Irregular holonomic kernels and Laplace transform

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Abstract

Given a (not necessarily regular) holonomic $\mathscr{D}$-module $\mathscr{L}$ defined on the product of two complex manifolds, we prove that the correspondence associated with $\mathscr{L}$ commutes (in some sense) with the De Rham functor. We apply this result to the study of the classical Laplace transform. The main tools used here are the theory of ind-sheaves and its enhanced version.

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1 Introduction

Perhaps the most popular integral transform in Mathematics is the Fourier transform, or its complex version, the Laplace transform. It interchanges objects living on a finite-dimensional vector space V with objects living on the dual space V^*. The kernel of this transform is $\mathrm{e}^{\langle x,y\rangle}$ and all the subtlety and difficulty of this transform comes from the fact that the $\mathscr{D}$-module on $\mathsf{V}\times\mathsf{V}^*$ generated by this kernel is holonomic but is not regular. In this paper, we shall give tools to treat integral transforms associated with general holonomic kernels and apply them to the particular case of the Laplace transform.

Let us be more precise. For a complex manifold $(X,\mathscr{O}_X)$ we denote by Ω_X the sheaf of differential forms of top degree, by $\mathscr{D}_X$ the sheaf of (finite-order) differential operators and by d_X the complex dimension of X. We use the usual six operations for sheaves and denote by $\mathrm{D}g_*$, $\mathrm{D}f^*$ and $\overset{\mathrm{D}}{\otimes}$ the operations of direct image, inverse image and tensor product for $\mathscr{D}$-modules.

Here, a sheaf or a $\mathscr{D}$-module should be understood in the derived sense, that is, in the bounded derived category of sheaves or $\mathscr{D}$-modules.

All along this paper, we shall use the language of ind-sheaves of [KS01] and the six operations for ind-sheaves, $\overset{\mathrm{L}}{\otimes}$, $\mathrm{R}\mathscr{I}hom$, f^{-1}, $f^{\,!}$, $\mathrm{R}f_*$ and $\mathrm{R}f_{!!}$. The main object of interest will be the ind-sheaf $\mathcal{O}_X^{\mathrm{t}}$ of holomorphic functions with tempered growth, realized as the Dolbeault complex of the ind-sheaf $\mathcal{C}_X^{\infty,\mathrm{t}}$ of C^∞-functions with tempered growth. On an open subanalytic subset U the sections of this last sheaf are functions which have polynomial growth at the boundary, as well as all their derivatives. The history of the ind-sheaf $\mathcal{O}_X^{\mathrm{t}}$ is closely related to the solution of the Riemann-Hilbert problem for regular holonomic modules of [Ka80, Ka84]. Recall that the main tool to solve this problem was the functor $Thom$, which in the language of ind-sheaves reads as $\mathrm{R}\mathscr{H}om\,(\,\bullet\,,\mathcal{O}_X^{\mathrm{t}})$.

We shall use the Sol and tempered Sol functors and the De Rham and tempered De Rham functors for $\mathscr{D}_X$-modules. Denoting by Ω_X^{t} the sheaf of tempered differential forms of top degree, these functors are given by:

$$\mathcal{S}ol_X(\mathscr{M}) = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X), \qquad \mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}) = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathcal{O}_X^{\mathrm{t}}),$$
$$\mathcal{DR}_X(\mathscr{M}) = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \qquad \mathcal{DR}_X^{\mathrm{t}}(\mathscr{M}) = \Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}.$$

Consider a correspondence of complex manifolds:

$$(1.1)\qquad \begin{array}{ccccc} & & S & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ X & & & & Y. \end{array}$$

For a $\mathscr{D}_S$-module $\mathscr{L}$ and a $\mathscr{D}_X$-module $\mathscr{M}$ one sets:

$$\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L} := \mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}).$$

For ind-sheaves L on S, F on X and G on Y one sets

$$(1.2)\qquad L\circ G := \mathrm{R}f_{!!}(L\otimes g^{-1}G), \quad \Psi_L(F) = \mathrm{R}g_*\,\mathrm{R}\mathscr{I}hom\,(L, f^{\,!}\,F).$$

For the notion of being quasi-good or good and the categories $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and $\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, see § 3.1.

Theorem 1.1. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and let $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_S)$. Set $L := \mathcal{S}ol_S(\mathscr{L})$ and assume*

(i) $\mathscr{L}$ *is regular holonomic,*

(ii) $f^{-1}\operatorname{supp}(\mathscr{M})\cap\operatorname{supp}(\mathscr{L})$ *is proper over* Y*.*

Then there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Y)$*:*

$$\Psi_L(\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}))\,[d_X-d_S]\simeq\mathcal{DR}^{\mathrm{t}}_Y(\mathscr{M}\overset{\mathrm{D}}{\circ}\mathscr{L}). \tag{1.3}$$

This result (which, to our knowledge, never appeared in the literature under this form) is an immediate consequence of three deep results:
(i) the direct image functor $\mathrm{D}g_*$ commutes, under an hypothesis of properness, with the tempered De Rham functor,
(ii) the inverse image functor $\mathrm{D}f^*$ commutes, up to a shift, with the tempered De Rham functor,
(ii) the formula, in which $\mathscr{N}$ is regular holonomic and $\mathscr{M}$ is coherent on X:

$$\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X(\mathscr{N}),\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}))\;\simeq\;\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}\overset{\mathrm{D}}{\otimes}\mathscr{N}). \tag{1.4}$$

As a corollary, one gets (under the same hypotheses) the adjunction formula of [KS01], in which G is an ind-sheaf on Y:

$$\mathrm{RHom}\,(L\circ G,\Omega^{\mathrm{t}}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathscr{M})\,[d_X-d_S]\simeq\mathrm{RHom}\,(G,\Omega^{\mathrm{t}}_Y\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y}(\mathscr{M}\overset{D}{\circ}\mathscr{L})). \tag{1.5}$$

Note that a similar formula holds when replacing $\mathcal{O}^{\mathrm{t}}_X$ and $\mathcal{O}^{\mathrm{t}}_Y$ with their non tempered versions $\mathscr{O}_X$ and $\mathscr{O}_Y$ (and ind-sheaves with usual sheaves), but the hypotheses are different. Essentially, $\mathscr{M}$ has to be coherent, f non characteristic for $\mathscr{M}$ and $\mathrm{D}f^*\mathscr{M}$ has to be transversal to the holonomic module $\mathscr{L}$. On the other hand, we do not need the regularity assumption on $\mathscr{L}$. See [DS96] for such a non tempered formula (in a more particular setting).

However, if one removes the hypothesis that the holonomic module $\mathscr{L}$ is regular in Theorem 1.1, formulas (1.3) and (1.5) do not hold anymore and we have to replace $\mathcal{O}^{\mathrm{t}}_X$ with its enhanced version, the object $\mathscr{O}^{\mathsf{E}}_X$ of [DK13], and this is one of the main purpose of this paper. Let us briefly explain what is $\mathscr{O}^{\mathsf{E}}_X$.

In order to keep in mind the behavior at infinity of our objects, one considers bordered spaces $M_\infty=(M,\widehat{M})$, where M is open in $\widehat{M}$. A typical example, which will play an essential role here, is the bordered space

$$\mathbb{R}_\infty=(\mathbb{R},\overline{\mathbb{R}})\text{ where }\overline{\mathbb{R}}:=\mathbb{R}\sqcup\{+\infty,-\infty\}.$$

For a commutative ring $\mathbf{k}$ and a "good" topological space M, denote by $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})$ the bounded derived category of ind-sheaves of $\mathbf{k}$-modules. Then one introduces the quotient category

$$\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) := \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})/\{\mathrm{F}; \mathbf{k}_{\mathrm{M}} \otimes \mathrm{F} \simeq 0\}.$$

(Note that when working with usual sheaves, one would recover the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ but the situation is different with ind-sheaves.) The six operations on ind-sheaves are easily extended to ind-sheaves on bordered spaces.

Now consider the bordered space $M \times \mathbb{R}_\infty = (M \times \mathbb{R}, M \times \overline{\mathbb{R}})$. Denote by $\pi\colon M \times \mathbb{R}_\infty \to M$ the projection. One defines the new category of enhanced ind-sheaves on M by setting:

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}) := \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty})/\{\mathrm{F}; \pi^{-1}\mathrm{R}\pi_*\mathrm{F} \xrightarrow{\sim} \mathrm{F}\}.$$

The quotient functor $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})$ admits a right and a left adjoint, denoted by R^{E} and L^{E}, respectively. This category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})$ is closely related to constructions initiated in Tamarkin [Ta08] (see also [GS12] for a detailed exposition and complements to Tamarkin's work). In particular it is endowed with a new tensor product, denoted by $\overset{+}{\otimes}$, and a new internal hom, denoted by $\mathscr{I}hom^+$. The four operations for enhanced ind-sheaves associated with a morphism of manifolds f are denoted by $\mathsf{E}f_*$, $\mathsf{E}f_{!!}$, $\mathsf{E}f^{-1}$ and $\mathsf{E}f^{\,!}$ and one also uses the bifunctor $\mathrm{RHom}^{\mathrm{E}}$ with values in $\mathsf{D}^{\mathrm{b}}(\mathbf{k})$ (see Definition 2.15). These operations enjoy similar properties to the one for sheaves.

Let X be a complex manifold, $Y \subset X$ a complex hypersurface and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}_X(*Y)$, one sets

$$\mathscr{D}_X \mathrm{e}^{\varphi} = \mathscr{D}_X/\{P; P\mathrm{e}^{\varphi} = 0 \text{ on } U\}, \quad \mathscr{E}^{\varphi}_{U|X} = \mathscr{D}_X \mathrm{e}^{\varphi}(*Y).$$

Then one introduces the object $\mathscr{O}^{\mathsf{E}}_X$ of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$ which plays a role analogous to the objects $\mathcal{O}^{\mathrm{t}}_X$ but contains more information. Denote by $i\colon X\times\mathbb{R}_\infty \to X\times\mathbb{P}$ the natural morphism and denote by $\tau \in \mathbb{C} \subset \mathsf{P}$ the affine variable in the complex projective line P. One sets:

$$\mathscr{O}^{\mathsf{E}}_X = i^{\,!}\,\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathbb{P}}}(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}})[2] \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}),$$
$$\Omega^{\mathsf{E}}_X = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}^{\mathsf{E}}_X.$$

One defines the enhanced De Rham and Sol functors by

$$\mathcal{DR}^{\mathsf{E}}_X \;:\; \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \quad \mathscr{M} \mapsto \Omega^{\mathsf{E}}_{\mathrm{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathrm{X}}} \mathscr{M},$$
$$\mathcal{S}ol^{\mathsf{E}}_X \;:\; (\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X))^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}\mathrm{om}_{\mathscr{D}_{\mathrm{X}}}(\mathscr{M}, \mathscr{O}^{\mathsf{E}}_{\mathrm{X}}).$$

Then our main result is the following (see Theorem 4.9) which generalizes Theorem 1.1 to the case where $\mathscr{L}$ is no more regular.

Theorem 1.2. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$*,* $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{good}}(\mathscr{D}_S)$ *and set* $L := \mathcal{S}ol^{\mathsf{E}}_S(\mathscr{L})$*. Assume that* $\mathscr{L}$ *is holonomic and that* $f^{-1}\operatorname{supp}(\mathscr{M}) \cap \operatorname{supp}(\mathscr{L})$ *is proper over* Y*. Then there is a natural isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$*:*

$$\Psi^{\mathsf{E}}_L(\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}))\,[d_X - d_S] \simeq \mathcal{DR}^{\mathsf{E}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}).$$

Note that in the course of the proof, we shall need to strengthen the Riemann-Hilbert theorem of [DK13] and to prove the isomorphism for holonomic $\mathscr{M}$ (see Theorem 4.5):

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}^{\mathsf{E}}_X \xrightarrow{\sim} \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X) \text{ in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}).$$

The proof of this isomorphism, which follows from the same lines as in loc. cit., is a main technical part of this paper. As an easy application of our theorem, we obtain:

Corollary 1.3. *In the situation as in* Theorem 1.2*, let* $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$*. Then there is a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C})$

$$\begin{aligned} &\mathrm{RHom}^{\mathrm{E}}(L \overset{E}{\circ} G, \Omega^{\mathsf{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})\,[d_X - d_S] \\ &\qquad\qquad \simeq \mathrm{RHom}^{\mathrm{E}}(G, \Omega^{\mathsf{E}}_Y \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} (\mathscr{M} \overset{D}{\circ} \mathscr{L})). \end{aligned}$$

Here $\overset{E}{\circ}$ is an enhanced version of the convolution $\circ$ in (1.2).

Next, we shall apply these results to the Laplace transform. For that purpose, we need to treat first the Fourier-Sato transform, and its enhanced version. Let V be a real finite-dimensional vector space of dimension n, V^* its dual. Recall that the Fourier-Sato transform, denoted here by ${}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}}$, is an equivalence of categories between conic sheaves on V and conic sheaves on V^*. References are made to [KS90]. In [Ta08], D. Tamarkin has extended the Fourier-Sato transform to no more conic (usual) sheaves, by adding an extra variable. Here we generalize this last transform to enhanced ind-sheaves on the bordered space $\mathsf{V}_\infty = (\mathsf{V}, \overline{\mathsf{V}})$, where $\overline{\mathsf{V}}$ is the projective compactification of V. We introduced the kernels $L_{\mathsf{V}} := \mathbf{k}_{\{t=\langle x,y\rangle\}}$ and $L^a_{\mathsf{V}} := \mathbf{k}_{\{t=-\langle x,y\rangle\}}$ and define the enhanced Fourier-Sato functors

$$\begin{aligned} {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}} &\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}), \quad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathrm{F}) = \mathrm{F} \overset{\mathrm{E}}{\circ} \mathrm{L}_{\mathsf{V}}, \\ {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}} &\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}), \quad {}^{\mathsf{E}}\mathscr{F}^{\mathrm{a}}_{\mathsf{V}}(\mathrm{F}) = \mathrm{F} \overset{\mathrm{E}}{\circ} \mathrm{L}^{\mathrm{a}}_{\mathsf{V}}. \end{aligned}$$

We easily prove that the functors ${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}$ and ${}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*}$ are equivalences of categories, quasi-inverse to each other up to shift (see Theorem 5.2). Moreover the enhanced Fourier-Sato transform is compatible with the classical one, that is, we have a quasi-commutative diagram of categories and functors (in which the vertical arrows are fully faithful functors):

$$\begin{array}{ccc} \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) & \xrightarrow{{}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}} & \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}) \\ \uparrow{\scriptstyle \varepsilon_{\mathsf{V}}} & & \uparrow{\scriptstyle \varepsilon_{\mathsf{V}^*}} \\ \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}}) & \xrightarrow{{}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}}} & \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}^*}). \end{array} \tag{1.6}$$

Let now $\mathbb{V}$ be a complex vector space of complex dimension $d_{\mathbb{V}}$ and let $\mathbb{V}^*$ be its dual. We denote here by $\overline{\mathbb{V}}$ the projective compactification of $\mathbb{V}$ and set $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$, $\mathbb{H} = \overline{\mathbb{V}} \setminus \mathbb{V}$. We set for short $X = \overline{\mathbb{V}} \times \overline{\mathbb{V}}^*$, $U = \mathbb{V} \times \mathbb{V}^*$ and consider the Laplace kernel

$$\mathscr{L} \quad := \quad \mathscr{E}^{\langle x,y\rangle}_{U|X}. \tag{1.7}$$

Denote by $\mathsf{D}_{\mathbb{V}}$ the Weyl algebra on $\mathbb{V}$. As it is well-known, the Laplace kernel induces an isomorphism $\mathsf{D}_{\mathbb{V}} \simeq \mathsf{D}_{\mathbb{V}^*}$, hence an isomorphism

$$\mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H}) \overset{\mathrm{D}}{\circ} \mathscr{L} \simeq \mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes \det \mathbb{V}^*. \tag{1.8}$$

Then our main theorem on the Laplace transform (see Theorem 6.3) is:

Theorem 1.4. *Isomorphism* (1.8) *induces an isomorphism*

$${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) \simeq \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty} \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}] \ \textit{in}\ \mathsf{D}^{\mathrm{b}}((\mathrm{I}\mathsf{D}_{\mathbb{V}})_{\mathbb{V}^*_\infty}). \tag{1.9}$$

As an immediate application (see Corollary 6.5), we obtain

Corollary 1.5. *Isomorphism* (1.9) *induces an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathsf{D}_{\mathbb{V}})$*, functorial in* $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$*:*

$$\mathrm{RHom}^{\mathrm{E}}(F, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) \simeq \mathrm{RHom}^{\mathrm{E}}({}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(F), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}) \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}]. \tag{1.10}$$

When restricting this isomorphism to conic sheaves on $\mathbb{V}$, we recover the main theorem of [KS97] which asserts that for an $\mathbb{R}$-constructible and conic sheaf $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{V}})$, the Laplace transform induces an isomorphism

$$\mathrm{RHom}\,(F, \mathcal{O}^{\mathrm{t}}_{\mathbb{V}}) \simeq \mathrm{RHom}\,({}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(F), \mathcal{O}^{\mathrm{t}}_{\mathbb{V}^*}) \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}]. \tag{1.11}$$

Several applications are given in loc. cit. and, by adding a variable, some non conic situations are also treated with the help of this isomorphism in [Da12].

Here, as an application of Corollary 1.5, we obtain the following result. For an open subset U of $\mathbb{V}$ subanalytic in $\overline{\mathbb{V}}$ and a continuous function φ on U with subanalytic graph in $\overline{\mathbb{V}}$, we can define the ind-sheaf $\mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}}_{M}$. Roughly speaking, it is the ind-sheaf of distributions u such that $\mathrm{e}^{-\varphi}u$ is tempered. Consider the Dolbeault complex

$$\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\mathbb{V}_\infty} := 0 \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,0)}_{\overline{\mathbb{V}}} \xrightarrow{\bar\partial} \cdots \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,d_{\mathbb{V}})}_{\overline{\mathbb{V}}} \to 0. \tag{1.12}$$

Assume that U is convex, φ is a convex function and denote by φ^* its Legendre transform. Then, under some hypotheses, we prove that the complex $\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\mathbb{V}_\infty}(U)$ is concentrated in degree 0, the complex $\mathrm{e}^{-\varphi^*}\mathcal{O}^{\mathrm{t}}_{\mathbb{V}_\infty}(\mathbb{V}^*)$ is concentrated in degree $d_{\mathbb{V}}$ and the Laplace transformation interchanges these two complexes (Corollary 6.15).

2 Enhanced ind-sheaves

2.1 Ind-sheaves

In this subsection and the next one, we recall some results of [KS01].

Let M be a locally compact space countable at infinity and let $\mathbf{k}$ be a commutative Noetherian ring with finite global dimension. (In this paper, all rings are unital.) Recall that $\mathrm{Mod}(\mathbf{k}_M)$ denotes the abelian category of sheaves of $\mathbf{k}$-modules on M. We denote by $\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$ the full subcategory consisting of sheaves with compact support. The category of ind-sheaves on M is the category of ind-objects of $\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$. We set for short:

$$\mathrm{I}\mathbf{k}_{\mathrm{M}} := \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_{\mathrm{M}})),$$

and call an object of this category an *ind-sheaf* on M. We denote by "$\varinjlim$" the inductive limit in the category $\mathrm{I}\mathbf{k}_{\mathrm{M}}$.

The prestack $U \mapsto \mathrm{I}(\mathbf{k}_U)$, U open in M, is a stack.

We have two pairs (α_M, ι_M) and (β_M, α_M) of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\beta_M}{\overset{\iota_M}{\underset{\xleftarrow{\;\alpha_M\;}}{\rightrightarrows}}} \mathrm{I}(\mathbf{k}_M).$$

If F has compact support, $\iota_M(F) = F$ after identifying a category $\mathscr{C}$ with a full subcategory of $\mathrm{Ind}(\mathscr{C})$. More generally, $\iota_M(F) = \text{“}\varinjlim_U\text{”}\, F_U$ where U ranges over the family of relatively compact open subsets of M. The functor α_M associates $\varinjlim_i F_i$ ($F_i \in \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$, $i \in I$, I small and filtrant) to the object “$\varinjlim_i$” F_i. If $\mathbf{k}$ is a field, $\beta_M(F)$ is the functor $G \mapsto \Gamma(M; H^0(\mathrm{D}'G) \otimes F)$.

- ι_M is exact, fully faithful, and commutes with $\varprojlim$,
- α_M is exact and commutes with $\varprojlim$ and $\varinjlim$,
- β_M is right exact, fully faithful and commutes with $\varinjlim$,
- α_M is left adjoint to ι_M,
- α_M is right adjoint to β_M,
- $\alpha_M \circ \iota_M \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_M)}$ and $\alpha_M \circ \beta_M \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_M)}$.

One denotes by $\overset{\mathrm{L}}{\otimes}$ and $\mathrm{R}\mathscr{I}hom$ the (derived) operations of tensor product and internal $\mathscr{H}om$. If $f\colon M \to N$ is a continuous map, one denotes by f^{-1}, $f^{\,!}$, $\mathrm{R}f_*$ and $\mathrm{R}f_{!!}$ the (derived) operations of inverse and direct images. One also sets

$$\mathrm{R}\mathscr{H}om = \alpha_M \circ \mathrm{R}\mathscr{I}hom : \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathrm{M}}).$$

2.2 Subanalytic topology

Here again, we recall some results of [KS01].

Assume that M is a real analytic manifold. Denote by Op_M the category of its open subsets (the morphisms being the inclusions) and by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the full subcategory of Op_M consisting of subanalytic and relatively compact open subsets. The site M_{sa} is obtained by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of U if there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$. One denotes by

$$\rho_M\colon M \to M_{\mathrm{sa}} \tag{2.1}$$

the natural morphism of sites. Here again, we have two pairs of adjoint functors (ρ_M^{-1}, ρ_{M*}) and $(\rho_{M!}, \rho_M^{-1})$:

$$\mathrm{Mod}(\mathbf{k}_M) \begin{array}{c} \xrightarrow{\ \rho_{M*}\ } \\ \xleftarrow{\ \rho_M^{-1}\ } \\ \xrightarrow[\ \rho_{M!}\ ]{} \end{array} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{M!}F$ is the sheaf associated to the presheaf $U \mapsto F(\overline{U})$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

One proves that the restriction of ρ_{M*} to the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ of $\mathbb{R}$-constructible sheaves is exact and fully faithful. By this result, we shall consider the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ as a subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ or as of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Denote by $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ consisting of sheaves with compact support and set:

$$\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) = \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)).$$

One defines the functor

$$\alpha_{M_{\mathrm{sa}}} \colon \mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \longrightarrow \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \tag{2.2}$$

similarly as the functor α_M.

