3** ([GS22, Conjecture 10.1]). *For any $\epsilon > 0$, we have*

$$h(m) \ll_\epsilon |m|^{\epsilon}.$$

**Conjecture A.2.4** ([GS22, Conjecture 10.2]). *The number of Hasse failures for $0 \leqslant m \leqslant K$ satisfies*

$$\#\{0 \leqslant m \leqslant K : h(m) = 0, \mathcal{U}_m(\boldsymbol{A}_{\mathbb{Z}}) \neq \emptyset\} \sim C_0 K^{\gamma},$$

*for some $C_0 > 0$ and $\frac{1}{2} < \gamma < 1$.*

*More generally, for $t \geqslant 1$,*

$$\#\{0 \leqslant m \leqslant K : h(m) = t\} \sim C_t K^{\gamma},$$

*with $C_t > 0$ following an exponential decay in $t$.*

Now we consider the Brauer–Manin obstruction, and we are interested in how often, the integral Hasse principle can be explained by this cohomological obstruction. We recall two significant results about the number of the Hasse failures, following [LM20] and [CWX20]. In these two papers, we have seen that the integral Brauer–Manin obstruction can be used to explain all the Hasse failures in [GS22]. The first result, from [LM20], is the asymptotic calculation of the number of counterexamples explained by this obstruction.

**Theorem A.2.5** ([LM20, Theorem 1.4]). *We have*

$$N_1(K) := \#\{m \in \mathbb{Z} : |m| \leqslant K,\ \mathcal{U}_m(\mathbf{A}_{\mathbb{Z}}) \neq \emptyset \textit{ but } \mathcal{U}_m(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} = \emptyset\} \asymp \frac{K^{1/2}}{(\log K)^{1/2}}.$$

Next we look at a result in [CWX20] that gives us a family of surfaces for which the Hasse failures, if they exist, cannot be explained by the Brauer–Manin obstruction. This will be useful for the counting result after that.

**Proposition A.2.6** ([CWX20, Theorem 5.11]). *Let $\mathcal{U}$ be the affine scheme over $\mathbb{Z}$ defined by*

$$x^2 + y^2 + z^2 - xyz = 4 + 2a^2\ell^2,$$

*where $a$ is an odd integer and $\ell$ is a prime with $\ell \equiv \pm 3 \bmod 8$. If $a\ell \equiv \pm 4 \bmod 9$, then $\mathcal{U}(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} \neq \emptyset$.*

The last result, from [CWX20], shows that there are infinitely many of Markoff surfaces such that the failures of the integral Hasse principal cannot be explained by the Brauer–Manin obstruction, using the above proposition. Furthermore, by the numerical evidence for the lower bound $C_0K^\gamma$ mentioned above, the previous theorem implies that almost all of these hypothetical Hasse failures in [GS22] cannot be explained by the Brauer–Manin obstruction. Loughran and Mitankin also gave a result for the lower bound of the number of Markoff surfaces for which that happens in [LM20, Theorem 1.5], that is $K^{1/2}/\log K$, but we can see below that the number given in [CWX20] is slightly better, although it is still far from close to the expected result in [GS22].

**Theorem A.2.7** ([CWX20, Theorem 5.14]). *We have*

$$N_2^+(K) := \#\{m \in \mathbb{Z} : 0 < m < K,\ \mathcal{U}_m(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} \neq \emptyset \text{ but } \mathcal{U}_m(\mathbb{Z}) = \emptyset\} \gg \frac{K^{1/2}}{(\log K)^{1/2}},$$

$$N_2^-(K) := \#\{m \in \mathbb{Z} : -K < m < 0,\ \mathcal{U}_m(\mathbf{A}_{\mathbb{Z}})^{\mathrm{Br}} \neq \emptyset \text{ but } \mathcal{U}_m(\mathbb{Z}) = \emptyset\} \gg \frac{K^{1/2}}{(\log K)^{1/2}}$$

*as $K \to +\infty$.*

As also noted in [LM20], it would be interesting to find a new lower bound which is of the form $K^{1/2}/(\log K)^\gamma$ for some $0 < \gamma < 1/2$, and by combining with the previous theorem, we could show that almost all counterexamples to the integral Hasse principle for Markoff surfaces cannot be explained by the Brauer–Manin obstruction. Of course, the big goal is to surpass the power $1/2$ to reach a number close to the one Ghosh and Sarnak predicted. This problem in the number field case remains open to be explored in later work.

