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VANISHING OF TEMPERATE COHOMOLOGY ON COMPLEX MANIFOLDS

ANNULATION DE LA COHOMOLOGIE TEMPÉRÉE SUR LES VARIÉTÉS COMPLEXES

Abstract. — Consider a complex Stein manifold X and a subanalytic relatively compact Stein open subset U of X. We prove the vanishing on U of the holomorphic temperate cohomology.

Résumé. — Considérons une variété complexe de Stein X et un ouvert sous-analytique relativement compact de Stein U de X. Nous démontrons l'annulation de la cohomologie holomorphe tempérée sur U.

Introduction

The theory of ind-sheaves and, as a byproduct, the subanalytic topology on a real analytic manifold M and the site M_{sa}, have been introduced in [KS01] after the construction by M. Kashiwara in [Kas84] of the functor Thom of temperate cohomology.

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On the site M_{sa} one easily obtains various sheaves which would have no meaning in the classical setting such that the sheaves of $\mathcal{C}^\infty$-functions or distributions with temperate growth or the sheaf of Whitney $\mathcal{C}^\infty$-functions. Notice that this topology has been refined in [GS16] in which the linear subanalytic topology and the site M_{sal} were introduced but we shall not use this topology here.

On a complex manifold X, by considering the Dolbeault complex with coefficients in these sheaves, we get various (derived) sheaves of tempered holomorphic functions on the site X_{sa} and in particular the sheaf $\mathcal{O}^{\mathrm{t}p}_{X_{\mathrm{sa}}}$ of temperate holomorphic functions.

A natural question is then to prove the vanishing of the cohomology of $\mathcal{O}^{\mathrm{t}p}_{X_{\mathrm{sa}}}$ on a subanalytic relatively compact Stein open subset of a complex manifold. Many specialists of complex analysis consider the answer to this question as easy or well-known but we have not found any proof of it in the literature, despite the fact of its many applications (see e.g., [Pet17]). The aim of this Note is to provide a short proof to this result, by combining the fundamental and classical vanishing theorem of Hörmander [Hör65] together with [KS96, Theorem 10.5].

Acknowledgments

(i) I warmly thank Henri Skoda who is at the origin of this paper. Indeed, this is when discussing together of the problem that I realized that Theorem 2.10 could be easily deduced from Hörmander vanishing theorem. On his side, Henri will soon publish a paper (see [Sko21]) in which he obtains a similar and more precise result to this theorem with a weaker hypothesis, namely without any subanalyticity assumption.

(ii) I also warmly thank Daniel Barlet who explained me how the result of Lemma 2.3 could easily be deduced from [GR65, Chapter 8, Section C, Theorem 8].

1. Review on the subanalytic site (after [KS01])

In this paper, M denotes a real analytic manifold endowed with a distance, denoted dist, locally Lipschitz equivalent to the Euclidian distance on $\mathbb{R}^n$. We set $\mathrm{dist}(x, \varnothing) = D_M + 1$, where D_M is the diameter of M. We also choose a measure $d\lambda$ on M locally isomorphic to the Lebesgue measure on $\mathbb{R}^n$. For a relatively compact open subset $U \subset M$, we shall denote by $\|\cdot\|_{L^p}$ the L^p-norm ($p = 1, 2, \infty$) for this measure on U. Since U is relatively compact, this norm does not depend on the choice of $d\lambda$, up to constants.

1.1. Sheaves

We fix a field $\mathbf{k}$. We denote by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ its bounded derived category. References for sheaf theory are made to [KS90].

In particular, we shall use the duality functor

$$\mathrm{D}'_M := \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbf{k}_M).$$

A sheaf F on M is $\mathbb{R}$-constructible if there exists a stratification $M = \bigsqcup_\alpha M_\alpha$ by locally closed subanalytic subsets M_α such that $F|_{M_\alpha}$ is locally constant of finite rank. We denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on M. Its bounded derived category is equivalent to $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects with $\mathbb{R}$-constructible cohomology. The derived categories of $\mathbb{R}$-constructible sheaves satisfy the formalism of Grothendieck's six operations.

1.2. Sheaves on the subanalytic site

Let us denote by $\mathrm{SA}(M)$ the family of subanalytic subsets of the real analytic manifold M.

This family contains that of semi-analytic sets and satisfies:

- the property of being subanalytic is local on M,
- $\mathrm{SA}(M)$ is stable by finite intersections, finite unions, complementary in M, closure, interior,
- a compact subanalytic set is topologically isomorphic to a finite CW-complex,
- for $f\colon M \to N$ a morphism, $Z \in \mathrm{SA}(N)$ and $S \in \mathrm{SA}(M)$, $f^{-1}(Z) \in \mathrm{SA}(M)$ and $f(S) \in \mathrm{SA}(N)$ as soon as f is proper on $\overline{S}$.

References are made to the pioneering work of Lojasiewicz, followed by those of Gabrielov and Hironaka and, for a modern treatment, to the paper [BM88] by E. Bierstone and P. Milman.

The main property of subanalytic sets are the so called Lojasiewicz inequalities which play an essential role here.

Lemma 1.1 (Lojasiewicz inequalities). — *Let $U = \bigcup_{j\in J} U_j$ be a finite covering in $\mathrm{Op}_{M_{\mathrm{sa}}}$. Then there exist a constant $C > 0$ and a positive integer N such that*

$$\mathrm{dist}(x, M \setminus U)^N \leqslant C \cdot \left(\max_{j\in J} \mathrm{dist}\,(x, M \setminus U_j)\right). \tag{1.2.1}$$

The subanalytic site M_{sa} associated to a real analytic manifold M is defined as follows.