Theorem 2.1. *The functor $\alpha_{M_{\mathrm{sa}}}$ in (2.2) is an equivalence of categories.*

In other words, ind-$\mathbb{R}$-constructible sheaves are "usual sheaves" on the subanalytic site. By this result, the embedding $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) \hookrightarrow \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$ gives a fully faithful functor $I_M \colon \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{I}(\mathbf{k}_M)$. Hence, in the diagram of categories

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) & \longrightarrow & \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \\ \big\downarrow & \overset{\rho_{M*}}{\nearrow} & \big\downarrow {\scriptstyle I_M} \\ \mathrm{Mod}(\mathbf{k}_M) & \xrightarrow{\ \iota_M\ } & \mathrm{I}(\mathbf{k}_M), \end{array} \tag{2.3}$$

all solid arrows are exact and fully faithful. One shall be aware that the square and the upper triangle quasi-commute but $\iota_M \neq I_M \circ \rho_{M*}$ in general. Moreover, ρ_{M*} is not right exact in general.

From now on, we shall identify sheaves on M_{sa} with ind-sheaves on M.

2.3 Ind-sheaves on bordered spaces

In this subsection, we mainly follow [DK13].

A topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension.

Definition 2.2. The category of *bordered spaces* is the category whose objects are pairs $(M, \widehat{M})$ with $M \subset \widehat{M}$ an open embedding of good topological spaces. Morphisms $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ are continuous maps $f\colon M \to N$ such that

$$\overline{\Gamma}_f \to \widehat{M} \text{ is proper.} \tag{2.4}$$

Here Γ_f is the graph of f and $\overline{\Gamma}_f$ its closure in $\widehat{M} \times \widehat{N}$.

The composition $(L, \widehat{L}) \xrightarrow{e} (M, \widehat{M}) \xrightarrow{f} (N, \widehat{N})$ is given by $f \circ e\colon L \to N$ (see Lemma 2.3 below), and the identity $\mathrm{id}_{(M,\widehat{M})}$ is given by id_M.

If there is no risk of confusion, we shall often denote by M_∞ a bordered space $(M, \widehat{M})$.

Lemma 2.3. *Let $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ and $e\colon (L, \widehat{L}) \to (M, \widehat{M})$ be morphisms of bordered spaces. Then the composition $f \circ e$ is a morphism of bordered spaces.*

One shall identify a space M and the bordered space (M, M). Then, by using the identifications $M = (M, M)$ and $\widehat{M} = (\widehat{M}, \widehat{M})$, there are natural morphisms

$$M \to (M, \widehat{M}) \to \widehat{M}.$$

Note however that $(M, \widehat{M}) \to M$ is not necessarily a morphism of bordered spaces.

One often denotes by j_M or simply j the natural morphism $M_\infty \to \widehat{M}$.

For two bordered spaces $(M, \widehat{M})$ and $(N, \widehat{N})$ their product in the category of bordered spaces is the bordered space $(M \times N, \widehat{M} \times \widehat{N})$.

Definition 2.4. Let $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ be a morphism of bordered spaces. We say that f is semi-proper if the map $\overline{\Gamma}_f \to \widehat{N}$ is proper.

Any isomorphism of bordered spaces is semi-proper.

Lemma 2.5. *Let $f\colon (M,\widehat{M}) \to (N,\widehat{N})$ and $e\colon (L,\widehat{L}) \to (M,\widehat{M})$ be morphisms of bordered spaces. If both f and e are semi-proper, then $f \circ e$ is semi-proper.*

Proof. Set for short $\mathrm{Im}(\overline{\Gamma_e} \times_{\widehat{M}} \overline{\Gamma}_f) := \mathrm{Im}(\overline{\Gamma}_e \times_{\widehat{M}} \overline{\Gamma}_f \to \overline{L} \times \overline{N})$ and consider the commutative diagrams:

$$
\begin{array}{ccc}
\overline{\Gamma_e} \times_{\widehat{M}} \overline{\Gamma}_f & \xrightarrow{c} & \overline{\Gamma}_f \\
\downarrow & \square & \downarrow \\
\overline{\Gamma}_e & \xrightarrow{a} & \overline{M}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
 & & \overline{\Gamma_e} \times_{\widehat{M}} \overline{\Gamma}_f & \xrightarrow{c} & \overline{\Gamma}_f \\
 & & \downarrow & & \downarrow{\scriptstyle b} \\
\overline{\Gamma}_{f\circ e} & \longrightarrow & \mathrm{Im}(\overline{\Gamma_e} \times_{\widehat{M}} \overline{\Gamma}_f) & \xrightarrow{d} & \overline{N}.
\end{array}
$$

The diagram on the left is Cartesian. Since the map a is proper, the map c is proper. Since the maps b and c are proper, the map d is proper. Q.E.D.

Let $M_\infty = (M,\widehat{M})$ be a bordered space. Denote by $i\colon \widehat{M}\setminus M \to \widehat{M}$ the closed embedding. Identifying $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}\setminus M})$ with its essential image in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}})$ by the fully faithful functor $\mathrm{R}i_! \simeq \mathrm{R}i_*$, the restriction functor $F \mapsto F|_M$ induces an equivalence $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}})/\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}\setminus M}) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. This is no longer true for ind-sheaves. Therefore one introduces

$$
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) := \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})/\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}}). \tag{2.5}
$$

where $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}})$ is identified with its essential image in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})$.

The fully faithful functor $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}}) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})$ induces a fully faithful functor

$$
\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}). \tag{2.6}
$$

We sometimes write $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_\infty})$ for the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ regarded as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$.

Recall that if $\mathscr{T}$ is a triangulated category and $\mathscr{I}$ a subcategory, one denotes by ${}^\perp\mathscr{I}$ and $\mathscr{I}^\perp$ the left and right orthogonal to $\mathscr{I}$ in $\mathscr{T}$, respectively.

Proposition 2.6. *Let $M_\infty = (M,\widehat{M})$ be a bordered space. One has*

$$
\begin{aligned}
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}}) &= \left\{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})\,;\, \mathbf{k}_{\mathrm{M}} \otimes \mathrm{F} \simeq 0\right\} \\
&= \left\{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})\,;\, \mathrm{R}\mathscr{I}\!\mathrm{hom}\,(\mathbf{k}_{\mathrm{M}}, \mathrm{F}) \simeq 0\right\}, \\
{}^\perp\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}}) &= \left\{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})\,;\, \mathbf{k}_{\mathrm{M}} \otimes \mathrm{F} \xrightarrow{\sim} \mathrm{F}\right\}, \\
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}})^\perp &= \left\{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})\,;\, \mathrm{F} \xrightarrow{\sim} \mathrm{R}\mathscr{I}\!\mathrm{hom}\,(\mathbf{k}_{\mathrm{M}}, \mathrm{F})\right\}.
\end{aligned}
$$

Moreover, there are equivalences

$$\begin{aligned}
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) &\xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}})^{\perp}, \quad \mathrm{F}\mapsto \mathrm{R}\mathscr{I}\mathrm{hom}\,(\mathbf{k}_{\mathrm{M}},\mathrm{F}),\\
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) &\xrightarrow{\sim} {}^{\perp}\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}\setminus\mathrm{M}}), \quad \mathrm{F}\mapsto \mathbf{k}_{\mathrm{M}}\otimes\mathrm{F},
\end{aligned}$$

quasi-inverse to the functor induced by the quotient functor.

The functors $\otimes$ and $\mathrm{R}\mathscr{I}hom$ in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})$ induce well defined functors (we keep the same notations)

$$\begin{aligned}
\otimes\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})\times\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})\to\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}),\\
\mathrm{R}\mathscr{I}hom : \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})^{\mathrm{op}}\times\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})\to\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}).
\end{aligned}$$

Let $f\colon (M,\widehat{M})\to(N,\widehat{N})$ be a morphism of bordered spaces, and recall that Γ_f denotes the graph of the associated map $f\colon M\to N$. Since Γ_f is locally closed in $\widehat{M}\times\widehat{N}$, one can consider the sheaf $\mathbf{k}_{\Gamma_f}$ on $\widehat{M}\times\widehat{N}$.

Definition 2.7. Let $f\colon (M,\widehat{M})\to(N,\widehat{N})$ be a morphism of bordered spaces. For $F\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{M}}})$ and $G\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{\mathrm{N}}})$, one sets

$$\begin{aligned}
\mathrm{R}f_{!!}F &= \mathrm{R}q_{2!!}(\mathbf{k}_{\Gamma_f}\otimes q_1^{-1}F),\\
\mathrm{R}f_{*}F &= \mathrm{R}q_{2*}\mathrm{R}\mathscr{I}hom\,(\mathbf{k}_{\Gamma_f}, q_1^{!}F),\\
f^{-1}G &= \mathrm{R}q_{1!!}(\mathbf{k}_{\Gamma_f}\otimes q_2^{-1}G),\\
f^{!}G &= \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom\,(\mathbf{k}_{\Gamma_f}, q_2^{!}G),
\end{aligned}$$

where $q_1\colon \widehat{M}\times\widehat{N}\to\widehat{M}$ and $q_2\colon \widehat{M}\times\widehat{N}\to\widehat{N}$ are the projections.

Remark 2.8. Considering a continuous map $f\colon M\to N$ as a morphism of bordered spaces with $\widehat{M}=M$ and $\widehat{N}=N$, the above functors are isomorphic to the usual operations for ind-sheaves.

Lemma 2.9. *The above definition induces well-defined functors*

$$\begin{aligned}
\mathrm{R}f_{!!},\mathrm{R}f_{*}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(\mathrm{M},\widehat{\mathrm{M}})})\to\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(\mathrm{N},\widehat{\mathrm{N}})}),\\
f^{-1},f^{!} : \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(\mathrm{N},\widehat{\mathrm{N}})})\to\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(\mathrm{M},\widehat{\mathrm{M}})}).
\end{aligned}$$

The operations for ind-sheaves on bordered spaces satisfy similar properties as for usual sheaves that we do not recall here.

Note that if $f\colon M_\infty \to N_\infty$ is semi-proper, then the diagram below quasi-commutes:

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) & \xrightarrow{\mathrm{R}f_!} & \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \\ \downarrow & & \downarrow \\ \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) & \xrightarrow{\mathrm{R}f_{!!}} & \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty}). \end{array} \tag{2.7}$$

2.4 Enhanced ind-sheaves

Tamarkin's constructions of [Ta08] are extended to ind-sheaves on bordered spaces. We refer to [GS12] for a detailed exposition and some complements to Tamarkin's paper.

In this subsection, we recall results of [DK13] with a slight generalisation. In loc. cit. the authors consider the bordered space $M \times \mathbb{R}_\infty$ where M is an usual space. However, we shall need to consider situations in which M is itself a bordered space.

Consider the 2-point compactification of the real line $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{\pm\infty\}$. Denote by $\mathsf{P} := \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \sqcup \{\infty\}$ the real projective line. Then $\overline{\mathbb{R}}$ has a structure of subanalytic space such that the natural map $\overline{\mathbb{R}} \to \mathsf{P}$ is a subanalytic map.

Notation 2.10. We will consider the bordered space

$$\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}).$$

Note that $\mathbb{R}_\infty$ is isomorphic to $(\mathbb{R}, \mathsf{P})$ as a bordered space.

Consider the morphisms of bordered spaces

$$\mu, q_1, q_2\colon \mathbb{R}^2_\infty \to \mathbb{R}_\infty, \tag{2.8}$$

where $\mu(t_1, t_2) = t_1 + t_2$ and q_1, q_2 are the natural projections.

For a bordered space M_∞, we will use the same notations for the associated morphisms

$$\mu, q_1, q_2\colon M_\infty \times \mathbb{R}^2_\infty \to M_\infty \times \mathbb{R}_\infty.$$

Consider also the natural morphisms

$$\begin{array}{ccc} M_\infty \times \mathbb{R}_\infty & \xrightarrow{j} & M_\infty \times \overline{\mathbb{R}} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \overline{\pi}} & \\ & M_\infty\ . & \end{array}$$

Definition 2.11. Let M_∞ be a bordered space. The functors

$$\overset{+}{\otimes}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}),$$
$$\mathscr{I}hom^{+}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$$

are defined by

$$K_1 \overset{+}{\otimes} K_2 = \mathrm{R}\mu_{!!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2),$$
$$\mathscr{I}hom^{+}(K_1, K_2) = \mathrm{R}q_{1*}\,\mathrm{R}\mathscr{I}hom\,(q_2^{-1}K_1, \mu^{!}\,K_2).$$

One sets

$$\mathbf{k}_{\{t\geq 0\}} = \mathbf{k}_{\left\{(x,t)\in\widehat{M}\times\overline{\mathbb{R}}\,;\,x\in M,\ t\in\mathbb{R},\ t\geq 0\right\}},$$

and one uses similar notations for $\mathbf{k}_{\{t=0\}}$, $\mathbf{k}_{\{t>0\}}$, $\mathbf{k}_{\{t\leq 0\}}$, $\mathbf{k}_{\{t<0\}}$, $\mathbf{k}_{\{t\neq 0\}}$, etc. These are sheaves on $\widehat{M}\times\overline{\mathbb{R}}$ whose stalks vanish on $(\widehat{M}\times\overline{\mathbb{R}})\setminus(M\times\mathbb{R})$. We also regard them as objects of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$.

The category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$ has a structure of commutative tensor category with $\overset{+}{\otimes}$ as tensor product and $\mathbf{k}_{\{t=0\}}$ as unit object.

One defines the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$:

$$\begin{aligned}
\mathrm{IC}_{\mathrm{t}^*=0} &= \left\{K\,;\,\pi^{-1}\mathrm{R}\pi_*K \xrightarrow{\sim} K\right\}\\
&= \left\{K\,;\,K \xrightarrow{\sim} \pi^{!}\,\mathrm{R}\pi_!K\right\}\\
&= \left\{K\,;\ \text{there exists } L\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \text{ with } \pi^{-1}L\simeq K\right\}\\
&= \left\{K\,;\,(\mathbf{k}_{\{t\geq 0\}}\oplus\mathbf{k}_{\{t\leq 0\}})\overset{+}{\otimes}K\simeq 0\right\}\\
&= \left\{K\,;\,\mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}}\oplus\mathbf{k}_{\{t\leq 0\}}, K)\simeq 0\right\}.
\end{aligned}$$

Definition 2.12. The triangulated category of *enhanced ind-sheaves*, denoted by $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$, is the quotient category

$$\begin{aligned}
\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) &= \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})/\mathrm{IC}_{\mathrm{t}^*=0}\\
&= \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})/\left\{\mathrm{K}\,;\,\pi^{-1}\mathrm{R}\pi_*\mathrm{K}\xrightarrow{\sim}\mathrm{K}\right\}.
\end{aligned}$$

Similarly, one defines the category $\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$ as

$$\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M) = \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})/\left\{K\,;\,\pi^{-1}\mathrm{R}\pi_*K\xrightarrow{\sim}K\right\}. \tag{2.9}$$

Then $\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$ is a full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$.

One has the equivalences

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \simeq {}^{\perp}\mathrm{IC}_{\mathrm{t}^*=0} \simeq (\mathrm{IC}_{\mathrm{t}^*=0})^{\perp}. \tag{2.10}$$

Hence, we may regard $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$ in two different ways.

Notation 2.13. The functors $\mathrm{L}^{\mathrm{E}}, \mathrm{R}^{\mathrm{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$ are given by:

$$\begin{aligned}\mathrm{L}^{\mathrm{E}} &= (\mathbf{k}_{\{t\geq 0\}} \oplus \mathbf{k}_{\{t\leq 0\}}) \overset{+}{\otimes} (\,\bullet\,),\\ \mathrm{R}^{\mathrm{E}} &= \mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}} \oplus \mathbf{k}_{\{t\leq 0\}}, \bullet\,).\end{aligned}$$

The functors L^{E} and R^{E} are the left and right adjoint of the quotient functor $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$, respectively. They are fully faithful. Note that for $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ we have

$$\mathrm{R}\pi_*\mathrm{R}^{\mathrm{E}}K \simeq 0, \qquad \mathrm{R}\pi_{!!}\mathrm{L}^{\mathrm{E}}K \simeq 0,$$
$$\mathrm{R}\pi_*\mathrm{L}^{\mathrm{E}}K \simeq \mathrm{R}\pi_{!!}\mathrm{R}^{\mathrm{E}}K.$$

2.5 Operations on enhanced ind-sheaves

As in the preceding subsection, we recall results of [DK13] with a slight generalisation, replacing usual spaces with bordered spaces.

The bifunctors

$$\overset{+}{\otimes} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}),$$
$$\mathscr{I}hom^{+} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$$

are those induced by the bifunctors $\overset{+}{\otimes}$ and $\mathscr{I}hom^{+}$ defined on $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty})$.

Note that for any $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ the composition $\mathbf{k}_{t\geq 0} \overset{+}{\otimes} K \to K \to \mathscr{I}hom^{+}(\mathbf{k}_{t\geq 0}, K)$ induces an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$

$$\mathbf{k}_{t\geq 0} \overset{+}{\otimes} K \xrightarrow{\sim} \mathscr{I}hom^{+}(\mathbf{k}_{t\geq 0}, K).$$

Such an isomorphism does not hold when replacing $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ with $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$.

Let $f\colon M_\infty \to N_\infty$ be a morphism of bordered spaces. Denote by $\tilde f\colon M_\infty\times\mathbb{R}_\infty \to N_\infty\times\mathbb{R}_\infty$ the associated morphism. Then the compositions of functors

$$\mathrm{R}\tilde f_{!!},\ \mathrm{R}\tilde f_{*}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty}), \tag{2.11}$$

$$\tilde f^{-1},\ \tilde f^{\,!}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \tag{2.12}$$

factor through $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ and $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty})$, respectively.

Definition 2.14. One denotes by

$$\begin{aligned}\mathsf{E}f_{!!},\ \mathsf{E}f_{*}\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty}),\\ \mathsf{E}f^{-1},\ \mathsf{E}f^{\,!}\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})\end{aligned}$$

the functors induced by (2.11) and (2.12), respectively.

The above operations satisfy analogous properties to the operations for sheaves.

One also defines similarly the external product functor

$$\begin{aligned}\bullet \overset{+}{\boxtimes} \bullet \quad &: \quad \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty\times\mathrm{N}_\infty}),\\ &F \overset{+}{\boxtimes} G = \mathsf{E}p_1^{-1}F \overset{+}{\otimes} \mathsf{E}p_2^{-1}G,\end{aligned}$$

where p_1 and p_2 denote the projections from $M_\infty \times N_\infty$ to M_∞ and N_∞, respectively.

Definition 2.15. One defines the hom-functor

$$\begin{aligned}\mathscr{I}hom^{\mathrm{E}}\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})\\ \mathscr{I}hom^{\mathrm{E}}(K_1,K_2) = \mathrm{R}\pi_*\,\mathrm{R}\mathscr{I}hom\,(\mathrm{L}^{\mathrm{E}}(K_1),\mathrm{R}^{\mathrm{E}}(K_2)),\end{aligned} \tag{2.13}$$

and one sets

$$\begin{aligned}&\mathscr{H}om^{\mathrm{E}} = \alpha_M \circ \mathscr{I}hom^{\mathrm{E}} \ :\ \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathrm{M}}),\\ &\mathrm{RHom}^{\mathrm{E}}(K_1,K_2) = \mathrm{R}\Gamma(M;\mathscr{H}om^{\mathrm{E}}(K_1,K_2)).\end{aligned}$$

Note that

$$\begin{aligned}\mathscr{I}hom^{\mathrm{E}}(K_1,K_2) &\simeq \mathrm{R}\pi_*\,\mathrm{R}\mathscr{I}hom\,(\mathrm{L}^{\mathrm{E}}(K_1),\mathrm{L}^{\mathrm{E}}(K_2))\\ &\simeq \mathrm{R}\pi_*\,\mathrm{R}\mathscr{I}hom\,(\mathrm{R}^{\mathrm{E}}(K_1),\mathrm{R}^{\mathrm{E}}(K_2))\end{aligned}$$

and

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})}(K_1,K_2) \simeq H^0\big(\mathrm{RHom}^{\mathrm{E}}(K_1,K_2)\big).$$

Remark 2.16. For $x \in M$, let ι_x be the embedding $\mathrm{pt} \hookrightarrow M$ given by $\iota_x(\mathrm{pt}) = x$. Let $F \in \mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$. Then $F \simeq 0$ if and only if $\mathsf{E}\iota_x^{-1}F \simeq 0$ for all $x \in M$.