Nevertheless, we are in a more convenient situation in the function field case. First, the theoretical results in [HV13] say that *the Brauer–Manin obstruction is the only one obstruction* to the existence of integral points on affine varieties over global function fields of positive characteristic. More precisely, the Brauer–Manin obstruction is associated with the $p$-primary part of the Brauer groups of the affine varieties. Next, the results in [LM20] and [CWX20] show that the quotient of the Brauer group of the Markoff surfaces $\mathrm{Br}\, U/\mathrm{Br}_0\, U$ is 2-torsion over a number field $k$, and in another work, we will show an analogue of these results in the function field case and construct some similar Brauer–Manin obstructions (of order 2) to the integral Hasse principle for Markoff surfaces over $\mathbb{F}_3(t)$ for instance. Of course, similar to the situation over $\mathbb{Q}$, the 2-part of the Brauer groups will not be enough to explain all the counterexamples to the integral Hasse principle there; however, we now know that the 3-primary part will help explain *everything*.

Furthermore, we can check easily that for any prime $p > 3$ and $M$ of any degree, there always exists a family of solutions $(x, y, z, M)$ of the following form:

$$\left(a+\frac{1}{a}, az+b, z, b\left(a-\frac{1}{a}\right)z+b^2+\left(a+\frac{1}{a}\right)^2\right),$$

for any $a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p[t]$ and $z \in \mathbb{F}_p[t]$ such that $b$ divides $M-\left(a+\frac{1}{a}\right)^2$. Indeed, by letting $a=b=2$, then for any $M \in \mathbb{F}_p[t]$ with $p>3$, we always have an integral solution $(x, y, z)$ where $x=5/2, y=2z+2, z=\frac{M-41/4}{3}$. In other words, the integral Hasse principle for Markoff surfaces **always holds** over $\mathbb{F}_p(t)$ for any $p>3$. In fact, it holds over any commutative ring $R$ which, for example, contains $\{2,3\}$ or, more generally, $\{a, a^2-1\}$ for some $a \in R$ as a subset of two units. In terms of finite fields of odd characteristic, only $\mathbb{F}_3$ is the *same* as $\mathbb{Z}$ with regard to the property that their only units are $\pm 1$, making it the only exception here.

Thus, we are only left with the cases $p=2,3$. Due to the particular form of the Markoff equation, we will only consider $p \neq 2$, i.e., the case $p=3$ is **our main problem**. Since the only units in $\mathbb{F}_3$ are $-1$ and 1, the above formula only holds for $M$ of the form $b^2+1$, which is not really of our main interest here (because of the same reason as the one given by [GS22]). In fact, we will mainly concentrate on $M$ of *odd degree*.

By using the computer, we first try to construct a family of counterexamples which can be explained by elementary methods, and then apply some descent arguments.

**Example A.2.2.** Let $X$ be the Markoff surface defined by

$$x^2+y^2+z^2-xyz=M,$$

where $M \in \mathbb{F}_3[t]$ such that $M$ is of *odd* degree. Let $\mathcal{X}$ be the integral model of $X$ defined over $\mathbb{F}_3[t]$ by the same equation. We consider $M=t^{2k+1}(t+1)^2(t-1)^2$ with any integer $k \geqslant 1$. Then this gives a potential candidate to be a family of counterexamples to the integral Hasse principle for $\mathcal{X}$. Moreover, this family does not come from any analogue over $\mathbb{Z}$ (in [GS22], [LM20] and [CWX20]) which was explained by the Brauer–Manin obstruction from quaternion algebras (of order 2).

We begin by a general reduction process for Markoff-type cubic equations over global function fields inspired by [CKV20], whose result is similar to the one given by reduction theory in [GS22] for the number field case. More precisely, we find a compact set such that every integral solution of the equation is $\Gamma$-equivalent to a solution in that set. We claim that the set is given by $\deg xyz = \deg M$.

*Proof of the claim.* Indeed, let $(x, y, z)$ be an integral point on $\mathcal{X}$. Without loss of generality, assume that $-\infty = \deg 0 < 0 \leqslant \deg x \leqslant \deg y \leqslant \deg z$ (note that $x, y, z \neq 0$ as we are only interested in $M$ of odd degree in $\mathbb{F}_3[t]$). From the equation $x^2+y^2+z^2=xyz+M$, if $\deg xyz > \deg M$ then we have the following two cases:

(1) If $\deg y < \deg z$, then $\deg xyz = \deg z^2$ with the same leading coefficient, so $\deg z = \deg x + \deg y$ and by the $\Gamma$-action $(x, y, z) \mapsto (x, y, xy-z)$, we have another solution $(x', y', z') = (x, y, z')$ with $\deg z' < \deg z$. This procedure must stop at some point, that is when we have either $\deg y = \deg z$ or $\deg xyz \leqslant \deg M$.

