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A finiteness theorem for holonomic DQ-modules on Poisson manifolds

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On a complex symplectic manifold, we prove a finiteness result for the global sections of solutions of holonomic DQ-modules in two cases: (a) by assuming that there exists a Poisson compactification, (b) in the algebraic case. This extends our previous result in which the symplectic manifold was compact. The main tool is a finiteness theorem for $\mathbb{R}$-constructible sheaves on a real analytic manifold in a nonproper situation.

1. Introduction and statement of the results

Consider a complex Poisson manifold X of complex dimension d_X endowed with a DQ-algebroid $\mathscr{A}_X$. Recall that $\mathscr{A}_X$ is a $\mathbb{C}[\![\hbar]\!]$-algebroid locally isomorphic to a star algebra $(\mathscr{O}_X[\![\hbar]\!], \star)$ to which the Poisson structure is associated. Denote by $\mathscr{A}_X^{\mathrm{loc}}$ the localization of $\mathscr{A}_X$ with respect to $\hbar$, a $\mathbb{C}((\hbar))$-algebroid. For short, we set

$$\mathbb{C}^{\hbar} := \mathbb{C}[\![\hbar]\!], \quad \mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar)).$$

Hence $\mathscr{A}_X^{\mathrm{loc}} \simeq \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^{\hbar}} \mathscr{A}_X$. The algebroids $\mathscr{A}_X$ and $\mathscr{A}_X^{\mathrm{loc}}$ are right and left Noetherian (in particular coherent) and if $\mathscr{M}$ is a (say left) coherent $\mathscr{A}_X^{\mathrm{loc}}$-module, then its support is a closed complex analytic subvariety of X and it follows from Gabber's theorem that it is coisotropic. In the extreme case where X is symplectic and the support is Lagrangian, one says that $\mathscr{M}$ is holonomic.

Recall the following definitions (see [Kashiwara and Schapira 2012, Definitions 2.3.14, 2.3.16 and 2.7.2]).

(a) A coherent $\mathscr{A}_X$-submodule $\mathscr{M}_0$ of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called an $\mathscr{A}_X$-lattice of $\mathscr{M}$ if $\mathscr{M}_0$ generates $\mathscr{M}$.

(b) A coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is good if, for any relatively compact open subset U of X, there exists an $(\mathscr{A}_X|_U)$-lattice of $\mathscr{M}|_U$.

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(c) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects with good cohomology.

(d) In the algebraic case (see below) a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called algebraically good if there exists an $\mathscr{A}_X$-lattice of $\mathscr{M}$. One still denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects with algebraically good cohomology.

Let $Y \subset X$. We shall consider the hypothesis

$$Y \text{ is open, relatively compact, subanalytic in } X \text{ and the Poisson structure on } X \text{ is symplectic on } Y. \tag{1-1}$$

Example 1.1. Denote by X_{ns} the closed complex subvariety of X consisting of points where the Poisson bracket is not symplectic and set $Y = X \setminus X_{\mathrm{ns}}$. Hence Y is an open subanalytic subset of X and is symplectic. If X is compact, then Y satisfies hypothesis (1-1).

In this paper we shall prove the following theorem which extends [Kashiwara and Schapira 2012, Theorem 7.2.3] in which X was symplectic, that is, $Y = X$.

Theorem 1.2. *Assume that Y satisfies hypothesis (1-1). Let $\mathscr{M}$ and $\mathscr{L}$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and assume that both $\mathscr{M}|_Y$ and $\mathscr{L}|_Y$ are holonomic. Then the two complexes*

$$\mathrm{RHom}_{\mathscr{A}_Y^{\mathrm{loc}}}(\mathscr{M}|_Y, \mathscr{L}|_Y) \quad \textit{and} \quad \mathrm{R}\Gamma_c\big(Y; \mathrm{R}\mathscr{H}om_{\mathscr{A}_Y^{\mathrm{loc}}}(\mathscr{L}|_Y, \mathscr{M}|_Y)\big)[d_X]$$

have finite dimensional cohomology over $\mathbb{C}^{\hbar,\mathrm{loc}}$ and are dual to each other.

We shall also obtain a similar conclusion under rather different hypotheses, namely that $X = Y$ is symplectic and all data are algebraic (see [Kashiwara and Schapira 2012, §2.7]). Let X be a smooth algebraic variety and let $\mathscr{A}_X$ be a DQ-algebroid on X. We denote by X_{an} the associated complex analytic manifold and $\mathscr{A}_{X_{\mathrm{an}}}$ the associated DQ-algebroid on X_{an} (see Lemma 5.1). For a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ we denote by $\mathscr{M}_{\mathrm{an}}$ its image by the natural functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}})$.

Theorem 1.3. *Let X be a quasicompact separated smooth symplectic algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. Let $\mathscr{M}$ and $\mathscr{L}$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$. Then the two complexes*

$$\mathrm{RHom}_{\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}}}(\mathscr{M}_{\mathrm{an}}, \mathscr{L}_{\mathrm{an}}) \quad \textit{and} \quad \mathrm{R}\Gamma_c\big(X_{\mathrm{an}}; \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}}}(\mathscr{L}_{\mathrm{an}}, \mathscr{M}_{\mathrm{an}})\big)[d_X]$$

have finite dimensional cohomology over $\mathbb{C}^{\hbar,\mathrm{loc}}$ and are dual to each other.

The main tool in the proof of both theorems is Theorem 2.2 below which gives a finiteness criterion for $\mathbb{R}$-constructible sheaves on a real analytic manifold in a nonproper situation.

This note is motivated by the paper of Sam Gunningham, David Jordan and Pavel Safronov [Gunningham et al. 2019] on Skein algebras, whose main theorem is based on such a finiteness result (see [loc. cit., §3]). The proof of these authors uses a kind of Nakayama theorem in the case where $\mathscr{M}$ and $\mathscr{L}$ are simple modules over smooth Lagrangian varieties.

2. Finiteness results for constructible sheaves

In this paper, $\boldsymbol{k}$ is a Noetherian commutative ring of finite global homological dimension.

We denote by $\mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$ consisting of objects with finitely generated cohomology. We denote by D the duality functor $\mathrm{RHom}(\,\bullet\,, \boldsymbol{k})$ and we say that two objects A and B of $\mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$ are dual to each other if $\mathrm{D}A \simeq B$, which is equivalent to $\mathrm{D}B \simeq A$.

For a sheaf of rings $\mathscr{R}$, one denotes by $\mathrm{D}(\mathscr{R})$ the derived category of left $\mathscr{R}$-modules. We shall also encounter the full triangulated subcategory $\mathrm{D}^{+}(\mathscr{R})$ or $\mathrm{D}^{\mathrm{b}}(\mathscr{R})$ of complexes whose cohomology is bounded from below or is bounded.

For a real analytic manifold M, one denotes by $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ the bounded derived category of sheaves of $\boldsymbol{k}$-modules on M. We shall use the six Grothendieck operations. In particular, we denote by ω_M the dualizing complex. We also use the notation for $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_M)$:

$$\mathrm{D}'_M F := \mathrm{R}\mathscr{H}om(F, \boldsymbol{k}_M), \quad \mathrm{D}_M F := \mathrm{R}\mathscr{H}om(F, \omega_M).$$

Recall that an object F of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ is weakly $\mathbb{R}$-constructible if condition (i) below is satisfied. If moreover condition (ii) is satisfied, then one says that F is $\mathbb{R}$-constructible.

(i) There exists a subanalytic stratification $M = \bigsqcup_{a \in A} M_a$ such that $H^j(F)|_{M_a}$ is locally constant for all $j \in \mathbb{Z}$ and all $a \in A$.

