

# Operations on constructible functions

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## *Abstract*

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A constructible function  $\varphi$  on a real analytic manifold  $X$  is a  $\mathbb{Z}$ -valued function such that the partition  $X = \bigsqcup_{m \in \mathbb{Z}} \varphi^{-1}(m)$  is a subanalytic stratification.

Here, we define new operations on constructible functions (inverse or direct images, duality) and prove some theorems related to these operations (e.g., duality commutes to direct image).

As an application we solve the convolution equation  $\varphi * \psi = \delta$ , when  $\varphi$  is the characteristic function of a convex compact set. This problem which seems to have some utility in robotics, was first considered by Guibas et al. and this paper may be considered as a new approach, and an extension to higher dimension of the material contained in their paper.

## 1. Introduction

Convolution of two compact convex subsets of  $\mathbb{R}^n$  is a classical matter and corresponds to the Minkowski sum. In [4], Guibas, Ramschaw and Stolfi introduce convolution of planar oriented polygonal tracings endowed with multiplicities and show how to use it in some problems of robotics.

The aim of this paper is to give an alternative approach to their construction, based on the notion of constructible functions.

In our opinion, this new approach has many advantages. First, it is extremely simple and works in any dimension. Secondly, we can relate it to the sophisticated, but well known, theory of sheaves, and by this method obtain quite immediately complete proofs of our results.

Let us briefly describe the contents of this paper.

We recall first the classical notion of a constructible function on a real analytic manifold. We have chosen the framework of real analytic geometry but most of

the constructions still hold in the framework of semi-algebraic, as well as piecewise-linear geometry.

Then we define the main operations on these functions. If the sums, products, composition (i.e., inverse images) of constructible functions are the usual ones, 'integration' (i.e., direct image) is completely different. For example, the integral of the characteristic function of a compact contractible subset is always equal to one. We also introduce a new operation, 'duality', which corresponds to Poincaré–Verdier duality in sheaf theory. For example, if  $\varphi$  is the characteristic function of a convex open subset  $\Omega$  of  $\mathbb{R}^n$ , its dual is  $(-1)^n$  times the characteristic function of the closure of  $\Omega$ .

We prove various compatibility results between these operations, and in particular we prove that duality commutes to direct image.

All these constructions are motivated by sheaf theory. In fact, if, to a constructible sheaf  $F$  on  $X$  (or more generally, to an object  $F$  of the derived category of constructible sheaves on  $X$ ), one associates the constructible function  $\chi(F)$ , the local Euler–Poincaré characteristic of  $F$ , a theorem of Kashiwara asserts that one obtains an isomorphism between the Grothendieck group of constructible sheaves on  $X$  and the group of constructible functions on  $X$ . Then the operations on constructible functions correspond to similar operations on sheaves, and the proofs of the results on constructible functions immediately follow from the corresponding results in sheaf theory.

As an application we study convolution equations on  $\mathbb{R}^n$  (of course the convolution of two constructible functions is not the classical one). First we solve the equation  $\varphi * \psi = \delta$ , where  $\delta$  is the unit (i.e., the characteristic function of  $\{0\}$  in  $\mathbb{R}^n$ ) and  $\varphi$  is the characteristic function of a compact convex subset. This problem is related to those considered in [4] in the two-dimensional case. Next we give an easy (necessary) test to find all translations which send a closed subset  $A$  into a closed subset  $B$ . In fact to the pair  $(A, B)$  we associate a constructible function  $\varphi_{A,B}$  with the property that  $x + A \subset B$  implies that  $\varphi_{A,B}(x)$  is equal to the Euler–Poincaré index of  $\text{Int } A$ .

## 2. Constructible functions

We shall systematically use Hironaka's theory of subanalytic sets [5].

Recall that if  $X$  is a real analytic manifold, the family of subanalytic subsets of  $X$  is closed by finite intersection, finite union, difference, closure, as well as (for a morphism  $f: X \rightarrow Y$  of real analytic manifolds) inverse images or proper direct images. Moreover, semi-algebraic sets (or an algebraic manifold) are subanalytic and to be subanalytic is a local property on  $X$ .

