Université Paris-Sud
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DOCTORAT D'ÉTAT THESIS IN MATHEMATICS

Prym Varieties, the Trisecant Conjecture, and Andreotti–Mayer Loci

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To Rebecca

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About this edition

This edition retains the text and historical context of the 1987 thesis. Corrections to notation and mathematics, and editorial interventions, are documented in the accompanying report. Statements historically presented as conjectures retain that status in the author's narrative.

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When one admits to working as a researcher in mathematics, the usual response is surprise, accompanied by a comment generally limited to a rather unenthusiastic "Oh?".

Few people take their enquiry any further: mathematics, which everyone knows, or thinks they know, more often arouses indifference than curiosity. Mentioning it brings back buried memories of endless additions and obscure properties of triangles, or, more recently, arrows between potato-shaped objects with strange names. Although the subjects taught and the teaching methods have often changed in recent years, the outcome has, alas, often remained the same.

This makes it very difficult to convey the fascination mathematics can exert. One thus falls victim to the old image of the slightly eccentric scholar, lost in a world of mysterious symbols and usually unable to answer the simple question: "What is it for?".

Although mathematicians may sometimes feel more at ease with their equations than with life, this view reflects only part of the reality: mathematics is a constantly evolving world in which one encounters people often willing to give their time without counting the hours and to share their enthusiasm for progress in their discipline, people passionate about their work.

In this way I had the good fortune to meet, in chronological order, Daniel Bennequin, whose enthusiasm for Phillip Griffiths and Joseph Harris's *Principles of Algebraic Geometry* undoubtedly influenced my choice of this speciality. Joseph Le Potier then introduced me to a subject dear to him: vector bundles on projective spaces. Arnaud Beauville, who supervised this thesis, has for almost eight years always been available to answer my questions with expertise and precision. He also suggested fascinating, and approachable, research topics. I thank him sincerely for his help. Thanks to a Harvard University fellowship, I was then able, in long discussions, to benefit from the extensive knowledge of Phillip Griffiths, whose kindness and generosity are appreciated by all who know him. For me, Fabrizio Catanese personifies the Italian school. During the Italian cooking lessons at his home in Princeton, "foreigners never put enough water for pasta", and later during my stays in Pisa, we dissected together the fascinating properties of surfaces of general type with $K^2 = 2$, $p_g = 1$, and $q = 0$. I express here my warm friendship to him. On returning from the United States, I encountered the harsh climate of the Moulon plateau at the Centre de mathématiques of the École polytechnique, directed by Michel Demazure. Fortunately, the atmosphere was warmed by the energy of Marie-Jo Lécuyer, who helped me on many occasions. I then moved to the Centre d'Orsay at Université Paris-Sud; although the architecture is austere, the setting is pleasant and the staff welcoming. I thank Luc Illusie in particular for accepting me into his team.

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Introduction

A principally polarized abelian variety is a pair (A, Θ), where A is a complex torus and Θ is an ample divisor on A, defined up to translation, such that $H^0(A, O_A(\Theta))$ has dimension 1. The associated Kummer variety is the image of the morphism defined by the global sections of the line bundle $O_A(2\Theta)$. It is thus embedded in the projective space $\mathbf{P}H^0(A, O_A(2\Theta))^\vee$, of dimension $2^{\dim A} - 1$.

Jacobians of smooth projective curves provide examples of principally polarized abelian varieties. In this case, many lines in the projective space meet the associated Kummer variety in at least three points; we call them *trisecants*. This phenomenon led Mumford to conjecture that the existence of sufficiently many trisecants to the Kummer variety might characterize Jacobians of curves among all principally polarized abelian varieties.

This question has prompted much work, notably by E. Arbarello, C. De Concini, B. A. Dubrovin, R. C. Gunning, Z. Ran, T. Shiota, and G. Welters. In particular, no indecomposable principally polarized abelian variety is known whose Kummer variety admits a trisecant but which is not a Jacobian of a curve. This led G. Welters to sharpen Mumford's formulation by conjecturing that an *indecomposable* principally polarized abelian variety is a Jacobian of a curve if and only if the associated Kummer variety admits a trisecant. The infinitesimal version of this problem is Novikov's conjecture, recently proved by T. Shiota. It is useful to note that the existence of a trisecant is equivalent to an inclusion of the form

$$\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y,$$

where Θ_a denotes the translate of Θ by the element a of the abelian variety A, and x, y, and a are distinct and nonzero. In particular, this implies that the intersection $\Theta \cap \Theta_a$ is reducible.