(ii) $H^j(F)_x$ is finitely generated for all $x \in M$ and all $j \in \mathbb{Z}$.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_M)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ consisting of $\mathbb{R}$-constructible objects.

If X is a complex analytic manifold, one defines similarly the notions of (weakly) $\mathbb{C}$-constructible sheaf, replacing "subanalytic" with "complex analytic" and one denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ consisting of $\mathbb{C}$-constructible objects.

We shall use the following classical result (see [Kashiwara and Schapira 1990, Proposition 8.4.8 and Exercise VIII.3]).

Proposition 2.1. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_M)$ and assume that F has compact support. Then both objects $\mathrm{R}\Gamma(M; F)$ and $\mathrm{R}\Gamma(M; \mathrm{D}_M F)$ belong to $\mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$ and are dual to each other.*

For $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_M)$, one denotes by $\mathrm{SS}(F)$ its microsupport [Kashiwara and Schapira 1990, Definition 5.1.2], a closed $\mathbb{R}^+$-conic (i.e., invariant by the $\mathbb{R}^+$-action on T^*M) subset of T^*M. Recall that this set is involutive (one also says *coisotropic*), see [loc. cit., Definition 6.5.1].

Theorem 2.2. *Let $j : U \hookrightarrow M$ be the embedding of an open subanalytic subset U of M and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_U)$. Assume that $\mathrm{SS}(F)$ is contained in a closed subanalytic $\mathbb{R}^+$-conic Lagrangian subset Λ of T^*U which is subanalytic in T^*M. Then $\mathrm{R}j_*F$ and $j_!F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_M)$.*

Proof. (i) Let us treat first $j_!F$. The set Λ is a locally closed subanalytic subset of T^*M and is isotropic. By [Kashiwara and Schapira 1990, Corollary 8.3.22], there exists a μ-stratification $M = \bigsqcup_{a\in A} M_a$ such that $\Lambda \subset \bigsqcup_{a\in A} T^*_{M_a}M$.

Set $U_a = U \cap M_a$. Then $U = \bigsqcup_{a\in A} U_a$ is a μ-stratification and one can apply [loc. cit., Proposition 8.4.1]. Hence, for each $a \in A$, $F|_{U_a}$ is locally constant of finite rank. Hence $(j_!F)|_{U_a}$ as well as $(j_!F)_{M\setminus U} \simeq 0$ is locally constant of finite rank. Hence $j_!F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_M)$.

(ii) Set $G = j_!F$. Then $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_M)$ by (i) and also $\mathrm{R}j_*F \simeq \mathrm{R}\mathcal{H}om(\boldsymbol{k}_U, G)$ (apply [Kashiwara and Schapira 1990, Proposition 8.4.10]). □

Remark 2.3. One has $\mathrm{SS}(\mathrm{D}_M F) = \mathrm{SS}(F)^a$, where $(\bullet)^a$ is the antipodal map. Hence $\mathrm{D}_M F$ satisfies the same hypotheses as F.

Corollary 2.4. *In the preceding situation, assume moreover that U is relatively compact in M. Then $\mathrm{R}\Gamma(U; F)$ and $\mathrm{R}\Gamma_c(U; \mathrm{D}_U F)$ belong to $\mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$ and are dual to each other.*

Proof. One has

$$\mathrm{R}\Gamma(U; F) \simeq \mathrm{R}\Gamma(M; \mathrm{R}j_*F) \quad \text{and} \quad \mathrm{R}\Gamma_c(U; \mathrm{D}_U F) \simeq \mathrm{R}\Gamma(M; \mathrm{D}_M \mathrm{R}j_*F).$$

Since $\mathrm{R}j_*F$ is $\mathbb{R}$-constructible and has compact support, the result follows from Proposition 2.1. □

In this paper, a smooth algebraic variety X means a quasicompact smooth algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. We denote by X_{an} the complex analytic manifold underlying X. If X is a smooth algebraic variety, we keep the notation $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_X)$ to denote the category of algebraically constructible sheaves, that is, object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_{X_{\mathrm{an}}})$ locally constant on an algebraic stratifications. Hence, for an algebraic variety X, one shall not confuse $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_{X_{\mathrm{an}}})$, although $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_X)$ is a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_{X_{\mathrm{an}}})$.

Corollary 2.5. *Let X be a smooth algebraic variety and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\boldsymbol{k}_X)$. Then $\mathrm{R}\Gamma(X_{\mathrm{an}}; F)$ and $\mathrm{R}\Gamma_c(X_{\mathrm{an}}; \mathrm{D}_{X_{\mathrm{an}}} F)$ have finite dimensional cohomology over $\boldsymbol{k}$ and are dual to each other.*

Proof. Let Z be a smooth algebraic compactification of X with X open in Z. By the hypothesis, Λ is a closed algebraic subvariety of T^*X. Hence, its closure in T^*Z is a closed algebraic subvariety of T^*Z. Therefore Λ is subanalytic in T^*Z_{an}.
Then the result follows from Corollary 2.4 with $M = Z_{\mathrm{an}}$ and $U = X_{\mathrm{an}}$. □

3. Reminders on DQ-modules, after [Kashiwara and Schapira 2012]

3A. ***Cohomologically complete modules.*** In this subsection,

$$X \text{ denotes a topological space and } \mathscr{R} \text{ is a sheaf of } \mathbb{Z}[\hbar]\text{-algebras on } X \text{ with no } \hbar\text{-torsion.} \tag{3-1}$$

Let $\mathscr{M}$ be an $\mathscr{R}$-module. (Hence, a $\mathbb{Z}_X[\hbar]$-module.) One sets

$$\begin{aligned} \mathscr{R}^{\mathrm{loc}} &:= \mathbb{Z}_X[\hbar,\hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{R}, \\ \mathscr{M}^{\mathrm{loc}} &:= \mathscr{R}^{\mathrm{loc}} \otimes_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X[\hbar,\hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{M}, \\ \mathrm{gr}_\hbar(\mathscr{R}) &:= \mathscr{R}/\hbar\mathscr{R}, \\ \mathrm{gr}_\hbar(\mathscr{M}) &:= \mathrm{gr}_\hbar(\mathscr{R}) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathscr{M}. \end{aligned}$$

Definition 3.1 [Kashiwara and Schapira 2012, Definition 1.5.5]. One says that an object $\mathscr{M}$ of $\mathrm{D}(\mathscr{R})$ is cohomologically complete if it belongs to $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$, that is, $\mathrm{Hom}_{\mathrm{D}(\mathscr{R})}(\mathscr{N},\mathscr{M}) \simeq 0$ for any $\mathscr{N} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$.

Proposition 3.2 [Kashiwara and Schapira 2012, Proposition 1.5.6]. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$. Then the conditions below are equivalent.*

(a) *$\mathscr{M}$ is cohomologically complete,*

(b) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}},\mathscr{M}) \simeq 0$,

(c) $\varinjlim_{U \ni x} \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], H^i(U;\mathscr{M})\big) \simeq 0$ *for any $x \in X$, $j = 0, 1$ and any $i \in \mathbb{Z}$. Here, U ranges over an open neighborhood system of x.*

Denote by $\mathrm{D}_{\mathrm{cc}}(\mathscr{R})$ the full subcategory of $\mathrm{D}(\mathscr{R})$ consisting of cohomologically complete modules. Then clearly $\mathrm{D}_{\mathrm{cc}}(\mathscr{R})$ is triangulated.