Recall that a function  $\varphi: X \rightarrow \mathbb{Z}$  is said to be constructible if for each  $m \in \mathbb{Z}$ , the set  $\varphi^{-1}(m)$  is subanalytic and the family  $\{\varphi^{-1}(m)\}_{m \in \mathbb{Z}}$  is locally finite.

By the 'triangulation theorem' it is equivalent to say that there exists a locally

finite covering  $X = \bigcup_{\alpha \in A} X_\alpha$ , where the  $X_\alpha$ 's are compact, subanalytic, contractible subsets of  $X$  and

$$\varphi = \sum_{\alpha} C_{\alpha} \mathbb{1}_{X_{\alpha}}, \tag{2.1}$$

$\mathbb{1}_A$  denoting the characteristic function of a subset  $A$ .

Denote by  $\text{Cons}(X)$  the space of constructible functions on  $X$ . This space is an algebra for the usual operations of addition and multiplication of functions. This follows immediately from the definition and the properties of subanalytic sets.

Now let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds. If  $\psi$  is a constructible function on  $Y$ , one defines the function  $f^*\psi$  on  $X$  by

$$f^*\psi(x) = \psi(f(x)). \tag{2.2}$$

Since the inverse image by  $f$  of a subanalytic subset of  $Y$  is subanalytic in  $X$ ,  $f^*\psi$  is a constructible function on  $X$ . It is called the *inverse image* of  $\psi$  by  $f$ .

If  $g: Y \rightarrow Z$  is another morphism of real analytic manifolds, and  $\theta$  is a constructible function on  $Z$ , one obviously has

$$f^*g^*(\theta) = (g \circ f)^*(\theta). \tag{2.3}$$

Let  $\varphi$  be a constructible function on  $X$ , and assume  $\varphi$  has compact support. One can write  $\varphi$  as in (2.1) where all  $C_{\alpha}$ 's but a finite number are zero.

**Lemma 2.1.** *Let  $\varphi = \sum_{\alpha} C_{\alpha} \mathbb{1}_{X_{\alpha}}$ , where the sum is finite and the  $X_{\alpha}$ 's are compact subanalytic contractible subsets of  $X$ . Then the number  $\sum_{\alpha} C_{\alpha}$  depends only on  $\varphi$ .*

**Definition 2.2.** Let  $\varphi \in \text{Cons}(X)$  with compact support. One sets

$$\int_X \varphi = \sum_{\alpha} C_{\alpha} \tag{2.4}$$

where  $\varphi = \sum_{\alpha} C_{\alpha} \mathbb{1}_{X_{\alpha}}$ , as in Lemma 2.1.

Then consider a morphism  $f: X \rightarrow Y$  of real analytic manifolds, a constructible function  $\varphi$  on  $X$ , and assume  $f$  is proper on the support of  $\varphi$ . This implies in particular that for each  $y \in Y$ ,  $\varphi|_{f^{-1}(y)}$ , the restriction of  $\varphi$  to  $f^{-1}(y)$ , has compact support, and this function, considered as a function on  $X$  (which is zero outside of  $f^{-1}(y)$ ) is constructible. One defines the function  $f_*\varphi$  on  $Y$ , called the *direct image* of  $\varphi$  by  $f$ , by the formula

$$(f_*\varphi)(y) = \int_X \varphi \cdot \mathbb{1}_{f^{-1}(y)}. \tag{2.5}$$

**Theorem 2.3.** (i) Let  $\varphi$  be a constructible function on  $X$  and assume  $f$  is proper on  $\text{supp}(\varphi)$ . Then  $f_*\varphi$  is constructible on  $Y$ .

(ii) Let  $g: Y \rightarrow Z$  be another morphism of real analytic manifolds and assume  $\varphi$  is constructible on  $X$  and  $g \circ f$  is proper on  $\text{supp}(\varphi)$ . Then:

$$g_*f_*\varphi = (g \circ f)_*\varphi. \quad (2.6)$$

We shall now define the dual  $D_X\varphi$  of a constructible function  $\varphi$  on  $X$ .