(2) If $\deg y = \deg z$, then the degree of the left hand side of the equation is at most $2\deg z$ while that of the right hand side is $\deg x + 2\deg z$, which implies that $\deg x = 0$, and so $x$ is a nonzero constant in $\mathbb{F}_3$. That leads to $M = 1 + (y \pm z)^2$, a contradiction.

Therefore, we may reduce the set of integral solutions to the *box*: $\deg xyz \leqslant \deg M$, similar to the one given in [GS22, Theorem 1.1]. In fact, we can do even better with $M$ of odd degree: If $\deg xyz < \deg M$, we have $\deg(x^2+y^2+z^2) = \deg M$, and so if $\deg y < \deg z$ then $\deg M = 2\deg z$ is even which is a contradiction; otherwise, $\deg y = \deg z$ gives $\deg M > 2\deg z \geqslant \deg(x^2+y^2+z^2)$, also a contradiction. Therefore, the final box of our reduction process is given by: $\deg xyz = \deg M$.

As a result, if $(x,y,z)$ is an integral solution of the Markoff equation, we can assume that it satisfies $0 \leqslant \deg x \leqslant \deg y \leqslant \deg z$ with $\deg xyz = \deg M$. This implies that $\deg M \geqslant \deg(x^2+y^2+z^2)$, so if $\deg z > \frac{\deg M}{2}$ then $\deg(x^2+y^2+z^2) = 2\deg z > \deg M$, a contradiction. Thus, with $M$ of odd degree, we must have $\deg z \leqslant \frac{\deg M - 1}{2}$, or $\deg x + \deg y \geqslant \frac{\deg M+1}{2}$. In summary, we have the following estimations: $1 \leqslant \deg x \leqslant \frac{\deg M}{3} \leqslant \deg z \leqslant \frac{\deg M-1}{2}$ (if $\deg x = 0$ then $M$ will be the sum of 1 and a square, the same contradiction as before), $\frac{\deg M+1}{4} \leqslant \deg y \leqslant \deg z \leqslant \frac{\deg M-1}{2}$. QED

Back to our example where $M = t^{2k+1}(t+1)^2(t-1)^2$ with $k > 0$, by the same method using the *Fricke trace identity* in [GS22, Section 6], we can deduce that the Markoff cubic equation in question has local integral points everywhere. Let $(x_v, y_v, z_v)_v$ be such a local integral point. Now we assume by contradiction that it comes from a global integral point $(x,y,z)$. Applying the action of $\Gamma$ continuously $s$ times for some $s \geqslant 0$, we can reduce this point into an element of the box: $\deg x \leqslant \deg y \leqslant \deg z$ **and** $\deg xyz = \deg M$. It is easy to verify by using reduction modulo $t, t+1, t-1$ that all the variables $x, y, z$ must be divisible by these three linear polynomials. If we write $x = t(t+1)(t-1)x_1 = (t^3-t)x_1, y = (t^3-t)y_1, z = (t^3-t)z_1$ with $\deg x_1, \deg y_1, \deg z_1 \in \mathbb{F}_3[t]$, then we must have $3 \leqslant \deg x \leqslant \deg y \leqslant \deg z$ and the equation reduces to

$$x_1^2 + y_1^2 + z_1^2 - (t^3-t)x_1y_1z_1 = t^{2k-1},$$

where $\deg x_1y_1z_1 = \deg M - 3$. In this case, computer computations provide ample evidence that it gives an *infinite* family of counterexamples to the integral Hasse principle for Markoff surfaces over the global function field $\mathbb{F}_3(t)$. Furthermore, it is easy to see that their number behaves asymptotically like a power of $\log N$ where $N = 3^n$ for any odd integer $n \geqslant 7$, with $\deg M$ running up to $n$. Inspired by Ghosh–Sarnak's conjectures in [GS22] for the number field case, it is natural to search for a much larger result in the function field case as well. Furthermore, we look forward to being able to apply the Artin–Schreier torsors in [HV13] to prove more counterexamples theoretically without using the computer.

As mentioned above, Ghosh and Sarnak provided in their paper a sample of the percentages of the Hasse failures, and the data suggests that

$$\mathcal{A}_{HF}(K) \sim CK^{0.8875\cdots+o(1)},$$

where $\mathcal{A}_{HF}(K)$ denotes the number of Hasse failures in the interval $[5, K]$ for $K \geqslant 5$, and $C$ is some positive constant, at least for $K$ about 564 million in their numerical experiments. We also would like to deduce a similar result in the function field case. From the arguments in the above example, we have proved the following result.