Proposition 3.3 [Kashiwara and Schapira 2012, Proposition 1.5.10, Corollary 1.5.9]. *Let $\mathscr{M} \in \mathrm{D}_{\mathrm{cc}}(\mathscr{R})$. Then*

(a) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{N},\mathscr{M}) \in \mathrm{D}(\mathbb{Z}_X[\hbar])$ *is cohomologically complete for any $\mathscr{N} \in \mathrm{D}(\mathscr{R})$.*

(b) *If $\mathrm{gr}_\hbar(\mathscr{M}) \simeq 0$, then $\mathscr{M} \simeq 0$.*

Proposition 3.4 [Kashiwara and Schapira 2012, Proposition 1.5.12]. *Let $f : X \to Y$ be a continuous map and let $\mathscr{M} \in \mathrm{D}(\mathbb{Z}_X[\hbar])$. If $\mathscr{M}$ is cohomologically complete, then so is $\mathrm{R}f_*\mathscr{M}$.*

Proposition 3.5. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ be a cohomologically complete object and $a \in \mathbb{Z}$. If $H^i(\mathrm{gr}_\hbar(\mathscr{M})) = 0$ for any $i \geq a$, then $H^i(\mathscr{M}) = 0$ for any $i > a$.*

Proof. The proof is exactly the same as that of [Kashiwara and Schapira 2012, Proposition 1.5.8] when replacing $i > a$ with $i < a$. □

3B. *Microsupport and constructible sheaves.* Let M be a *real analytic* manifold and let $\boldsymbol{k}$ be a Noetherian commutative ring of finite global homological dimension.

We shall need the next result which does not appear in [Kashiwara and Schapira 2012].

Proposition 3.6. *Let* $F \in \mathrm{D^b}(\mathbb{Z}_M[\hbar])$. *Then* $\mathrm{SS}(F^{\mathrm{loc}}) \subset \mathrm{SS}(F)$.

Proof. By using one of the equivalent definitions of the microsupport given in [Kashiwara and Schapira 1990, Proposition 5.11], it is enough to check that for K compact, $\mathrm{R}\Gamma(K;F)^{\mathrm{loc}} \simeq \mathrm{R}\Gamma(K;F^{\mathrm{loc}})$ which follows from [loc. cit., Proposition 2.6.6] and the fact that $\mathbb{Z}[\hbar,\hbar^{-1}]$ is flat over $\mathbb{Z}[\hbar]$. □

Proposition 3.7 [Kashiwara and Schapira 2012, Proposition 7.1.6]. *Assume that* $F \in \mathrm{D^b}(\mathbb{Z}_M[\hbar])$ *is cohomologically complete. Then*

$$\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F)). \tag{3-2}$$

Proof. Let us recall the proof of [loc. cit.]. The inclusion

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) \subset \mathrm{SS}(F)$$

follows from the distinguished triangle $F \xrightarrow{\hbar} F \to \mathrm{gr}_\hbar(F) \xrightarrow{+1}$. Let us prove the converse inclusion.

Using the definition of the microsupport, it is enough to prove that given two open subsets $U \subset V$ of M, $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F)$ is an isomorphism as soon as $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F))$ is an isomorphism. Consider a distinguished triangle $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F) \to G \to [+1]$. Then we get a distinguished triangle $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F)) \to \mathrm{gr}_\hbar(G) \xrightarrow{+1}$. Therefore, $\mathrm{gr}_\hbar(G) \simeq 0$. On the other hand, G is cohomologically complete, thanks to Proposition 3.4 (applied to $F|_U$ and $F|_V$) and then $G \simeq 0$ by Proposition 3.3(b). □

Proposition 3.8 [Kashiwara and Schapira 2012, Proposition 7.1.7]. *Let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{C}^\hbar)$. *Then* F *is cohomologically complete.*

Proof. Let us recall the proof of [loc. cit.]. One has

$$\begin{aligned}\text{“}\varinjlim_{U\ni x}\text{”} \operatorname{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], H^i(U;F)\big) &\simeq \operatorname{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], \text{“}\varinjlim_{U\ni x}\text{”} H^i(U;F)\big)\\ &\simeq \operatorname{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], F_x\big) \simeq 0,\end{aligned}$$

where the last isomorphism follows from the fact that F_x is cohomologically complete.

Hence, hypothesis (c) of Proposition 3.2 is satisfied. □

3C. DQ-*modules*. In this subsection, X will be a complex manifold (not necessarily symplectic) of complex dimension d_X.

Set $\mathscr{O}_X^\hbar := \mathscr{O}_X[\![\hbar]\!] = \varprojlim_n \mathscr{O}_X \otimes_{\mathbb{C}} (\mathbb{C}^\hbar/\hbar^n\mathbb{C}^\hbar)$. An associative multiplication law $\star$ on $\mathscr{O}_X^\hbar$ is a star-product if it is $\mathbb{C}^\hbar$-bilinear and satisfies

$$f \star g = \sum_{i\geq 0} P_i(f, g)\hbar^i \quad \text{for } f, g \in \mathscr{O}_X, \tag{3-3}$$

where the P_i are bidifferential operators, $P_0(f, g) = fg$ and $P_i(f, 1) = P_i(1, f) = 0$ for $f \in \mathscr{O}_X$ and $i > 0$.

We call $(\mathscr{O}_X[\![\hbar]\!], \star)$ a *star-algebra*. A star-product defines a Poisson structure on $(X, \mathscr{O}_X)$ by the formula

$$\{f, g\} = P_1(f, g) - P_1(g, f) \equiv \hbar^{-1}(f \star g - g \star f) \mod \hbar\mathscr{O}_X[\![\hbar]\!]. \tag{3-4}$$

Definition 3.9. A DQ-algebroid $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebroid locally isomorphic to a star-algebra as a $\mathbb{C}^\hbar$-algebroid.

Remark 3.10. The data of a DQ-algebroid $\mathscr{A}_X$ on X endows X with a structure of a complex Poisson manifold and one says that $\mathscr{A}_X$ is a quantization of the Poisson manifold. Kontsevich's famous theorem [2001; 2003] (see also [Kashiwara 1996] for the case of contact manifolds) asserts that any complex Poisson manifold may be quantized.

Example 3.11. Assume that M is an open subset of $\mathbb{C}^n$, $X = T^*M$ and denote by $(x; u)$ the symplectic coordinates on X. In this case there is a canonical star-algebra $\mathscr{A}_X$ that is usually denoted by $\widehat{\mathscr{W}}_X(0)$, its localization with respect to $\hbar$ being denoted by $\widehat{\mathscr{W}}_X$.

Let $f, g \in \mathscr{O}_X[\![\hbar]\!]$. Then the DQ-algebra $\widehat{\mathscr{W}}_X(0)$ is the star-algebra $(\mathscr{O}_X[\![\hbar]\!], \star)$ where

$$f \star g = \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{3-5}$$

This product is similar to the product of the total symbols of differential operators on X and indeed, there is a natural morphism of $\mathbb{C}$-algebras $\pi_M^{-1}\mathscr{D}_M \to \widehat{\mathscr{W}}_X$ given by

$$f(x) \mapsto f(x), \quad \partial_{x_i} \mapsto \hbar^{-1}u_i,$$

where, as usual, $\mathscr{D}_M$ denotes the ring of finite order holomorphic differential operators and $\pi_M : T^*M \to M$ is the projection.

For a DQ-algebroid $\mathscr{A}_X$, there is locally an isomorphism of $\mathbb{C}$-algebroids

$$\mathrm{gr}_\hbar(\mathscr{A}_X) := \mathscr{A}_X/\hbar\mathscr{A}_X \xrightarrow{\sim} \mathscr{O}_X.$$

Moreover there exists a unique isomorphism of $\mathbb{C}$-algebras

$$\mathscr{E}nd(\mathrm{id}_{\mathrm{gr}_\hbar \mathscr{A}_X}) \simeq \mathscr{O}_X. \tag{3-6}$$

Therefore, there is a well-defined functor

$$\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \bullet : \mathrm{D}^{\mathrm{b}}(\mathscr{O}_X) \times \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar \mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar \mathscr{A}_X). \tag{3-7}$$

Theorem 3.12 [Kashiwara and Schapira 2012, Theorem 1.2.5]. *For a* DQ*-algebroid* $\mathscr{A}_X$, *both* $\mathscr{A}_X$ *and* $\mathscr{A}_X^{\mathrm{loc}}$ *are right and left Noetherian* (*in particular*, *coherent*).