**Lemma and Definition 2.4.** Let  $\varphi \in \text{Cons}(X)$  and let  $x_0 \in X$ . Choose a local chart in a neighborhood of  $x_0$  and denote by  $B(x_0, \varepsilon)$  the open ball with center  $x_0$  and radius  $\varepsilon$  in this chart. Then for  $\varepsilon > 0$  small enough, the integer  $\int_X \mathbb{1}_{B(x_0, \varepsilon)}\varphi$  depends only on  $\varphi$ . One sets

$$(D_X\varphi)(x_0) = \int_X \mathbb{1}_{B(x_0, \varepsilon)}\varphi. \quad (2.7)$$

**Theorem 2.5.** Let  $\varphi \in \text{Cons}(X)$ . Then:

- (i)  $D_X\varphi \in \text{Cons}(X)$ ,
- (ii)  $D_X \circ D_X\varphi = \varphi$ ,
- (iii) if  $f: X \rightarrow Y$  is a morphism of real analytic manifolds and if  $f$  is proper on  $\text{supp}(\varphi)$ , we have:

$$f_*(D_X\varphi) = D_Y(f_*\varphi). \quad (2.8)$$

**Example 2.6.** Let  $Z$  be a (locally closed) real analytic submanifold of  $X$  with dimension  $d$ . Assume that, locally on  $X$ ,  $Z$  is homeomorphic to a convex open subset of  $\mathbb{R}^d$  (i.e.,  $\bar{Z}$  is a  $C^0$ -manifold with boundary). Then:

$$D_X(\mathbb{1}_Z) = (-1)^d \mathbb{1}_{\bar{Z}} \quad (2.9)$$

(hence  $D_X(\mathbb{1}_{\bar{Z}}) = (-1)^d \mathbb{1}_Z$ ).

Assume moreover that  $\int \mathbb{1}_{\bar{Z}} = 1$ . Applying (2.8) we get

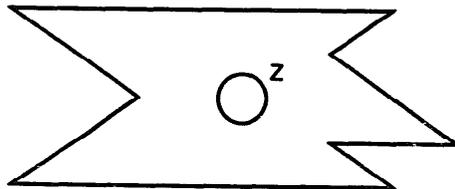


Fig. 1.

$$\int_X \mathbb{1}_{\partial Z} = \int_X \mathbb{1}_Z - (-1)^d \int_X D_X \mathbb{1}_Z = 1 - (-1)^d.$$

This is the ‘Euler formula’.

In Fig. 1,  $Z$  is a closed subset of  $\mathbb{R}^2$ , locally homeomorphic to a closed convex subset of  $\mathbb{R}^2$ .

### 3. Proofs

All proofs are based on deep, but almost classical results of sheaf theory. We refer to [6] and [11] for a general exposition of this theory, and to [11] for a short exposition of constructible sheaves.

Let  $k$  be a commutative field, fixed for this paper. One denotes by  $\text{Sh}(X)$  the abelian category of sheaves of  $k$ -vector spaces on  $X$ , where  $X$  is, as before, a real analytic manifold. One denotes by  $\mathbb{R}\text{-C}(X)$  the full abelian subcategory of  $\text{Sh}(X)$  whose objects are the  $\mathbb{R}$ -constructible sheaves on  $X$ . Recall that a sheaf  $F$  on  $X$  is  $\mathbb{R}$ -constructible if there exists a subanalytic stratification  $X = \bigcup_{\alpha} X_{\alpha}$  such that for each  $\alpha$ ,  $F|_{X_{\alpha}}$  is locally constant of finite rank.