2.6 The functor $\mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} (\,\bullet\,)$

Consider the objects of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty \times \mathbb{R}_\infty})$

$$\mathbf{k}_{\{t \gg 0\}} := \text{“}\varinjlim_{a \to +\infty}\text{”}\, \mathbf{k}_{\{t \geq a\}}, \qquad \mathbf{k}_{\{t < *\}} := \text{“}\varinjlim_{a \to +\infty}\text{”}\, \mathbf{k}_{\{t < a\}}.$$

There are a distinguished triangle and isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty \times \mathbb{R}_\infty})$

$$\begin{aligned}
&\mathbf{k}_{M \times \mathbb{R}} \to \mathbf{k}_{\{t \gg 0\}} \to \mathbf{k}_{\{t < *\}}[1] \xrightarrow{+1}, \\
&\mathbf{k}_{\{t \geq -a\}} \overset{+}{\otimes} \mathbf{k}_{\{t \gg 0\}} \xrightarrow{\sim} \mathbf{k}_{\{t \gg 0\}} \xrightarrow{\sim} \mathbf{k}_{\{t \geq a\}} \overset{+}{\otimes} \mathbf{k}_{\{t \gg 0\}}, \quad (a \in \mathbb{R}_{\geq 0}).
\end{aligned}$$

Denote by $\mathbf{k}_{M_\infty}^{\mathrm{E}}$ the object of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ associated with the ind-sheaf $\mathbf{k}_{\{t \gg 0\}} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty \times \mathbb{R}_\infty})$. Note that

$$\mathrm{L}^{\mathrm{E}}(\mathbf{k}_{M_\infty}^{\mathrm{E}}) \simeq \mathbf{k}_{\{t \gg 0\}}, \quad \mathrm{R}^{\mathrm{E}}(\mathbf{k}_{M_\infty}^{\mathrm{E}}) \simeq \mathbf{k}_{\{t < *\}}[1].$$

Lemma 2.17. *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_\infty \times \mathbb{R}_\infty})$ and $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$, there is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$*

$$\mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} \mathscr{I}hom^{+}(F, K) \xrightarrow{\sim} \mathscr{I}hom^{+}(F, \mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} K).$$

Proposition 2.18. *Let $f \colon M_\infty \to N_\infty$ be a morphism of bordered spaces.*

(i) *For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$ one has*

$$\mathsf{E}f_{!!}(\mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} K) \simeq \mathbf{k}_{N_\infty}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f_{!!}K.$$

(ii) *For $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{N}_\infty})$ one has*

$$\begin{aligned}
\mathsf{E}f^{-1}(\mathbf{k}_{N_\infty}^{\mathrm{E}} \overset{+}{\otimes} L) &\simeq \mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f^{-1}L, \\
\mathsf{E}f^{!}(\mathbf{k}_{N_\infty}^{\mathrm{E}} \overset{+}{\otimes} L) &\simeq \mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f^{!}L.
\end{aligned}$$

Definition 2.19. One defines the functors

$$(2.14)\qquad \begin{aligned} &e_M, \varepsilon_M \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty}),\\ &\qquad e_M(F) = \mathbf{k}^{\mathrm{E}}_{M_\infty} \otimes \pi^{-1}F, \quad \varepsilon_M(F) = \mathbf{k}_{\{t\geq 0\}} \otimes \pi^{-1}F. \end{aligned}$$

Note that

$$e_M(F) \simeq \mathbf{k}^{\mathrm{E}}_{M_\infty} \overset{+}{\otimes} \varepsilon_M(F).$$

Proposition 2.20. *The functors e_M and ε_M are fully faithful.*

2.7 Enhanced ind-sheaves over an algebra

We need to generalize some definitions and results of the preceding subsections to the case where $\mathbf{k}$ is replaced by a sheaf of algebras. For simplicity we only consider the case where M is a good topological space, not a bordered space.

In the sequel, if $\mathscr{A}$ is a sheaf of $\mathbf{k}$-algebras on a space X, we shall denote by $\mathsf{D}^{\mathrm{b}}(\mathrm{I}(\mathscr{A}))$, or simply $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A})$, the derived category of ind-sheaves of $\mathscr{A}$-modules on X (see [KS01]). (In [KS01], it was denoted by $\mathsf{D}^{\mathrm{b}}(\mathrm{I}(\beta\mathscr{A}))$.)

Consider a good topological space M and the bordered space $M \times \mathbb{R}_\infty$. As above, we denote by $\pi\colon M \times \mathbb{R}_\infty \to M$ and $\overline{\pi}\colon M \times \overline{\mathbb{R}} \to M$ the projections. Assume to be given a $\mathbf{k}$-flat sheaf of algebras $\mathscr{A}$ on M. For short, we set

$$\mathscr{A}_{M\times\overline{\mathbb{R}}} = \overline{\pi}^{-1}\mathscr{A}.$$

One defines the categories:

$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) &= \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\overline{\mathbb{R}}}) / \left\{\mathrm{F} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\overline{\mathbb{R}}})\,;\, \mathrm{F} \otimes \mathbf{k}_{\mathrm{M}\times\mathbb{R}} \simeq 0\right\},\\ \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}) &= \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) / \left\{\mathrm{K}\,;\, \pi^{-1}\mathrm{R}\pi_*\mathrm{K} \xrightarrow{\sim} \mathrm{K}\right\}. \end{aligned}$$

We keep the notations q_1, q_2, μ as in (2.8) and denote by $q\colon M \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M$ the projection.

Definition 2.21. One defines the functors

$$(2.15)\qquad \begin{aligned} &\bullet \overset{+}{\otimes}_{\mathscr{A}} \bullet \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}^{\mathrm{op}}_{\mathrm{M}\times\mathbb{R}_\infty}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty})\\ &\qquad \mathscr{N} \overset{+}{\otimes}_{\mathscr{A}} \mathscr{M} := \mathrm{R}\mu_{!!}(q_1^{-1}\mathscr{N} \overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{A}} q_2^{-1}\mathscr{M}), \end{aligned}$$

$$(2.16)\qquad \begin{aligned} &\mathscr{I}hom^+_{\mathscr{A}}(\bullet, \bullet) \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty})\\ &\qquad \mathscr{I}hom^+_{\mathscr{A}}(\mathscr{M}_1, \mathscr{M}_2) := \mathrm{R}q_{1*}\,\mathrm{R}\mathscr{I}hom_{q^{-1}\mathscr{A}}(q_2^{-1}\mathscr{M}_1, \mu^!\,\mathscr{M}_2). \end{aligned}$$

If $\mathscr{A}$ is commutative, these functors take their values in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty})$.

On easily checks that the functors (2.15) and (2.16) induce functors (we keep the same notation):

$$\bullet \overset{+}{\otimes}_{\mathscr{A}} \bullet \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}^{\mathrm{op}}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}), \tag{2.17}$$

$$\mathscr{I}hom^{+}_{\mathscr{A}}(\bullet, \bullet) \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}). \tag{2.18}$$

If $\mathscr{A}$ is commutative, these functors take their values in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A})$.

We shall also need to consider the functors:

$$\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \bullet \colon \mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\mathrm{op}}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty}) \tag{2.19}$$

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\bullet, \bullet) \colon \mathsf{D}^{\mathrm{b}}(\mathscr{A})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}\times\mathbb{R}_\infty}) \tag{2.20}$$

which induce functors

$$\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \bullet \colon \mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\mathrm{op}}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}) \tag{2.21}$$

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\bullet, \bullet) \colon \mathsf{D}^{\mathrm{b}}(\mathscr{A})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}). \tag{2.22}$$

If $\mathscr{A}$ is commutative, these functors take their values in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}_{\mathrm{M}\times\mathbb{R}_\infty})$ or in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{A})$.

3 Holomorphic solutions of $\mathscr{D}$-modules

3.1 $\mathscr{D}$-modules

Reference are made to [Ka03] for the theory of $\mathscr{D}$-modules. The aim of this subsection is simply to fix a few notations.

Let $(X, \mathscr{O}_X)$ be a *complex* manifold. We introduce the following notations (most of them are classical).

- d_X is the complex dimension of X, $\mathscr{D}_X$ the sheaf of $\mathbb{C}$-algebras of holomorphic finite-order differential operators, Ω_X the invertible sheaf of differential forms of top degree, $\mathrm{Mod}(\mathscr{D}_X)$ the category of left $\mathscr{D}_X$-modules, $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ its bounded derived category.

- $\mathrm{r}\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$ is the equivalence of categories given by

$$\mathscr{M}^{\mathrm{r}} = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{M}.$$

- $\overset{\mathrm{D}}{\otimes}$ and $\overset{\mathrm{D}}{\boxtimes}$ are the (derived) operations of tensor product and external product for $\mathscr{D}$-modules. Recall that $\mathscr{N} \overset{\mathrm{D}}{\otimes} \mathscr{M} = \mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{O}_X)$.

- For $f\colon X \to Y$ a morphism of complex manifolds, $\mathrm{D}f^*$ and $\mathrm{D}f_*$ are the (derived) operations of inverse image and direct images for $\mathscr{D}$-modules.

- The dual of $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ is given by

$$ {}^{\mathrm{D}}\mathbb{D}_X \mathscr{M} = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X]. $$

- $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, $\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ are the full subcategories of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have coherent, quasi-good and good cohomologies, respectively. Here, a $\mathscr{D}_X$-module $\mathscr{M}$ is called *quasi-good* if, for any relatively compact open subset $U \subset X$, $\mathscr{M}|_U$ is a sum of coherent $(\mathscr{O}_X|_U)$-submodules. A $\mathscr{D}_X$-module $\mathscr{M}$ is called *good* if it is quasi-good and coherent.

- For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, $\mathrm{char}(\mathscr{M})$ is its characteristic variety, a closed conic involutive subset of the cotangent bundle T^*X. If $\mathrm{char}(\mathscr{M})$ is Lagrangian, $\mathscr{M}$ is called holonomic. For the notion of regular holonomic $\mathscr{D}_X$-module, refer e.g. to [Ka03, §5.2]. We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ the full subcategories of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have holonomic and regular holonomic cohomologies, respectively.

Note that $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, $\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$, $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ are triangulated categories.

If $Y \subset X$ is a closed hypersurface, denote by $\mathscr{O}_X(*Y)$ the sheaf of meromorphic functions with poles at Y. It is a regular holonomic $\mathscr{D}_X$-module. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, set

$$ \mathscr{M}(*Y) = \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X(*Y). $$

3.2 Tempered functions and distributions

In this subsection we recall some results of [KS96, KS01]. Here, M is a real analytic manifold and $\mathbf{k} = \mathbb{C}$. As usual, we denote by $\mathscr{C}^\infty_M$ (resp. $\mathscr{C}^\omega_M$) the sheaf of $\mathbb{C}$-valued functions of class C^∞ (resp. real analytic), by $\mathcal{D}b_M$ (resp. $\mathscr{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathscr{D}_M$ the sheaf of real analytic finite-order differential operators.

Definition 3.1. Let U be an open subset of M and $f \in \mathscr{C}_M^\infty(U)$. One says that f has *polynomial growth* at $p \in M$ if it satisfies the following condition. For a local coordinate system $(x_1, \ldots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,, \tag{3.1}$$

with the convention that if $K \cap U = \emptyset$ or if $K \subset U$, then the left-hand side of (3.1) is understood to be 0. It is obvious that f has polynomial growth at any point of $U \cup (M \setminus \overline{U})$. We say that f is *tempered* at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point.

For an open subanalytic set U in M, denote by $\mathcal{C}_M^{\infty,\mathrm{t}}(U)$ the subspace of $\mathscr{C}_M^\infty(U)$ consisting of tempered C^∞-functions.

Denote by $\mathcal{D}b_M^{\mathrm{t}}(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{M \setminus U}(M; \mathcal{D}b_M) \to \Gamma(M; \mathscr{D}b_M) \to \mathcal{D}b_M^{\mathrm{t}}(U) \to 0.$$

Using Lojasiewicz's inequalities, one easily proves that

- the presheaf $\mathcal{C}_M^{\infty,\mathrm{t}} := U \mapsto \mathcal{C}_M^{\infty,\mathrm{t}}(U)$ is a sheaf on M_{sa}, hence an ind-sheaf on M. One calls it the *ind-sheaf of tempered* C^∞*-functions*.
- the presheaf $\mathcal{D}b_M^{\mathrm{t}} := U \mapsto \mathcal{D}b_M^{\mathrm{t}}(U)$ is a sheaf on M_{sa}, hence an ind-sheaf on M. One calls it the *ind-sheaf of tempered distributions*.

Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. One has the isomorphism

$$\alpha_M \, \mathrm{R}\mathscr{I}hom\,(F, \mathcal{D}b_M^{\mathrm{t}}) \simeq Thom(F, \mathcal{D}b_M), \tag{3.2}$$

where the right-hand side was defined by Kashiwara as the main tool for his proof of the Riemann-Hilbert correspondence in [Ka80, Ka84].

Definition 3.2 (See [DK13, Def. 5.4.1])**.** The category of *real analytic bordered spaces* is defined as follows.
The objects are pairs $(M, \widehat{M})$ where $\widehat{M}$ is a real analytic manifold and $M \subset \widehat{M}$ is an open subanalytic subset.
Morphisms $f \colon (M, \widehat{M}) \to (N, \widehat{N})$ are real analytic maps $f \colon M \to N$ such that

(i) Γ_f is a subanalytic subset of $\widehat{M} \times \widehat{N}$, and

(ii) $\overline{\Gamma}_f \to \widehat{M}$ is proper.

Hence a morphism of real analytic bordered spaces is a morphism of bordered spaces. Recall that $j_M \colon (M, \widehat{M}) \to \widehat{M}$ denotes the natural morphism.

Definition 3.3. Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space. One sets $\mathcal{D}b^{\mathrm{t}}_{M_\infty} := j_M^{-1} \mathcal{D}b^{\mathrm{t}}_{\widehat{M}}$.

If $f \colon M_\infty \to N_\infty$ is an isomorphism of real analytic bordered spaces, then $\mathcal{D}b^{\mathrm{t}}_{M_\infty} \simeq f^{-1} \mathcal{D}b^{\mathrm{t}}_{N_\infty}$ as object of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}_\infty})$.

We say that S is a subanalytic subset of M_∞ if S is a subset of M subanalytic in $\widehat{M}$.

3.3 Holomorphic functions with tempered growth

References for this subsection are made to [KS01]. We slightly change our notations and write $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \bullet)$ instead of $\mathrm{R}\mathscr{H}om_{\beta_X \mathscr{D}_X}(\beta_X \mathscr{M}, \bullet)$.

One defines the ind-sheaf of tempered holomorphic functions $\mathcal{O}^{\mathrm{t}}_X$ as the Dolbeault complex with coefficients in $\mathcal{C}^{\infty,\mathrm{t}}_X$. More precisely, denoting by X^c the complex conjugate manifold to X and by $X_\mathbb{R}$ the underlying real analytic manifold, we set:

$$\mathcal{O}^{\mathrm{t}}_X = \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c}}(\mathscr{O}_{X^c}, \mathcal{C}^{\infty,\mathrm{t}}_{X_\mathbb{R}}). \tag{3.3}$$

One proves the isomorphism

$$\mathcal{O}^{\mathrm{t}}_X \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c}}(\mathscr{O}_{X^c}, \mathcal{D}b^{\mathrm{t}}_{X_\mathbb{R}}). \tag{3.4}$$

Note that the object $\mathcal{O}^{\mathrm{t}}_X$ is not concentrated in degree zero in dimension > 1. Indeed, with the subanalytic topology, only finite coverings are allowed. If one considers for example the open set $U \subset \mathbb{C}^n$, the difference of an open ball of radius $R > 0$ and a closed ball of radius $0 < r < R$, then the Dolbeault complex will not be exact after any finite covering.

Still denote by $\mathscr{O}_X$ the image of this sheaf in $\mathrm{Mod}(\mathrm{I}\mathbb{C}_\mathrm{X})$. We have then the morphism in the category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_\mathrm{X})$:

$$\mathcal{O}^{\mathrm{t}}_X \to \mathscr{O}_X.$$

Example 3.4. Let Z be a closed complex analytic subset of the complex manifold X and let M be a real analytic manifold such that X is a complexification of M. We have the isomorphisms

$$\begin{aligned}
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathbb{C}_Z, \mathcal{O}_X^{\mathrm{t}}) \simeq \mathrm{R}\Gamma_{[Z]}(\mathscr{O}_X) \text{ (algebraic cohomology)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathbb{C}_Z, \mathscr{O}_X) \simeq \mathrm{R}\Gamma_Z(\mathscr{O}_X),\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}_X^{\mathrm{t}}) \simeq \mathcal{D}b_M,\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X) \simeq \mathscr{B}_M,
\end{aligned}$$

where $\mathrm{D}'_X \colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ is the duality functor $\mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbb{C}_X)$.

Notice that with this approach, the sheaf $\mathcal{D}b_M$ of Schwartz's distributions is constructed similarly as the sheaf $\mathscr{B}_M$ of Sato's hyperfunctions.

The classical de Rham and solution functors are given by

$$\begin{aligned}
&\mathcal{DR}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), && \mathscr{M} \mapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M},\\
&\mathcal{S}ol_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), && \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X),
\end{aligned}$$

and the tempered de Rham and solution functors are

$$\begin{aligned}
&\mathcal{DR}_X^{\mathrm{t}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), && \mathscr{M} \mapsto \Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M},\\
&\mathcal{S}ol_X^{\mathrm{t}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), && \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X^{\mathrm{t}}).
\end{aligned}$$

One has

$$\mathcal{S}ol_X \simeq \alpha_X \mathcal{S}ol_X^{\mathrm{t}}, \quad \mathcal{DR}_X \simeq \alpha_X \mathcal{DR}_X^{\mathrm{t}},$$

and it follows from [Ka84] that for $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ one has

$$\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{L}) \simeq \mathcal{S}ol_X(\mathscr{L}), \quad \mathcal{DR}_X^{\mathrm{t}}(\mathscr{L}) \simeq \mathcal{DR}_X(\mathscr{L}).$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has

$$\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}) \simeq \mathcal{DR}_X^{\mathrm{t}}({}^{\mathrm{D}}\mathbb{D}_X\mathscr{M})[-d_X].$$

Let us recall some functorial properties of the tempered de Rham and solution functors.

Theorem 3.5 ([KS01, Theorems 7.4.1, 7.4.6 and 7.4.12])**.** *Let $f \colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}(f^{-1}\mathscr{D}_Y))$

$$f^{\,!}\,\mathcal{O}_Y^{\mathrm{t}}[d_Y] \simeq \mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}_X^{\mathrm{t}}[d_X].$$

(ii) *For any* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$*, there is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$

$$\mathcal{DR}_X^{\mathrm{t}}(\mathrm{D}f^*\mathscr{N})[d_X] \simeq f^{\,!}\,\mathcal{DR}_Y^{\mathrm{t}}(\mathscr{N})[d_Y].$$

(iii) *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$*, and assume that* $\operatorname{supp}\mathscr{M}$ *is proper over* Y*. Then there is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$

$$\mathcal{DR}_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}) \simeq \mathrm{R}f_{!!}\mathcal{DR}_X^{\mathrm{t}}(\mathscr{M}).$$

(iv) *Let* $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$*. Then there is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}})$

$$\mathcal{O}_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{L} \simeq \mathrm{R}\mathscr{I}\!hom\,(\mathcal{S}ol_X(\mathscr{L}), \mathcal{O}_X^{\mathrm{t}}).$$

In particular, for a closed hypersurface $Y \subset X$*, one has*

$$\mathcal{O}_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}_X(*Y) \simeq \mathrm{R}\mathscr{I}\!hom\,(\mathbb{C}_{X\setminus Y}, \mathcal{O}_X^{\mathrm{t}}).$$

3.4 Enhanced solutions of $\mathscr{D}$-modules

References for this subsection are made to [DK13].

Let X be a complex analytic manifold, $Y \subset X$ a complex analytic hypersurface and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}_X(*Y)$, one sets

$$\begin{aligned} \mathscr{D}_X \mathrm{e}^{\varphi} &= \mathscr{D}_X / \{P\,;\, P\mathrm{e}^{\varphi} = 0 \text{ on } U\}\,, \\ \mathscr{E}^{\varphi}_{U|X} &= \mathscr{D}_X \mathrm{e}^{\varphi}(*Y). \end{aligned}$$

Hence $\mathscr{D}_X \mathrm{e}^{\varphi}$ is a $\mathscr{D}_X$-submodule of $\mathscr{E}^{\varphi}_{U|X}$ and $\mathscr{E}^{\varphi}_{U|X}$ is a holonomic $\mathscr{D}_X$-module. Moreover

$$({}^{\mathrm{D}}\mathbb{D}_X \mathscr{E}^{\varphi}_{U|X})(*Y) \simeq \mathscr{E}^{-\varphi}_{U|X}. \tag{3.5}$$

For $c \in \mathbb{R}$, set for short

$$\{\operatorname{Re}\varphi < c\} := \{x \in U\,;\, \operatorname{Re}\varphi(x) < c\} \subset X.$$

Notation 3.6. One sets

$$\mathbb{C}_{\{\operatorname{Re}\varphi<*\}} := \text{“}\varinjlim_{c\to+\infty}\text{”}\, \mathbb{C}_{\{\operatorname{Re}\varphi<c\}} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}),$$
$$E^{\varphi}_{U|X} := \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathbb{C}_{\{\operatorname{Re}\varphi<*\}}) \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}).$$

The next result (see [DK13, Prop. 6.2.2]) generalizes [KS03, Proposition 7.3] in which the case $X = \mathbb{C}$ and $\varphi(z) = 1/z$ was treated.

Proposition 3.7. *Let $Y \subset X$ be a closed complex analytic hypersurface, and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}_X(*Y)$, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$*

$$\mathcal{DR}^{\mathrm{t}}_X(\mathscr{E}^{-\varphi}_{U|X}) \simeq E^{\varphi}_{U|X}[d_X].$$

Recall that we have set $\mathsf{P} := \mathbb{P}^1(\mathbb{R})$. In the sequel, one sets for short

$$\mathbb{P} := \mathbb{P}^1(\mathbb{C}).$$

We denote by $\tau \in \mathbb{C} \subset \mathbb{P}$ the affine coordinate such that $\tau|_{\mathbb{R}} = t$, the affine coordinate of $\mathbb{R}$.

Consider the natural morphism of bordered spaces

$$i \colon X \times \mathbb{R}_\infty \to X \times \mathbb{P}.$$

Recall that $\mathrm{r} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathrm{op}}_{\mathbb{P}})$ is the functor given by $\mathscr{M}^{\mathrm{r}} = \Omega_{\mathbb{P}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{\mathbb{P}}} \mathscr{M}$.