- $\mathrm{Op}_{M_{\mathrm{sa}}}$ is the category of relatively compact subanalytic open subsets of M,
- the coverings are the finite coverings, meaning that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ if $U_i \subset U$ for all i and there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$.

Hence, we get the abelian category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ of sheaves on M_{sa} and its bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}})$.

On denotes by $\rho_M\colon M \to M_{\mathrm{sa}}$ the natural morphism of sites. As usual, one gets a pair of adjoint functors

$$\rho_M^{-1}\colon\ \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \rightleftarrows \mathrm{Mod}(\mathbf{k}_M)\colon \rho_{M*}. \tag{1.2.2}$$

The functor ρ_{M*} is fully faithful and the functor ρ_M^{-1} also admits a left adjoint:

$$\rho_{M!}\colon \mathrm{Mod}(\mathbf{k}_M) \rightleftarrows \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})\colon \rho_M^{-1}. \tag{1.2.3}$$

For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{M!}F$ is the sheaf associated with the presheaf $U \mapsto F(\overline{U})$. Moreover, the functor $\rho_{M!}$ is exact.

Recall that the functor ρ_{M*} is exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and this last category may be considered as a full thick subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ as well as of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Similarly, $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ can be considered as a full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ as well as of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}})$.

We shall need the next result, already proved in [KS01, Corollary 4.3.7] in the more general framework of indsheaves. For the reader's convenience, we give a direct proof.

Lemma 1.2. — *Let $f\colon M \to N$ be a morphism of manifolds. There is a natural isomorphism of functors $f^{-1}\rho_{N!} \simeq \rho_{M!}f^{-1}$.*

Proof. — By adjunction, it is enough to check the isomorphism of functors

$$\rho_N^{-1} f_* \simeq f_*\rho_M^{-1}.$$

Denote by $\rho_M^{\dagger}$ the inverse image functor for presheaves associated to the morphism ρ_M and similarly for ρ_N. Denote by $(\cdot)^a$ the functor which associates a sheaf to a presheaf. Since direct images commute with $(\cdot)^a$ and $\mathrm{Op}_{N_{\mathrm{sa}}}$ is a basis of open subsets of N, it is enough to prove the isomorphism of functors of presheaves on N_{sa}

$$\rho_N^{\dagger} f_* \simeq f_*\rho_M^{\dagger}.$$

Let $F \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ and let V be open in N_{sa}. Then

$$\left(\rho_N^{\dagger} f_* F\right)(V) \simeq f_*F(V) \simeq F\left(f^{-1}V\right) \simeq \left(\rho_N^{\dagger}F\right)\left(f^{-1}V\right) \simeq \left(f_*\rho_M^{\dagger}F\right)(V). \qquad \square$$

Recall that one says after [GS16] that a sheaf F on M_{sa} is Γ-acyclic if $\mathrm{R}\Gamma(U;F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

Proposition 1.3 (see [KS01, Proposition 6.4.1] and [GS16, Proposition 2.14]).

(i) *A presheaf F on M_{sa} is a sheaf as soon as $F(\varnothing) = 0$ and, for any $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence below is exact:*

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2). \tag{1.2.4}$$

(ii) *If moreover, for any $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$, the sequence*

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0 \tag{1.2.5}$$

is exact, then F is Γ-acyclic.

1.3. Some subanalytic sheaves on real manifolds

Classical sheaves

Let M be a real analytic manifold. One denotes as usual by $\mathscr{A}_M, \mathscr{C}_M^\infty, \mathcal{D}b_M, \mathscr{B}_M$ the sheaves of complex valued real analytic functions, C^∞-functions, distributions and hyperfunctions on M. We also denote by Ω_M the sheaf of real analytic differential forms of maximal degree, by or_M the orientation sheaf and by $\mathscr{V}_M$ the sheaf of real analytic densities, $\mathscr{V}_M = \Omega_M \otimes \mathrm{or}_M$. Finally, we denote by $\mathscr{D}_M$ the sheaf of finite order differential operators with coefficients in $\mathscr{A}_M$.

Sheaves of temperate functions and distributions

Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Set for short

$$\mathrm{dist}_U(x) = \mathrm{dist}(x, M \setminus U) \tag{1.3.1}$$

and denote as usual by $\|\cdot\|_{L^\infty}$ the sup-norm.

(i) One says that $f \in \mathscr{C}^\infty_M(U)$ has *polynomial growth* if there exists $N \geqslant 0$ such that

$$\Big\|\mathrm{dist}_U(x)^N f(x)\Big\|_{L^\infty} < \infty\,.$$

(ii) One says that $f \in \mathscr{C}^\infty_M(U)$ is temperate if all its derivatives (in local charts) have polynomial growth.

(iii) One says that a distribution $u \in \mathcal{D}b_M(U)$ is temperate if u extends as a distribution on M.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, denote by

- $\mathcal{C}^{\infty,\mathrm{t}p}_M(U)$ the subspace of $\mathcal{C}^\infty_M(U)$ consisting of temperate functions on U,
- $\mathcal{D}b^{\mathrm{t}p}_M(U)$ the space of temperate distributions on U.

Denote by $\mathcal{C}^{\infty,\mathrm{t}p}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{t}p}_{M_{\mathrm{sa}}}$ the presheaves so defined. Using Lojasiewicz's inequalities, one checks that these presheaves are sheaves on M_{sa}. Moreover, $\mathcal{C}^{\infty,\mathrm{t}p}_{M_{\mathrm{sa}}}$ is a sheaf of rings and $\mathcal{D}b^{\mathrm{t}p}_{M_{\mathrm{sa}}}$ is a sheaf of $\mathcal{C}^{\infty,\mathrm{t}p}_{M_{\mathrm{sa}}}$-modules.