If  $A$  is an abelian category one denotes by  $D(A)$  the derived category of  $A$ , and by  $D^b(A)$  the full subcategory consisting of complexes with bounded cohomologies. For short, one sets

$$D^b(X) = D^b(\text{Sh}(X)). \tag{3.1}$$

One denotes by  $D_{\mathbb{R}\text{-C}}^b(X)$  the full subcategory of  $D^b(X)$  consisting of objects with cohomology in  $\mathbb{R}\text{-C}(X)$ . Recall Kashiwara’s theorem:

**Theorem 3.1** [8]. *The natural functor  $D^b(\mathbb{R}\text{-C}(X)) \rightarrow D_{\mathbb{R}\text{-C}}^b(X)$  is an equivalence of categories.  $\square$*

If  $A$  denotes an abelian category (respectively a triangulated category), one denotes by  $K(A)$  its Grothendieck’s group. It is the quotient of the free abelian group generated by the objects of  $A$  by the relation  $F = F' + F''$ , if there exists an exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  (respectively a distinguished triangle  $F' \rightarrow F \rightarrow F'' \xrightarrow{+1}$ ) in  $A$ . If  $A$  is an abelian category, there is a natural group isomorphism

$$K(A) \xrightarrow{\sim} K(D^b(A)). \tag{3.2}$$

In particular, choosing  $A = \mathbb{R}\text{-C}(X)$ , we get by Theorem 3.1

$$K(\mathbb{R}\text{-C}(X)) \xrightarrow{\sim} K(D_{\mathbb{R}\text{-C}}^b(X)). \tag{3.3}$$

Let  $F$  be an object of  $D_{\mathbb{R}\text{-}\mathbb{C}}^b(X)$ . Its local Euler–Poincaré index at  $x \in X$ , denoted  $\chi(F)(x)$ , is defined by

$$\chi(F)(x) = \sum_i (-1)^i \dim_k(H^i(F)_x). \quad (3.4)$$

By the definition of  $D_{\mathbb{R}\text{-}\mathbb{C}}^b(X)$ , one sees that  $\chi(F)$  is a constructible function on  $X$ . Moreover if  $F' \rightarrow F \rightarrow F'' \xrightarrow{+1}$  is a distinguished triangle in  $D_{\mathbb{R}\text{-}\mathbb{C}}^b(X)$ , one has

$$\chi(F) = \chi(F') + \chi(F''). \quad (3.5)$$

Hence  $\chi$  defines a function, still denoted by  $\chi$ ,

$$K(D_{\mathbb{R}\text{-}\mathbb{C}}^b(X)) \xrightarrow{\chi} \text{Cons}(X). \quad (3.6)$$

**Example 3.2.** If  $F$  is a constructible sheaf on  $X$ ,  $\chi(F)(x)$  is simply the dimension of the stalk  $F_x$  of  $F$  at  $x$ . In particular, if  $A$  is a subanalytic subset on  $X$ , and  $F = k_A$  is the sheaf on  $X$  which is zero outside of  $A$ , and whose inverse image on  $A$  is the constant sheaf on  $A$  with stalk  $k$ , we get

$$\chi(k_A) = \mathbb{1}_A. \quad (3.7)$$

**Remark 3.3.** It follows immediately from the definition that if  $F$  and  $G$  are objects of  $D_{\mathbb{R}\text{-}\mathbb{C}}^b(X)$ , then:

$$\begin{aligned} \chi(F \oplus G) &= \chi(F) + \chi(G), \\ \chi(F \otimes G) &= \chi(F) \cdot \chi(G). \end{aligned} \quad (3.8)$$

**Theorem 3.4** (Kashiwara [9]). *The morphism  $\chi$  in (3.6) is an isomorphism.*  $\square$

Note that Kashiwara’s theorem is already implicitly contained in his previous work [7], as well as in MacPherson’s work [12] (cf. also [2]). In the complex case, a similar result has been announced by Ginsburg [3].

Now we shall compare the operations on constructible sheaves with those on constructible functions, and at the same time we shall prove the results of Section 2.

Let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds.