After a first chapter devoted to terminology and notation, the second proves the trisecant conjecture for a class of principally polarized abelian varieties larger than that of Jacobians: Prym varieties, attached to double covers of smooth (or possibly stable) curves. In dimension at most 5, every principally polarized abelian variety is a Prym variety, so this proves the conjecture in those dimensions.

Our approach is as follows. After a brief introduction (II.A), in II.B we revisit and simplify A. Weil's proof of Torelli's theorem for curves, based on determining the cases in which the intersection $\Theta \cap \Theta_a$ is reducible in a Jacobian.

In II.C, written jointly with A. Beauville, we show that the existence of a trisecant implies the Andreotti–Mayer condition $\dim \operatorname{Sing} \Theta \geq \dim A - 4$. In 1963, these authors showed that the family of Jacobians is an irreducible component of the family N_{g-4}^g of principally polarized abelian varieties satisfying this condition. We thus obtain a weak form of the trisecant conjecture: Jacobians form an irreducible component of the locus of principally polarized abelian varieties whose Kummer variety admits a trisecant.

In II.D and II.E we turn to the study of N_{g-4}^g. Beauville's work describes the Prym varieties belonging to N_{g-4}^g. Some are special cases of a very simple construction, detailed in Appendix 1, which yields new irreducible components of N_{g-4}^g consisting of principally polarized abelian varieties isogenous to products. A detailed study of $\operatorname{Sing} \Theta$ for the others shows that they too form components of N_{g-4}^g, consisting, therefore, of Prym varieties.

We then have all the ingredients needed to prove the trisecant conjecture for Prym varieties in II.F. We proceed by induction on dimension, starting from work of Z. Ran in dimension 4.

Chapter III applies the results of II.D and II.E in dimension 4. More precisely, in this dimension we describe the subvarieties of the moduli space of principally polarized abelian varieties on which a given number of theta constants vanish. The most remarkable fact is that, apart from hyperelliptic Jacobians, there is a unique indecomposable principally polarized abelian fourfold with 10 vanishing theta constants. It was discovered by R. Varley. We also describe the principally polarized abelian varieties with respectively 2, 3, or 9 vanishing theta constants.

The fourth and final chapter concerns the relations between Prym varieties, the loci N^g_{g-6} of principally polarized abelian varieties (A, Θ) of dimension g with $\dim \operatorname{Sing} \Theta \geq g-6$, and the existence of a quadrisecant plane to the Kummer variety. We first adapt Andreotti and Mayer's method to show in IV.A that the family of Prym varieties is a component of N^g_{g-6} for $g \geq 7$. Section IV.B is the counterpart of II.C for Prym varieties. The Kummer variety of a Prym variety generally has no trisecant line, but does have quadrisecant planes. With a theorem of Andreotti–Mayer type available (IV.A), we can prove that Prym varieties form an irreducible component of the locus of principally polarized abelian varieties whose Kummer variety admits a quadrisecant plane. Finally, although no counterexample to the trisecant conjecture is known, one can construct abelian varieties which are not Prym varieties but whose Kummer variety nevertheless admits a quadrisecant plane; in fact, it contains a conic.

Abstract

The trisecant conjecture states that a complex indecomposable principally polarized abelian variety is the Jacobian of an algebraic curve if and only if the associated Kummer variety (the image of the morphism associated with the linear system of divisors linearly equivalent to twice the theta divisor), embedded in projective space, has a line meeting it in at least three points. We prove this conjecture when the principally polarized abelian variety under consideration is a (generalized) Prym variety. This proves the conjecture, in particular, in dimension at most 5. Our techniques also describe the family of principally polarized abelian fourfolds whose theta divisor has a given number of singular points of order 2 (so-called "vanishing thetanulls"). In particular, we show that there is only one indecomposable principally polarized abelian fourfold with 10 vanishing thetanulls, apart, of course, from hyperelliptic Jacobians. Finally, we show that in dimension at least 7, Prym varieties form an irreducible component of the family of principally polarized abelian varieties for which the singular locus of the theta divisor has codimension at most 6. It follows that, still in dimension at least 7, Prym varieties form an irreducible component of the family of principally polarized abelian varieties whose Kummer variety has a quadrisecant plane (a result obtained jointly with A. Beauville).