The next lemma below is an essential tool for our study.

Lemma 1.4 (see [Hör83]). — *Let Z_0 and Z_1 be two closed subanalytic subsets of M. There exists $\psi \in \mathcal{C}^{\infty,\mathrm{t}p}(M \setminus (Z_0 \cap Z_1))$ such that $\psi = 0$ in a neighborhood of Z_0 and $\psi = 1$ in a neighborhood of Z_1.*

Lemma 1.5. — *Any sheaf of $\mathcal{C}^{\infty,\mathrm{t}p}_{M_{\mathrm{sa}}}$-modules on M_{sa} is Γ-acyclic.*

Proof. — Apply Lemma 1.4 as in [GS16, Proposition 4.18]. ☐

In particular, the sheaf $\mathcal{D}b^{\mathrm{t}p}_{M_{\mathrm{sa}}}$ is Γ-acyclic.

We shall also use the sheaf

$$\mathcal{D}b^{\mathrm{t}p\vee}_{M_{\mathrm{sa}}} := \mathcal{D}b^{\mathrm{t}p}_{M_{\mathrm{sa}}} \otimes_{\rho_{M!}\mathscr{A}_M} \rho_{M!}\mathscr{V}_M.$$

Sheaf of Whitney functions

For a closed subanalytic subset S in M, denote by $\mathrm{I}^\infty_{M,S}$ the space of C^∞-functions defined on M which vanish up to infinite order on S. In [KS96], one introduced the sheaf:

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M := V \longmapsto \mathrm{I}^\infty_{V,V\setminus U}$$

and showed that it uniquely extends to an exact functor

$$\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M, \quad \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}_M).$$

One denotes by $\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ the sheaf on M_{sa} given by

$$\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}(U) = \Gamma\left(M; H^0\left(\mathrm{D}'_M\mathbf{k}_U\right) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M\right), U \in \mathrm{Op}_{M_{\mathrm{sa}}}.$$

If $\mathrm{D}'_M\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$, then $\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}(U) \simeq \mathcal{C}^\infty(M)/\mathrm{I}^\infty_{M,\overline{U}}$ is the space of Whitney functions on $\overline{U}$. It is thus natural to call $\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ the sheaf of Whitney $\mathcal{C}^\infty$-functions on M_{sa}.

Recall that a $\mathbb{C}$-vector space is of type FN (resp. DFN) if it is a Fréchet-nuclear space (resp. dual of a Fréchet-nuclear space).

Proposition 1.6 (see [KS96, Proposition 2.2]). — *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. There exist natural topologies of type* FN *on* $\Gamma(M; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)$ *and of type* DFN *on* $\Gamma(U; \mathcal{D}b^{\mathrm{tp}\vee}_{M_{\mathrm{sa}}}))$ *and they are dual to each other.*

Note that the sheaf $\rho_{M*}\mathscr{D}_M$ does not operate on the sheaves $\mathcal{C}^{\infty,t}_{M_{\mathrm{sa}}}$, $\mathcal{D}b^{tp}_{M_{\mathrm{sa}}}$, $\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ but $\rho_{M!}\mathscr{D}_M$ does.

Remark 1.7. — It would have been more natural to consider the cosheaf $U \mapsto \Gamma(M; \mathbf{k}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M)$. We didn't since cosheaf theory is still not well-established.

Sheaves of temperate L^2-functions

Recall that M is endowed with a measure $d\lambda$ locally equivalent to the Lebesgue measure. Denote by L^0_M the sheaf of measurable functions on M and recall that $\|\cdot\|_{L^2}$ denotes the L^2-norm. Also recall notation (1.3.1). For U open and relatively compact in M and $s \in \mathbb{R}_{\geqslant 0}$, set

$$L^{2,s}(U) = \left\{f \in L^0_M(U); \|\mathrm{dist}^s_U(x)f(x)\|_{L^2} < \infty\right\}. \tag{1.3.2}$$

Also set

$$L^{2,\mathrm{tp}}(U) := \varinjlim_{s \geqslant 0} L^{2,s}(U). \tag{1.3.3}$$

Denote by $\mathscr{L}^{2,\mathrm{tp}}_{M_{\mathrm{sa}}}$ the presheaf $U \mapsto L^{2,\mathrm{tp}}(U)$ on M_{sa}.

Lemma 1.8. —
(i) *The presheaf $\mathscr{L}^{2,\mathrm{tp}}_{M_{\mathrm{sa}}}$ is a sheaf on M_{sa}.*
(ii) *The sheaf $\mathscr{L}^{2,\mathrm{tp}}_{M_{\mathrm{sa}}}$ is a $\mathcal{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$-module and in particular is Γ-acyclic.*

Proof.
(i) follows immediately from Lojasiewicz's inequalities (Lemma 1.1) and the fact that coverings are finite coverings.
(ii) Let $\varphi \in L^{2,s}(U)$ and let $\theta \in \mathcal{C}^{\infty,tp}_{M_{\mathrm{sa}}}(U)$. Then $\|\mathrm{dist}_U(x)^t \cdot \theta(x)\|_{L^\infty} < \infty$ for some $t \geqslant 0$ and we get

$$\left\|\mathrm{dist}_U(x)^{t+s}\theta(x)\varphi(x)\right\|_{L^2} \leqslant C \cdot \left\|\mathrm{dist}_U(x)^t\theta(x)\right\|_{L^\infty} \cdot \|\mathrm{dist}_U(x)^s\varphi(x)\|_{L^2} < \infty.$$