Let  $G \in \text{Ob}(D_{\mathbb{R}\text{-}\mathbb{C}}^b(Y))$ . Then:

$$\chi(f^{-1}G) = f^* \chi(G). \quad (3.9)$$

In fact (3.9) follows from

$$(H^j(f^{-1}(G)))_x = (H^j(G))_{f(x)}.$$

Let us prove Lemma 2.1. Let  $\varphi = \sum_{\alpha} C_{\alpha} \mathbb{1}_{X_{\alpha}}$  be a constructible function on  $X$ , the sum being finite and the  $X_{\alpha}$ 's being subanalytic, compact and contractible. Set  $\varepsilon_{\alpha} = \text{sgn}(C_{\alpha})$  (consider only the  $X_{\alpha}$ 's with  $C_{\alpha} \neq 0$ ), and set

$$F = \bigoplus_{\alpha} k_{X_{\alpha}}^{|C_{\alpha}|} [(1 - \varepsilon_{\alpha})/2].$$

Then  $\chi(F) = \varphi$  and moreover, if  $p$  denotes the map  $X \rightarrow \{p'\}$  (where  $\{p'\}$  is the zero-dimensional manifold reduced to a single point), we have

$$\chi(Rp_*F) = \sum_{\alpha} C_{\alpha}.$$

Hence Lemma 2.1 follows and

$$\chi(Rp_*F) = p_*(\varphi). \quad (3.10)$$

Now let  $F \in \text{Ob}(D_{\mathbb{R}\text{-C}}^b(X))$  and assume  $f$  is proper on  $\text{supp}(F)$ . Then:

$$\chi(Rf_*F) = f_*\chi(F). \quad (3.11)$$

In fact it is enough to check (3.11) at each  $y \in Y$ , and then it follows from (3.10).

By (3.11) we get Theorem 2.3, since it is known that  $Rf_*F$  is constructible, and  $Rg_*Rf_*F \simeq R(g \circ f)_*F$ .

Let  $F \in \text{Ob}(D_{\mathbb{R}\text{-C}}^b(X))$ . One sets (cf. [14]),

$$D_X F = R \text{Hom}(F, \omega_X)$$

where  $\omega_X$  is the dualizing complex on  $X$ .

It is known that, with the notations of Lemma 2.4, we have the isomorphism

$$\begin{aligned} R\Gamma_c(B(x_0, \varepsilon); F) &\simeq R\Gamma_{\{x_0\}}(X; F) \\ &\simeq \text{Hom}((D_X F)_{x_0}; k). \end{aligned}$$

Set  $\varphi = \chi(F)$ . Since

$$R\Gamma_c(B(x_0, \varepsilon); F) \cong Rp_*(k_{B(x_0, \varepsilon)} \otimes F),$$

we get

$$\begin{aligned} \int \mathbb{1}_{B(x_0, \varepsilon)} \varphi &= \chi(R\Gamma_c(B(x_0, \varepsilon); F)) \\ &= \chi(\text{Hom}((D_X F)_{x_0}; k)) \\ &= \chi(D_X F)(x_0). \end{aligned}$$

In other words,

$$\chi(D_X F) = D_X(\chi(F)). \quad (3.12)$$

Since it is known that  $D_X \circ D_X F = F$  and  $D_X$  commutes to direct image, Theorem 2.5 follows.

**Remark 3.5.** It would be possible to prove the results of Section 2 without making use of sheaf theory, by a careful study of subanalytic stratifications, with the help for example, of Teissier's work [13].

**Remark 3.6.** Let us denote by  $L_X$  the group of Lagrangian cycles on  $T^*X$  (cf. [9]). To an object  $F$  of  $D^b(X)$  one can associate  $SS(F)$  (cf. [10]), its micro-support, a closed conic involutive subset of  $T^*X$ , which is subanalytic and Lagrangian if  $F$  is  $\mathbb{R}$ -constructible, and in this case one can also associate to  $F$  a cycle,  $\underline{SS}(F)$ , supported by  $SS(F)$ . Then Kashiwara proves that the three groups,  $K(D_{\mathbb{R},C}^b(X))$ ,  $\text{Cons}(X)$ , and  $L_X$  are isomorphic. Therefore it would be possible to develop a calculus on  $L_X$  equivalent to that obtained here on  $\text{Cons}(X)$ . In fact in [4] some results (e.g. Theorem 6.4) could easily be interpreted in  $L_X$  (i.e. in the language of 'conormal geometry').