One defines the functors

$$\begin{aligned}
\mathrm{gr}_\hbar &: \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar \mathscr{A}_X), \qquad \mathscr{M} \mapsto \mathbb{C}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{M}, \\
(\bullet)^{\mathrm{loc}} &: \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X^{\mathrm{loc}}), \qquad \mathscr{M} \mapsto \mathbb{C}_X^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}_X^\hbar} \mathscr{M}, \\
\mathit{for} &: \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar(\mathscr{A}_X)) \to \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \quad \text{associated with } \sigma_0 : \mathscr{A}_X \to \mathrm{gr}_\hbar(\mathscr{A}_X).
\end{aligned}$$

The functor $(\bullet)^{\mathrm{loc}}$ is exact on $\mathrm{Mod}(\mathscr{A}_X)$. The category $\mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}_X)$ consisting of objects $\mathscr{M}$ such that $\hbar : \mathscr{M} \to \mathscr{M}$ vanishes.

Theorem 3.13 [Kashiwara and Schapira 2012, Theorems 1.6.1 and 1.6.4]. *Let* $\mathscr{M} \in \mathrm{D}^+(\mathscr{A}_X)$. *Then the two conditions below are equivalent*:

(a) $\mathscr{M}$ *is cohomologically complete and* $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D}^+_{\mathrm{coh}}(\mathrm{gr}_\hbar \mathscr{A}_X)$,

(b) $\mathscr{M} \in \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A}_X)$.

The next result follows from Gabber's theorem [1981].

Proposition 3.14 [Kashiwara and Schapira 2012, Proposition 2.3.18]. *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. *Then* $\mathrm{supp}(\mathscr{M})$ (*the support of* $\mathscr{M}$) *is a closed complex analytic subset of* X, *involutive* (*i.e.*, *coisotropic*) *for the Poisson bracket on* X.

Remark 3.15. One should be aware that the support of a coherent $\mathscr{A}_X$-module is not involutive in general. Indeed, any coherent $\mathrm{gr}_\hbar \mathscr{A}_X$-module may be regarded as an $\mathscr{A}_X$-module. Hence any closed analytic subset can be the support of a coherent $\mathscr{A}_X$-module.

4. DQ-modules along Λ

4A. ***A variation on a theorem of*** [Kashiwara 2003]**.** In order to prove Lemma 4.6 below, we need a slight modification of a result of [Kashiwara 2003].

Let $\mathscr{R}$ be a ring on a topological space X, and let $\{F_n(\mathscr{R})\}_{n\in\mathbb{Z}}$ be a filtration of $\mathscr{R}$ which satisfies

(a) $\mathscr{R} = \bigcup_{n\in\mathbb{Z}} F_n(\mathscr{R})$,

(b) $1 \in F_0(\mathscr{R})$,

(c) $F_m(\mathscr{R}) \cdot F_n(\mathscr{R}) \subset F_{m+n}(\mathscr{R})$.

We set

$$\mathrm{gr}^F_{\geq 0}(\mathscr{R}) = \bigoplus_{n \geq 0} \mathrm{gr}^F_n(\mathscr{R}).$$

Proposition 4.1. *Assume that*

(a) *$F_0(\mathscr{R})$ and $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$ are Noetherian rings,*

(b) *$\mathrm{gr}^F_n(\mathscr{R})$ is a coherent $F_0(\mathscr{R})$-module for any $n \geq 0$.*

Then $\mathscr{R}$ is Noetherian.

Proof. Define $\widetilde{F}_n(\mathscr{R})$ by

$$\widetilde{F}_n(\mathscr{R}) = \begin{cases} F_n(\mathscr{R}) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We shall apply [Kashiwara 2003, Theorem A.20] to $\widetilde{F}_k(\mathscr{R})$. Hence in order to prove the theorem, it is enough to show:

$$\text{(4-1)}\quad \begin{array}{l} \text{for any positive integer } m \text{ and an open subset } U \text{ of } X, \text{ if an} \\ \mathscr{R}|_U\text{-submodule } \mathscr{N} \text{ of } \mathscr{R}^{\oplus m}|_U \text{ has the property that } F_k(\mathscr{N}) := \\ \mathscr{N} \cap F_k(\mathscr{R})^{\oplus m}|_U \text{ is a coherent } F_0(\mathscr{R})|_U\text{-module for any } k \geq 0, \\ \text{then } \mathscr{N} \text{ is a locally finitely generated } \mathscr{R}|_U\text{-module.} \end{array}$$

Since $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$ is a Noetherian ring, $\mathrm{gr}^F_{\geq 0}(\mathscr{N}) := \bigoplus_{n \geq 0} \mathrm{gr}^F_n(\mathscr{N})$ is a coherent $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$-module. Hence there exists locally a finitely generated $\mathscr{R}$-submodule $\mathscr{N}'$ of $\mathscr{N}$ such that $\mathrm{gr}^F_{\geq 0}(\mathscr{N}') = \mathrm{gr}^F_{\geq 0}(\mathscr{N})$. Hence we have $\mathscr{N} = \mathscr{N}' + F_0(\mathscr{N})$. Since $F_0(\mathscr{N})$ is a locally finitely generated $F_0(\mathscr{R})$-module, $\mathscr{N}$ is a locally finitely generated $\mathscr{R}$-module. □

4B. *The algebroid $\mathscr{A}_{\Lambda/X}$.* From now on, X is a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$.

Definition 4.2 [Kashiwara and Schapira 2012, Definition 2.3.10]. Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathscr{L}$ is simple along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is concentrated in degree 0 and $H^0(\mathrm{gr}_\hbar(\mathscr{L}))$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_X)$-module. (In particular, $\mathscr{L}$ has no $\hbar$-torsion.)

Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module simple along Λ. We set for short

$$\mathscr{O}^\hbar_\Lambda := \mathscr{O}_\Lambda[\![\hbar]\!], \quad \mathscr{O}^{\hbar,\mathrm{loc}}_\Lambda := \mathscr{O}_\Lambda((\hbar)),$$
$$\mathscr{D}^\hbar_\Lambda := \mathscr{D}_\Lambda[\![\hbar]\!], \quad \mathscr{D}^{\hbar,\mathrm{loc}}_\Lambda := \mathscr{D}_\Lambda((\hbar)).$$

One proves that there is a natural isomorphism of algebroids $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L}) \simeq \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_\Lambda^\hbar)$ [Kashiwara and Schapira 2012, Lemma 2.1.12]. Then the subalgebroid of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L})$ corresponding to the subring $\mathscr{D}_\Lambda[[\hbar]]$ of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_\Lambda^\hbar)$ is well-defined. We denote it by $\mathscr{D}_\mathscr{L}$. Then (see [Kashiwara and Schapira 2012, Lemma 7.1.1]):

(a) $\mathscr{D}_\mathscr{L}$ is isomorphic to $\mathscr{D}_\Lambda^\hbar$ as a $\mathbb{C}^\hbar$-algebroid and $\mathrm{gr}_\hbar(\mathscr{D}_\mathscr{L}) \simeq \mathscr{D}_\Lambda$.

(b) The $\mathbb{C}^\hbar$-algebra $\mathscr{D}_\mathscr{L}$ is right and left Noetherian.