**Proposition A.2.8.** *If $M \in \mathbb{F}_3[t]$ is a* ***monic*** *polynomial* of odd degree*, denoting by* $\mathrm{lc}(P)$ *the leading coefficient of an arbitrary polynomial $P$, we consider the compact set*

$$\Delta_M := \{(a,b,c) \in \mathbb{F}_3[t] : 1 \leqslant \deg a \leqslant \deg b \leqslant \deg c,\ \mathrm{lc}(a) = \mathrm{lc}(b) = \mathrm{lc}(c) = 1,$$
$$a^2+b^2+c^2+abc = M,\ \deg abc = \deg M\}.$$

*Then any point $(x,y,z) \in \mathcal{X}(\mathbb{F}_3[t])$ is $\Gamma$-equivalent to a point $(a,b,c)$ in $\Delta_M$.*

*On the other hand, if we also consider $M$ of even degree, then the condition* $\deg abc = \deg M$ *can be replaced by* $\deg abc \leqslant \deg M$.

We expect to get similar consequences as in the number field case. However, in the function field case, there is an absolute change to make: we consider polynomials not individually, but based on their respective degrees. Previously, for $m \in \mathbb{Z}$ generic, we have $h(m)$ as the number of $\Gamma$-orbits in the set of integer solutions, i.e., $h(m) = |\Delta_m(\mathbb{Z})|$; now we define the number $h(M)$ for any monic polynomial $M \in \mathbb{F}_3[t]$ to be

$$h(M) = h(\deg M) := \sum_{\deg M = k} |\Delta_M|$$

for any integer $k \geqslant 1$. There is an asymptotic formula for the sum of the $h(m)$ in the number field case which is of the form $CK(\log K)^2$, as given in [GS22].

If $\deg M = k$, then the number of $(\deg a, \deg b, \deg c)$ satisfying $\deg abc = \deg M$ is $\begin{pmatrix} k+2 \\ 2 \end{pmatrix} = \dfrac{(k+1)(k+2)}{2}$, and for each $(a, b, c)$ of such degrees there are totally $3^a \times 3^b \times 3^c = 3^k$ possibilities. All of these imply that $|\Delta_M| \leqslant A_0 3^k k^2$ for some real constant $A_0 > 0$. Summing them up over (odd degrees) $\deg M \leqslant K$, we get

$$\sum_{\deg M \leqslant K} h(M) \leqslant A_0 3^K K^2,$$

which can be viewed as an analogue of the result over the integers. In fact, we see that working over global function fields can be more straightforward and so it can give some *new ideas* for the number field case as well, which we hope to explore in the near future.

# Bibliography


[Alp] D. Alpern. *Dario Alpern's Web site : Generic two integer variable equation solver.* URL: `https://www.alpertron.com/QUAD.HTM`.

[Apo76] T. M. Apostol. *Introduction to Analytic Number Theory.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1976.

[BGS16a] J. Bourgain, A. Gamburd, and P. Sarnak. "Markoff surfaces and strong approximation: 1". In: `arxiv:1607.01530` (2016).

[BGS16b] J. Bourgain, A. Gamburd, and P. Sarnak. "Markoff triples and strong approximation". In: *Comptes Rendus Mathématique* **354**.2 (2016), pp. 131–135.

[Bil97] H. Billard. "Propriétés arithmétiques d'une famille de surfaces K3". In: *Compositio Math.* **108**.3 (1997), pp. 247–275.

[BL07] A. Baragar and R. van Luijk. "K3 surfaces with Picard number three and canonical heights". In: *Comptes Rendus Mathématique* **259**.2 (2007), pp. 1493–1498.

[BLL20] M. Bright, A. Logan, and R. van Luijk. "Finiteness theorems for K3 surfaces over arbitrary fields". In: *European Journal of Mathematics* **6** (2020), pp. 336–366.

[Bor96] M. Borovoi. "The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer". In: *Journal für die reine und angewandte Mathematik* **473** (1996), pp. 181–194.

[Can09] S. Cantat. "Bers and Hénon, Painlevé and Schroedinger". In: *Duke Math. Journal* **149**.3 (2009), pp. 411–460.

[Cao20] Y. Cao. "Sous-groupe de Brauer invariant et obstruction de descente itérée". In: *Algebra and Number Theory* **14**.8 (2020), pp. 2151–2183.