CHAPTER I

Terminology and notation

This chapter fixes the notation used throughout this work. We also review the basic results on algebraic curves and abelian varieties that we will need.

I.1 General conventions

1.1. We work over the field $\mathbb{C}$ of complex numbers, except in II.C, where the base field is assumed only to be algebraically closed. We adopt the convention that a set of dimension < 0 is empty.

1.2. The singular locus of a scheme X is denoted by $\operatorname{Sing} X$, and the locus of points of multiplicity n by $\operatorname{Sing}_n X$.

We also write $X^{(d)}$ for the symmetric product of d copies of X, that is, the quotient of X^d by the symmetric group $\mathfrak{S}_d$.

1.3. For a coherent sheaf $\mathcal{F}$ on a projective variety X, we write $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$ for $i \geq 0$.

1.4. If $\mathcal{F}$ is also invertible and X is reduced, $|\mathcal{F}|$ denotes the set of effective Cartier divisors D such that $\mathcal{O}_X(D) \simeq \mathcal{F}$. This is an open subset of $\mathbf{P}H^0(X, \mathcal{F})$, and may be a proper subset if X is reducible.

One associates with $\mathcal{F}$ a rational map

$$\phi_{|\mathcal{F}|} \colon X \dashrightarrow \mathbf{P}H^0(X, \mathcal{F})^\vee, \qquad \phi_{|\mathcal{F}|}(x) = \{s \in H^0(X, \mathcal{F}) \mid s(x) = 0\}.$$

1.5. If X is a projective Cohen–Macaulay scheme of finite type over $\mathbb{C}$, its dualizing sheaf is denoted by ω_X.

I.2 Curves

2.1. A curve means a reduced projective variety C of dimension 1. Its genus is defined by $g(C) = 1 - \chi(\mathcal{O}_C)$. We consider only connected curves with at most ordinary double points as singularities. The Jacobian JC of C is then a smooth commutative algebraic group of dimension $g(C)$, an extension of an abelian variety by a torus. Its points are naturally identified with the isomorphism classes of line bundles on C whose restriction to each component has degree 0. An ordering of the components of C is always understood to have been chosen. We write $\operatorname{Pic}(C)$ for its Picard group and $\operatorname{Pic}^{\boldsymbol{d}}(C)$, where $\boldsymbol{d} = (d_1, \dots, d_r)$, for the set of isomorphism classes of line bundles on C whose restriction to the jth component has degree d_j (that is, of multidegree $\boldsymbol{d}$). The total degree is $d = \sum_j d_j$.

The dualizing sheaf of C is a line bundle of total degree $2g(C) - 2$.

2.2. A stable curve of genus $g \geq 2$ ([De–M], Definition 1.1) is a connected nodal curve such that each rational component of its normalization contains at least three points lying over

nodes. In particular, each smooth rational component meets the other components in at least three points.

There is a moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$, an irreducible projective variety of dimension $3g-3$. It contains the moduli space $\mathcal{M}_g$ of smooth curves of genus g as a dense Zariski open subset. For $g = 0, 1$, the usual moduli spaces have dimensions $0, 1$, respectively; their compactification is distinct from the unpointed stable-curve construction for $g \geq 2$.

2.3. A hyperelliptic curve is a curve admitting a morphism of degree 2 to $\mathbf{P}^1$.

A trigonal curve is a curve admitting a morphism of degree 3 to $\mathbf{P}^1$.

A bielliptic curve is a smooth curve admitting a morphism of degree 2 to an elliptic curve. The French original calls such a curve *superelliptique*; throughout this translation, that historical term is rendered as *bielliptic*.

2.4. When C is smooth, $C^{(d)}$ (see 1.2) is the set of effective divisors of degree d on C. We also write

$$\begin{aligned} C_d^r &= \{D \in C^{(d)} \mid h^0(C, O_C(D)) \geq r+1\}, \\ W^d &= \{L \in J^d C \mid h^0(C, L) > 0\}, \\ W_d^r &= \{L \in J^d C \mid h^0(C, L) \geq r+1\}. \end{aligned}$$

The natural morphism $C_d^r \to W_d^r$ restricts over $W_d^r \setminus W_d^{r+1}$ to a projective-space bundle with fibre $\mathbf{P}^r$. In particular, for $r = 0$, it induces an isomorphism

$$C_d^0 \setminus C_d^1 \xrightarrow{\sim} W_d^0 \setminus W_d^1.$$

The loci W_d^r and C_d^r have natural scheme structures (see, for example, [A–C–G–H 1], Chapter IV), for which the following hold:

- If $L \in W_d^r \setminus W_d^{r+1}$, the tangent space

  $$T_L W_d^r \subset T_L J^d C \simeq H^0(C, \omega_C)^\vee$$

  is the annihilator of the image of the natural map

  $$H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C, \omega_C).$$

- If C is a general curve of genus g, then W_d^r has dimension $\rho = g - (r+1)(g-d+r)$, is irreducible if $\rho > 0$, is smooth away from W_d^{r+1}, and is Cohen–Macaulay (see [A–C–G–H 1], Chapter VII).

2.5. If p is a point of a smooth curve C and Z is a subvariety of $J^d C$, define $Z_p \subset J^{d+1} C$ to be $\{L \otimes O_C(p) \mid L \in Z\}$. Likewise, for $p, q \in C$, set $Z_{p,q} = (Z_p)_q$. If Z is a subvariety of $C^{(d)}$, also define

$$Z_p = \{D \in C^{(d+1)} \mid (D - p) \in Z\}.$$

For an effective divisor D on C, s_D denotes the canonical section of $O_C(D)$ with divisor D. For $p, q \in C$, we write s_{pq} instead of s_{p+q}.

2.6. A double cover $\pi\colon \widetilde{C} \to C$ of smooth connected projective curves is a finite morphism of degree 2. Giving it is equivalent to giving the class of a divisor δ on C not equivalent to 0, and a smooth divisor $\Delta \in |2\delta|$. We have

$$\begin{aligned} K_{\widetilde{C}} &\equiv \pi^*(K_C + \delta), \\ \pi_* O_{\widetilde{C}} &\simeq O_C \oplus O_C(-\delta), \\ H^i(\widetilde{C}, \pi^* L) &\simeq H^i(C, L) \oplus H^i(C, L \otimes O_C(-\delta)) \qquad (L \in \operatorname{Pic}(C)). \end{aligned}$$

There is a norm morphism $\operatorname{Nm}_\pi\colon \operatorname{Pic}\widetilde{C} \to \operatorname{Pic} C$, satisfying

$$\operatorname{Nm}_\pi O_{\widetilde{C}}(x) = O_C(\pi x) \qquad (x \in \widetilde{C}).$$

I.3 Abelian varieties

3.1. An abelian variety is a complete algebraic group. The group of points killed by $n \geq 2$ on an abelian variety X is denoted by $X[n]$. We write $\widehat{X} = \operatorname{Pic}^0 X$ for the group of isomorphism classes of line bundles on X algebraically equivalent to 0. This is an abelian variety, called the dual abelian variety of X ([Mu 6], p. 74). Each morphism $f\colon X \to Y$ has a functorially associated dual morphism $\hat{f}\colon \widehat{Y} \to \widehat{X}$.

3.2. For an element x of an abelian variety X, let τ_x be the automorphism $z \mapsto z + x$ of X. For a subvariety Z of X, write $Z_x = \tau_x(Z)$. For a sheaf $\mathcal{F}$ on X, also write $\mathcal{F}_x = \tau_{-x}^*(\mathcal{F})$.

3.3. Every line bundle M on an abelian variety X defines a group morphism ([Mu 6], p. 59)

$$\phi_M\colon X \longrightarrow \widehat{X}, \qquad x \longmapsto M_x \otimes M^{-1},$$

whose kernel is denoted by $H(M) = \{x \in X \mid M \simeq M_x\}$.

The set $\mathcal{G}(M) = \{(x, \alpha) \mid x \in X,\ \alpha\colon M \xrightarrow{\sim} M_x\}$ has a natural group structure ([Mu 5], p. 289) giving an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(M) \xrightarrow{p} H(M) \longrightarrow 0. \tag{3.4}$$

An alternating bilinear pairing is defined by

$$\begin{aligned} e^M\colon H(M) \times H(M) &\longrightarrow \mathbb{C}^*, \\ (x_1, x_2) &\longmapsto \tilde{x}_1 \tilde{x}_2 \tilde{x}_1^{-1} \tilde{x}_2^{-1}, \end{aligned} \qquad p(\tilde{x}_j) = x_j.$$

For example, if $M \in \operatorname{Pic}^0 X$, then $H(M) = X$ ([Mu 6], p. 74), $\mathcal{G}(M)$ is commutative, and $e^M \equiv 1$.