#### 4. Convolution

Let  $V$  be a real vector space of dimension  $n$ . Let  $s$  be the map  $V \times V \rightarrow V$  given by

$$s(x, y) = x + y. \quad (4.1)$$

If  $\varphi$  and  $\psi$  are two constructible functions on  $V$ , one denotes by  $\varphi \otimes \psi$  the constructible function on  $V \times V$  defined by

$$(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y). \quad (4.2)$$

Now assume

$$s \text{ is proper on } \text{supp}(\varphi) \times \text{supp}(\psi). \quad (4.3)$$

One sets

$$\varphi * \psi = s_*(\varphi \otimes \psi). \quad (4.4)$$

We also introduce the notation

$$\delta = \mathbb{1}_{\{0\}}. \quad (4.5)$$

We call  $\varphi * \psi$  the *convolution product* of  $\varphi$  and  $\psi$ .

**Example 4.1.** Let  $\gamma$  be a closed convex proper cone of  $V$  ( $\gamma$  is proper if  $\gamma$  contains no line). Then:

$$\mathbb{1}_\gamma * \mathbb{1}_\gamma = \mathbb{1}_\gamma, \quad \mathbb{1}_\gamma * \mathbb{1}_{\text{Int } \gamma} = 0. \quad (4.6)$$

For simplicity we shall now restrict our attention to the space of constructible functions with compact support. We denote this space by  $\text{Cons}_c(V)$ .

**Theorem 4.2.** (i) *The space  $\text{Cons}_c(V)$  is endowed with a structure of a commutative algebra for the operation  $*$ , with unit  $\delta$ .*

(ii) *Convolution commutes to duality, that is,  $D_V(\varphi * \psi) = \mathcal{D}_V \varphi * D_V \psi$ .*

(iii) *We have  $\int_V (\varphi * \psi) = (\int_V \varphi)(\int_V \psi)$ .  $\square$*

The proof is obvious, using the method of Section 3.

**Example 4.3.** Let  $A$  and  $B$  be two compact convex subanalytic subsets of  $V$ . Then:

$$\mathbb{1}_A * \mathbb{1}_B = \mathbb{1}_{A+B}. \quad (4.7)$$

In fact if  $x \in V$ ,  $(\mathbb{1}_A * \mathbb{1}_B)(x)$  is the index of the complex  $R\Gamma(A \cap (x - B); k_V)$ . This complex is zero if  $x \notin A + B$ . Otherwise it is isomorphic to  $k$  since  $A \cap (x - B)$  is compact and contractible.

**Example 4.4.** Let  $A$  be a compact convex subanalytic subset of  $V$ . Then, setting  $A^a = -A$ ,

$$\mathbb{1}_A * D_V(\mathbb{1}_{A^a}) = \delta. \quad (4.8)$$

In fact, the value at  $x \in V$  of the left-hand side of (4.8) is the index of the complex  $R\Gamma_c(A \cap (x + \text{Int } A); k_V) [d]$ , where  $d$  denotes the dimension of the vector space generated by  $A$ , and  $\text{Int } A$  denotes the interior of  $A$  in this vector space. This complex is isomorphic to zero if  $x \neq 0$ , to  $k$  if  $x = 0$ .

**Example 4.5.** If one wants to interpret in our language the constructions of [4], it is enough to decompose a tracing as a finite sum of convex tracings, then, after having chosen an orientation on  $\mathbb{R}^2$ , to a direct (respectively reverse) convex tracing one associates the characteristic function of the closed (respectively open) convex underlying set. Let us treat an example which already appears in [4].

In  $\mathbb{R}^2$ , let  $A$  be the closed square with vertices at  $(2, 2)$ ,  $(-2, 2)$ ,  $(-2, -2)$ ,  $(2, -2)$  and let  $B$  be the open square with vertices  $x_1 = (1, 0)$ ,  $x_2 = (0, 1)$ ,  $x_3 = (-1, 0)$ ,  $x_4 = (0, -1)$ . Setting  $x_5 = x_1$ , we get

$$\mathbb{1}_{\bar{B}} = \mathbb{1}_B - \sum_{j=1}^4 \mathbb{1}_{[x_j, x_{j+1}]} + \sum_{j=1}^4 \mathbb{1}_{\{x_j\}}.$$

Set  $A_x = x + A$ ,  $A_{x,y} = [x, y] + A$ .