Definition 3.8. One sets:

$$\mathscr{O}^{\mathsf{E}}_X = i^!\,((\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}})[1] \simeq i^!\,\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathbb{P}}}(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}})[2] \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}),$$
$$\Omega^{\mathsf{E}}_X = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}^{\mathsf{E}}_X \simeq i^!\,(\Omega^{\mathrm{t}}_{X\times\mathbb{P}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} \mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})[1] \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}^{\mathrm{op}}_{\mathrm{X}}).$$

One defines the *enhanced de Rham functor* and the *enhanced solution functor* by

$$\mathcal{DR}^{\mathsf{E}}_X \;:\; \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \quad \mathscr{M} \mapsto \Omega^{\mathsf{E}}_{\mathrm{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathrm{X}}} \mathscr{M},$$
$$\mathcal{S}ol^{\mathsf{E}}_X \;:\; \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}\mathrm{om}_{\mathscr{D}_{\mathrm{X}}}(\mathscr{M}, \mathscr{O}^{\mathsf{E}}_{\mathrm{X}}).$$

One defines similarly the functors $\mathcal{DR}^{\mathsf{E}}_X$ and $\mathcal{S}ol^{\mathsf{E}}_X$ for right modules.

Note that

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}) \simeq \mathcal{DR}^{\mathsf{E}}_X({}^{\mathrm{D}}\mathbb{D}_X\mathscr{M})[-d_X] \quad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X).$$

Theorem 3.9. *There is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}\times\mathbb{R}_\infty})$

$$\mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X \simeq i^{!}\big((\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}}\big)[1],$$

and there are isomorphisms in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}})$

$$\begin{aligned} \mathscr{O}^{\mathsf{E}}_X &\xrightarrow{\sim} \mathscr{I}hom^{+}(\mathbb{C}_{\{t\geq 0\}}, \mathscr{O}^{\mathsf{E}}_X) \\ &\xleftarrow{\sim} \mathscr{I}hom^{+}(\mathbb{C}_{\{t\geq a\}}, \mathscr{O}^{\mathsf{E}}_X) \quad \text{for any } a \geq 0 \\ &\simeq \mathscr{I}hom^{+}(\mathbb{C}^{\mathsf{E}}_X, \mathscr{O}^{\mathsf{E}}_X) \simeq \mathbb{C}^{\mathrm{E}}_X \overset{+}{\otimes} \mathscr{O}^{\mathsf{E}}_X. \end{aligned}$$

Let $\varphi \in \mathscr{O}_X(*Y)$ be as above. By [DK13, Corollary 9.4.12], one has:

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{E}^{-\varphi}_{U|X}) \simeq \mathbb{C}^{\mathrm{E}}_X \overset{+}{\otimes} \mathbb{C}_{\{t=\mathrm{Re}\,\varphi\}} \simeq \text{“}\varinjlim_{a\to+\infty}\text{”}\ \mathbb{C}_{\{t\geq \mathrm{Re}\,\varphi+a\}}. \tag{3.6}$$

Here $\{t = \mathrm{Re}\,\varphi\}$ denotes $\{(x,t) \in M \times \mathbb{R}\,;\, x \in U,\ t = \mathrm{Re}\,\varphi(x)\}$ and similarly for $\{t \geq \mathrm{Re}\,\varphi + a\}$.

One can recover $\mathcal{O}^{\mathrm{t}}_X$ from $\mathscr{O}^{\mathsf{E}}_X$. Indeed, one has:

Proposition 3.10. *For* $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$*, one has the isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$

$$\begin{aligned} \mathrm{R}\mathscr{I}hom(F, \mathcal{O}^{\mathrm{t}}_X) &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq 0\}} \otimes \pi^{-1}F, \mathscr{O}^{\mathsf{E}}_X) \\ &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t=0\}} \otimes \pi^{-1}F, \mathscr{O}^{\mathsf{E}}_X) \\ &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}^{\mathrm{E}}_X \otimes \pi^{-1}F, \mathscr{O}^{\mathsf{E}}_X). \end{aligned}$$

In particular, we have

$$\mathcal{O}^{\mathrm{t}}_X \simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq 0\}}, \mathscr{O}^{\mathsf{E}}_X).$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ recall that one sets

$$\mathcal{S}ol_X(\mathscr{M}) = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X).$$

The next result follows from Theorem 3.5.

Theorem 3.11. *Let $f\colon X\to Y$ be a morphism of complex manifolds.*

(i) *There is an isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}(f^{-1}\mathscr{D}_Y))$

$$\mathsf{E} f^{\,!}\mathscr{O}_Y^{\mathsf{E}}\,[d_Y]\simeq\mathscr{D}_{Y\leftarrow X}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathscr{O}_X^{\mathsf{E}}\,[d_X].$$

(ii) *Let $\mathscr{N}\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$. There is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$, functorial in $\mathscr{N}$:*

$$\mathcal{DR}_X^{\mathsf{E}}(\mathrm{D}f^*\mathscr{N})\,[d_X]\simeq\mathsf{E} f^{\,!}\mathcal{DR}_Y^{\mathsf{E}}(\mathscr{N})\,[d_Y].$$

If moreover $\mathscr{N}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$, there is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathrm{D}f^*\mathscr{N})\simeq\mathsf{E} f^{-1}\mathcal{S}ol_Y^{\mathsf{E}}(\mathscr{N}).$$

(iii) *Let $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, and assume that $\operatorname{supp}\mathscr{M}$ is proper over Y. There is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$, functorial in $\mathscr{M}$:*

$$\mathcal{DR}_Y^{\mathsf{E}}(\mathrm{D}f_*\mathscr{M})\simeq\mathsf{E} f_{!!}\mathcal{DR}_X^{\mathsf{E}}(\mathscr{M}).$$

If moreover $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$, then

$$\mathcal{S}ol_Y^{\mathsf{E}}(\mathrm{D}f_*\mathscr{M})\,[d_Y]\simeq\mathsf{E} f_{!!}\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M})\,[d_X].$$

(iv) *Let $\mathscr{L}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ and $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$. Then*

$$\mathcal{DR}_X^{\mathsf{E}}(\mathscr{L}\overset{\mathrm{D}}{\otimes}\mathscr{M})\simeq\mathrm{R}\mathscr{I}hom\,(\pi^{-1}\mathcal{S}ol_X(\mathscr{L}),\mathcal{DR}_X^{\mathsf{E}}(\mathscr{M})).$$

The next result will be of constant use in the next section.

Lemma 3.12. *Let $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$, $\mathscr{L}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, $\mathscr{K}\in\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$ and assume that $\mathscr{I}hom^+(\mathbb{C}_X^{\mathrm{E}},\mathscr{K})\simeq\mathscr{K}$. Then we have the natural isomorphism*

$$\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}),\mathscr{K})\xrightarrow{\sim}\mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}),\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathscr{K}).$$

Proof. Since the morphism is well-defined, we may argue locally. We know by [DK13] that there exists an object $F\in\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times\mathbb{R}})$ such that

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L})\simeq F\overset{+}{\otimes}\mathbb{C}_X^{\mathrm{E}}.$$

It follows from the hypothesis on $\mathscr{K}$ that

$$\mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathscr{K}) \simeq \mathscr{I}hom^+(F, \mathscr{K}),$$
$$\mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \simeq \mathscr{I}hom^+(F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).$$

Indeed, we have

$$\mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \simeq \mathscr{I}hom^+\big(F, \mathscr{I}hom^+(\mathbb{C}_X^{\mathrm{E}}, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K})\big).$$

By [DK13, Proposition 4.7.5], we have $\mathbb{C}_X^{\mathrm{E}} \otimes (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \simeq \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathbb{C}_X^{\mathrm{E}} \otimes \mathscr{K}) \simeq \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}$, and $\mathscr{I}hom^+(\mathbb{C}_X^{\mathrm{E}}, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \simeq \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}$.

Hence, it is enough to prove the isomorphism

$$\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^+(F, \mathscr{K}) \xrightarrow{\sim} \mathscr{I}hom^+(F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).$$

Let $\bar{q}_k \colon X \times \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to X \times \overline{\mathbb{R}}$ ($k = 1, 2$) be the projection, $\sigma \colon X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$ the map $(t_1, t_2) \mapsto t_2 - t_1$ and $i \colon X \times \mathbb{R} \times \mathbb{R} \to X \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ the inclusion. Then we have

$$\begin{aligned}
\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^+(F, \mathscr{K}) &\simeq \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathrm{R}\bar{q}_{1!!} \mathrm{R}\mathscr{I}hom\,(\mathrm{R}i_! \sigma^{-1} F, \bar{q}_2^{\,!} \mathscr{K}) \\
&\underset{(1)}{\simeq} \mathrm{R}\bar{q}_{1!!}\Big(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathrm{R}\mathscr{I}hom\,(\mathrm{R}i_! \sigma^{-1} F, \bar{q}_2^{\,!} \mathscr{K})\Big) \\
&\underset{(2)}{\simeq} \mathrm{R}\bar{q}_{1!!} \mathrm{R}\mathscr{I}hom\,(\mathrm{R}i_! \sigma^{-1} F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \bar{q}_2^{\,!} \mathscr{K}) \\
&\underset{(3)}{\simeq} \mathrm{R}\bar{q}_{1!!} \mathrm{R}\mathscr{I}hom\,\big(\mathrm{R}i_! \sigma^{-1} F, \bar{q}_2^{\,!}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K})\big) \\
&\simeq \mathscr{I}hom^+(F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).
\end{aligned}$$

Here the isomorphisms (1), (2) and (3) follow from Theorem 5.5.4, Theorem 5.6.1 (ii) and Theorem 5.6.3 in [KS01], respectively. Q.E.D.

4 Integral transform for De Rham

4.1 An enhanced Riemann-Hilbert correspondence

Theorem 7.4.12 of [KS01] (that is, Theorem 3.5 (iv) in this paper) is a reformulation of a result of Björk [Bj93]. The aim of this subsection is to extend this result to the case where $\mathscr{L}$ is holonomic but not necessarily regular. Note that we do not any more use Björk's result in the proof of Theorem 4.5 below. As we shall see, this last theorem generalizes the reconstruction theorem (Riemann-Hilbert) of [DK13] but of course, its proof deeply uses the tools of loc. cit.

Lemma 4.1. *Let $\iota\colon X \to Y$ be a closed embedding of complex manifolds. There is a natural isomorphism*

$$\mathsf{E}\,\iota^{-1}(\mathscr{D}_{X\to Y}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y}\mathscr{O}_Y^{\mathsf{E}}) \simeq \mathscr{O}_X^{\mathsf{E}}.$$

Proof. Applying Theorem 3.11 (iii) with $\mathscr{M} = \mathscr{D}_X$ we get

$$\mathsf{E}\iota_*\mathscr{O}_X^{\mathsf{E}} \simeq \mathscr{D}_{X\to Y}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y}\mathscr{O}_Y^{\mathsf{E}},$$

and the result follows. Q.E.D.

Lemma 4.2. *There is a canonical morphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}})$

$$\mathscr{O}_X^{\mathsf{E}}\overset{+}{\otimes}_{\mathscr{O}_X}\mathscr{O}_X^{\mathsf{E}} \to \mathscr{O}_X^{\mathsf{E}}.$$

Proof. Consider the diagram in which q_k and $\widetilde{p}_k$ ($k = 1,2$) denote the projections.

$$\begin{array}{ccccc}
X & \xleftarrow{\;q\;} & X\times\mathbb{R}_\infty\times\mathbb{R}_\infty & \overset{q_k}{\underset{\mu}{\rightrightarrows}} & X\times\mathbb{R}_\infty \\
 & & \downarrow{\scriptstyle\widetilde{\delta}} & & \downarrow{\scriptstyle\delta} \\
 & & X\times\mathbb{R}_\infty\times X\times\mathbb{R}_\infty & \xrightarrow{\;\widetilde{\mu}\;} & X\times X\times\mathbb{R}_\infty \\
 & & \downarrow{\scriptstyle p} & \searrow^{\widetilde{p}_k} & \\
 & & X\times X & & X\times\mathbb{R}_\infty.
\end{array}$$

Denote by $q\colon X\times\mathbb{R}_\infty\times\mathbb{R}_\infty \to X$ the projection. One has

$$\begin{aligned}
&q_1^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{O}_X} q_2^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \\
&\qquad\qquad \simeq (q_1^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\otimes q_2^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}})\overset{\mathrm{L}}{\otimes}_{q^{-1}(\mathscr{O}_X\otimes\mathscr{O}_X)} q^{-1}\mathscr{O}_X \\
&\qquad\qquad \simeq \widetilde{\delta}^{-1}\big(\widetilde{p}_1^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\otimes\widetilde{p}_2^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\big)\overset{\mathrm{L}}{\otimes}_{q^{-1}(\mathscr{O}_X\otimes\mathscr{O}_X)} q^{-1}\mathscr{O}_X.
\end{aligned}$$

On the other hand, by [DK13, Prop.8.2.4], there is a canonical morphism

$$\begin{aligned}\mathscr{O}_X^{\mathsf{E}} \overset{+}{\boxtimes} \mathscr{O}_X^{\mathsf{E}} &\simeq \mathrm{R}\widetilde{\mu}_{!!}(\widetilde{p_1}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \otimes \widetilde{p_2}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}) \\ &\to \mathscr{O}_{X\times X}^{\mathsf{E}}.\end{aligned}$$

By adjunction, we get the morphism

$$\widetilde{p_1}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \otimes \widetilde{p_2}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \to \widetilde{\mu}^{!}\,\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times X}^{\mathsf{E}}.$$

Therefore, we have the morphisms

$$\begin{aligned}&\widetilde{q_1}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{O}_X} \widetilde{q_2}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \\ &\qquad\simeq \widetilde{\delta}^{-1}(\widetilde{p_1}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \otimes \widetilde{p_2}^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}) \overset{\mathrm{L}}{\otimes}_{q^{-1}(\mathscr{O}_X\otimes\mathscr{O}_X)} q^{-1}\mathscr{O}_X \\ &\qquad\to \widetilde{\delta}^{-1}\widetilde{\mu}^{!}\,\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times X}^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{q^{-1}(\mathscr{O}_X\otimes\mathscr{O}_X)} q^{-1}\mathscr{O}_X \\ &\qquad\to \widetilde{\delta}^{-1}\widetilde{\mu}^{!}\,\big(\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times X}^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{p^{-1}\mathscr{O}_{X\times X}} p^{-1}\mathscr{O}_X\big) \\ &\qquad\simeq \mu^{!}\,\delta^{-1}\big(\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times X}^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{p^{-1}\mathscr{O}_{X\times X}} p^{-1}\mathscr{O}_X\big) \\ &\qquad\simeq \mu^{!}\,\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}},\end{aligned}$$

where $\mathscr{O}_X$ is identified with $\mathscr{O}_{\Delta_X}$ by the diagonal embedding $\delta\colon X \hookrightarrow X\times X$, and the last isomorphism follows from Lemma 4.1.

The result then follows by adjunction. Q.E.D.

Let M be a good topological space, $\mathbf{k}$ a commutative unital ring and $\mathscr{A}$ a flat $\mathbf{k}$-algebra on M as above.

Lemma 4.3. *Let $M \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$, $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A}^{\mathrm{op}})$ and $G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{A})$. Then there is a canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})$:*

$$(F\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}M) \overset{\mathrm{L}}{\otimes}_{\mathbf{k}} \mathrm{R}\mathscr{H}om_{\mathscr{A}}(M,G) \to F \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} G.$$

Proof. We have

$$\begin{aligned}(F\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}M) \overset{\mathrm{L}}{\otimes}_{\mathbf{k}} \mathrm{R}\mathscr{H}om_{\mathscr{A}}(M,G) &\simeq F\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\big(M \overset{\mathrm{L}}{\otimes}_{\mathbf{k}} \mathrm{R}\mathscr{H}om_{\mathscr{A}}(M,G)\big) \\ &\to F \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} G.\end{aligned}$$

Q.E.D.

Lemma 4.4. *There exists a canonical morphism functorial with respect to $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:*

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}} \to \mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{E}}) \text{ in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}). \tag{4.1}$$

Proof. It is enough to construct the morphism

$$(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}}) \overset{+}{\otimes} \mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}) \;\to\; \mathscr{O}_X^{\mathsf{E}}.$$

With the notations as in Lemma 4.2 we have

$$\begin{aligned}
(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}}) \overset{+}{\otimes} \mathcal{S}ol^{\mathsf{E}}(\mathscr{M}) &\simeq \mathrm{R}\mu_{!!}\big(q_1^{-1}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}) \otimes q_2^{-1}\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M})\big)\\
&\simeq \mathrm{R}\mu_{!!}\big(q_1^{-1}(\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{D}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}) \otimes q_2^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}})\big)\\
&\to \mathrm{R}\mu_{!!}\big(q_1^{-1}(\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{D}_X) \overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{D}_X} q^{-1}\mathscr{M}\\
&\qquad\qquad \otimes \mathrm{R}\mathscr{H}om_{q^{-1}\mathscr{D}_X}(q^{-1}\mathscr{M}, q_2^{1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}})\big)\\
&\to \mathrm{R}\mu_{!!}\big((q_1^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{q_1^{-1}\mathscr{O}_X} q_1^{-1}\mathscr{D}_X) \overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{D}_X} q_2^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\big)\\
&\simeq \mathrm{R}\mu_{!!}\big(q_1^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{q^{-1}\mathscr{O}_X} q_2^{-1}\mathrm{R}^{\mathrm{E}}\mathscr{O}_X^{\mathsf{E}}\big)\\
&\simeq \mathscr{O}_X^{\mathsf{E}} \overset{+}{\otimes}_{\mathscr{O}_X} \mathscr{O}_X^{\mathsf{E}} \to \mathscr{O}_X^{\mathsf{E}}.
\end{aligned}$$

Here, we have used Lemma 4.3. Q.E.D.

Theorem 4.5 (Extended Riemann-Hilbert theorem)**.** *There exists a canonical isomorphism functorial with respect to $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$:*

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}} \xrightarrow{\sim} \mathscr{I}hom^+(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{E}}) \text{ in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}). \tag{4.2}$$

Proof. We shall apply [DK13, Lemma 7.3.7]. Recall that this lemma summarizes deep results of Mochizuki [Mo09, Mo11] in the algebraic case completed by those of Kedlaya [Ke10, Ke11] in the analytic case (see also [Sa00]).

All conditions of this lemma are easily verified except conditions (e) and (f) that we shall check now.

(e) Let $f\colon X \to Y$ be a projective morphism and let $\mathscr{M}$ be a good holonomic $\mathscr{D}$-module such that (4.1) is an isomorphism. We shall prove the isomorphism:

$$\mathrm{D}f_*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_Y^{\mathsf{E}} \xrightarrow{\sim} \mathscr{I}hom^+(\mathcal{S}ol_Y^{\mathsf{E}}(\mathrm{D}f_*\mathscr{M}), \mathscr{O}_Y^{\mathsf{E}}). \tag{4.3}$$

By Theorem 3.11 (i) we have

$$\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^{\mathsf{E}} \simeq \mathsf{E} f^{\,!} \mathscr{O}_Y^{\mathsf{E}}[d_Y - d_X].$$

Therefore, using Lemma 3.12, we get:

$$\begin{aligned}&\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{E}})\\ &\qquad\simeq \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathsf{E} f^{\,!} \mathscr{O}_Y^{\mathsf{E}}[d_Y - d_X]).\end{aligned}$$

Hence, using Theorem 3.11 (iii) :

$$\begin{aligned}&\mathsf{E} f_*\big(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{E}})\big)\\ &\qquad\simeq \mathscr{I}hom^{+}(\mathsf{E} f_{!!} \mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_Y^{\mathsf{E}}[d_Y - d_X])\\ &\qquad\simeq \mathscr{I}hom^{+}(\mathcal{S}ol_Y^{\mathsf{E}}(\mathrm{D} f_* \mathscr{M}), \mathscr{O}_Y^{\mathsf{E}}).\end{aligned}$$

Hence, the right hand side of (4.3) is calculated as

$$\begin{aligned}&\mathscr{I}hom^{+}(\mathcal{S}ol_Y^{\mathsf{E}}(\mathrm{D} f_* \mathscr{M}), \mathscr{O}_Y^{\mathsf{E}})\\ &\qquad\simeq \mathsf{E} f_*\big(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{E}})\big).\end{aligned} \tag{4.4}$$

On the other hand we have:

$$\begin{aligned}\Omega_Y^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_Y} \mathrm{D} f_* \mathscr{M} &\simeq \Omega_Y^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D} f_* \mathscr{M}\\ &\simeq \mathcal{DR}_Y^{\mathsf{E}}(\mathscr{D}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D} f_* \mathscr{M})\\ &\simeq \mathcal{DR}_Y^{\mathsf{E}}(\mathrm{D} f_*(\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})),\end{aligned}$$

where the last isomorphism follows from the projection formula of [Ka03, Th. 4.2.8]. Hence

$$\begin{aligned}\Omega_Y^{\mathsf{E}} \overset{\mathrm{D}}{\otimes} \mathrm{D} f_* \mathscr{M} &\simeq \mathcal{DR}_Y^{\mathsf{E}}\big(\mathrm{D} f_*(\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})\big)\\ &\simeq \mathsf{E} f_*\big(\mathcal{DR}_X^{\mathsf{E}}(\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})\big),\end{aligned}$$

where the last isomorphism follows from Theorem 3.11 (iii). Since

$$\begin{aligned}\mathcal{DR}_X^{\mathsf{E}}(\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M}) &\simeq \Omega_X^{\mathsf{E}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})\\ &\simeq (\Omega_X^{\mathsf{E}} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y},\end{aligned}$$

we obtain

$$\Omega_Y^{\mathsf{E}} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_*\mathscr{M} \simeq \mathsf{E}f_*\big((\Omega_X^{\mathsf{E}} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big),$$

which is equivalent to

$$\mathrm{D}f_*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_Y^{\mathsf{E}} \simeq \mathsf{E}f_*\big(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}})\big). \tag{4.5}$$

By comparing (4.4) and (4.5) and assuming that (4.1) is an isomorphism, we get isomorphism (4.3).

(f)–(1) Let $Y \subset X$ be a normal crossing divisor, $U = X \setminus Y$ and $\varphi \in \Gamma(X; \mathscr{O}_X(*Y))$ a meromorphic function on X with poles on Y. We shall first prove that (4.1) is an isomorphism when $\mathscr{M} = \mathscr{E}_{U|X}^{\varphi}$.