Hence, $\theta\varphi$ belongs to $L^{2,s+t}(U)$. □

Proposition 1.9. — *One has natural monomorphisms $\mathcal{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}} \hookrightarrow \mathscr{L}^{2,\mathrm{tp}}_{M_{\mathrm{sa}}} \hookrightarrow \mathcal{D}b^{tp}_{M_{\mathrm{sa}}}$.*

Proof. — The monomorphism $\mathscr{L}^{2,\mathrm{tp}}_{M_{\mathrm{sa}}} \hookrightarrow \mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ is obvious.
(ii) Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $\varphi \in \mathcal{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}(U)$. There exists $t \in \mathbb{R}_{\geqslant 0}$ such that

$$\left\|\mathrm{dist}_U(x)^t \varphi(x)\right\|_{L^\infty} < \infty.$$

Hence, we have for some constant $C > 0$

$$\left\|\mathrm{dist}_U(x)^t \varphi(x)\right\|_{L^2} \leqslant C \cdot \left\|\mathrm{dist}_U(x)^t \varphi(x)\right\|_{L^\infty} < \infty$$ □

1.4. Some subanalytic sheaves on complex manifolds

Temperate holomorphic functions

Now let X be a *complex* manifold of complex dimension n. One defines the (derived) sheaf of temperate holomorphic functions $\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ as the Dolbeault complex with coefficients in $\mathcal{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sa}}}$. In other words

$$\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} := 0 \to \mathscr{C}^{\infty,\mathrm{tp}(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{tp}(0,n)}_{X_{\mathrm{sa}}} \to 0. \tag{1.4.1}$$

If $n > 1$, this object is no more concentrated in degree 0.

Also consider the object $\widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$

$$\widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}} := 0 \to \mathcal{D}b^{\mathrm{tp}(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{tp}(0,n)}_{X_{\mathrm{sa}}} \to 0. \tag{1.4.2}$$

Theorem 1.10 (see [KS96, Theorem 10.5]). — *The natural morphism $\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \to \widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$.*

Proof. — Let $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$. The sheaves $\mathcal{C}^{\infty,tp}_{X_{\mathrm{sa}}}$ and $\mathcal{D}b^{tp}_{X_{\mathrm{sa}}}$ being Γ-acyclic, $\mathrm{R}\Gamma(U;\mathcal{O}^{tp}_{X_{\mathrm{sa}}})$ and $\mathrm{R}\Gamma(U;\widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ are represented by the complexes

$$\begin{aligned}\mathrm{R}\Gamma\left(U;\mathcal{O}^{tp}_{X_{\mathrm{sa}}}\right)&\colon 0 \to \mathscr{C}^{\infty,\mathrm{tp}(0,0)}_{X_{\mathrm{sa}}}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{tp}(0,n)}_{X_{\mathrm{sa}}}(U) \to 0,\\ \mathrm{R}\Gamma\left(U;\widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}}\right)&\colon 0 \to \mathcal{D}b^{\mathrm{tp}(0,0)}_{X_{\mathrm{sa}}}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{tp}(0,n)}_{X_{\mathrm{sa}}}(U) \to 0.\end{aligned} \tag{1.4.3}$$

When $X = \mathbb{C}^n$, it is proved in [KS96, Theorem 10.5] that these two complexes are quasi-isomorphic and this is enough for our purpose since the statement is of local nature. □

Whitney holomorphic functions

One also defines the (derived) sheaf of Whitney holomorphic functions by taking the Dolbeault complex of the sheaf $\mathcal{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$

$$\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}} := 0 \to \mathscr{C}^{\infty,\mathrm{w}(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{w}(0,n)}_{X_{\mathrm{sa}}} \to 0. \tag{1.4.4}$$

Following [KS96], we shall use the quasi-abelian categories of FN or DFN spaces (see [Sch99]). The topological duality functor induces an equivalence of triangulated categories

$$\mathsf{D}^{\mathrm{b}}(\mathrm{FN})^{\mathrm{op}} \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{DFN}).$$

By applying Proposition 1.6, one gets:

Proposition 1.11. — *Let $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$. The two objects $\mathrm{R}\Gamma(U;\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}})$ and $\mathrm{R}\Gamma(U;\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ are well-defined in the categories $\mathsf{D}^{\mathrm{b}}(\mathrm{FN})$ and $\mathsf{D}^{\mathrm{b}}(\mathrm{DFN})$ respectively, and are dual to each other.*

See [KS96, Theorem 6.1] for a more general statement.

Example 1.12. —

(i) Let Z be a closed complex analytic subset of the complex manifold X. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:

$$\rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}\left(\mathrm{D}'_X\mathbb{C}_Z,\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}\right)\simeq \mathcal{O}_X\widehat{|}_Z(\text{formal completion}),$$

$$\rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}\left(\mathbb{C}_Z,\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}\right)\simeq \mathrm{R}\Gamma_{[Z]}(\mathcal{O}_X)(\text{algebraic cohomology}).$$

(ii) Let M be a real analytic manifold such that X is a complexification of M. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_M)$:

$$\rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}\left(\mathrm{D}'_X\mathbb{C}_M,\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}\right)|_M\simeq \mathcal{C}^{\infty}_M(\mathrm{C}^{\infty}\text{-functions}),$$

$$\rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}\left(\mathrm{D}'_X\mathbb{C}_M,\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}\right)|_M\simeq \mathcal{D}b_M(\text{distributions}).$$

2. The vanishing theorem

2.1. Stein subanalytic sets

The next lemmas will be useful in the sequel.