We denote by $I_\Lambda \subset \mathscr{O}_X$ the defining ideal of Λ. Let $\mathscr{I}$ be the kernel of the composition

$$\hbar^{-1}\mathscr{A}_X \xrightarrow{\hbar} \mathscr{A}_X \to \mathrm{gr}_\hbar \mathscr{A}_X \to \mathscr{O}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathrm{gr}_\hbar \mathscr{A}_X. \tag{4-2}$$

Then we have

$$\mathscr{I}/\mathscr{A}_X \simeq I_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar \mathscr{A}_X. \tag{4-3}$$

Remark 4.3. In [Kashiwara and Schapira 2012, Ch. 7, § 1] we have used the symbol map $\sigma : \mathscr{A}_X \to \mathscr{O}_X$. This map is only defined locally, but all results of this chapter are of a local nature. If, nevertheless, one wants a global construction, then one has to replace the sequence two lines above Definition 7.1.2 of [loc. cit.] with (4-2).

Definition 4.4 [Kashiwara and Schapira 2012, Definition 7.1.2]. One denotes by $\mathscr{A}_{\Lambda/X}$ the $\mathbb{C}^\hbar$-subalgebroid of $\mathscr{A}_X^{\mathrm{loc}}$ generated by $\mathscr{I}$.

The ideal $\hbar\mathscr{I}$ is contained in $\mathscr{A}_X$, hence acts on $\mathscr{L}$ and one sees easily that $\hbar\mathscr{I}$ sends $\mathscr{L}$ to $\hbar\mathscr{L}$. Hence, $\mathscr{I}$ acts on $\mathscr{L}$ and defines a functor $\mathscr{A}_{\Lambda/X} \to \mathscr{D}_\mathscr{L}$. We thus have the morphisms of algebroids

$$\begin{array}{ccccc}
\mathscr{A}_X|_\Lambda & \longrightarrow & \mathscr{A}_{\Lambda/X}|_\Lambda & \longrightarrow & \mathscr{A}_X^{\mathrm{loc}}|_\Lambda \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mathscr{D}_\mathscr{L} & \longrightarrow & \mathscr{D}_\mathscr{L}^{\mathrm{loc}}
\end{array}$$

In particular, $\mathscr{L}$ is naturally an $\mathscr{A}_{\Lambda/X}$-module and $\mathrm{gr}_\hbar(\mathscr{D}_\mathscr{L}) \simeq \mathscr{D}_\Lambda$ is a $\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})$-module.

Example 4.5. We follow the notation of Example 3.11. Let $\Lambda = M$. Then $\mathscr{L} := \widehat{\mathscr{W}}_X(0)/\big(\sum_i \widehat{\mathscr{W}}_X(0)u_i\big) \simeq \mathscr{O}_\Lambda^\hbar$ is simple along Λ and $\mathscr{I} \subset \hbar^{-1}\mathscr{A}_X = \mathscr{A}_X(-1)$ is generated by $\hbar^{-1}u = (\hbar^{-1}u_1, \dots, \hbar^{-1}u_n)$. Identifying $\hbar^{-1}u_i$ with $\frac{\partial}{\partial x_i}$ we get an isomorphism $\mathscr{D}_\mathscr{L} \simeq \mathscr{D}_\Lambda[[\hbar]]$.

From now on, and until the end of the proof of Proposition 4.8 we work locally on X and thus we may assume that there is an isomorphism $\mathrm{gr}_\hbar \mathscr{A}_X \xrightarrow{\sim} \mathscr{O}_X$.

We introduce a filtration $F\mathscr{A}_X^{\mathrm{loc}}$ on $\mathscr{A}_X^{\mathrm{loc}}$ by setting

$$F_k \mathscr{A}_X^{\mathrm{loc}} = \hbar^{-k}\mathscr{A}_X \quad \text{for } k \in \mathbb{Z}. \tag{4-4}$$

Therefore, there is a natural isomorphism

$$\mathrm{gr}^F_k \mathscr{A}^{\mathrm{loc}}_X \simeq \hbar^{-k}\mathscr{O}_X. \tag{4-5}$$

We endow $\mathscr{A}_{\Lambda/X}$ with the induced filtration, that is,

$$F_k\mathscr{A}_{\Lambda/X} = \mathscr{A}_{\Lambda/X} \cap F_k\mathscr{A}^{\mathrm{loc}}_X.$$

Recall (see [Kashiwara and Schapira 2012, §1.4]) that for a left Noetherian $\mathbb{C}^\hbar$-algebra $\mathscr{R}$, one says that a coherent $\mathscr{R}$-module $\mathscr{P}$ is locally projective if the functor

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{P}, \bullet) : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{R}) \to \mathrm{Mod}(\mathbb{C}^\hbar_X)$$

is exact. This is equivalent to each of the following conditions: (i) for each $x \in X$, the stalk $\mathscr{P}_x$ is projective as an $\mathscr{R}_x$-module, (ii) for each $x \in X$, the stalk $\mathscr{P}_x$ is a flat $\mathscr{R}_x$-module, (iii) $\mathscr{P}$ is locally a direct summand of a free $\mathscr{R}$-module of finite rank.

Recall that one says that a ring R has global homological dimension $\leq d$ if both $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{\mathrm{op}})$ have homological dimension $\leq d$ (see [Kashiwara and Schapira 1990, Exercise I.28]). In such a case, we shall write for short $\mathrm{ghd}(R) \leq d$.

Also recall that d_X denotes the complex dimension of X.

Lemma 4.6. *One has*

(a) $(\mathscr{A}_{\Lambda/X})^{\mathrm{loc}} \simeq \mathscr{A}^{\mathrm{loc}}_X$.

(b) *The algebra* $\mathrm{gr}^F\mathscr{A}_{\Lambda/X}$ *is a graded commutative subalgebra of* $\mathrm{gr}^F\mathscr{A}^{\mathrm{loc}}_X$.

(c) *There are natural isomorphisms*

$$\mathrm{gr}^F\mathscr{A}_{\Lambda/X} \simeq \bigoplus_{k\in\mathbb{Z}} T^{-k} I^k_\Lambda \quad \textit{and} \quad \mathrm{gr}^F_{\geq 0}\mathscr{A}_{\Lambda/X} \simeq \bigoplus_{k\geq 0} T^{-k} I^k_\Lambda,$$

where $I^k_\Lambda := \mathscr{O}_X$ *for* $k \leq 0$.

(d) *The sheaves of algebras* $\mathrm{gr}^F\mathscr{A}_{\Lambda/X}$ *and* $\mathrm{gr}^F_{\geq 0}\mathscr{A}_{\Lambda/X}$ *are Noetherian.*

(e) *For any* $x \in X$, *one has* $\mathrm{ghd}(\mathrm{gr}^F\mathscr{A}_{\Lambda/X})_x \leq d_X + 1$.

Proof. (a) This is obvious since $\mathscr{A}_X \subset \mathscr{A}_{\Lambda/X} \subset \mathscr{A}^{\mathrm{loc}}_X$.

(b) This is obvious.

(c) $\mathrm{gr}^F_1(\mathscr{A}_{\Lambda/X}) \simeq I_\Lambda$. Hence, $\mathrm{gr}^F_k\mathscr{A}_{\Lambda/X} \simeq I^k_\Lambda$.

(d) The commutative algebras $\mathrm{gr}^F\mathscr{A}_{\Lambda/X}$ and $\mathrm{gr}^F_{\geq 0}\mathscr{A}_{\Lambda/X}$ are locally finitely presented $\mathscr{O}_X$-algebras. Hence they are Noetherian. (Note that the associated variety with $\mathrm{gr}^F\mathscr{A}_{\Lambda/X}$ is the normal deformation of Λ in X.)