Recall that $\mathbb{P} = \mathbb{P}^1(\mathbb{C})$ and denote by $\mathbb{P}_1$ and $\mathbb{P}_2$ two copies of $\mathbb{P}$. Let τ_k be the inhomogeneous coordinate of $\mathbb{P}_k$ $(k = 1, 2)$. Let $q_{kX}\colon X \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to X \times \mathbb{R}_\infty$ be the projection $(k = 1, 2)$. Consider the maps

$$\begin{array}{ccccc}
X \times \mathbb{R}_\infty \times \mathbb{R}_\infty & \xrightarrow{\ j\ } & X \times \mathbb{P}_1 \times \mathbb{R}_\infty & \xrightarrow{\ i_{X\times\mathbb{P}}\ } & X \times \mathbb{P}_1 \times \mathbb{P}_2 \\
\downarrow{\scriptstyle q_{1X}} & \square & \downarrow{\scriptstyle \pi_{X\times\mathbb{P}_1}} & \searrow{\scriptstyle q} & \\
X \times \mathbb{R}_\infty & \xrightarrow{\ i_X\ } & X \times \mathbb{P}_1 & & X \times \mathbb{R}_\infty
\end{array}$$

Set $K := \mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M})$. Then by [DK13, Prop. 9.6.5], we have

$$\mathrm{R}\pi_{X\times\mathbb{P}_1*}\mathrm{R}\mathscr{I}hom\,(q^{-1}\mathrm{L}^{\mathrm{E}}(K), \mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times\mathbb{P}_1}^{\mathsf{E}}) \simeq \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_{X\times\mathbb{P}_1}^{\mathrm{t}}.$$

On the other hand, we have

$$\begin{aligned}
&i_X^{\,!}\,\mathrm{R}\pi_{X\times\mathbb{P}_1*}\mathrm{R}\mathscr{I}hom\,(q^{-1}\mathrm{L}^{\mathrm{E}}(K), \mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times\mathbb{P}}^{\mathsf{E}}) \\
&\qquad \simeq \mathrm{R}q_{1X*}j^{\,!}\,\mathrm{R}\mathscr{I}hom\,(q^{-1}\mathrm{L}^{\mathrm{E}}(K), \mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times\mathbb{P}}^{\mathsf{E}}) \\
&\qquad \simeq \mathrm{R}q_{1X*}\mathrm{R}\mathscr{I}hom\,(j^{-1}q^{-1}\mathrm{L}^{\mathrm{E}}(K), j^{\,!}\,\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times\mathbb{P}_1}^{\mathsf{E}}).
\end{aligned}$$

Since

$$\mathrm{R}^{\mathrm{E}}\mathscr{O}_{X\times\mathbb{P}_1}^{\mathsf{E}} \ \simeq\ i_{X\times\mathbb{P}}^{\,!}\big((\mathscr{E}_{\mathbb{C}|\mathbb{P}_2}^{-\tau_2})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_2}} \mathcal{O}_{X\times\mathbb{P}_1\times\mathbb{P}_2}^{\mathrm{t}}\big)[1],$$

we obtain

$$\begin{aligned}
&i_X^{\,!}\,(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_{X\times\mathbb{P}_1}^{\mathrm{t}}) \\
&\quad \simeq \mathrm{R}q_{1X*}\mathrm{R}\mathscr{I}hom\,\Big(j^{-1}q^{-1}\mathrm{L}^{\mathrm{E}}(K), j^{\,!}\,i_{X\times\mathbb{P}}^{\,!}\big((\mathscr{E}_{\mathbb{C}|\mathbb{P}_2}^{-\tau_2})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_2}} \mathcal{O}_{X\times\mathbb{P}_1\times\mathbb{P}_2}^{\mathrm{t}}\big)[1]\Big).
\end{aligned}$$

By applying the functor $(\mathscr{E}^{-\tau_1}_{\mathbb{C}|\mathbb{P}_1})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1}} \bullet$, we get the isomorphism

$$\begin{aligned}\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X &\simeq i_X^! \Big((\mathscr{E}^{-\tau_1}_{\mathbb{C}|\mathbb{P}_1})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1}} (\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}_1}) \Big)[1] \\ &\simeq \mathrm{R}q_{1X*}\mathrm{R}\mathscr{I}hom\Big(q_{2X}^{-1}\mathrm{L}^{\mathrm{E}}(K), j^!\, i^!_{X\times\mathbb{P}_1}\,((\mathscr{E}^{-\tau_1-\tau_2}_{\mathbb{C}^2|\mathbb{P}_1\times\mathbb{P}_2})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1\times\mathbb{P}_2}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}_1\times\mathbb{P}_2})[2]\Big).\end{aligned}$$

Let $\tilde{\mu}\colon X\times(\mathbb{C},\mathbb{P}_1)\times(\mathbb{C},\mathbb{P}_2) \to X\times(\mathbb{C},\mathbb{P})$ be the morphism given by $(\tau_1,\tau_2)\mapsto \tau_1+\tau_2$, and let $\mu\colon X\times\mathbb{R}_\infty\times\mathbb{R}_\infty \to X\times\mathbb{R}_\infty$ be its restriction. Since

$$\begin{aligned}(\mathscr{E}^{-\tau_1-\tau_2}_{\mathbb{C}^2|\mathbb{P}_1\times\mathbb{P}_2})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1\times\mathbb{P}_2}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}_1\times\mathbb{P}_2}[2] &\simeq (\mathrm{D}\tilde{\mu}^*\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1\times\mathbb{P}_2}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}_1\times\mathbb{P}_2}[2] \\ &\simeq \tilde{\mu}^!\Big((\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}}\Big)[1],\end{aligned}$$

we get

$$\begin{aligned}&j^!\, i^!_{X\times\mathbb{P}}\Big((\mathscr{E}^{-\tau_1-\tau_2}_{\mathbb{C}^2|\mathbb{P}_1\times\mathbb{P}_2})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}_1\times\mathbb{P}_2}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}_1\times\mathbb{P}_2})\Big)[2] \\ &\qquad\simeq \mu^!\, i^!_X\Big((\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}}\Big)[1] \simeq \mu^!\,\mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X &\simeq \mathrm{R}q_{1X*}\mathrm{R}\mathscr{I}hom\,(q_{2X}^{-1}\mathrm{L}^{\mathrm{E}}(K), \mu^!\,\mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X) \\ &\simeq \mathscr{I}hom^{+}(\mathrm{L}^{\mathrm{E}}(K), \mathrm{R}^{\mathrm{E}}\mathscr{O}^{\mathsf{E}}_X).\end{aligned}$$

(f)–(2) Let $Y\subset X$ be a normal crossing divisor, $U = X\setminus Y$ as in (f)–(1). We shall prove that (4.1) is an isomorphism when $\mathscr{M}$ has a normal form along Y. We keep the notations of [DK13, Lemma 9.6.6]. Recall from [DK13, § 7.1, 7.2] that $\widetilde{X} := \widetilde{X}_Y$ is the real blow-up of X along Y and $\varpi\colon \widetilde{X}\to X$ is the projection.

Similarly to Lemma 4.4, we have a morphism

$$\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\widetilde{X}}} \mathscr{O}^{\mathsf{E}}_{\widetilde{X}} \to \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{L}), \mathscr{O}^{\mathsf{E}}_{\widetilde{X}}) \quad \text{for any } \mathscr{L}\in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathscr{A}}_{\widetilde{X}}). \tag{4.6}$$

Let us first show that (4.6) is an isomorphism when $\mathscr{L} = \mathscr{M}^{\mathscr{A}}$. Note that by [DK13, (9.6.6)],

$$\pi^{-1}\varpi^{-1}\mathbb{C}_U \otimes \mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathscr{A}}) \simeq \mathsf{E}\varpi^{-1}\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}). \tag{4.7}$$

Since $\mathscr{M}^{\mathscr{A}}$ is isomorphic to a direct sum of modules of type $(\mathscr{E}^{\varphi}_{U|X})^{\mathscr{A}}$, we may assume that $\mathscr{M}^{\mathscr{A}} = \mathscr{N}^{\mathscr{A}}$ with $\mathscr{N} = \mathscr{E}^{\varphi}_{U|X}$. In this case we have proved the isomorphism

$$\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}^{\mathsf{E}}_X \xrightarrow{\sim} \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{N}), \mathscr{O}^{\mathsf{E}}_X) \tag{4.8}$$

in (f)–(1). By [DK13, Th. 9.2.2],

$$\mathscr{O}^{\mathsf{E}}_{\widetilde{X}} \simeq \mathsf{E}\varpi^! \mathrm{R}\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, \mathscr{O}^{\mathsf{E}}_X).$$

Applying the functor $\mathsf{E}\varpi^! \mathrm{R}\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, \bullet)$ to (4.8), we obtain

$$\mathscr{M}^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\widetilde{X}}} \mathscr{O}^{\mathsf{E}}_{\widetilde{X}} \simeq \mathscr{N} \overset{\mathrm{D}}{\otimes} \mathscr{O}^{\mathsf{E}}_{\widetilde{X}} \xrightarrow{\sim} \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathscr{A}}), \mathscr{O}^{\mathsf{E}}_{\widetilde{X}}).$$

Therefore (4.6) is an isomorphism for $\mathscr{L} = \mathscr{M}^{\mathscr{A}}$.

Hence, we obtain

$$\begin{aligned}
\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{\widetilde{X}}} \mathsf{E}\varpi^! \mathscr{I}hom^+(\pi^{-1}\mathbb{C}_U, \mathscr{O}^{\mathsf{E}}_X) &\simeq \mathscr{M}^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\widetilde{X}}} \mathscr{O}^{\mathsf{E}}_{\widetilde{X}} \\
&\simeq \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathscr{A}}), \mathsf{E}\varpi^! \mathrm{R}\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, \mathscr{O}^{\mathsf{E}}_X)) \\
&\simeq \mathscr{I}hom^+(\pi^{-1}_{\widetilde{X}}\varpi^{-1}\mathbb{C}_U \otimes \mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathscr{A}}), \mathsf{E}\varpi^! \mathscr{O}^{\mathsf{E}}_X) \\
&\simeq \mathscr{I}hom^+(\mathsf{E}\varpi^{-1}\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathsf{E}\varpi^! \mathscr{O}^{\mathsf{E}}_X) \\
&\simeq \mathsf{E}\varpi^! \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X).
\end{aligned}$$

Here, we have used (4.7). Applying the functor $\mathsf{E}\varpi_*$ we obtain

$$\begin{aligned}
&\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathrm{R}\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, \mathscr{O}^{\mathsf{E}}_X) \\
&\qquad \simeq \mathrm{R}\mathscr{I}hom(\pi^{-1}\mathrm{R}\varpi_*\mathbb{C}_{\widetilde{X}}, \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X)) \\
&\qquad \simeq \mathscr{I}hom^+(\pi^{-1}\mathrm{R}\varpi_*\mathbb{C}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes} \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X).
\end{aligned}$$

Since $\pi^{-1}\mathbb{C}_U \otimes \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}) \simeq \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M})$, we also have $\pi^{-1}\mathrm{R}\varpi_*\mathbb{C}_{\widetilde{X}} \otimes \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}) \simeq \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M})$.

Since $\mathrm{R}\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, \mathscr{O}^{\mathsf{E}}_X) \simeq \mathscr{O}_X(*Y) \otimes_{\mathscr{O}_X} \mathscr{O}^{\mathsf{E}}_X$ and $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(*Y) \simeq \mathscr{M}$, we obtain isomorphism (4.2).

Thus condition (f) is checked, and this completes the proof of isomorphism (4.2) for an arbitrary $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Q.E.D.

Corollary 4.6 (cf. [DK13, Th. 9.6.1]). *There are isomorphisms, functorial with respect to* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$:

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{t}}_X \xrightarrow{\sim} \mathscr{I}hom^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X) \quad \text{in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X),$$
$$\mathscr{M} \xrightarrow{\sim} \mathscr{H}om^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{E}}_X) \quad \text{in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).$$

Proof. We shall apply the functor $\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t=0\}}, \bullet)$ to isomorphism (4.2). The first isomorphism follows by Proposition 3.10. The second isomorphism is deduced from the first one by applying the functor α_X. Q.E.D.

4.2 Functoriality of the De Rham functor

Proposition 4.7. *The enhanced de Rham functor has the following properties.*

(i) *Let* $f\colon X \to Y$ *be a morphism of complex manifolds and let* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_Y)$. *There is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_X(\mathrm{D}f^*\mathscr{N})\,[d_X] \simeq \mathsf{E}\,f^{\,!}\mathcal{DR}^{\mathsf{E}}_Y(\mathscr{N})\,[d_Y].$$

(ii) *Let* $f\colon X \to Y$ *be a morphism of complex manifolds and let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$. *Assume that* f *is proper on* $\operatorname{supp}(\mathscr{M})$. *Then there is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_Y(\mathrm{D}f_*\mathscr{M}) \simeq \mathsf{E}f_*\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}).$$

(iii) *Let* X *be a complex manifold and let* $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ *and* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$. *There is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_X(\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})).$$

Proof. (i) and (ii) follow from Theorem 3.11. Let us prove (iii). By Theorem 4.5, we have an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}})$:

$$\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{O}^{\mathsf{E}}_X \xrightarrow{\sim} \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathscr{O}^{\mathsf{E}}_X). \tag{4.9}$$

Let us apply $\mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \bullet$ to both sides of (4.9). We have

$$\begin{aligned} \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X^{\mathsf{E}}) &\simeq (\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^{\mathsf{E}} \\ &\simeq \mathcal{DR}_X^{\mathsf{E}}(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}), \end{aligned}$$

and, using Lemma 3.12,

$$\begin{aligned} \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathscr{O}_X^{\mathsf{E}}) &\simeq \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^{\mathsf{E}}) \\ &\simeq \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}), \mathcal{DR}_X^{\mathsf{E}}(\mathscr{M})). \end{aligned}$$

Q.E.D.

Consider morphisms of complex manifolds

$$\begin{array}{ccc} & S & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y. \end{array} \tag{4.10}$$

Notation 4.8. (i) For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_S)$ one sets

$$\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L} := \mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}). \tag{4.11}$$

(ii) For $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{S}})$, $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$ one sets

$$\begin{gathered} L \overset{E}{\circ} G := \mathsf{E}f_{!!}(L \overset{+}{\otimes} \mathsf{E}g^{-1}G), \quad F \overset{E}{\circ} L := \mathsf{E}g_{!!}(\mathsf{E}f^{-1}F \overset{+}{\otimes} L), \\ \Phi_L^{\mathsf{E}}(G) = L \overset{E}{\circ} G, \quad \Psi_L^{\mathsf{E}}(F) = \mathsf{E}g_*\mathscr{I}hom^{+}(L, \mathsf{E}f^{!}F). \end{gathered} \tag{4.12}$$

Note that we have a pair of adjoint functors

$$\Phi_L^{\mathsf{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}}) \rightleftarrows \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}) \colon \Psi_{\mathrm{L}}^{\mathrm{E}} \tag{4.13}$$

Theorem 4.9. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_S)$ and let $L := \mathcal{S}ol_S^{\mathsf{E}}(\mathscr{L})$. Assume that $f^{-1}\operatorname{supp}(\mathscr{M}) \cap \operatorname{supp}(\mathscr{L})$ is proper over Y. Then there is a natural isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$:*

$$\Psi_L^{\mathsf{E}}\big(\mathcal{DR}_X^{\mathsf{E}}(\mathscr{M})\big)\,[d_X - d_S] \simeq \mathcal{DR}_Y^{\mathsf{E}}(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}).$$

Proof. Applying Proposition 4.7, we get:

$$\begin{aligned}
\mathcal{DR}^{\mathsf{E}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}) &= \mathcal{DR}^{\mathsf{E}}_Y(\mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L})) \\
&\simeq \mathsf{E}g_*\mathcal{DR}^{\mathsf{E}}_S(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}) \\
&\simeq \mathsf{E}g_*\mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_S(\mathscr{L}), \mathcal{DR}^{\mathsf{E}}_S(\mathrm{D}f^*\mathscr{M})) \\
&\simeq \mathsf{E}g_*\mathscr{I}hom^+(L, \mathsf{E}f^!\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}))\,[d_X - d_S] \\
&= \Psi^{\mathrm{E}}_L(\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}))\,[d_X - d_S].
\end{aligned}$$

Q.E.D.

Corollary 4.10. *In the situation of* Theorem 4.9*, let* $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$*. Then there is a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C})$

$$\begin{aligned}
&\mathrm{RHom}^{\mathrm{E}}(L \overset{E}{\circ} G, \Omega^{\mathsf{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})\,[d_X - d_S] \\
&\qquad\qquad \simeq \mathrm{RHom}^{\mathrm{E}}(G, \Omega^{\mathsf{E}}_Y \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} (\mathscr{M} \overset{D}{\circ} \mathscr{L})).
\end{aligned}$$

Proof. Applying Theorem 4.9 and the adjunction (4.13), we get

$$\begin{aligned}
\mathrm{RHom}^{\mathrm{E}}(\Phi^{\mathrm{E}}_L(G), \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})) &\simeq \mathrm{RHom}^{\mathrm{E}}(G, \Psi^{\mathrm{E}}_L(\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}))) \\
&\simeq \mathrm{RHom}^{\mathrm{E}}(G, \mathcal{DR}^{\mathsf{E}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}))\,[d_S - d_X].
\end{aligned}$$

Q.E.D.

Note that Corollary 4.10 is a generalisation of [KS01, Th.7.4.12] to not necessarily regular holonomic $\mathscr{D}$-modules.

4.3 Generalization to complex bordered spaces

It is possible to generalize all the preceding results when replacing complex manifolds with complex bordered spaces, as defined below.

Definition 4.11. The category of *complex bordered spaces* (complex bordered spaces for short) is defined as follows.
The objects are pairs $(X, \widehat{X})$ where $\widehat{X}$ is a complex manifold and $X \subset \widehat{X}$ is an open subset such that $\widehat{X} \setminus X$ is a complex analytic subset of $\widehat{X}$.
Morphisms $f\colon (X, \widehat{X}) \to (Y, \widehat{Y})$ are complex analytic maps $f\colon X \to Y$ such that

(i) Γ_f is a complex analytic subset of $\widehat{X} \times \widehat{Y}$ and

(ii) $\overline{\Gamma}_f \to \widehat{X}$ is proper.

Hence a morphism of complex bordered spaces is a morphism of real analytic bordered spaces.

Let $X_\infty = (X, \widehat{X})$ be a complex bordered space. One sets

$$\begin{aligned}
\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty}) &:= \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\widehat{X}})/\left\{\mathscr{M}\,;\mathrm{supp}(\mathscr{M}) \subset \widehat{X} \setminus X\right\} \simeq \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X),\\
\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty}) &:= \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{\widehat{X}})/\left\{\mathscr{M}\,;\mathrm{supp}(\mathscr{M}) \subset \widehat{X} \setminus X\right\},\\
\mathsf{D}^{\mathrm{b}}_{\text{good}}(\mathscr{D}_{X_\infty}) &:= \mathsf{D}^{\mathrm{b}}_{\text{good}}(\mathscr{D}_{\widehat{X}})/\left\{\mathscr{M}\,;\mathrm{supp}(\mathscr{M}) \subset \widehat{X} \setminus X\right\},\\
\mathsf{D}^{\mathrm{b}}_{\text{hol}}(\mathscr{D}_{X_\infty}) &:= \mathsf{D}^{\mathrm{b}}_{\text{hol}}(\mathscr{D}_{\widehat{X}})/\left\{\mathscr{M}\,;\mathrm{supp}(\mathscr{M}) \subset \widehat{X} \setminus X\right\},\\
\mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_{X_\infty}) &:= \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_{X_\infty})/\left\{\mathscr{M}\,;\mathrm{supp}(\mathscr{M}) \subset \widehat{X} \setminus X\right\}.
\end{aligned}$$

Let $f\colon X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ be a morphism of complex bordered spaces. One sets

$$\mathscr{B}_f := \mathrm{R}\Gamma_{[\Gamma_f]}(\mathscr{O}_{\widehat{X}\times\widehat{Y}})\,[d_Y].$$

Denote by p_1 and p_2 the first and second projection defined on $\widehat{X} \times \widehat{Y}$. By representing an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty})$ by an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\widehat{X}})$ and similarly with $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{Y_\infty})$, we define the functors:

$$\begin{aligned}
\mathrm{D}f^*\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{Y_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty}), &\quad \mathscr{N} \mapsto \mathrm{D}p_{1*}(\mathrm{D}p_2{}^*\mathscr{N} \overset{\mathrm{D}}{\otimes} \mathscr{B}_f),\\
\mathrm{D}f_*\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{Y_\infty}), &\quad \mathscr{M} \mapsto \mathrm{D}p_{2*}(\mathrm{D}p_1{}^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{B}_f).
\end{aligned}$$

Also set:

$$\text{(4.14)}\quad \mathscr{D}_{X_\infty\to Y_\infty} := \mathscr{B}_f \otimes_{p_2^{-1}\mathscr{O}_{\widehat{Y}}} p_2^{-1}\Omega_{\widehat{Y}}, \quad \mathscr{D}_{Y_\infty\leftarrow X_\infty} := \mathscr{B}_f \otimes_{p_1^{-1}\mathscr{O}_{\widehat{X}}} p_1^{-1}\Omega_{\widehat{X}}.$$

Lemma 4.12. *Let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{Y_\infty})$ and let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty})$. Then*

$$\begin{aligned}
\mathrm{D}f^*\mathscr{N} &\simeq \mathrm{R}p_{1*}(\mathscr{D}_{X_\infty\to Y_\infty} \overset{\mathrm{L}}{\otimes}_{p_2^{-1}\mathscr{D}_{\widehat{Y}}} p_2^{-1}\mathscr{N}),\\
\mathrm{D}f_*\mathscr{M} &\simeq \mathrm{R}p_{2*}(\mathscr{D}_{Y_\infty\leftarrow X_\infty} \overset{\mathrm{L}}{\otimes}_{p_1^{-1}\mathscr{D}_{\widehat{X}}} p_1^{-1}\mathscr{M}).
\end{aligned}$$

Proof. After replacing the notations X and Y by $\widehat{X}$ and $\widehat{Y}$, this follows from the formula, valid for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times Y})$:

$$\mathscr{D}_{Y\leftarrow X\times Y} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times Y}} (p_1^{-1}\mathscr{M} \overset{\mathrm{L}}{\otimes}_{p_1^{-1}\mathscr{O}_X} \mathscr{L}) \simeq p_1^{-1}\mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{p_1^{-1}\mathscr{D}_X} \mathscr{L}.$$

Q.E.D.