[CD20] S. Cantat and R. Dujardin. "Finite orbits for large groups of automorphisms of projective surfaces". In: `arXiv:2012.01762` (2020).

[CD22] S. Cantat and R. Dujardin. "Random dynamics on real and complex projective surfaces". In: *J. Reine Ange. Math.* (2022). to appear.

[CDX19] Y. Cao, C. Demarche, and F. Xu. "Comparing descent obstruction and Brauer–Manin obstruction for open varieties". In: *Trans. Amer. Math. Soc.* **371**.12 (2019), pp. 8625–8650.

[Čes15] K. Česnavičius. "Poitou–Tate without restrictions on the order". In: *Math. Res. Lett.* **22** (2015), pp. 1621–1666.

[CF67] J. Cassels and A. Fröhlich. *Algebraic Number Theory.* Proc. Instructional Conf. Academic Press, 1967.

[Che22] W. Chen. "Nonabelian level structures, Nielsen equivalence, and Markoff triples". In: *Annals of Mathematics* (2022). To appear. URL: `https://arxiv.org/abs/2011.12940`.

[CKV20] R. Conceição, R. Kelly, and S. Vanfossen. "Markoff equation over polynomial rings". In: `arXiv:2003.05584` (2020).

[CL09] S. Cantat and F. Loray. "Dynamics on Character Varieties and Malgrange irreducibility of Painlevé VI equation". In: *Ann. Inst. Fourier* **59**.7 (2009), pp. 2927–2978.

[CO15] S. Cantat and K. Oguiso. "Birational automorphism groups and the movable cone theorem for Calabi-Yau manifolds of Wehler type via universal Coxeter groups". In: *Amer. J. Math.* **137**.4 (2015), pp. 1013–1044.

[Cona] K. Conrad. *Galois groups of cubics and quartics (not in characteristic 2)*. Expository paper. URL: `https://kconrad.math.uconn.edu/blurbs/galoistheory/cubicquartic.pdf`.

[Conb] K. Conrad. *Hensel's lemma*. Expository paper. URL: `https://kconrad.math.uconn.edu/blurbs/gradnumthy/hensel.pdf`.

[Con12] B. Conrad. "Weil and Grothendieck approaches to adelic points". In: *Enseign. Math.* **58**.2 (2012), pp. 61–97.

[CS21] J.-L. Colliot-Thélène and A. N. Skorobogatov. *The Brauer–Grothendieck group*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. Springer Cham, 2021.

[CS87] J.-L. Colliot-Thélène and J.-J. Sansuc. "La descente sur les variétés rationnelles II". In: *Duke Math. J.* **54** (1987), pp. 375–492.

[CW12] J.-L. Colliot-Thélène and O. Wittenberg. "Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines". In: *Am. J. Math.* **134**.5 (2012), pp. 1303–1327.

[CWX20] J.-L. Colliot-Thélène, D. Wei, and F. Xu. "Brauer–Manin obstruction for Markoff surfaces". In: *Annali della Scuola Normale Superiore di Pisa* **XXI**.5 (2020), pp. 1257–1313.

[CX09] J.-L. Colliot-Thélène and F. Xu. "Brauer–Manin obstruction for integral points of homoheneous spaces and representation by integral quadratic forms". In: *Compos. Math.* **145** (2009), pp. 309–363.

[Dao23] Q.-D. Dao. "Brauer–Manin obstruction for Wehler K3 surfaces of Markoff type". In: `arxiv:2302.11515 [math.NT]` (2023). Submitted. URL: `https://arxiv.org/abs/2302.11515`.

[Dao24] Q.-D. Dao. "Brauer–Manin obstruction for integral points on Markoff-type cubic surfaces". In: *Journal of Number Theory* **254** (2024), pp. 65–102. DOI: `https://doi.org/10.1016/j.jnt.2023.07.007`.

[Eke91] T. Ekedahl. "An Infinite Version of the Chinese Remainder Theorem". In: *Comment. Math. Univ. St. Paul.* **40**.1 (1991), pp. 53–59.

[Fuc+22] E. Fuchs et al. "Orbits on K3 Surfaces of Markoff Type". In: `arXiv:2201.12588 [math.AG]` (2022). Submitted. URL: `https://arxiv.org/abs/2201.12588`.

[Ful98] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. Second Edition. Springer New York, NY, 1998.

[GLN22] D. Gvirtz, D. Loughran, and M. Nakahara. "Quantitative arithmetic of diagonal degree 2 K3 surfaces". In: *Mathematische Annalen* **384**.1-2 (2022), pp. 135–209.