(e) For $x \in X$, set $R_x = (\mathrm{gr}^F\mathscr{A}_{\Lambda/X})_x$. If $x \notin \Lambda$, then $R_x \simeq \mathscr{O}_{X,x}[T, T^{-1}]$ and $\mathrm{ghd}(R_x) \leq d_X + 1$. Assume now that $x \in \Lambda$. Then $R_x/TR_x \simeq \mathscr{O}_{\Lambda,x}[y_1, \dots, y_n]$

(with $n = \operatorname{codim}_X \Lambda$) and $R_x[T^{-1}] \simeq \mathscr{O}_{X,x}[T, T^{-1}]$ have global homological dimensions d_X and $d_X + 1$, respectively. Hence, $\operatorname{ghd}(R_x) \le d_X + 1$ by the classical Lemma 4.7 below. □

Lemma 4.7. *Let R be a commutative Noetherian ring and let $t \in R$ be a nonzero divisor. Assume that R/tR has global homological dimension $\le d$ and the localization $R[t^{-1}]$ has global homological dimension $\le d+1$. Then R has global homological dimension $\le d+1$.*

Proof. (i) Let $\operatorname{Spec}(R)$ denote as usual the set of prime ideals of R. For $\mathfrak{p} \in \operatorname{Spec}(R)$, denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$. It is well-known that R has global homological dimension $\le d$ if and only if for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ has global homological dimension $\le d$.

(ii) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and assume that $t \notin \mathfrak{p}$. Then $R_\mathfrak{p} \simeq (R[t^{-1}])_\mathfrak{p}$ has global homological dimension $\le d+1$.

(iii) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and assume that $t \in \mathfrak{p}$. In this case, $R_\mathfrak{p}/tR_\mathfrak{p} \simeq (R/tR)_\mathfrak{p}$ has global homological dimension $\le d$. This implies that $R_\mathfrak{p}$ is a regular local ring of global homological dimension $\le d+1$. □

Proposition 4.8 (see [Kashiwara and Schapira 2012, Lemma 7.1.3] in the symplectic case). *One has*

(a) *the $\mathbb{C}^\hbar$-algebroid $\mathscr{A}_{\Lambda/X}$ is right and left Noetherian,*

(b) $\operatorname{gr}_\hbar(\mathscr{N}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\operatorname{gr}_\hbar \mathscr{A}_{\Lambda/X})$ *for any* $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$.

Proof. (a) This follows from Proposition 4.1 since $\mathscr{A}$ is Noetherian by Theorem 3.12, $\operatorname{gr}_{\ge 0}\mathscr{A}_{\Lambda/X}$ is Noetherian by Lemma 4.6 and the I^k_Λ are coherent $\mathscr{A}_X$-modules since they are coherent $\mathscr{O}_X$-modules.

(b) Let us represent $\mathscr{N}$ by a complex $\mathscr{L}^\bullet$ bounded from above of locally free $\mathscr{A}_{\Lambda/X}$-modules of finite rank. Then $H^i(\mathscr{L}^\bullet) \simeq 0$ for $i \ll 0$. Replacing $\mathscr{L}^\bullet$ with $\tau^{\ge j}\mathscr{L}^\bullet$ for $j \ll 0$ we find a bounded complex $\mathscr{L}^\bullet$ of coherent $\mathscr{A}_{\Lambda/X}$-modules for which $\hbar$ is injective. Now $\operatorname{gr}_\hbar(\mathscr{N})$ is represented by the complex $\mathscr{L}^\bullet/\hbar\mathscr{L}^\bullet$ and the result follows. □

In the sequel, for $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{\Lambda/X})$ we set

$$\operatorname{gr}_\Lambda(\mathscr{N}) := \operatorname{gr}_\hbar(\mathscr{D}_\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}) \simeq \mathscr{D}_\Lambda \overset{\mathrm{L}}{\otimes}_{\operatorname{gr}_\hbar(\mathscr{A}_{\Lambda/X})} \operatorname{gr}_\hbar(\mathscr{N}). \tag{4-6}$$

Corollary 4.9. *If* $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$, *then* $\operatorname{gr}_\Lambda(\mathscr{N}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_\Lambda)$ *and* $\operatorname{char}(\operatorname{gr}_\Lambda(\mathscr{N}))$ *is a closed $\mathbb{C}^\times$-conic complex analytic subset of $T^*\Lambda$.*

Proof. By Proposition 4.8(b) and Lemma 4.6(e), $\operatorname{gr}_\hbar\mathscr{N}$ is locally quasi-isomorphic to a bounded complex of projective $\operatorname{gr}_\hbar\mathscr{A}_{\Lambda/X}$-modules of finite type. To conclude, note that if $\mathscr{P}$ is a projective $\mathrm{gr}_\hbar \mathscr{A}_{\Lambda/X}$-modules of finite type, then

$$\mathscr{D}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})} \mathrm{gr}_\hbar(\mathscr{P})$$

is concentrated in degree 0 and is $\mathscr{D}_\Lambda$-coherent. The result for $\mathrm{char}(\mathrm{gr}_\Lambda(\mathscr{N}))$ follows. □

Proposition 4.10 (see [Kashiwara and Schapira 2012, Proposition 7.1.8] in the symplectic case). *Let $\mathscr{N}$ be a coherent $\mathscr{A}_{\Lambda/X}$-module. Then*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L}) \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X), \tag{4-7}$$

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L})) = \mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N}). \tag{4-8}$$

Proof. (i) One has

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}, \mathscr{L}).$$

Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}, \mathscr{L})$. Then $F \in \mathrm{D}^+(\mathbb{C}^\hbar_X)$, F is cohomologically complete by Proposition 3.3 and $\mathrm{gr}_\hbar(F) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\Lambda}(\mathrm{gr}_\Lambda \mathscr{N}, \mathscr{O}_\Lambda)$.

(ii) We have $\mathrm{gr}_\hbar F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$ by Lemma 4.6(c). This implies (4-7) by Proposition 3.5.

(iii) We have $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F))$ by Proposition 3.7. On the other hand, $\mathrm{gr}_\hbar(F) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\Lambda}(\mathrm{gr}_\Lambda \mathscr{N}, \mathscr{O}_\Lambda)$ and the microsupport of this complex is equal to $\mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ by [Kashiwara and Schapira 1990, Theorem 11.3.3]. □

Definition 4.11. A coherent $\mathscr{A}_{\Lambda/X}$-submodule $\mathscr{N}$ of a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$ is called an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$ if $\mathscr{N}$ generates $\mathscr{M}$ as an $\mathscr{A}^{\mathrm{loc}}_X$-module.

One easily proves that if $\mathscr{N}$ is an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$, then $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{N})$ depends only on $\mathscr{M}$.

Notation 4.12. For a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$, one sets $\mathrm{char}_\Lambda(\mathscr{M}) := \mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ for $\mathscr{N}$ a (locally defined) $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$.

4C. *Reminders on holonomic* DQ-*modules.* We shall recall here the main results of [Kashiwara and Schapira 2012, Chapter 7].

In this subsection, we assume that X is symplectic and that Λ is Lagrangian. In this case, $\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X}) \simeq \mathscr{D}_\Lambda$ as an algebroid and thus $\mathrm{gr}_\Lambda(\mathscr{N}) \simeq \mathrm{gr}_\hbar(\mathscr{N})$.

Definition 4.13. Assume that X is symplectic and Λ is Lagrangian. An object $\mathscr{N}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$ is holonomic if $\mathrm{gr}_\hbar(\mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_\Lambda)$.

Theorem 4.14 (see [Kashiwara and Schapira 2012, Theorem 7.1.10]). *Assume that X is symplectic. Let $\mathscr{N}$ be a holonomic $\mathscr{A}_{\Lambda/X}$-module.*

(a) *The objects $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L})$ and $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L}, \mathscr{N})$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^\hbar_\Lambda)$ and their microsupports are contained in* $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{N})$.