Recall that a compact subset K of a complex manifold is said to be Stein if K admits a fundamental neighborhood system consisting of Stein open subsets.

Lemma 2.1. — *Let X be a Stein manifold and let K be a compact subset. Then there exist an open Stein subanalytic subset U of X and a compact Stein subanalytic subset L of X with $K \subset L \subset U$.*

Proof. — One embeds X as a closed smooth complex submanifold of $\mathbb{C}^N$ for some N. Then choose for U the intersection of X with an open ball which contains K and similarly choose a closed ball for L. □

Lemma 2.2. — *Let X be a complex manifold and let K be a Stein compact subset. Then there exists a fundamental neighborhood system of K consisting of open Stein subanalytic subsets as well as a fundamental neighborhood system of K consisting of compact Stein subanalytic subsets.*

Proof. — Choose a Stein open neighborhood W of K and apply Lemma 2.1. □

The proof of Lemma 2.3 below was suggested to us by Daniel Barlet. Note that this lemma can be compared to [BMF11] which treats complex neighborhoods of real manifolds.

Lemma 2.3. — *Let X be a closed smooth Stein submanifold of $Y=\mathbb{C}^N$ and let $U\subset X$ be an subanalytic relatively compact Stein open subset of X. Then for each open neighborhood W of X in Y, there exists a subanalytic relatively compact Stein open subset V of W such that $V\cap X=U$.*

Proof. — Denote by T_XY the normal bundle to X in Y and identify X with the zero-section of T_XY. By [GR65, Chapter 8, Section C, Theorem 8], there exist an open neighborhood Ω of X in Y, an open neighborhood $\widetilde{\Omega}$ of X in T_XY and a holomorphic isomorphism $\Omega \xrightarrow{\sim} \widetilde{\Omega}$. It is thus enough to construct V in $\widetilde{\Omega} \subset T_XY$. Since $U \times_X T_XY$ is a vector bundle over a Stein manifold, it is Stein by loc. cit. Th. 9. Moreover, since T_XY is Stein, there exists an open relatively compact subanalytic subset W of T_XY which contains $\overline{U}$. Recalling that $\mathbb{R}^+$ acts on the vector bundle T_XY we get that the open set $V = (c \cdot W) \cap (U \times_X T_XY)$ is Stein, subanalytic and contained in $\widetilde{\Omega}$ for $c > 0$ small enough. ☐

Let K be a Stein subanalytic compact subset of a complex manifold X and consider the ring $A := \mathscr{O}_X(K)$. Let $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X|_K)$ denote the category of coherent $\mathscr{O}_X$-modules defined in a neighborhood of K and let $\mathrm{Mod}^f(A)$ denote the category of finitely generated A-modules. It is well-known after the work of Frisch and Siu (see [Fri67, Siu69] and [Tay02, Theorem 11.9.2]) that A is Noetherian and that the functor $\Gamma(K; \bullet)$ induces an equivalence of categories

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X|_K) \xrightarrow{\sim} \mathrm{Mod}^f(A). \tag{2.1.1}$$

Lemma 2.4. — *Let K be a Stein subanalytic subset of the complex manifold X. The category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X|_K)$ has finite homological dimension. Moreover, any object of this category admits a finite resolution by projective objects and projective objects are direct factors of finite free $\mathscr{O}_X$-modules.*

Proof. — The functor $\mathscr{H}om_{\mathscr{O}_X}$ has finite homological dimension and the functor $\Gamma(K; \bullet)$ is exact. Therefore, the functor $\mathrm{Hom}_{\mathscr{O}_X}$ has finite homological dimension. This proves the first assertion. The second one follows from the equivalence (2.1.1). ☐

2.2. A vanishing theorem on the affine space

The sheaf $\mathcal{O}^{2,\mathrm{tp}}_{X_{\mathrm{sa}}}$

Recall the spaces $L^{2,s}(U)$ and $L^{2,\mathrm{tp}}(U)$ of (1.3.2) and (1.3.3). On a complex manifold we denote by $L^{2,s,(p,q)}(U)$ and $L^{2,\mathrm{tp},(p,q)}(U)$ the spaces of differential forms with coefficients in these spaces. We set

$$\begin{aligned} L_0^{2,s,(p,q)}(U) &:= \left\{ f \in L^{2,s,(p,q)}(U); \overline{\partial} f \in L^{2,s,(p,q+1)}(U) \right\}, \\ L_0^{2,\mathrm{tp},(p,q)}(U) &:= \left\{ f \in L^{2,\mathrm{tp},(p,q)}(U); \overline{\partial} f \in L^{2,\mathrm{tp},(p,q+1)}(U) \right\}, \\ L_0^{2,\mathrm{tpst},(p,q)}(U) &:= \varinjlim_s L_0^{2,s,(p,q)}(U). \end{aligned} \tag{2.2.1}$$

Lemma 2.5. — *The natural morphism $L_0^{2,\mathrm{tpst},(p,q)}(U) \to L_0^{2,\mathrm{tp},(p,q)}(U)$ is an isomorphism.*

The proof is obvious but for the reader's convenience, we develop it.