[Gol09] W. M. Goldman. "Trace coordinates on Fricke spaces of some simple hyperbolic surfaces". In: *Handbook of Teichmüller theory. Vol. **II**, IRMA Lect. Math. Theor. Phys., **13**, Eur. Math. Soc., Zürich* (2009), pp. 611–684.

[Gon09] C. Gonzalez-Avilés. "Arithmetic duality theorems for 1-motives over function fields". In: *J. Reine Angew. Math.* **632** (2009), pp. 203–231.

[GS09] A. Garbagnati and A. Sarti. "Elliptic fibrations and symplectic automorphisms on K3 surfaces". In: *Comm. in Algebra* **37** (2009), pp. 3601–3631.

[GS17] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology.* Cambridge Studies in Advanced Mathematics **101**. Second Edition. Cambridge University Press, 2017.

[GS22] A. Ghosh and P. Sarnak. "Integral points on Markoff type cubic surfaces". In: *Invent. math.* **229** (2022), pp. 689–749.

[GW20] U. Görtz and T. Wedhorn. *Algebraic Geometry I: Schemes with Examples and Exercises.* Springer Studium Mathematik - Master. Springer Fachmedien Wiesbaden GmbH - Springer Nature, 2020.

[Har08] D. Harari. "Le défaut d'approximation forte pour les groupes algébriques commutatifs". In: *Algebra and Number Theory* **2**.5 (2008), pp. 595–611.

[Har17] Y. Harpaz. "Geometry and arithmetic of certain log K3 surfaces". In: *Annales de l'Institut Fourier* **67**.5 (2017), pp. 2167–2200.

[Har19] Y. Harpaz. "Integral points on conic log K3 surfaces". In: *Journal of the European Mathematical Society* **21**.3 (2019), pp. 627–664.

[Har20] D. Harari. *Galois Cohomology and Class Field Theory.* Universitext. English translation by Andrei Yafaev of the original French edition *Cohomologie galoisienne et théorie du corps de classes* published by EDP Sciences, Les Ulis, 2017. Springer Cham, 2020.

[Har77] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer-Verlag New York, 1977.

[Huy16] D. Huybrechts. *Lectures on K3 surfaces.* Cambridge Stud. Adv. Math. **158**. Cambridge Univ. Press, 2016.

[HV10] D. Harari and J. F. Voloch. "The Brauer–Manin obstruction for integral points on curves". In: *Math. Proc. Cambridge Philos. Soc.* **149** (2010), pp. 413–421.

[HV13] D. Harari and J. F. Voloch. "Descent obstructions and Brauer–Manin obstruction in positive characteristic". In: *Journal de l'Institut de Mathématiques de Jussieu* **12**.3 (2013), pp. 545–551.

[JS17] Jörg Jahnel and Damaris Schindler. "On integral points on degree four del Pezzo surfaces". In: *Israel Journal of Mathematics* **222**.1 (2017), pp. 21–62.

[Klo07] R. Kloosterman. "Elliptic K3 surfaces with geometric Mordell-Weil rank 15". In: *Canad. Math. Bull.* **50** (2007), pp. 215–226.

[Kol02] J. Kollár. "Unirationality of cubic hypersurfaces". In: *J. Inst. Math. Jussieu* **1**.3 (2002), pp. 467–476.

[KOS75] M. A. Knus, M. Ojanguren, and D. J. Saltman. *On Brauer groups in characteristic $p$ - Brauer Groups Evanston 1975.* Lecture Notes in Math. **549**. Springer-Verlag Berlin Heidelberg New York, 1975.

[KT08] Andrew Kresch and Yuri Tschinkel. “Two examples of brauer–manin obstruction to integral points”. In: *Bull. Lond. Math. Soc.* **40**.6 (2008), pp. 995–1001.

[LM20] D. Loughran and V. Mitankin. “Integral Hasse Principle and Strong Approximation for Markoff Surfaces”. In: *International Mathematics Research Notices* **2021**.18 (2020), pp. 14086–14122.

[Lui07a] R. van Luijk. “An elliptic K3 surface associated to Heron triangles”. In: *Journal of Number Theory* **123**.3 (2007), pp. 92–119.

[Lui07b] R. van Luijk. “K3 surfaces with Picard number one and infinitely many rational points”. In: *Algebra and Number Theory* **1** (2007), pp. 1–15.

[Man71] Yu. I. Manin. *Le groupe de Brauer–Grothendieck en géométrie diophantienne.* Actes Congr. Int. Math. (Nice, 1970). Gauthier-Villars, 1971, pp. 14086–14122.