(b) *There is a natural isomorphism in* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^{\hbar}_{\Lambda})$

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}) \xrightarrow{\sim} \mathrm{D}'_X\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L},\mathscr{N})\big)[d_X]. \tag{4-9}$$

The crucial result in order to prove Theorem 4.16 below is the following.

Proposition 4.15 (see [Kashiwara and Schapira 2012, Proposition 7.1.16]). *Assume that X is symplectic and Λ is Lagrangian. For a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$, we have*

$$\operatorname{codim}\operatorname{char}_{\Lambda}(\mathscr{M}) \geq \operatorname{codim}\operatorname{Supp}(\mathscr{M}).$$

The next result is a variation on a classical theorem of [Kashiwara 1975] on holonomic D-modules.

Theorem 4.16 (see [Kashiwara and Schapira 2012, Theorem 7.2.3]). *Assume that X is symplectic. Let $\mathscr{M}$ and $\mathscr{N}$ be two holonomic $\mathscr{A}^{\mathrm{loc}}_X$-modules. Then*

(i) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M},\mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^{\hbar,\mathrm{loc}}_X)$,*

(ii) *there is a canonical isomorphism*:

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M},\mathscr{N}) \xrightarrow{\sim} \big(\mathrm{D}'_X\mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{N},\mathscr{M})\big)[d_X], \tag{4-10}$$

(iii) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M},\mathscr{N})[d_X/2]$ is perverse.*

5. Proof of the main theorems and an example

5A. *Proof of Theorem 1.2.* In this subsection, X is again a complex Poisson manifold endowed with a DQ-algebroid $\mathscr{A}_X$.

By using the diagonal procedure [Kashiwara and Schapira 2012, Lemma 2.4.10], we may assume that $\mathscr{L} = \mathscr{L}_0^{\mathrm{loc}}$ with $\mathscr{L}_0$ an $\mathscr{A}_X$-module simple along Λ. By the hypothesis, we may find an $\mathscr{A}_{\Lambda/X}$-lattice $\mathscr{N}$ of $\mathscr{M}$. Set

$$F_0 := \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}_0), \quad F := \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M},\mathscr{L}) \simeq F^{\mathrm{loc}}. \tag{5-1}$$

One knows by Theorem 4.16 that $F|_Y \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^{\hbar,\mathrm{loc}}_{Y\cap\Lambda})$ and from Proposition 4.10 and Corollary 4.9 that $\mathrm{SS}(F_0)\times_{\Lambda}(\Lambda\cap Y)$ is Lagrangian and subanalytic in $T^*\Lambda$. Since $\mathrm{SS}(F)\subset\mathrm{SS}(F_0)$ by Proposition 3.6, it remains to apply Corollary 2.4.

5B. *Proof of Theorem 1.3.* In this subsection, X is a quasicompact separated smooth algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. For an algebraic variety X, one denotes by X_{an} the complex analytic manifold associated with X and by $\rho\colon X_{\mathrm{an}}\to X$ the natural map. There is a natural morphism $\rho^{-1}\mathscr{O}_X\to\mathscr{O}_{X_{\mathrm{an}}}$ and it is well-known that this morphism is faithfully flat [Serre 1956].

Lemma 5.1. *Let $\mathscr{A}_X$ be a* DQ-*algebroid on X. Then there exists a* DQ-*algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an} together with a functor $\rho^{-1}\mathscr{A}_X\to\mathscr{A}_{X_{\mathrm{an}}}$. Moreover such an $\mathscr{A}_{X_{\mathrm{an}}}$ is unique up to a unique isomorphism.*

Proof. First, consider a star-algebra $\mathscr{A} = (\mathscr{O}^{\hbar}_X, \star)$ on a smooth algebraic variety X. The star product is defined by a sequence of algebraic bidifferential operators $\{P_i\}_i$ (see [Kashiwara and Schapira 2012, Definition 2.2.2]) and one defines a star algebra $\mathscr{A}^{\mathrm{an}} = (\mathscr{O}^{\hbar}_{X_{\mathrm{an}}}, \star)$ on X_{an} by using the same bidifferential operators.

There exists an open (for the Zariski topology) covering $X = \bigcup_{i\in I} U_i$ such that, for each i, there exists an object s_i of the category $\mathscr{A}_X(U_i)$. Then $\mathscr{A}_i := \mathscr{E}nd(s_i)$ is a star algebra. For $i, j \in I$, since $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$ are locally isomorphic, there exists an open covering $U_i \cap U_j = \bigcup_{a\in A_{ij}} U^a_{ij}$ such that setting $U_{ij} = \bigsqcup_{a\in A_{ij}} U^a_{ij}$, there exist an isomorphism $\alpha_{ij} : s_i|_{U_{ij}} \xrightarrow{\sim} s_j|_{U_{ij}}$. Then we have

$$a_{ijk} := \alpha_{ij}\alpha_{jk}\alpha_{ki} \in \mathrm{End}(s_i|_{U_{ijk}}) = \mathscr{A}_i(U_{ijk}),$$

where $U_{ijk} = U_{ij} \times_X U_{jk} \times_X U_{ki}$.

Hence we have an isomorphism $\beta_{ij} : \mathscr{A}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{A}_j|_{U_{ij}}$ defined by

$$\mathscr{A}_i \ni a \mapsto \alpha_{ij} \circ a \circ \alpha^{-1}_{ij} \in \mathscr{A}_j.$$

Moreover they satisfy the compatibility condition:

$$\beta_{ij}\beta_{jk}\beta_{ki} = \mathrm{Ad}(a_{ijk}) \in \mathrm{End}(\mathscr{A}_i|_{U_{ijk}}).$$

Then the data $(\{U_i\}, \{U_{ij}\}, \{\mathscr{A}_i\}, \{\beta_{i.j}\}, \{a_{ijk}\})$ satisfies the compatibility condition. Conversely, we can recover $\mathscr{A}_X$ from such data (see [Kashiwara and Schapira 2012]).

On $(U_i)_{\mathrm{an}}$ we can define $\mathscr{A}^{\mathrm{an}}_i$. Similarly we can extend β_{ij} to

$$\beta^{\mathrm{an}}_{ij} : \mathscr{A}^{\mathrm{an}}_i|_{(U_{ij})_{\mathrm{an}}} \xrightarrow{\sim} \mathscr{A}^{\mathrm{an}}_j|_{(U_{ij})_{\mathrm{an}}}$$

Finally we have $a_{ijk} \in \mathscr{A}_i(U_{ijk}) \subset \mathscr{A}^{\mathrm{an}}_i\big((U_{ijk})_{\mathrm{an}}\big)$, Then the data

$$\big(\{(U_i)_{\mathrm{an}}\}, \{(U_{ij})_{\mathrm{an}}\}, \{\mathscr{A}^{\mathrm{an}}_i\}, \{\beta^{\mathrm{an}}_{i.j}\}, \{a_{ijk}\}\big)$$

satisfies the compatibility condition, and it defines a DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an}. □

Proposition 5.2. *The algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ is faithfully flat over $\rho^{-1}\mathscr{A}_X$.*

Proof. It is enough to prove that for each $x \in X$, $\mathscr{A}_{X_{\mathrm{an}},x}$ is faithfully flat over $\mathscr{A}_{X,x}$. This follows from [Kashiwara and Schapira 2012, Corollary 1.6.7] since

$$\mathscr{A}_{X,x}/\hbar\mathscr{A}_{X,x} \simeq \mathscr{O}_{X,x}$$

is Noetherian, $\mathscr{A}_{X_{\mathrm{an}},x}$ is cohomologically complete and finally $\mathscr{A}_{X_{\mathrm{an}},x}/\hbar\mathscr{A}_{X_{\mathrm{an}},x} \simeq \mathscr{O}_{X_{\mathrm{an}},x}$ is faithfully flat over $\mathscr{O}_{X,x}$. □