Proof. — Set for short

$$E^s = L^{2,s,(p,q)}(U), \quad E^{\mathrm{tp}} = \varinjlim E^s, \quad F^s = L^{2,s,(p,q+1)}(U), \quad F^{\mathrm{tp}} = \varinjlim F^s,$$
$$G = \mathcal{D}b^{\mathrm{tp},(p,q+1)}(U), \quad u = \overline{\partial}\colon E^{\mathrm{tp}} \to G,$$
$$E_0^s = \{x \in E^s; u(x) \in F^s\}, \quad E_0^{\mathrm{tpst}} = \varinjlim E_0^s, \quad E_0^{\mathrm{tp}} = \left\{x \in E^{\mathrm{tp}}; u(x) \in F^{\mathrm{tp}}\right\}.$$

Notice that we have monomorphisms $E^s \hookrightarrow E^t$ and $F^s \hookrightarrow F^t$ for $s \leqslant t$. The morphism $E_0^{\mathrm{tpst}} \to E_0^{\mathrm{tp}}$ is a monomorphism since both spaces are contained in E^{tp}. Let us show that it is an epimorphism. Let $x \in E_0^{\mathrm{tp}}$. There exists s and $t \geqslant s$ such that $x \in E^s$ and $u(x) \in F^t$. Therefore, $x \in E_0^t$. □

We consider the complexes

$$\mathcal{O}^{2,s}(U) := 0 \to L_0^{2,s,(0,0)}(U) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} L_0^{2,s,(0,n)}(U) \to 0, \tag{2.2.2}$$

$$\mathcal{O}^{2,\mathrm{tp}}(U) := 0 \to L_0^{2,\mathrm{tp},(0,0)}(U) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} L_0^{2,\mathrm{tp},(0,n)}(U) \to 0. \tag{2.2.3}$$

We shall first recall a fundamental result due to Hörmander.

Theorem 2.6 (see [Hör65, Theorem 2.2.1']). — *Assume that $X = \mathbb{C}^n$ and $U \subset X$ is a relatively compact open subset and is Stein. Then the complex* (2.2.2) *is concentrated in degree* 0.

Note that Hörmander's theorem applies since the function $\varphi := -\ln(\mathrm{dist}(x, X \setminus U))$ is plurisubharmonic on the Stein open subset U.

Example 2.7. — Let $X = A^1(\mathbb{C})$, the complex line with coordinate z, and let $U = D^\times$ the disc minus $\{0\}$. The map $\overline{\partial}\colon L_0^{2,1}(U) \to L^{2,1}(U)$ is surjective (note that $L^{2,1}(U) \simeq L_0^{2,1,(0,1)}(U)$). For example, $1/\overline{z} \in L^{2,1}(U)$ and $\ln|z| \in L^{2,1}(U)$ is a solution of the equation $\overline{\partial}u = 1/\overline{z}$.

Corollary 2.8. — *Assume that $X = \mathbb{C}^n$ and $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ is Stein. Then the complex* (2.2.3) *is concentrated in degree* 0.

Proof. — By Lemma 2.5, one has $\varinjlim L_0^{2,s,(p,q)}(U) \xrightarrow{\sim} L_0^{2,\mathrm{tp},(p,q)}(U)$. Since the inductive limit is filtrant, it commutes with the functor of cohomology and the result follows from Theorem 2.6. □

Denote by $\mathscr{L}_{X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q)}$ and $\mathscr{L}_{0,X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q)}$ the presheaves $U \mapsto L^{2,\mathrm{tp},(p,q)}(U)$ and $U \mapsto L_0^{2,\mathrm{tp},(p,q)}(U)$ on X_{sa}, respectively.

It follows from Lemma 1.8 that the presheaves $\mathscr{L}_{X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q)}$ are sheaves of $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,tp}$-modules.

Lemma 2.9. — *The presheaves $\mathscr{L}_{0,X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q)}$ are sheaves and are $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,tp}$-modules. In particular, these sheaves are Γ-acyclic.*

Proof. — The fact that $\mathscr{L}_{0,X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q)}$ is a sheaf follows from the fact that $\mathscr{L}_{X_{\mathrm{sa}}}^{2,\mathrm{tp},(p,q+1)}$ is a sheaf. It is a $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,tp}$-module since $L^{2,\mathrm{tp},(p,q)}(U)$ is a $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,tp}(U)$-module and for $\theta \in \mathcal{C}_{X_{\mathrm{sa}}}^{\infty,tp}(U)$ and $\varphi \in L_0^{2,\mathrm{tp},(p,q)}(U)$, $\overline{\partial}(\theta\varphi) = (\overline{\partial}\theta)\varphi + \theta(\overline{\partial}\varphi)$. □

We define the object $\mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}} \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ by the complex of sheaves on X_{sa}:

$$\mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}} := 0 \to \mathscr{L}^{2,\,\mathrm{tp},\,(0,0)}_{0,\,X_{\mathrm{sa}}} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{L}^{2,\,\mathrm{tp},\,(0,n)}_{0,\,X_{\mathrm{sa}}} \to 0. \tag{2.2.4}$$

It follows from Lemma 2.9 that, for $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$, the object $\mathrm{R}\Gamma(U; \mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}})$ is represented by the complex (2.2.3)

Theorem 2.10. — *Assume that $X = \mathbb{C}^n$ and $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ is Stein. Then $\mathrm{R}\Gamma(U; \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is concentrated in degree 0.*

Proof. — It follows from Proposition 1.9 that there are natural morphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$

$$\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \xrightarrow{\alpha} \mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}} \xrightarrow{\beta} \widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$$

which induce

$$\mathrm{R}\Gamma\left(U; \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}\right) \xrightarrow{\alpha} \mathrm{R}\Gamma\left(U; \mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}}\right) \xrightarrow{\beta} \mathrm{R}\Gamma\left(U; \widetilde{\mathcal{O}}^{\mathrm{tp}}_{X_{\mathrm{sa}}}\right).$$