By Example 4.3, we find (cf. Fig. 2):

$$\mathbb{1}_A * \mathbb{1}_B = \mathbb{1}_{A+\bar{B}} - \sum_{j=1}^4 \mathbb{1}_{A_{x_j, x_{j+1}}} + \sum_{j=1}^4 \mathbb{1}_{A_{x_j}}.$$

**Remark 4.6.** Let  $A$  and  $B$  be two closed subanalytic subsets of  $V$  and assume  $A$  is compact and locally on  $V$ ,  $A$  is homeomorphic to a closed convex subset which generates a subspace of dimension  $d$  (i.e.,  $A$  is a  $d$ -dimensional topological manifold with boundary).

With the preceding notation introduce the function

$$\varphi_{A,B}(x) = ((D_X \mathbb{1}_A) * \mathbb{1}_B)(x). \tag{4.9}$$

We get, as in Example 4.4,

$$\varphi_{A,B}(x) = (-1)^d \chi(R\Gamma_c((x + \text{Int } A) \cap B; k_V)).$$

Hence

$$x + A \subset B \Rightarrow \varphi_{A,B}(x) = (-1)^d \chi(R\Gamma_c(\text{Int } A; k_V)). \tag{4.10}$$

This implication is not an equivalence in general. However, if  $A$  and  $B$  are convex and generate  $V$ , one proves that  $\varphi_{A,B}(x) = 1$  if and only if  $x + A \subset B$  or  $x + B \subset A$  (cf. [4, Theorem 7.3]). Moreover, making sufficiently many ‘holes’ in  $A$  (i.e., replacing  $A$  by  $A \setminus \bigcup_j B(x_j, \varepsilon_j)$ , where the  $B(x_j, \varepsilon_j)$  are open balls with center  $x_j \in \text{Int } A$ , and radius  $\varepsilon_j \ll 1$ ), the necessary condition (4.10) becomes ‘almost sufficient’. (We leave it to the reader to give a precise meaning to this sentence.)

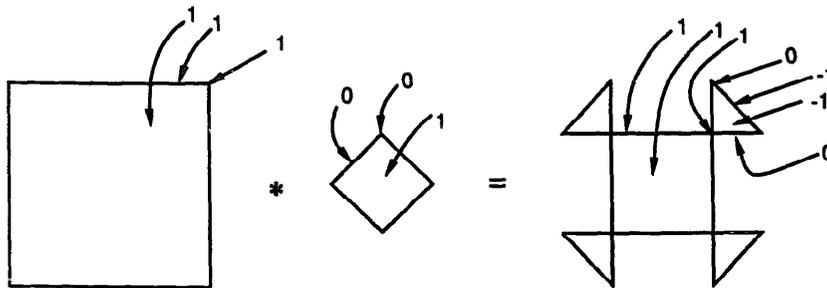


Fig. 2.

The following simple example shows that the general situation is rather intricate.

**Example 4.7.** Let  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ . Set  $A = \{x: x_1^2 + x_2^2 \leq 1, x_3 = 0\}$  and  $B = \{x: x_1^2 + x_2^2 \leq x_3^2, 0 \leq x_3 \leq 2\}$ . Then  $\varphi_{A,B}(x) = \mathbb{1}_C$ , where  $C = \{x: x_1^2 + x_2^2 < (x_3 - 1)^2, 0 \leq x_3 < 1\} \cup \{x: x_1^2 + x_2^2 \leq (x_3 - 1)^2, 1 \leq x_3 \leq 2\}$ .

**Concluding remark 4.8.** The important problem of finding all translations and rotations which send a compact convex set  $A$  into a closed set  $B$  is far from being solved, but one can treat rotations by adding new variables (cf. [1]) and making  $A$  ‘turn’. Hence it is important, even for planar problems, to handle higher-dimensional situations and we hope that constructible functions is an appropriate tool for that purpose.

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