Lemma 4.13. (a) *The functor* $\mathrm{D}f^*$ *above induces well-defined functors*

$$\mathrm{D}f^*\colon \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{Y_\infty}) \to \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty}),$$
$$\mathrm{D}f^*\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{Y_\infty}) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_\infty}).$$

(b) *Assume that the morphism* f *is semi-proper. Then the functor* $\mathrm{D}f_*$ *above induces well-defined functors*

$$\mathrm{D}f_*\colon \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty}) \to \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{Y_\infty}),$$
$$\mathrm{D}f_*\colon \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_{X_\infty}) \to \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_{Y_\infty}).$$

The proof is obvious.

Definition 4.14. Let $X_\infty = (X, \widehat{X})$ be a complex bordered space and denote by $j\colon X_\infty \to \widehat{X}$ the natural morphism. We set

$$\mathscr{O}^{\mathsf{E}}_{X_\infty} := \mathsf{E}j^{-1}\mathscr{O}^{\mathsf{E}}_{\widehat{X}}. \tag{4.15}$$

We define similarly $\Omega^{\mathsf{E}}_{X_\infty}$ and we define the functors

$$\mathcal{DR}^{\mathsf{E}}_{X_\infty}\colon \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty}),$$
$$\mathcal{S}ol^{\mathsf{E}}_{X_\infty}\colon \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty}),$$

as in Definition 3.8.

Proposition 4.15. (i) *Let* $f\colon X_\infty \to Y_\infty$ *be a morphism of complex bordered spaces and let* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{Y_\infty})$*. There is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathrm{D}f^*\mathscr{N})\,[d_X] \simeq \mathsf{E}f^{!}\mathcal{DR}^{\mathsf{E}}_{Y_\infty}(\mathscr{N})\,[d_Y].$$

(ii) *Let* $f\colon X_\infty \to Y_\infty$ *be a morphism of complex bordered spaces and let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$*. Assume that* f *is semi-proper. Then there is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_{Y_\infty}(\mathrm{D}f_*\mathscr{M}) \simeq \mathsf{E}f_*\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{M}).$$

(iii) *Let* X_∞ *be a complex bordered space and let* $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_\infty})$ *and* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$*. There is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_{X_\infty}(\mathscr{L}), \mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{M})).$$

Proof. There exist a complex manifold Z and a proper morphism $h\colon Z \to \overline{\Gamma}_f$ such that $h^{-1}X \to X$ is an isomorphism. Hence, replacing $(X, \widehat{X})$ with (X, Z) we may assume from the beginning that $f\colon X \to Y$ extends to a morphism of complex manifolds $\widetilde{f}\colon \widehat{X} \to \widehat{Y}$.
(i) Choose a representative $\mathscr{N}' \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{\widehat{Y}})$ of $\mathscr{N}$ and apply Proposition 4.7 (i).
(ii) Choose a representative $\mathscr{M}' \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{\widehat{X}})$ of $\mathscr{M}$. Then $\mathrm{D}f_*\mathscr{M}$ is represented by $\mathrm{D}\widetilde{f}_*(\mathscr{M}' \overset{\mathrm{D}}{\otimes} \mathrm{R}\Gamma_{[X]}(\mathscr{O}_{\widetilde{X}}))$. We have

$$\begin{aligned}
\mathcal{DR}^{\mathsf{E}}_{\widehat{X}}(\mathscr{M}' \overset{\mathrm{D}}{\otimes} \mathrm{R}\Gamma_{[X]}(\mathscr{O}_{\widetilde{X}})) &\simeq \mathsf{E}j_{X*}\mathsf{E}j_X^{-1}\mathcal{DR}^{\mathsf{E}}_{\widehat{X}}(\mathscr{M}') \\
&\simeq \mathsf{E}j_{X*}\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{M}).
\end{aligned}$$

Applying Proposition 4.7 (ii), we get

$$\begin{aligned}
\mathcal{DR}^{\mathsf{E}}_{Y_\infty}(\mathrm{D}f_*\mathscr{M}) &\simeq \mathsf{E}\widetilde{f}_*\mathcal{DR}^{\mathsf{E}}_{\widehat{X}}(\mathscr{M}' \overset{\mathrm{D}}{\otimes} \mathrm{R}\Gamma_{[X]}(\mathscr{O}_{\widetilde{X}})) \\
&\simeq \mathsf{E}\widetilde{f}_*\mathsf{E}j_{X*}\mathsf{E}j_X^{-1}\mathcal{DR}^{\mathsf{E}}_{\widehat{X}}(\mathscr{M}') \\
&\simeq \mathsf{E}f_*\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{M}).
\end{aligned}$$

(iii) Choose a representative $\mathscr{M}'$ of $\mathscr{M}$ as in (ii), choose a representative $\mathscr{L}'$ of $\mathscr{L}$ and apply Proposition 4.7 (iii) to $\mathscr{M}'$ and $\mathscr{L}'$. Q.E.D.

Consider morphisms of bordered spaces.

(4.16)
$$\begin{array}{ccc}
 & S_\infty & \\
{}^{f}\swarrow & & \searrow^{g} \\
X_\infty & & Y_\infty\,.
\end{array}$$

Notation 4.16. (i) For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{S_\infty})$ one sets

$$\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L} := \mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}). \tag{4.17}$$

(ii) For $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{S}_\infty})$, $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}_\infty})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}_\infty})$ one sets

$$\begin{gathered} L \overset{E}{\circ} G := \mathsf{E}f_{!!}(L \overset{+}{\otimes} \mathsf{E}g^{-1}G), \\ \Phi^{\mathrm{E}}_L(G) = L \overset{E}{\circ} G, \quad \Psi^{\mathrm{E}}_L(F) = \mathsf{E}g_*\mathscr{I}hom^+(L, \mathsf{E}f^!F). \end{gathered} \tag{4.18}$$

Here again, we get a pair of adjoint functors

$$\Phi^{\mathrm{E}}_L \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}_\infty}) \rightleftarrows \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}_\infty}) \colon \Psi^{\mathrm{E}}_{\mathrm{L}} \tag{4.19}$$

Theorem 4.17. *Assume that g is semi-proper. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$, $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_{S_\infty})$ and let $L := \mathcal{S}ol^{\mathsf{E}}_{S_\infty}(\mathscr{L})$. Then there is a natural isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}_\infty})$*

$$\Psi^{\mathrm{E}}_L(\mathcal{DR}^{\mathsf{E}}_{X_\infty}(\mathscr{M}))\,[d_X - d_S] \simeq \mathcal{DR}^{\mathsf{E}}_{Y_\infty}(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}).$$

Note that, as particular cases of this result, we find a generalisation to complex bordered spaces of Theorem 4.5 and Proposition 4.7.

Proof. The proof goes as for Theorem 4.9, using Proposition 4.15 instead of Proposition 4.7. Q.E.D.

5 Enhanced Fourier-Sato transform

5.1 Enhanced Fourier-Sato transform

Let V be a real finite-dimensional vector space, V^* its dual. Recall that the Fourier-Sato transform is an equivalence of categories between conic sheaves on V and conic sheaves on V^*. References are made to [KS90]. In [Ta08], D. Tamarkin has extended the Fourier-Sato transform to no more conic (usual) sheaves, by adding an extra variable. Here we generalise this last transform to enhanced ind-sheaves on V_∞.

We set $n = \dim \mathsf{V}$ and we denote by or_{V} the orientation $\mathbf{k}$-module of V, *i.e.*, $\mathrm{or}_{\mathsf{V}} = H^n_c(\mathsf{V}; \mathbf{k}_{\mathsf{V}})$. We have a canonical isomorphism $\mathrm{or}_{\mathsf{V}} \simeq \mathrm{or}_{\mathsf{V}^*}$. We denote by Δ_{V} the diagonal of $\mathsf{V} \times \mathsf{V}$.

We consider the bordered space $\mathsf{V}_\infty = (\mathsf{V}, \overline{\mathsf{V}})$ where $\overline{\mathsf{V}}$ is the projective compactification of V, that is

$$\overline{\mathsf{V}} = \big((\mathsf{V} \oplus \mathbb{R}) \setminus \{0\}\big)/\mathbb{R}^\times.$$

We shall work in the categories of enhanced ind-sheaves. If M_∞ is a bordered space and $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$, recall that in Definition 2.19 we set:

$$\varepsilon_M(F) := \pi^{-1}F \otimes \mathbf{k}_{\{t \geq 0\}} \in \mathrm{E}^{\mathrm{b}}(\mathbf{k}_{M_\infty}). \tag{5.1}$$

Also recall that π is the projection $M_\infty \times \mathbb{R}_\infty \to M_\infty$ and t is the coordinate on $\mathbb{R}$.

Recall (see (2.9)) that for a bordered space M_∞, $\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$ is a full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}_\infty})$.

We introduced the kernels

$$\begin{aligned} L_{\mathsf{V}} &:= \mathbf{k}_{\{t=\langle x,y\rangle\}} \in \mathrm{E}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}\times\mathsf{V}^*}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty\times\mathsf{V}^*_\infty}), \\ L^a_{\mathsf{V}} &:= \mathbf{k}_{\{t=-\langle x,y\rangle\}} \in \mathrm{E}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}\times\mathsf{V}^*}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty\times\mathsf{V}^*_\infty}). \end{aligned} \tag{5.2}$$

Here, x and y denote points of V and V^*, respectively.

Lemma 5.1. *One has isomorphisms in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty\times\mathsf{V}_\infty})$

$$\begin{aligned} L_{\mathsf{V}} \overset{E}{\circ} L^a_{\mathsf{V}^*} &\xrightarrow{\sim} \mathbf{k}_{\Delta_{\mathsf{V}}\times\{t=0\}} \otimes \mathrm{or}_{\mathsf{V}}[-n], \\ L^a_{\mathsf{V}^*} \overset{E}{\circ} L_{\mathsf{V}} &\xrightarrow{\sim} \mathbf{k}_{\Delta_{\mathsf{V}^*}\times\{t=0\}} \otimes \mathrm{or}_{\mathsf{V}}[-n]. \end{aligned} \tag{5.3}$$

Proof. Of course, it is enough to prove the first isomorphism. Denote by (x, y, x') a point of $\mathsf{V}\times\mathsf{V}^*\times\mathsf{V}$ and denote by p the projection $\mathsf{V}_\infty \times \mathsf{V}^*_\infty \times \mathsf{V}_\infty \times \mathbb{R}_\infty \to \mathsf{V}_\infty \times \mathsf{V}_\infty \times \mathbb{R}_\infty$. We have in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty\times\mathsf{V}_\infty\times\mathbb{R}_\infty})$:

$$\begin{aligned} L_{\mathsf{V}} \overset{E}{\circ} L^a_{\mathsf{V}^*} &\simeq \mathrm{R}p_{!!}(\mathbf{k}_{t=\langle x,y\rangle} \overset{+}{\otimes} \mathbf{k}_{t=-\langle x',y\rangle}) \\ &\simeq \mathrm{R}p_{!!}(\mathbf{k}_{t=\langle x-x',y\rangle}) \\ &\simeq \mathrm{R}p_{!}(\mathbf{k}_{t=\langle x-x',y\rangle}), \end{aligned}$$

where the last isomorphism follows from the fact that p is semi-proper (see Diagram (2.7)).

The first morphism in (5.3) is deduced from the morphism $\mathbf{k}_{t=\langle x-x',y\rangle} \to \mathbf{k}_{\{x=x'\}\times\{t=0\}}$. To check it is an isomorphism it is thus enough to calculate the restriction of these sheaves at each fiber of $\mathsf{V}\times\mathsf{V}\times\mathbb{R} \to \mathsf{V}\times\mathsf{V}$ (see Remark 2.16).

The restriction of the left-hand side of (5.3). to $(x, x') \times \mathbb{R}$ is

$$\begin{cases} \mathbf{k}_{\mathbb{R}} \otimes \mathrm{or}_{\mathsf{V}}[1-n] & \text{if } x \neq x', \\ \mathbf{k}_{\{t=0\}} \otimes \mathrm{or}_{\mathsf{V}}[-n] & \text{if } x = x'. \end{cases} \tag{5.4}$$

Since the image of $\mathbf{k}_{\mathbb{R}}$ in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{pt}})$ is 0, we get the result. Q.E.D.

Now we introduce the enhanced Fourier-Sato functors

$$\begin{aligned} {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}} &\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}), \quad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathrm{F}) = \mathrm{F} \overset{\mathrm{E}}{\circ} \mathrm{L}_{\mathsf{V}}, \\ {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}} &\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}), \quad {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}}(\mathrm{F}) = \mathrm{F} \overset{\mathrm{E}}{\circ} \mathrm{L}^{\mathrm{a}}_{\mathsf{V}}. \end{aligned} \tag{5.5}$$

Applying Lemma 5.1, we obtain:

Theorem 5.2 (See [Ta08])**.** *The functors ${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}$ and ${}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*} \otimes \mathrm{or}_{\mathsf{V}}[n]$ are equivalences of categories, inverse to each other. In other words, one has the isomorphisms, functorial with respect to $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty})$:*

$$\begin{aligned} {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*} \circ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(F) &\simeq F \otimes \mathrm{or}_{\mathsf{V}}[-n], \\ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}} \circ {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*}(G) &\simeq G \otimes \mathrm{or}_{\mathsf{V}}[-n]. \end{aligned}$$

Corollary 5.3. *There is an isomorphism functorial in $F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty})$:*

$$\mathrm{RHom}^{\mathrm{E}}(F_1, F_2) \simeq \mathrm{RHom}^{\mathrm{E}}({}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(F_1), {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(F_2)). \tag{5.6}$$

We shall give an alternative construction of ${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}$. Denote by p_1 and p_2 the projections from $\mathsf{V} \times \mathsf{V}^*$ to V and V^*, respectively and recall Notation 4.16 in which $S_\infty = \mathsf{V}_\infty \times \mathsf{V}^*_\infty$, $X_\infty = \mathsf{V}_\infty$, $Y_\infty = \mathsf{V}^*_\infty$, $f = p_1$ and $g = p_2$.

Corollary 5.4. *The two functors ${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\bullet)$ and $\Psi^{\mathrm{E}}_{L^a_{\mathsf{V}}}(\bullet) \otimes \mathrm{or}_{\mathsf{V}}[-n]$ are isomorphic.*

Proof. The functor ${}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*}(\bullet) \otimes \mathrm{or}_{\mathsf{V}}[n]$ admits an inverse, namely the functor ${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\bullet)$, and also admits $\Psi^{\mathrm{E}}_{L^a_{\mathsf{V}}}(\bullet) \otimes \mathrm{or}_{\mathsf{V}}[-n]$ as a right adjoint. Therefore, these two last functors are isomorphic. Q.E.D.

For a bordered space M_∞, denote by $a_{M_\infty} \colon M_\infty \to \mathrm{pt}$ the unique morphism from M_∞ to the bordered space pt.

Corollary 5.5. *We have the isomorphism, functorial with respect to $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{pt}})$:*

$${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathscr{I}hom^+(F, \mathsf{E}a^!_{\mathsf{V}_\infty}(G))) \simeq \mathscr{I}hom^+({}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}}(F), \mathsf{E}a^{-1}_{\mathsf{V}^*_\infty}(G)).$$

Proof. One has the sequence of isomorphisms

$$\begin{aligned}\mathscr{I}hom^+({}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}}(F), \mathsf{E}a^{-1}_{\mathsf{V}^*_\infty}(G)) \otimes \mathrm{or}_{\mathsf{V}}[n] &\simeq \mathscr{I}hom^+({}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}}(F), \mathsf{E}a^!_{\mathsf{V}^*_\infty}G) \\ &\simeq \mathscr{I}hom^+(\mathsf{E}p_{2!!}(L^a_{\mathsf{V}} \overset{+}{\otimes} \mathsf{E}p_1^{-1}F), \mathsf{E}a^!_{\mathsf{V}^*_\infty}G) \\ &\simeq \mathsf{E}p_{2*}\mathscr{I}hom^+(L^a_{\mathsf{V}} \overset{+}{\otimes} \mathsf{E}p_1^{-1}F, \mathsf{E}p_2^!\mathsf{E}a^!_{\mathsf{V}^*_\infty}G) \\ &\simeq \mathsf{E}p_{2*}\mathscr{I}hom^+(L^a_{\mathsf{V}}, \mathscr{I}hom^+(\mathsf{E}p_1^{-1}F, \mathsf{E}p_1^!\mathsf{E}a^!_{\mathsf{V}_\infty}G)) \\ &\simeq \mathsf{E}p_{2*}\mathscr{I}hom^+(L^a_{\mathsf{V}}, \mathsf{E}p_1^!\mathscr{I}hom^+(F, \mathsf{E}a^!_{\mathsf{V}_\infty}G)) \\ &\simeq {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathscr{I}hom^+(F, \mathsf{E}a^!_{\mathsf{V}_\infty}G)) \otimes \mathrm{or}_{\mathsf{V}}[n].\end{aligned}$$

Here the last isomorphism follows from the preceding corollary. Q.E.D.

5.2 Operations

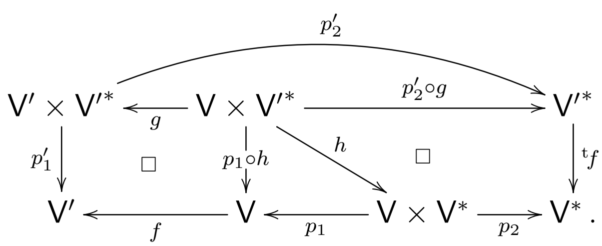
Let $f\colon \mathsf{V} \to \mathsf{V}'$ be a linear map of finite-dimensional vector spaces. We denote by n and n' the dimensions of V and V', respectively. We denote by ${}^{\mathrm{t}}f$ the transpose of f.

Proposition 5.6. *Let* $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty})$ *and let* $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}'_\infty})$*. Then:*

$$(5.7)\qquad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}'}(\mathsf{E}f_{!!}F) \simeq \mathsf{E}\,{}^{\mathrm{t}}f^{-1}\,{}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(F),$$

$$(5.8)\qquad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathsf{E}f^{-1}G \otimes \mathrm{or}_{\mathsf{V}'}[-n']) \simeq \mathsf{E}\,{}^{\mathrm{t}}f_{!!}\,{}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}'}(G) \otimes \mathrm{or}_{\mathsf{V}}[-n].$$

Proof. Consider the commutative diagram



(i) Let us prove (5.7). Using $\mathsf{E}g^{-1}L_{\mathsf{V}'} \simeq \mathsf{E}h^{-1}L_{\mathsf{V}}$, we have

$$\begin{aligned}\mathsf{E}p'_{2!!}(\mathsf{E}p'^{-1}_1\mathsf{E}f_{!!}F \otimes L_{\mathsf{V}'}) &\simeq \mathsf{E}p'_{2!!}(\mathsf{E}g_{!!}\mathsf{E}(p_1 \circ h)^{-1}F \otimes L_{\mathsf{V}'}) \\ &\simeq \mathsf{E}p'_{2!!}\mathsf{E}g_{!!}(\mathsf{E}h^{-1}\mathsf{E}p_1^{-1}F \otimes \mathsf{E}g^{-1}L_{\mathsf{V}'}) \\ &\simeq \mathsf{E}p'_{2!!}\mathsf{E}g_{!!}(\mathsf{E}h^{-1}\mathsf{E}p_1^{-1}F \otimes \mathsf{E}h^{-1}L_{\mathsf{V}}) \\ &\simeq \mathsf{E}(p'_2 \circ g)_{!!}\mathsf{E}h^{-1}(\mathsf{E}p_1^{-1}F \otimes L_{\mathsf{V}}) \\ &\simeq \mathsf{E}\,{}^{\mathrm{t}}f^{-1}\mathsf{E}p_{2!!}(\mathsf{E}p_1^{-1}F \otimes L_{\mathsf{V}}).\end{aligned}$$

(ii) Let us prove (5.8). Applying (5.7), we obtain

$$ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}^*} \mathsf{E}{}^{\mathrm{t}}f_{!!}({}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}'}G) \simeq \mathsf{E}\, f^{-1}\, {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}'^*}\, {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}'}G \simeq \mathsf{E}\, f^{-1}G \otimes \mathrm{or}_{\mathsf{V}'}[-n']. $$

Hence

$$ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}\, {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*} \mathsf{E}{}^{\mathrm{t}}f_{!!}({}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}'}G) \simeq {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}} \mathsf{E}\, f^{-1}G \otimes \mathrm{or}_{\mathsf{V}'}[-n'], $$

and the result follows since

$$ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}\, {}^{\mathsf{E}}\mathscr{F}^a_{\mathsf{V}^*} K \simeq K \otimes \mathrm{or}_{\mathsf{V}}[-n]. $$

Q.E.D.

5.3 Compatibility of Fourier-Sato transforms

We shall compare the enhanced Fourier-Sato transform with the classical one, for which we refer to [KS90, § 3.7].

Recall that one denotes by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}})$ the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}})$ consisting of conic sheaves. We shall denote here by ${}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}}(F)$ the Fourier-Sato transform of $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}})$, which was denoted by $F^{\wedge}$ in loc. cit. The functor ${}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}} \colon \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}}) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}^*})$ is an equivalence of categories.

Recall that one identifies the sheaf $\mathbf{k}_{\{t \geq 0\}}$ with its image in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}\times\mathbb{R}_\infty})$ and that the functor

$$ \varepsilon_{\mathsf{V}} \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}}) \hookrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}), \quad \varepsilon_{\mathsf{V}}(\mathrm{F}) = \mathbf{k}_{\{\mathrm{t}\geq 0\}} \otimes \pi^{-1}\mathrm{F} $$

is a fully faithful embedding (see Proposition 2.20).