The composition $\beta \circ \alpha$ is an isomorphism by Theorem 1.10 and the cohomology of the complex $\mathrm{R}\Gamma(U; \mathcal{O}^{2,\,\mathrm{tp}}_{X_{\mathrm{sa}}})$ is concentrated in degree 0 by Corollary 2.8. The result follows. □

Recall the functor $\rho_{X!}$ of (1.2.3) and consider a coherent $\mathscr{O}_X$-module $\mathscr{F}$. We define the sheaf

$$\mathscr{F}^{\mathrm{tp}} := \rho_{X!}\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\rho_{X!}\mathscr{O}_X} \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}. \tag{2.2.5}$$

Corollary 2.11. — *Assume that $X = \mathbb{C}^n$ and $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ is Stein. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module defined in a neighborhood of a Stein compact subset K of X such that $U \subset K$. Then $\mathrm{R}\Gamma(U; \mathscr{F}^{\mathrm{tp}})$ is concentrated in degree 0.*

Proof. — By Lemma 2.2, we may assume that K is a compact Stein subanalytic subset of X. By Lemma 2.4 there exists an exact sequence

$$0 \to \mathscr{L}_p \to \cdots \to \mathscr{L}_0 \to \mathscr{F} \to 0 \tag{2.2.6}$$

where all $\mathscr{L}_i$ are projective objects of the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X|_K)$, hence direct factors of finite free $\mathscr{O}_X$-modules. Since the functor $\rho_{X!}$ is exact, $\rho_{X!}\mathscr{F}$ is quasi-isomorphic to the complex

$$0 \to \rho_{X!}\mathscr{L}_p \to \cdots \to \rho_{X!}\mathscr{L}_0 \to 0$$

and thus $\mathscr{F}^{\mathrm{tp}}$ is quasi-isomorphic to the complex

$$0 \to \mathscr{L}^{\mathrm{tp}}_p \to \cdots \to \mathscr{L}^{\mathrm{tp}}_0 \to 0. \tag{2.2.7}$$

Note that for each i, $\mathscr{L}_i$ being a direct factor of finite free $\mathscr{O}_X$-modules, $\mathscr{L}^{\mathrm{tp}}_i$ is concentrated in degree 0.

We shall argue by induction on p. If $p = 0$ the result follows from Theorem 2.10. Now assume that the result holds for any coherent sheaf which admits a projective of length $\leqslant p - 1$. Define $\mathscr{G}$ by the exact sequence

$$0 \to \mathscr{L}_p \to \cdots \to \mathscr{L}_1 \to \mathscr{G} \to 0.$$

Then $\mathrm{R}\Gamma(U;\mathscr{G}^{\mathrm{tp}})$ is concentrated in degree 0 by the induction hypothesis. Moreover, $\mathscr{G}^{\mathrm{tp}}$ is quasi-isomorphic to the complex

$$0 \to \mathscr{L}_p'^{\,\mathrm{tp}} \to \cdots \to \mathscr{L}_1^{\mathrm{tp}} \to 0. \tag{2.2.8}$$

It follows that $0 \to \mathscr{G}^{\mathrm{tp}} \to \mathscr{L}_0^{\mathrm{tp}} \to \mathscr{F}^{\mathrm{tp}} \to 0$ is an exact sequence of sheaves on X_{sa} and the result follows from the long exact sequence obtained by applying the functor $\Gamma(U;\bullet)$. □

Remark 2.12. — On a complex manifold, it would be possible to replace the subanalytic topology with the Stein subanalytic topology for which the open sets are the finite union of Stein relatively compact subanalytic open subsets of X (see [Pet17]). With this new topology the sheaf of holomorphic functions and, thanks to Theorem 2.10, the sheaf of temperate holomorphic functions, are concentrated in degree 0.

2.3. A vanishing theorem on Stein manifolds

In this section, we shall extend Theorem 2.10 by replacing $\mathbb{C}^n$ with a complex manifold.

LEMMA 2.13. — *Let $j\colon X \hookrightarrow Y$ be a closed embedding of smooth complex manifolds. Then there is a natural isomorphism $(j_*\mathscr{O}_X)^{\mathrm{tp}} \xrightarrow{\sim} j_*\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$.*

Proof. — By Lemma 1.2, we have the isomorphism of functors

$$j^{-1}\rho_{Y!} \simeq \rho_{X!}j^{-1}$$

which induces the morphisms

$$\begin{aligned} j^{-1}\left(j_*\mathscr{O}_X\right)^{\mathrm{tp}} &\simeq j^{-1}\rho_{Y!}j_*\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{j^{-1}\rho_{Y!}\mathscr{O}_Y} j^{-1}\mathcal{O}^{tp}_{Y_{\mathrm{sa}}} \\ &\simeq \rho_{X!}j^{-1}j_*\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{j^{-1}\rho_{Y!}\mathscr{O}_Y} j^{-1}\mathcal{O}^{tp}_{Y_{\mathrm{sa}}} \to j^{-1}\mathcal{O}^{tp}_{Y_{\mathrm{sa}}}. \end{aligned}$$

Here, the last morphism is associated with $j^{-1}\rho_{Y!}\mathscr{O}_Y \simeq \rho_{X!}j^{-1}\mathscr{O}_Y \to \rho_{X!}\mathscr{O}_X$.