For an $\mathscr{A}_X$-module $\mathscr{M}$ we set

$$\mathscr{M}_{\mathrm{an}} := \mathscr{A}_{X_{\mathrm{an}}} \otimes_{\rho^{-1}\mathscr{A}_X} \rho^{-1}\mathscr{M}.$$

Proof of Theorem 1.3. As in the proof of Theorem 1.2, we may assume that $\mathscr{L} \simeq \mathscr{L}_0^{\mathrm{loc}}$ where $\mathscr{L}_0$ is a simple $\mathscr{A}_X$-module along a smooth algebraic Lagrangian manifold Λ, the module $\mathscr{M}$ remaining algebraically good. Choose an $\mathscr{A}_{\Lambda/X}$-lattice $\mathscr{N}$ of $\mathscr{M}$. Let

$$F_{\mathrm{an}} := \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X_{\mathrm{an}}}}(\mathscr{M}_{\mathrm{an}}, \mathscr{L}_{\mathrm{an}}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X_{\mathrm{an}}}}(\mathscr{N}_{\mathrm{an}}, (\mathscr{L}_0)_{\mathrm{an}})^{\mathrm{loc}}. \tag{5-2}$$

By Proposition 4.10 we know that $\mathrm{SS}(F_{\mathrm{an}}) \subset \mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N}_{\mathrm{an}})$ and this set is contained in $\mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ which is an algebraic Lagrangian subvariety of $T^*\Lambda$. To conclude, apply Corollary 2.5. □

Remark 5.3. (i) If one assumes that $\mathscr{M}$ and $\mathscr{L}$ are simple modules along two smooth algebraic varieties Λ_1 and Λ_2 of X, which is the situation appearing in [Gunningham et al. 2019], there is a much simpler proof. Indeed, it follows from [Kashiwara and Schapira 2012, Theorem 7.4.3] that in this case

$$\mathrm{SS}(F) \subset C(\Lambda_1, \Lambda_2), \tag{5-3}$$

the Whitney normal cone of Λ_1 along Λ_2 and this set is algebraic. Hence, it remains to apply Corollary 2.5. Note that [loc. cit., Theorem 7.4.3] is a variation on [Kashiwara and Schapira 2008].

(ii) Also remark that (5-3) is no longer true in the general case of irregular holonomic modules and until now, there is no estimate of $\mathrm{SS}(F)$, except of course, the fact that it is a Lagrangian set.

5C. ***An example.*** Consider coordinates (x_1, x_2, y_1, y_2) on the Poisson manifold $X = \mathbb{C}^4$ with the Poisson bracket defined by

$$\begin{gathered} \{x_1, x_2\} = 0, \quad \{y_1, x_1\} = \{y_2, x_2\} = x_1, \\ \{y_1, y_2\} = y_2, \quad \{y_1, x_2\} = y_2, \quad \{y_1, x_2\} = \{y_2, x_1\} = 0. \end{gathered} \tag{5-4}$$

Denote by $\mathscr{A}_X$ the DQ-algebra defined by the relations $y_1 = \hbar x_1 \partial_{x_1}$, $y_2 = \hbar x_1 \partial_{x_2}$, that is,

$$\begin{gathered} [x_1, x_2] = 0, \quad [y_1, x_1] = [y_2, x_2] = \hbar x_1, \quad [y_1, y_2] = \hbar y_2, \\ [y_1, x_2] = \hbar y_2, \quad [y_1, x_2] = [y_2, x_1] = 0. \end{gathered} \tag{5-5}$$

Hence, $Y = \{x_1 \neq 0\}$ is the symplectic locus $X \setminus X_{\mathrm{ns}}$ of the Poisson manifold X. Set $\Lambda = \{y_1 = y_2 = 0\}$. Then $\Lambda \cap Y$ is Lagrangian in Y.

Define the $\mathscr{A}_X$-module $\mathscr{L}$ by $\mathscr{L} = \mathscr{A}_X \cdot u$ with the relations $y_1 u = y_2 u = 0$. Then $\mathscr{L} \simeq \mathscr{O}_\Lambda^\hbar$ and for $a(x) \in \mathscr{O}_\Lambda^\hbar$, one has

$$\begin{cases} y_1 a(x) u = \hbar x_1 \dfrac{\partial a}{\partial x_1} u, \\ y_2 a(x) u = \hbar x_1 \dfrac{\partial a}{\partial x_2} u. \end{cases}$$

Now define the left $\mathscr{A}_X$ module $\mathscr{M}$ by $\mathscr{M} = \mathscr{A}_X \cdot v$ with the relations $(y_1 + \hbar)v = y_2 v = 0$. Then the complex below, in which the operators act on the right

$$0 \longleftarrow \mathscr{M} \longleftarrow \mathscr{A}_X \xleftarrow[\cdot\binom{y_1+\hbar}{y_2}]{} \mathscr{A}_X^{\oplus 2} \xleftarrow[\cdot(y_2,-y_1)]{} \mathscr{A}_X \longleftarrow 0 \tag{5-6}$$

is a free resolution of $\mathscr{M}$.

Hence, the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}})$ is represented by the complex (the operators act on the left)

$$0 \longrightarrow \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \xrightarrow[\binom{x_1\partial_{x_1}+1}{x_1\partial_{x_2}}\cdot]{} (\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}})^{\oplus 2} \xrightarrow[(x_1\partial_{x_2},-x_1\partial_{x_1})\cdot]{} \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \longrightarrow 0. \tag{5-7}$$

Since $x_1\partial_{x_1}\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} + x_1\partial_{x_2}\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} = x_1\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}}$ and $\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}}/x_1\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \simeq \mathscr{O}_{\Lambda\cap\{x_1=0\}}^{\hbar,\mathrm{loc}}$, we have

$$\mathscr{E}xt^2_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}}) \simeq \mathscr{O}_{\Lambda\cap\{x_1=0\}}^{\hbar,\mathrm{loc}}.$$

This example shows that $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}})$ does not belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^{\hbar,\mathrm{loc}})$.

References

[Gabber 1981] O. Gabber, "The integrability of the characteristic variety", *Amer. J. Math.* **103**:3 (1981), 445–468. MR Zbl

[Gunningham et al. 2019] S. Gunningham, D. Jordan, and P. Safronov, "The finiteness conjecture for skein modules", preprint, 2019. arXiv 1908.05233

[Kashiwara 1975] M. Kashiwara, "On the maximally overdetermined system of linear differential equations, I", *Publ. Res. Inst. Math. Sci.* **10**:2 (1975), 563–579. MR Zbl

[Kashiwara 1996] M. Kashiwara, "Quantization of contact manifolds", *Publ. Res. Inst. Math. Sci.* **32**:1 (1996), 1–7. MR Zbl

[Kashiwara 2003] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs **217**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Kashiwara and Schapira 1990] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Math. Wissenschaften **292**, Springer, 1990. MR Zbl

[Kashiwara and Schapira 2008] M. Kashiwara and P. Schapira, "Constructibility and duality for simple holonomic modules on complex symplectic manifolds", *Amer. J. Math.* **130**:1 (2008), 207–237. MR Zbl

[Kashiwara and Schapira 2012] M. Kashiwara and P. Schapira, *Deformation quantization modules*, Astérisque **345**, 2012. MR Zbl

[Kontsevich 2001] M. Kontsevich, "Deformation quantization of algebraic varieties", *Lett. Math. Phys.* **56**:3 (2001), 271–294. MR Zbl

[Kontsevich 2003] M. Kontsevich, "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66**:3 (2003), 157–216. MR Zbl

[Serre 1956] J.-P. Serre, "Géométrie algébrique et géométrie analytique", *Ann. Inst. Fourier* (*Grenoble*) **6** (1956), 1–42. MR Zbl

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