Consider the diagram of categories and functors

(5.9)
$$ \begin{array}{ccc} \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}_\infty}) & \xrightarrow{{}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}} & \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*_\infty}) \\ \uparrow{\scriptstyle \varepsilon_{\mathsf{V}}} & & \uparrow{\scriptstyle \varepsilon_{\mathsf{V}^*}} \\ \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}}) & \xrightarrow{{}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}}} & \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}^*}). \end{array} $$

Theorem 5.7. *Diagram (5.9) is quasi-commutative.*

Proof. Consider the morphism of diagrams (in which we denote by π any of the projections $X \times \mathbb{R}_\infty \to X$, with $X = \mathsf{V}, \mathsf{V}^*$, etc.):

$$ \begin{array}{ccc} & \mathsf{V}\times\mathsf{V}^*\times\mathbb{R}_\infty & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ \mathsf{V}\times\mathbb{R}_\infty & & \mathsf{V}^*\times\mathbb{R}_\infty \end{array} \xrightarrow{\ \pi\ } \begin{array}{ccc} & \mathsf{V}\times\mathsf{V}^* & \\ {\scriptstyle q_1}\swarrow & & \searrow{\scriptstyle q_2} \\ \mathsf{V} & & \mathsf{V}^*. \end{array} $$

(i) We shall first construct the morphism, functorial in $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathsf{V}})$:

$$ {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}} \circ \varepsilon_{\mathsf{V}}(F) \to \varepsilon_{\mathsf{V}^*} \circ {}^{\mathrm{S}}\mathscr{F}_{\mathsf{V}}(F). \tag{5.10} $$

Consider the sequence of morphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}^* \times \mathbb{R}})$:

$$
\begin{aligned}
\mathrm{R}p_{2!}(\mathbf{k}_{\{t=\langle x,y\rangle\}} \overset{+}{\otimes} p_1^{-1}(\mathbf{k}_{\{t\geq 0\}} \overset{\mathrm{L}}{\otimes} \pi^{-1}F)) &\simeq \mathrm{R}p_{2!}(\mathbf{k}_{\{t\geq\langle x,y\rangle\}} \otimes p_1^{-1}\pi^{-1}F)) \\
&\to \mathrm{R}p_{2!}(\mathbf{k}_{\{t\geq 0\geq\langle x,y\rangle\}} \otimes p_1^{-1}\pi^{-1}F)) \\
&\simeq \mathrm{R}p_{2!}(\mathbf{k}_{\{t\geq 0\}} \otimes \mathbf{k}_{\{0\geq\langle x,y\rangle\}} \otimes \pi^{-1}q_1^{-1}F) \\
&\simeq \mathbf{k}_{\{t\geq 0\}} \otimes \pi^{-1}\mathrm{R}q_{2!}(\mathbf{k}_{\{0\geq\langle x,y\rangle\}} \otimes q_1^{-1}F).
\end{aligned}
$$

The image of these morphisms in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{V}^*})$ gives the morphism (5.10). To prove that it is an isomorphism, we again argue in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{V}^* \times \mathbb{R}})$.

By the exact sequence

$$ 0 \to \mathbf{k}_{\{t\geq\langle x,y\rangle>0\}} \oplus \mathbf{k}_{\{0>t\geq\langle x,y\rangle\}} \to \mathbf{k}_{\{t\geq\langle x,y\rangle\}} \to \mathbf{k}_{\{t\geq 0\geq\langle x,y\rangle\}} \to 0, $$

it is enough to show that for any $(y,t) \in \mathsf{V}^* \times \mathbb{R}$,

$$
\begin{aligned}
&\mathrm{R}\Gamma_c(\mathsf{V}; \mathbf{k}_{\{0>t\geq\langle x,y\rangle\}} \otimes F) \simeq 0, \\
&\mathrm{R}\Gamma_c(\mathsf{V}; \mathbf{k}_{\{t\geq\langle x,y\rangle>0\}} \otimes F) \simeq 0.
\end{aligned}
$$

Denote by $h\colon \mathsf{V} \to \mathbb{R}$ the map $h(x) = \langle x,y\rangle$ and set $G = \mathrm{R}h_!F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$. Then

$$
\begin{aligned}
&\mathrm{R}\Gamma_c(\mathsf{V}; \mathbf{k}_{\{0>t\geq\langle x,y\rangle\}} \otimes F) \simeq \mathrm{R}\Gamma_c(\{\lambda \in \mathbb{R}; 0 > t \geq \lambda\}; G), \\
&\mathrm{R}\Gamma_c(\mathsf{V}; \mathbf{k}_{\{t\geq\langle x,y\rangle>0\}} \otimes F) \simeq \mathrm{R}\Gamma_c(\{\lambda \in \mathbb{R}; 0 < \lambda \leq t\}; G).
\end{aligned} \tag{5.11}
$$

Since F is $\mathbb{R}^+$-conic, so is G and the vanishing of the right-hand sides of (5.11) follows. Q.E.D.

5.4 Legendre transform

In this section, we shall make a link between the enhanced Fourier-Sato transform and the classical Legendre transform of convex functions.

As above, V is a real vector space of dimension n.

Definition 5.8. Let $f\colon \mathsf{V} \to \mathbb{R} \sqcup \{+\infty\}$ be a function.

(a) We say that f is a closed proper convex function on V if its epigraph $\{(x,t)\in\mathsf{V}\times\mathbb{R}\,;\,t\geq f(x)\}$ is closed, convex and non-empty.

(b) We denote by $\mathrm{Conv}(\mathsf{V})$ the space of closed proper convex functions on V.

(c) For $f\in\mathrm{Conv}(\mathsf{V})$, we denote by $\mathrm{dom}(f)$ the set $f^{-1}(\mathbb{R})$ and call it the domain of f. We denote by $\mathrm{H}(f)$ the affine space generated by $\mathrm{dom}(f)$ and by $\mathrm{dom}^\circ(f)$ the interior of $\mathrm{dom}(f)$ in $\mathrm{H}(f)$.

(d) We define $f^*\colon\mathsf{V}^*\to\mathbb{R}\sqcup\{+\infty\}$ by $f^*(y)=\sup_{x\in\mathrm{dom}(f)}\big(\langle x,y\rangle-f(x)\big)$ and call f^* the Legendre transform, or the Legendre-Fenchel conjugate or the convex conjugate of f.

   Note that

   - the set $\mathrm{dom}(f)$ is convex and non-empty,
   - the function f^* is also a closed proper convex function, that is, belongs to $\mathrm{Conv}(\mathsf{V}^*)$,
   - $f^{**}=f$. Hence, * gives an isomorphism $\mathrm{Conv}(\mathsf{V})\xrightarrow{\sim}\mathrm{Conv}(\mathsf{V}^*)$.

Now we introduce the set:

$$\begin{aligned}\mathrm{E}(f)&=\{v\in\mathsf{V}\,;\ \text{there exists } a\in\mathbb{R} \text{ such that } f(x+v)=f(x)+a\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for any } x\in\mathsf{V}\}\\&=\{v\in\mathsf{V}\ ;\ \text{there exists } a\in\mathbb{R} \text{ such that } f(x+tv)=f(x)+ta\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for any } x\in\mathsf{V},t\in\mathbb{R}\}.\end{aligned}$$

Denote by $\mathrm{H}^\perp$ the orthogonal vector space to an affine space $\mathrm{H}\subset\mathsf{V}^*$, that is, $\mathrm{H}^\perp=\{v\in\mathsf{V}^*\,;\,v|_{\mathrm{H}}\text{ is constant}\}$. Then

$$\mathrm{E}(f)=\mathrm{H}(f^*)^\perp.$$

In particular, $\dim\mathrm{E}(f)=\mathrm{codim}\,\mathrm{H}(f^*)$. In the sequel, we set:

$$\mathrm{d}(f)=\dim\mathrm{E}(f)=\mathrm{codim}\,\mathrm{H}(f^*).\tag{5.12}$$

In the theorem below,

$$\{t\geq f(x)\}:=\{(x,t)\in\mathsf{V}\times\mathbb{R}\,;\,t\geq f(x)\}$$

is a closed subset of $\mathsf{V} \times \mathbb{R}$ and

$$\{t \geq -f(x),\ x \in \operatorname{dom}^\circ(f)\} := \{(x,t) \in \mathsf{V} \times \mathbb{R}\,;\, t \geq -f(x),\ x \in \operatorname{dom}^\circ(f)\}$$

is a closed subset of $\operatorname{dom}^\circ(f) \times \mathbb{R}$, and hence it is a locally closed subset of $\mathsf{V} \times \mathbb{R}$.

Theorem 5.9. *For $f \in \operatorname{Conv}(\mathsf{V})$, we have isomorphisms*:

$$({5.13}) \qquad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t \geq f(x)\}}) \simeq \mathbf{k}_{\{t \geq -f^*(-y), -y \in \operatorname{dom}^\circ(f^*)\}} \otimes \operatorname{or}_{\mathrm{E}(f)}[-\mathrm{d}(f)],$$

$$({5.14}) \qquad {}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t \geq -f(x),\, x \in \operatorname{dom}^\circ(f)\}}) \simeq \mathbf{k}_{\{t \geq f^*(y)\}} \otimes \operatorname{or}_{\mathrm{H}(f)}[-\dim \mathrm{H}(f)].$$

Proof. It is enough to prove the second isomorphism. Equivalently, it is enough to prove:

$${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t < -f(x), x \in \operatorname{dom}^\circ(f)\}}) \simeq \mathbf{k}_{\{t < f^*(y)\}} \otimes \operatorname{or}_{\mathrm{H}(f)}[-\dim \mathrm{H}(f)].$$

By the definition

$${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t < -f(x), x \in \operatorname{dom}^\circ(f)\}}) \simeq \mathrm{R}p_{2!}\mathbf{k}_{\{t < -f(x) + \langle x,y\rangle; x \in \operatorname{dom}^\circ(f)\}},$$

and we have a natural morphism

$$\mathbf{k}_{\{x\,;\, t < -f(x) + \langle x,y\rangle,\, x \in \operatorname{dom}^\circ(f)\}} \to \mathbf{k}_{\{x \in \mathrm{H}(f),\, t < f^*(y)\}}.$$

This defines the morphism

$${}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t < -f(x), x \in \operatorname{dom}^\circ(f)\}}) \to \mathbf{k}_{\{t < f^*(y)\}} \otimes \operatorname{or}_{\mathrm{H}(f)}[-\dim \mathrm{H}(f)].$$

To check that it is an isomorphism, choose $(y,t) \in \mathsf{V}^* \times \mathbb{R}$. Then

$$\begin{aligned}
\big({}^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\mathbf{k}_{\{t < -f(x), x \in \operatorname{dom}^\circ(f)\}})\big)_{(y,t)} &\simeq \mathrm{R}\Gamma_c(\mathsf{V}; \mathbf{k}_{\{x \in \operatorname{dom}^\circ(f)\,;\, t < -f(x) + \langle x,y\rangle\}}) \\
&\simeq \begin{cases} \operatorname{or}_{\mathrm{H}(f)}[-\dim \mathrm{H}(f)] & \text{if } \{x \in \operatorname{dom}^\circ(f)\,;\, t < -f(x) + \langle x,y\rangle\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

To conclude, notice that $\{x \in \operatorname{dom}^\circ(f)\,;\, t < -f(x) + \langle x,y\rangle\} \neq \emptyset$ if and only if $t < f^*(y)$. Q.E.D.

Recall the notation (5.1).

Corollary 5.10. *Let $G \subset \mathsf{V}$ be a non empty compact convex subset. Define $p_G^-\colon \mathsf{V}^* \to \mathbb{R}$ by $p_G^-(y) = \inf_{x\in G}\langle x, y\rangle$. Then*

$$^{\mathsf{E}}\mathscr{F}_{\mathsf{V}}(\varepsilon_M(\mathbf{k}_G)) \simeq \mathbf{k}_{\{t \geq p_G^-(y)\}} \otimes \mathrm{or}_{\mathsf{V}}\,.$$

Proof. Define the function

$$f(x) = 0 \text{ if } x \in G \text{ and } f(x) = +\infty \text{ if } x \notin G.$$

Then $f \in \mathrm{Conv}(\mathsf{V})$, $f^*(y) = \sup_{x\in G}\langle x, y\rangle$, $\mathrm{dom}^\circ(f^*) = \mathsf{V}^*$, $\mathrm{d}(f) = 0$ and $-f^*(-y) = p_G^-(y)$. Then the result follows from isomorphism (5.13). Q.E.D.

Corollary 5.11. *Let $U \subset \mathsf{V}$ be a non empty open convex subset. Define $p_U^+\colon \mathsf{V}^* \to \mathbb{R} \sqcup +\infty$ by $p_U^+(y) = \sup_{x\in U}\langle x, y\rangle$. Then*

$$^{\mathsf{E}}\mathscr{F}_{\mathsf{V}_\infty}\big(\varepsilon_M(\mathbf{k}_U)\big) \simeq \mathbf{k}_{\{t \geq p_U^+(y)\}} \otimes \mathrm{or}_{\mathsf{V}}\,[-n].$$

Proof. As in the proof of Corollary 5.10, define the function

$$f(x) = 0 \text{ if } x \in \overline{U} \text{ and } f(x) = +\infty \text{ if } x \notin \overline{U}.$$

Then, $\mathrm{dom}^\circ(f) = U$, $\mathrm{H}(f) = \mathsf{V}$, $f^* = p_U^+$ and the result follows from isomorphism (5.14). Q.E.D.

We endow V with an Euclidean norm and denote by d the associated distance.

Lemma 5.12. *Let φ be a strictly decreasing convex function defined on $\mathbb{R}_{>0}$ (e.g. $\varphi(t) = -\log t$ or $\varphi(t) = t^{-\lambda}$ with $\lambda \in \mathbb{R}_{>0}$), and let Ω be an open convex subset of V such that $\Omega \neq \mathsf{V}$. Then the function $\varphi\big(\mathrm{d}(x, \mathsf{V} \setminus \Omega)\big)$ is a convex function on Ω.*

Proof. Let $x_j \in \Omega$ and $a_j \in \mathbb{R}_{>0}$ $(j = 1, 2)$ with $a_1 + a_2 = 1$. Set $x_0 = a_1x_1 + a_2x_2$ and $r_j = \mathrm{d}(x_j, \mathsf{V} \setminus \Omega)$ $(j = 1, 2)$. It is enough to show that $\varphi\big(\mathrm{d}(x_0, \mathsf{V} \setminus \Omega)\big) \leq a_1\varphi(r_1) + a_2\varphi(r_2)$ or equivalently

$$\mathrm{d}(x_0, \mathsf{V} \setminus \Omega) \geq r_0 := \varphi^{-1}\big(a_1\varphi(r_1) + a_2\varphi(r_2)\big).$$

This is equivalent to saying that

$$\{x\,;\, |x - x_0| < r_0\} \subset \Omega.$$

Since φ is convex, $\varphi(a_1 r_1 + a_2 r_2) \le a_1\varphi(r_1) + a_2\varphi(r_2) = \varphi(r_0)$. Since φ is strictly decreasing, we have $a_1 r_1 + a_2 r_2 \ge r_0$. Let $x = x_0 + y$ with $|y| < r_0$. Let us show that $x \in \Omega$.

Since $|\frac{r_j}{a_1 r_1 + a_2 r_2} y| < r_j$ $(j = 1, 2)$ and $\{x\,;\, |x - x_j| < r_j\} \subset \Omega$, we have

$$x_j + \frac{r_j}{a_1 r_1 + a_2 r_2} y \in \Omega.$$

Hence $\sum_{j=1}^{2} a_j(x_j + \frac{r_j}{a_1 r_1 + a_2 r_2} y) = x_0 + y$ belongs to Ω. Q.E.D.

6 Laplace transform

6.1 Laplace transform

Recall the $\mathscr{D}_X$-module $\mathscr{E}^{-\varphi}_{U|X}$ and Notation 3.6. We saw in (3.6)

(6.1) $$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{E}^{\varphi}_{U|X}) \simeq \mathbb{C}^{\mathrm{E}}_X \overset{+}{\otimes} \mathbb{C}_{\{t = -\operatorname{Re}\varphi\}}.$$

We shall apply this result in the following situation.

Let $\mathbb{V}$ be a complex finite-dimensional vector space of complex dimension $d_{\mathbb{V}}$, $\mathbb{V}^*$ its dual. Since $\mathbb{V}$ is a complex vector space, we shall identify $\mathrm{or}_{\mathbb{V}}$ with $\mathbb{C}$. We denote here by $\overline{\mathbb{V}}$ the projective compactification of $\mathbb{V}$, we set $\mathbb{H} = \overline{\mathbb{V}} \setminus \mathbb{V}$, and similarly with $\overline{\mathbb{V}}^*$ and $\mathbb{H}^*$. We also introduce the bordered spaces

$$\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}}), \quad \mathbb{V}^*_\infty = (\mathbb{V}^*, \overline{\mathbb{V}}^*).$$

We set for short

$$X = \overline{\mathbb{V}} \times \overline{\mathbb{V}}^*, U = \mathbb{V} \times \mathbb{V}^*, Y = X \setminus U.$$

We shall consider the function

$$\varphi\colon \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}, \quad \varphi(x, y) = \langle x, y\rangle.$$

We introduce the Laplace kernel

(6.2) $$\mathscr{L} \;:=\; \mathscr{E}^{\langle x,y\rangle}_{U|X}.$$

Recall from Corollary 5.4 that the kernel of the enhanced Fourier transform with respect to the underlying real vector spaces of $\mathbb{V}$ and $\mathbb{V}^*$ is given by

$$L^a_{\mathbb{V}} := \mathbf{k}_{\{t = -\operatorname{Re}\langle x,y\rangle\}} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}).$$

Lemma 6.1. *One has the isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{L}) \simeq \mathbb{C}_X^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}j_{!!}L_{\mathbb{V}}^a, \tag{6.3}$$

where $j\colon \mathbb{V}_\infty \times \mathbb{V}_\infty^* \to X$ *is the inclusion.*

Proof. This follows immediately from isomorphism (6.1). Q.E.D.

In the sequel, we denote by $\mathsf{D}_{\mathbb{V}}$ the Weyl algebra on $\mathbb{V}$. We also use the $(\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}, \mathsf{D}_{\mathbb{V}^*})$-bimodule $\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*\to\mathbb{V}^*}$ similar to the bimodule $\mathscr{D}_{X\to Y}$ of the theory of $\mathscr{D}$-modules, and finally we denote by $\mathsf{O}_{\mathbb{V}}$ the ring of polynomials on $\mathbb{V}$.

The next result is well-known and goes back to [KL85] or before.

Lemma 6.2. *There is a natural isomorphism*

$$\mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H}) \overset{\mathrm{D}}{\circ} \mathscr{L} \simeq \mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes \det \mathbb{V}^*. \tag{6.4}$$

Proof. Using the GAGA principle, we may replace $\mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H})$ with $\mathsf{D}_{\mathbb{V}}$, $\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H})$ with $\mathsf{D}_{\mathbb{V}^*}$, $\mathscr{L}$ with $\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}\mathrm{e}^{\langle x,y\rangle}$ and thus $\mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H}) \overset{\mathrm{D}}{\circ} \mathscr{L}$ with

$$\mathsf{D}_{\mathbb{V}^*\leftarrow\mathbb{V}\times\mathbb{V}^*} \overset{\mathrm{L}}{\otimes}_{\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}} (\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}\mathrm{e}^{\langle x,y\rangle} \overset{\mathrm{L}}{\otimes}_{\mathsf{O}_{\mathbb{V}\times\mathbb{V}^*}} \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*\to\mathbb{V}^*}). \tag{6.5}$$

This last object is isomorphic to

$$\big(\mathsf{D}_{\mathbb{V}^*\leftarrow\mathbb{V}\times\mathbb{V}^*} \overset{\mathrm{L}}{\otimes}_{\mathsf{O}_{\mathbb{V}\times\mathbb{V}^*}} \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*\to\mathbb{V}}\big) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}\times\mathbb{V}^*}} \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}\mathrm{e}^{\langle x,y\rangle}.$$

Since

$$\mathsf{D}_{\mathbb{V}^*\leftarrow\mathbb{V}\times\mathbb{V}^*} \overset{\mathrm{L}}{\otimes}_{\mathsf{O}_{\mathbb{V}\times\mathbb{V}^*}} \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*\to\mathbb{V}} \simeq \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*} \otimes \det \mathbb{V}^*,$$

the module (6.5) is isomorphic to $\mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}\mathrm{e}^{\langle x,y\rangle} \otimes \det \mathbb{V}^*$. Finally, one remarks that the natural morphism $\mathsf{D}_{\mathbb{V}^*} \to \mathsf{D}_{\mathbb{V}\times\mathbb{V}^*}\mathrm{e}^{\langle x,y\rangle}$ is an isomorphism. Q.E.D.

In the sequel, we shall identify $\mathsf{D}_{\mathbb{V}}$ and $\mathsf{D}_{\mathbb{V}^*}$ by the correspondence $x_i \leftrightarrow -\partial_{y_i}$, $\partial_{x_i} \leftrightarrow y_i$. (Of course, this does not depend on the choice of linear coordinates on $\mathbb{V}$ and the dual coordinates on $\mathbb{V}^*$.)