On the other hand, there is a natural morphism $j^{-1}\mathcal{C}^{\infty,tp}_{Y_{\mathrm{sa}}} \to \mathcal{C}^{\infty,tp}_{X_{\mathrm{sa}}}$ which induces the morphism $j^{-1}\mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}} \to \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$. These define the morphism

$$(j_*\mathscr{O}_{X_{\mathrm{sa}}})^{\mathrm{tp}} \to j_*\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}. \tag{2.3.1}$$

Let us prove that (2.3.1) is an isomorphism. This is a local problem and we may assume that $Y = X \times Z$ with $X = \mathbb{C}^n$, $Z = \mathbb{C}^p$ and j is the embedding identifying X and $X \times \{0\}$. By induction we may assume $p = 1$. Let z denote a holomorphic coordinate on $Z = \mathbb{C}$. For simplicity, we do not write X. Then we are reduced to prove that the complex $\mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}} \xrightarrow{z} \mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}}$ is quasi-isomorphic to $\mathbb{C}_{\{0\}}[1]$. Let us replace $\mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}}$ with the complex $\mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}} \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}}$ we get the double complex

$$\begin{array}{ccc} \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}} & \xrightarrow{\bar\partial} & \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}} \\ \Big\downarrow{\scriptstyle z} & & \Big\downarrow{\scriptstyle z} \\ \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}} & \xrightarrow{\bar\partial} & \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}}. \end{array} \tag{2.3.2}$$

Given $U, V \in \mathrm{Op}_{Y_{\mathrm{sa}}}$ with $U \subset V$ and $0 \in U$, the two complexes $\Gamma(W; \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}}) \xrightarrow{z} \Gamma(W; \mathscr{C}^{\infty,\mathrm{tp}}_{Y_{\mathrm{sa}}})$ with W being either U or V are quasi-isomorphic. Hence, in order to calculate the cohomology of the double complex (2.3.2) we may apply to it the functor $\Gamma(U; \bullet)$ for any $U \in \mathrm{Op}_{Y_{\mathrm{sa}}}$ with $0 \in U$. If we choose U convex, this complex is quasi-isomorphic to the complex $\mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}}(U) \xrightarrow{z} \mathscr{O}^{\mathrm{tp}}_{Y_{\mathrm{sa}}}(U)$ which is itself quasi-isomorphic to $\mathbb{C}_{\{0\}}[1]$. □

Lemma 2.14. — *Let $j\colon X \hookrightarrow Y$ be a closed embedding of smooth complex manifolds and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then there is a natural isomorphism $(j_*\mathscr{F})^{\mathrm{tp}} \xrightarrow{\sim} j_*\mathscr{F}^{\mathrm{tp}}$.*

Proof. — One constructs the natural morphism

$$(j_*\mathscr{F})^{\mathrm{tp}} \to j_*\mathscr{F}^{\mathrm{tp}} \tag{2.3.3}$$

by the same procedure as for $\mathscr{O}_X$ in (2.3.1). To prove that this morphism is an isomorphism, one may replace locally $\mathscr{F}$ with a free resolution. □

Theorem 2.15. — *Let X be a complex Stein manifold and let U be a subanalytic relatively compact Stein open subset of X contained in a Stein compact subset K of X. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module defined in a neighborhood of K. Then $\mathrm{R}\Gamma(U; \mathscr{F}^{\mathrm{tp}})$ is concentrated in degree 0.*

Proof. —

(i) Since X is Stein, there exist some integer N and a closed embedding $j\colon X \hookrightarrow \mathbb{C}^N$. Set $Y = \mathbb{C}^N$ for short.

(ii) The coherent $\mathscr{O}_Y$-module $j_*\mathscr{F}$ is defined in a neighborhood of K in Y and K admits a fundamental neighborhood system of Stein open subsets in Y by [Siu76]. Let W be such a Stein open subset on which $j_*\mathscr{F}$ is defined.

(iii) By applying Lemma 2.3, we find a relatively compact subanalytic Stein open subset V of Y such that $U = X \cap V$. By replacing V with $V \cap V'$ for a Stein open subanalytic subset containing $\overline{U}$, we may assume that $\overline{V} \subset W$.

(iv) Applying the result of Corollary 2.11, we get that

$$\mathrm{R}\Gamma\left(V; (j_*\mathscr{F})^{\mathrm{tp}}\right) \text{ is concentrated in degree } 0. \tag{2.3.4}$$

Applying Lemma 2.14, we get

$$\mathrm{R}\Gamma\left(V; j_*\mathscr{F}^{\mathrm{tp}}\right) \text{ is concentrated in degree } 0. \tag{2.3.5}$$

Since $\mathrm{R}\Gamma(V; j_*\mathscr{F}^{\mathrm{tp}}) \simeq \mathrm{R}\Gamma(U; \mathscr{F}^{\mathrm{tp}})$, the proof is complete. □

Remark 2.16. — Theorem 2.10 was deduced from Hörmander's Theorem 2.6 and the same argument would apply on a complex manifold if the Hörmander's theorem had been stated in such a framework. And indeed, according to H. Skoda, such a generalization of Hörmander's theorem should be possible when combining [Dem09, Chapter VIII §6, Theorem 6.5] and [Ele75]. This would provide an alternative proof to Theorem 2.15.

Corollary 2.17. — *Let X be a complex Stein manifold of pure dimension n and let U be a subanalytic relatively compact Stein open subset of X. Then $\mathrm{R}\Gamma(U; \mathcal{O}^{tp}_{X_{\mathrm{sa}}})$ is concentrated in degree 0 and $\mathrm{R}\Gamma(U; \mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}})$ is concentrated in degree n.*

Proof. — The first vanishing result is a particular case of Theorem 2.15 and the second result follows by applying Proposition 1.11. ☐

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