[Mil06] J. S. Milne. *Arithmetic duality theorems.* Journal de l’Institut de Mathématiques de Jussieu. Second Edition. BookSurge, LLC, Charleston, SC, 2006.

[Mil80] J. S. Milne. *Étale cohomology.* Princeton Mathematical Series **33**. Princeton University Press, 1980.

[Neu00] J. Neukirch. *Algebraic Number Theory.* Grundlehren der Math. **322**. Springer-Verlag Berlin Heidelberg, 2000.

[Neu13] J. Neukirch. *Class field theory.* The Bonn lectures. Springer-Verlag Berlin Heidelberg New York, 2013.

[Nik75] V. V. Nikulin. “On Kummer surfaces”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* **39**.2 (1975), pp. 278–293.

[Nik83] V. V. Nikulin. “Factor groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications”. In: *J. Soviet Math.* **22**.4 (1983), pp. 1401–1475.

[NSW15] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of Number Fields.* Grundlehren der Math. **323**. Second Edition. Springer-Verlag Berlin Heidelberg New York, 2015.

[Poo03] B. Poonen. “Squarefree Values of Multivariate Polynomials”. In: *Duke Math. J.* **118**.2 (2003), pp. 353–373.

[Poo10] B. Poonen. “Insufficiency of the Brauer-Manin obstruction applied to étale covers”. In: *Ann. of Math.* **171**.2 (2010), pp. 2157–2169.

[Poo17] B. Poonen. *Rational points on varieties.* Graduate Studies in Mathematics **186**. American Mathematical Society, 2017.

[Ros23] Z. Rosengarten. “Tate Duality In Positive Dimension Over Function Fields”. In: *Memoirs of the American Mathematical Society* (2023). to appear.

[Ser85] C. Series. “The Geometry of Markoff Numbers”. In: *The Mathematical Intelligencer* **7**.3 (1985), pp. 20–29.

[SJ05] W. Stein and D. Joyner. “SAGE: **S**ystem for **A**lgebra and **G**eometry **E**xperimentation”. In: *ACM SIGSAM Bulletin* **39**.2 (2005), pp. 61–64.

[Sko01] A. N. Skorobogatov. *Torsors and rational points.* Cambridge Tracts in Mathematics **140**. Cambridge University Press, 2001.

[Wei74] A. Weil. *Basic number theory.* Classics in Mathematics. Third Edition. Springer-Verlag Berlin Heidelberg, 1974.

[Wei82] A. Weil. *Adeles and algebraic groups.* Birkhäuser, Boston, MA, 1982.

[Wha20] J. P. Whang. “Nonlinear descent on moduli of local systems”. In: *Israel Journal of Mathematics* **240** (2020), pp. 935–1004.

[Yue02] Q. Yue. “Dyadic ideal, class group, and tame kernel in quadratic fields”. In: *Journal of Pure and Applied Algebra* **166** (2002), pp. 229–238.

**Local-global principle for integral points on certain algebraic surfaces**


**Abstract**

In this thesis, we study the problem of existence and local approximation of integral points on certain algebraic surfaces defined over number fields, particularly the field of rational numbers. In the first chapter, we introduce the history of the problem and some recent progress in the subject of our study, especially the recent work of Ghosh–Sarnak, Loughran–Mitankin, and Colliot-Thélène–Wei–Xu. In Chapter 2, we study the Brauer–Manin obstruction for Markoff-type cubic surfaces. We first provide some background on character varieties and the natural origin of the Markoff-type cubic surfaces, then we explicitly calculate the Brauer group of the smooth compactifications and the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer group to prove the failure of strong approximation which can be explained by the Brauer–Manin obstruction in an infinite family of surfaces, and then give some counting results for the frequency of the obstructions. Furthermore, we apply the reduction theory, similar to that of Markoff surfaces, in recent work by Whang to give an explicit counterexample to the integral Hasse principle for our Markoff-type cubic surfaces. We also give some analogous results to those on Markoff surfaces about the Brauer–Manin obstruction in some special cases of Markoff-type cubic surfaces. In Chapter 3, we study the Brauer–Manin obstruction for Wehler K3 surfaces of Markoff type and follow the same structure as the previous chapter. We first provide some background on Wehler K3 surfaces and a recent study of Fuchs et al. on Markoff-type K3 (MK3) surfaces, as well as introduce the *three* explicit families of MK3 surfaces that interest us. Next, we explicitly calculate the algebraic Brauer group of the projective closures for one smooth family, and then the algebraic Brauer group of the affine surfaces. Afterward, we use the Brauer groups to prove the failure of the integral Hasse principle which can be explained by the Brauer–Manin obstruction for three families of MK3 surfaces, and then give some counting results for the Hasse failures. In addition, we study some cases when the Brauer–Manin obstruction to the existence of integral points and rational points can vanish, then give some counterexamples to strong approximation which can be explained by the Brauer–Manin obstruction. Furthermore, we provide some explicit examples which show that rational points do exist on affine MK3 surfaces. To complete the thesis, in Appendix A, we give a brief introduction to the descent obstructions associated with Artin–Schreier torsors and their relation to the Brauer–Manin obstruction for integral points on affine varieties over global function fields, as studied by Harari and Voloch. Finally, we study some counterexamples to the integral Hasse principle on conics and Markoff surfaces.