Theorem 6.3. *We have an isomorphism in* $\mathsf{D}^{\mathrm{b}}((\mathrm{I}\mathsf{D}_{\mathbb{V}})_{\mathbb{V}_\infty^*})$

$${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\mathscr{O}_{\mathbb{V}_\infty}^{\mathsf{E}}) \simeq \mathscr{O}_{\mathbb{V}_\infty^*}^{\mathsf{E}} \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}]. \tag{6.6}$$

Proof. Set $K = \mathcal{S}ol^{\mathsf{E}}_{X_\infty}(\mathscr{L})$. By Theorem 4.17, we have

$$\Psi^{\mathrm{E}}_K(\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M}))\,[-d_{\mathbb{V}}] \simeq \mathcal{DR}^{\mathsf{E}}_{\mathbb{V}^*_\infty}(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}).$$

By Lemma 6.1, $K = \mathbb{C}^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{+}{\otimes} L_{\mathbb{V}^*}$ and by Corollary 5.4, the functor ${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}$ is isomorphic to the functor $\Psi^{\mathrm{E}}_{L^a_{\mathbb{V}}}[-2d_{\mathbb{V}}]$. Since

$$\mathscr{I}hom^+\big(\mathbb{C}^{\mathrm{E}}_{\mathbb{V}_\infty}, \mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M})\big) \simeq \mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M}),$$

we have

$$\Psi_K(\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M})) \simeq \Psi_{L^a_{\mathbb{V}}}(\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M})).$$

Therefore

$${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M})) \simeq \mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L})\,[-d_{\mathbb{V}}].$$

Now choose $\mathscr{M} = \mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H})$ and apply Lemma 6.2. Since $\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}_\infty}(\mathscr{M}) \simeq \Omega^{\mathsf{E}}_{\mathbb{V}_\infty}$ and $\mathcal{DR}^{\mathsf{E}}_{\mathbb{V}^*_\infty}(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}) \simeq \Omega^{\mathsf{E}}_{\mathbb{V}_\infty} \otimes \det \mathbb{V}^*$, we obtain

$${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\Omega^{\mathsf{E}}_{\mathbb{V}_\infty}) \simeq \Omega^{\mathsf{E}}_{\mathbb{V}^*_\infty} \otimes \det \mathbb{V}^*\,[-d_{\mathbb{V}}].$$

Hence, it is enough to remark that

$$\Omega^{\mathsf{E}}_{\mathbb{V}_\infty} \simeq \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty} \otimes \det \mathbb{V}^* \text{ and } \Omega^{\mathsf{E}}_{\mathbb{V}^*_\infty} \simeq \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty} \otimes \det \mathbb{V}.$$

Q.E.D.

Remark 6.4. (i) Symbolically, isomorphism (6.6) is given by

$${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) \otimes \det \mathbb{V}^* \ni \varphi(x)dx \longmapsto \int \varphi(x)\mathrm{e}^{-\langle x,y\rangle}dx \in \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}.$$

(ii) The identification of $\mathsf{D}_{\mathbb{V}}$ and $\mathsf{D}_{\mathbb{V}^*}$ is by:

$$\begin{aligned}\mathsf{D}_{\mathbb{V}} \ni P(x,\partial_x) &\leftrightarrow Q(y,\partial_y) \in \mathsf{D}_{\mathbb{V}^*}\\ &\Longleftrightarrow P(x,\partial_x)\mathrm{e}^{\langle x,y\rangle} = Q^*(y,\partial_y)\mathrm{e}^{\langle x,y\rangle}\\ &\Longleftrightarrow P^*(x,\partial_x)\mathrm{e}^{-\langle x,y\rangle} = Q(y,\partial_y)\mathrm{e}^{-\langle x,y\rangle}.\end{aligned}$$

Here $Q^*(y,\partial_y)$ denotes the formal adjoint operator of $Q(y,\partial_y) \in \mathsf{D}_{\mathbb{V}^*}$.

Applying Corollary 5.3, we get:

Corollary 6.5. *Isomorphism* (6.4) *together with the enhanced Fourier-Sato isomorphism induce an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathsf{D}_{\mathbb{V}})$*, functorial in* $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$:

$$\mathrm{RHom}^{\mathrm{E}}(F, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) \simeq \mathrm{RHom}^{\mathrm{E}}\big({}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(F), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}\big) \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}]. \tag{6.7}$$

As a consequence of Corollary 6.5, we recover the main result of [KS97]:

Corollary 6.6. *Isomorphism* (6.4) *together with the Fourier-Sato isomorphism induces an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathsf{D}_{\mathbb{V}})$*, functorial in* $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{\mathbb{V}})$:

$$\mathrm{RHom}\,(G, \mathcal{O}^{\mathrm{t}}_{\mathbb{V}}) \simeq \mathrm{RHom}\,\big({}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(G), \mathcal{O}^{\mathrm{t}}_{\mathbb{V}^*}\big) \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}]. \tag{6.8}$$

Proof. By Theorem 5.7 we have the isomorphism ${}^{\mathsf{E}}\mathscr{F}_{\mathbb{V}}(\varepsilon_{\mathbb{V}}(G)) = \varepsilon_{\mathbb{V}^*}{}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(G)$, where $\varepsilon_{\mathbb{V}}$ is given in (2.14). Applying isomorphism (6.7) with $F = \varepsilon_{\mathbb{V}}(G)$, we obtain

$$\mathrm{RHom}^{\mathrm{E}}(\varepsilon_{\mathbb{V}}(G), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) \simeq \mathrm{RHom}^{\mathrm{E}}\big(\varepsilon_{\mathbb{V}^*}{}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(G), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}\big) \otimes \det \mathbb{V}\,[-d_{\mathbb{V}}].$$

By Proposition 3.10, we have

$$\begin{aligned}\mathrm{RHom}^{\mathrm{E}}(\varepsilon_{\mathbb{V}}(G), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}) &\simeq \mathrm{R}\Gamma\big(\mathbb{V}; \alpha_{\mathbb{V}}\mathscr{I}hom^{\mathrm{E}}(\varepsilon_{\mathbb{V}}(G), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty})\big)\\ &\simeq \mathrm{R}\Gamma\big(\mathbb{V}; \alpha_{\mathbb{V}}\,\mathrm{R}\mathscr{I}hom\,(G, \mathcal{O}^{\mathrm{t}}_{\mathbb{V}})\big)\\ &\simeq \mathrm{RHom}\,(G, \mathcal{O}^{\mathrm{t}}_{\mathbb{V}}),\end{aligned}$$

and similarly $\mathrm{RHom}^{\mathrm{E}}(\varepsilon_{\mathbb{V}^*}{}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(G), \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}) \simeq \mathrm{RHom}\,({}^{\mathrm{S}}\mathscr{F}_{\mathbb{V}}(G), \mathcal{O}^{\mathrm{t}}_{\mathbb{V}^*})$. Q.E.D.

6.2 Enhanced distributions

Let M be a real analytic manifold and consider the natural morphism of bordered spaces

$$j\colon M \times \mathbb{R}_\infty \to M \times \mathsf{P}.$$

Definition 6.7 (See [DK13, Def. 8.1.1])**.** One sets

$$\mathcal{D}b^{\mathsf{T}}_M = j^{\,!}\,\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathsf{P}}}(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}})[1] \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}\times\mathbb{R}_\infty}),$$

and denotes by $\mathcal{D}b^{\mathsf{E}}_M$ the corresponding object of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{M}})$.

Proposition 6.8 (See [DK13, Pro. 8.1.3]). *There are isomorphisms*

$$\mathrm{R}^{\mathrm{E}}\mathcal{D}b^{\mathsf{E}}_M \simeq \mathcal{D}b^{\mathsf{T}}_M \quad \textit{in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}\times\mathbb{R}_\infty}),$$
$$\mathbb{C}^{\mathrm{E}}_M \overset{+}{\otimes} \mathcal{D}b^{\mathsf{E}}_M \simeq \mathcal{D}b^{\mathsf{E}}_M \quad \textit{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{M}}).$$

Remark 6.9. Let X be a complex manifold and denote as usual by $X_{\mathbb{R}}$ the underlying real analytic manifold to X and X^c the complex conjugate manifold. Then

$$\mathscr{O}^{\mathsf{E}}_X \simeq \mathrm{R}\mathscr{H}om_{\pi^{-1}\mathscr{D}_{X^c}}(\pi^{-1}\mathscr{O}_{X^c}, \mathcal{D}b^{\mathsf{E}}_{X_{\mathbb{R}}}). \tag{6.9}$$

Definition 6.10. Let $U \subset M$ be an open subanalytic subset and let $\varphi\colon U \to \mathbb{R}$ be a continuous map. We say that φ is of class (ASA) (almost subanalytic) on U if there exists a C^∞-function $\psi\colon U \to \mathbb{R}$ such that

(i) ψ is subanalytic on M, that is, the graph $\Gamma_\psi \subset U \times \mathbb{R}$ is subanalytic in $M \times \overline{\mathbb{R}}$,

(ii) there exists a constant $C > 0$ such that $|\varphi(x) - \psi(x)| \leq C$ for all $x \in U$.

In such a case, we say that ψ belongs to the (ASA)-class of φ.

Conjecture 6.11. Let $U \subset M$ be an open subanalytic subset and let $\varphi\colon U \to \mathbb{R}$ be a continuous map, subanalytic on M. We conjecture that such a φ is always of class (ASA).

Let U and φ be as above with φ of class (ASA) and let us choose ψ as in Definition 6.10. For an open subanalytic subset V of M, we set:

$$\mathrm{e}^{-\varphi}\mathcal{D}b^{\mathrm{t}}_M(V) = \left\{u \in \mathcal{D}b_M(V \cap U)\,;\, \mathrm{e}^{\psi}u \in \mathcal{D}b^{\mathrm{t}}_M(U \cap V)\right\}. \tag{6.10}$$

The correspondence $V \mapsto \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}}_M(V)$ defines a sheaf on M_{sa}, hence an ind-sheaf on M. We denote this ind-sheaf by $\mathrm{e}^{-\varphi}\mathcal{D}b^{\mathrm{t}}_M$.

Theorem 6.12. *Let φ be a continuous function on a subanalytic open subset U of M. Assume that φ is of class* (ASA)*. Then the right-hand side of* (6.10) *does not depend on the choice of ψ as soon as ψ belongs to the* (ASA)*-class of φ. Moreover, we have isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{M}})$

$$\begin{aligned}\mathrm{e}^{-\varphi}\mathcal{D}b^{\mathrm{t}}_M &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq\varphi(x), x\in U\}}, \mathcal{D}b^{\mathsf{E}}_M) \\ &\simeq \mathrm{R}\pi_*\,\mathrm{R}\mathscr{I}hom\big(\mathbb{C}_{\{t<\varphi(x)\,;\,x\in U\}}, \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathbb{P}}}(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}})\big).\end{aligned} \tag{6.11}$$

In particular, these objects are concentrated in degree 0.

Proof. (i) Since

$$\mathbb{C}_M^{\mathsf{E}} \overset{+}{\otimes} \mathbb{C}_{W_\varphi} \simeq \mathbb{C}_M^{\mathsf{E}} \overset{+}{\otimes} \mathbb{C}_{W_\psi},$$

we have

$$\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq\varphi(x),x\in U\}}, \mathcal{D}b_M^{\mathsf{E}}) \simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq\psi(x),x\in U\}}, \mathcal{D}b_M^{\mathsf{E}}).$$

Hence, the fact that the right-hand side of (6.10) does not depend on the choice of ψ will follow from (6.11). Therefore, we may assume from the beginning that φ is C^∞ and subanalytic on M (see Definition 6.10 (i)).
(ii) Set $W_\varphi = \{(x,t)\,;\, t < \varphi(x); x \in U\}$. It is enough to show

$$\mathrm{e}^{-\varphi}\mathcal{D}b_M^{\mathrm{t}}(U) \simeq \mathrm{RHom}\,(\mathbb{C}_{W_\varphi}[1], \mathcal{D}b_M^{\mathsf{T}}). \tag{6.12}$$

(iii) Since $\mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}}(M\times\mathsf{P}) \to \mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}}(W)$ is surjective, $\partial_t - 1$ acting on $\mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}}(W)$ is surjective. It follows that the right-hand side of (6.12) is concentrated in degree 0. Set for short

$$S(\{t<\varphi\}) := \big\{u \in \mathcal{D}b_M(U)\,;\, \mathrm{e}^t u|_{\{t<\varphi(x)\}} \text{ is tempered on } M\times\mathsf{P}\big\}. \tag{6.13}$$

Then

$$\mathrm{Hom}\,(\mathbb{C}_{\{t<\varphi(x)\}}, \mathrm{R}\mathscr{H}om\,(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{D}b^{\mathrm{t}}_{M\times\mathsf{P}})) \simeq S(\{t<\varphi\}).$$

(iv) Assume that $u \in \Gamma(U; \mathrm{e}^{-\varphi}\mathcal{D}b_M^{\mathrm{t}})$. Let $w \in \mathcal{D}b(M)$ such that $w_{|_V} = \mathrm{e}^{\varphi}u$. Then $\mathrm{e}^t u = \mathrm{e}^{t-\varphi(x)}w$ and $\mathrm{e}^{t-\varphi(x)}$ is a C^∞-function tempered as a distribution on the set $\{(x,t)\,;\, t<\varphi(x)\}$. Therefore, $u \in S(\{t<\varphi\})$.
(v) Assume that $u \in S(\{t<\varphi\})$. Let $w \in \mathcal{D}b(M\times\mathsf{P})$ such that $w|_{\{t<\varphi\}} = \mathrm{e}^t u|_{\{t<\varphi\}}$. Let us choose a C^∞-function $\chi(t)$ supported on $\{t\in\mathbb{R}\,;\,|t|<1\}$ with $\int_{\mathbb{R}}\chi(t+1)\mathrm{e}^t\mathrm{d}t = 1$. Then $\chi(t-\varphi(x)+1)w = \chi(t-\varphi(x)+1)\mathrm{e}^t u$ and

$$\mathrm{e}^{\varphi}u = \int_{\mathbb{R}} \chi(t-\varphi(x)+1)w\,\mathrm{d}t|_V$$

is tempered. Q.E.D.

Lemma 6.13. *Let U be an open subanalytic subset and let $f\colon U \to \mathbb{R}\sqcup\{+\infty\}$ be a map such that the set $\{(x,t)\in U\times\mathbb{R}\,;\, t\geq f(x)\}$ is closed in $U\times\mathbb{R}$ and is subanalytic in $M\times\overline{\mathbb{R}}$. Then the objects $\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b_M^{\mathsf{E}})$ and $\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b_M^{\mathsf{E}})$ are concentrated in degree 0.*

Proof. (i) We know that $\mathrm{R}\pi_* \mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{t}}_{M\times\overline{\mathbb{R}}})$ is concentrated in degree 0. Then $\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathsf{E}}_M)$ is represented by the complex

$$\mathrm{R}\pi_* \mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{t}}_{M\times\overline{\mathbb{R}}}) \xrightarrow{\partial_t - 1} \mathrm{R}\pi_* \mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{t}}_{M\times\overline{\mathbb{R}}}),$$

concentrated in degrees -1 and 0. Hence, it is enough to check that the operator $\partial_t - 1$ acting on $\mathrm{R}\pi_* \mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{t}}_{M\times\overline{\mathbb{R}}})$ is a monomorphism. This follows from

$$\{u \in \mathcal{D}b_M(U \times \mathbb{R})\,;\, (\partial_t - 1)u = 0,\ \mathrm{supp}(u) \subset \{t \geq f(x)\}\} = 0.$$

(ii) Similarly, the complex $\mathrm{R}\Gamma\big(M; \mathrm{R}\pi_*\mathrm{R}\mathscr{H}om(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{t}}_{M\times\overline{\mathbb{R}}})\big)$ is concentrated in degree 0 and the operator $\partial_t - 1$ acting on this object is injective. Q.E.D.

6.3 Examples

In this subsection, we shall give some applications of Corollary 6.5 by using Theorems 6.12 and 5.9. As above, $\mathbb{V}$ is a complex vector space of dimension $d_{\mathbb{V}}$. We shall often identify $\mathbb{V}$ with the underlying real vector space of real dimension $2d_{\mathbb{V}}$.

Theorem 6.14. *Let $f \in \mathrm{Conv}(\mathbb{V})$. Then*

(i) *We have an isomorphism*

$$\begin{aligned}&\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty})\,[d_{\mathbb{V}}]\\ &\qquad\simeq \mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq -f^*(-y), -y\in\mathrm{dom}^\circ(f^*)\}}, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}) \otimes \det\mathbb{V}\end{aligned}$$

by the Laplace isomorphism (6.7)*.*

(ii) *Assume further that f is subanalytic on $\overline{\mathbb{V}}$ and $\mathrm{dom}^\circ(f^*)$ is an open subset of $\mathbb{V}^*$. Then, these objects are concentrated in degree* 0*.*

Proof. (i) follows from Corollary 6.5 and Theorem 5.9.
(ii) The object $\mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty}$ is represented by the Dolbeault complex

$$\mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty} \simeq \mathcal{D}b^{\mathsf{T}(0,\bullet)}_{\overline{\mathbb{V}}} := 0 \to \mathcal{D}b^{\mathsf{T}(0,0)}_{\overline{\mathbb{V}}} \xrightarrow{\bar\partial} \cdots \to \mathcal{D}b^{\mathsf{T}(0,d_{\mathbb{V}})}_{\overline{\mathbb{V}}} \to 0,$$

in which $\mathcal{D}b^{\mathsf{T}(0,0)}_{\overline{\mathbb{V}}}$ stands in degree 0. Then it follows from Lemma 6.13 applied with $M = \overline{\mathbb{V}}$ that $\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}_\infty})\,[d_{\mathbb{V}}]$ is concentrated in degrees $[-d_{\mathbb{V}}, 0]$ and $\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq -f^*(-y), -y\in\mathrm{dom}^\circ(f^*)\}}, \mathscr{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty})$ is concentrated in degrees $[0, d_{\mathbb{V}}]$. Q.E.D.

Let $U \subset \mathbb{V}$ be an open convex subanalytic subset and let $\varphi\colon U \to \mathbb{R}$ be a continuous function. We assume that φ is of class (ASA) on $\overline{\mathbb{V}}$. We define the object $\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}})$ by the Dolbeault complex:

$$\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}} := 0 \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,0)}_{\overline{\mathbb{V}}} \xrightarrow{\overline{\partial}} \cdots \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,d_{\mathbb{V}})}_{\overline{\mathbb{V}}} \to 0. \tag{6.14}$$

Hence, $\mathrm{R}\Gamma(U; \mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}}) := \mathrm{RHom}\,(\mathbb{C}_U, \mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}})$ is represented by the complex

$$0 \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,0)}_{\overline{\mathbb{V}}}(U) \xrightarrow{\overline{\partial}} \cdots \to \mathrm{e}^{\varphi}\mathcal{D}b^{\mathrm{t}\,(0,d_{\mathbb{V}})}_{\overline{\mathbb{V}}}(U) \to 0.$$

Corollary 6.15. *Let U be an open convex and subanalytic subset of $\mathbb{V}$. Let $\varphi\colon U \to \mathbb{R}$ be a continuous function and assume that φ is convex and of class* (ASA)*on $\overline{\mathbb{V}}$ (see* Definition 6.10*). Let $\widetilde{\varphi} \in \mathrm{Conv}(\mathbb{V})$ be the unique function such that $\widetilde{\varphi}|_U = \varphi$ and $\mathrm{dom}^{\circ}(\widetilde{\varphi}) = U$. Then*

(a) $\mathrm{R}\Gamma(U; \mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}})$ *is concentrated in degree 0 and its 0-th cohomology is the space* $\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}}(U) := \{u \in \mathscr{O}_{\mathbb{V}}(U)\,;\, \mathrm{e}^{-\varphi}u \text{ is tempered in } \overline{\mathbb{V}}\}$.

(b) *Assume moreover that $\widetilde{\varphi}^*$ is of class* (ASA) *on $\overline{\mathbb{V}}$ and $\mathrm{dom}(\widetilde{\varphi}^*) = \mathbb{V}^*$. Then $\mathrm{R}\Gamma(\mathbb{V}^*; \mathrm{e}^{-\widetilde{\varphi}^*}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}^*}})$ is concentrated in degree $d_{\mathbb{V}}$ and the Laplace transform induces an isomorphism between the space $\mathrm{e}^{\varphi}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}}(U)$ and the space $H^{d_{\mathbb{V}}}\big(\mathrm{R}\Gamma(\mathbb{V}^*; \mathrm{e}^{-\widetilde{\varphi}^*}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}^*}})\big)$.*

Proof. This follows from Theorems 6.14 and 6.12 applied to the function $\widetilde{\varphi}^*$. Q.E.D.

For an open subset Ω of $\mathbb{V}$, we set for short

$$\mathrm{d}_{\Omega}(x) = \mathrm{d}(x, \mathbb{V} \setminus \Omega),$$

and consider the Dolbeault complex (6.14) in which $\varphi(x) = \mathrm{d}_{\Omega}(x)^{-\lambda}$ ($\lambda \in \mathbb{Q}_{>0}$). For a subanalytic open subset U of Ω, we get the complex

$$\begin{aligned} 0 \to \mathrm{e}^{\mathrm{d}_{\Omega}(x)^{-\lambda}}\mathcal{D}b^{\mathrm{t}\,(0,0)}_{\overline{\mathbb{V}}}(U) &\xrightarrow{\overline{\partial}} \cdots \\ \cdots &\xrightarrow{\overline{\partial}} \mathrm{e}^{\mathrm{d}_{\Omega}(x)^{-\lambda}}\mathcal{D}b^{\mathrm{t}\,(0,d_{\mathbb{V}})}_{\overline{\mathbb{V}}}(U) \to 0 \end{aligned} \tag{6.15}$$

Corollary 6.16. *Let Ω and U be open convex subanalytic subsets of $\mathbb{V}_{\infty}$ with $U \subset \Omega$, and let $\lambda \in \mathbb{Q}_{>0}$. Then, if $\mathrm{d}_{\Omega}(x)^{-\lambda}$ is of class* (ASA), the complex (6.15) is concentrated in degree 0 and its 0-th cohomology is the space

$$\mathrm{e}^{\mathrm{d}_{\Omega}^{-\lambda}}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}}(U) := \Big\{u \in \mathscr{O}_{\mathbb{V}}(U)\,;\, \mathrm{e}^{-\mathrm{d}_{\Omega}(x)^{-\lambda}}u \text{ is tempered in } \overline{\mathbb{V}}\Big\}.$$

Proof. Apply Corollary 6.15 with $\varphi(x) = \mathrm{d}_\Omega(x)^{-\lambda}$. This function is convex by Lemma 5.12. Q.E.D.

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