**Keywords:** integral points, rational points, brauer-manin obstruction, local-global (hasse) principle, strong approximation, log k3 surfaces, (wehler) k3 surfaces, markoff-type cubic surfaces, markoff-type k3 surfaces, reduction theory, descent.



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**Résumé**

Dans cette thèse, nous étudions le problème d'existence et d'approximation locale de points entiers sur certaines surfaces algébriques définies sur des corps de nombres, en particulier le corps des nombres rationnels. Dans le premier chapitre, nous introduisons l'historique du problème et quelques progrès récents dans le sujet de notre étude, en particulier les travaux récents de Ghosh–Sarnak, Loughran–Mitankin, et Colliot-Thélène– Wei–Xu. Dans le chapitre 2, nous étudions l'obstruction de Brauer–Manin pour les surfaces cubiques de type Markoff. Nous fournissons d'abord quelques informations sur les variétés de caractères et l'origine naturelle des surfaces cubiques de type Markoff, puis nous calculons explicitement le groupe de Brauer des compactifications lisses et le groupe de Brauer algébrique des surfaces affines. Ensuite, nous utilisons le groupe de Brauer pour prouver l'échec de l'approximation forte qui peut s'expliquer par l'obstruction de Brauer–Manin dans une famille infinie de surfaces, puis donnons des estimations asymptotiques pour la fréquence des obstructions. De plus, nous appliquons la théorie de la réduction, similaire à celle des surfaces de Markoff, dans les travaux récents de Whang pour donner un contre-exemple explicite au principe de Hasse entier pour nos surfaces cubiques de type Markoff. Nous donnons aussi des résultats analogues à ceux sur les surfaces de Markoff à propos de l'obstruction de Brauer–Manin dans quelques cas particuliers de surfaces cubiques de type Markoff. Dans le chapitre 3, nous étudions l'obstruction de Brauer–Manin pour les surfaces de Wehler K3 de type Markoff et suivons la même structure que le chapitre précédent. Nous fournissons d'abord quelques informations sur les surfaces K3 de Wehler et une étude récente de Fuchs et al. sur les surfaces K3 de type Markoff (MK3), ainsi que les *trois* familles explicites de surfaces MK3 qui nous intéressent. Puis, nous calculons explicitement le groupe de Brauer algébrique des clôtures projectives pour une famille lisse, puis le groupe de Brauer algébrique des surfaces affines. Ensuite, nous utilisons les groupes de Brauer pour prouver l'échec du principe de Hasse entier qui peut être expliqué par l'obstruction de Brauer–Manin pour trois familles de surfaces MK3, puis donnons quelques estimations asymptotiques pour les échecs de Hasse. Par ailleurs, nous étudions quelques cas où l'obstruction de Brauer–Manin à l'existence de points entiers et de points rationnels peut disparaître, puis donnons quelques contre-exemples à l'approximation forte qui peut s'expliquer par l'obstruction de Brauer–Manin. De plus, nous donnons quelques exemples explicites qui montrent que des points rationnels existent sur des surfaces MK3 affines. Pour compléter la thèse, dans l'annexe A, nous donnons une brève introduction aux obstructions de descente associées aux torseurs d'Artin–Schreier et à leur relation avec l'obstruction de Brauer–Manin pour les points entiers sur les variétés affines sur un corps de fonctions d'une courbe algébrique sur un corps fini, comme étudiées par Harari et Voloch. Enfin, nous étudions quelques contre-exemples au principe de Hasse entier sur des coniques et des surfaces de Markoff.

**Mots clés :** points entiers, points rationnels, obstruction de brauer–manin, principe local-global (hasse), approximation forte, surfaces log k3, surfaces k3 (de wehler), surfaces cubiques de type markoff, surfaces k3 de type markoff, théorie de la réduction, descente.

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