In contrast, if M is ample, $H(M)$ is finite, the centre of $\mathcal{G}(M)$ is $\mathbb{C}^*$, and e^M is nondegenerate ([Mu 5], Theorem 1, p. 293). There is a sequence of integers $\delta = (d_1, \dots, d_k)$, $k \leq g$, $1 < d_1 \mid d_2 \mid \cdots \mid d_k$, called the type of M, such that (3.4) is isomorphic to ([Mu 5], Corollary, p. 294)

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(\delta) \longrightarrow H(\delta) \longrightarrow 0,$$

where

$$\begin{aligned} K(\delta) &= \bigoplus_{i=1}^{k} \mathbb{Z}/d_i\mathbb{Z}, & \widehat{K}(\delta) &= \operatorname{Hom}(K(\delta), \mathbb{C}^*), \\ H(\delta) &= K(\delta) \oplus \widehat{K}(\delta), & \mathcal{G}(\delta) &= \mathbb{C}^* \times K(\delta) \times \widehat{K}(\delta), \end{aligned}$$

with

$$(\alpha, x, \ell)(\alpha', x', \ell') = (\alpha\alpha'\ell'(x),\ x + x',\ \ell\ell').$$

The integer $d = d_1 \cdots d_k$ is the degree of M and satisfies, with $g = \dim X$:

$$h^0(X, M^{\otimes n}) = dn^g, \qquad h^i(X, M^{\otimes n}) = 0 \quad (n > 0,\ i > 0). \tag{3.5}$$

$$M \simeq O_X(D) \quad \Longrightarrow \quad D^g = d\, g!. \tag{3.6}$$

$$d^2 = \#H(M) = \deg \phi_M. \tag{3.7}$$

Algebraic equivalence of line bundles on X is denoted by $\sim$. The preceding constructions, ϕ_M, $H(M)$, e^M, and, up to isomorphism, $\mathcal{G}(M)$, depend only on the algebraic equivalence class of M.

3.8. If M is ample, the line bundles algebraically equivalent to M are the translates M_x, for $x \in X$. A polarization of type δ on X is an algebraic equivalence class of ample line bundles of type δ. By abuse of notation, M also denotes the polarization defined by its class. A principal polarization is a polarization L of degree 1. The unique element of $|L|$ (see 3.5) is generally denoted by Θ; it is thus defined up to translation.

3.9. Let ι be the automorphism $x \mapsto -x$ of X. A line bundle M on X is symmetric if $\iota^* M \simeq M$. Every ample line bundle on X has a symmetric translate. Thus every polarization on X is the class of a symmetric line bundle, which is not uniquely determined.

3.10. For polarized abelian varieties (X_1, M_1) and (X_2, M_2), write $M_1 \boxtimes M_2$ for the polarization $\mathrm{pr}_1^* M_1 \otimes \mathrm{pr}_2^* M_2$ on $X_1 \times X_2$, where $\mathrm{pr}_j \colon X_1 \times X_2 \to X_j$, $j = 1, 2$, are the two projections. This is the product polarization of M_1 and M_2.

A polarized abelian variety is called irreducible if it is not isomorphic to the product of two nonzero polarized abelian varieties with the product polarization. In modern terminology, it is also called indecomposable.

A principally polarized abelian variety (A, Θ) is irreducible if and only if the divisor Θ is irreducible ([Cl–G]).

3.11. For a principally polarized abelian variety (A, Θ), the linear system $|2\Theta|$ is base-point-free. It therefore induces a morphism $\phi_{|2\Theta|} \colon A \to \mathbf{P}^N$, with $N = 2^{\dim A} - 1$, whose image is the Kummer variety $K(A)$ of (A, Θ). When (A, Θ) is irreducible, it is isomorphic to the quotient A/ι (see 3.9).

3.12. Polarized abelian varieties of type δ have a coarse moduli space $\mathcal{A}_{g,\delta}$, which is a normal quasi-projective variety of dimension $g(g+1)/2$. The open subset of irreducible varieties is denoted by $\mathcal{A}_{g,\delta}^{\mathrm{irr}}$. In the principally polarized case these spaces are denoted by $\mathcal{A}_g$ and $\mathcal{A}_g^{\mathrm{irr}}$, respectively.

The locus $\mathcal{J}_g$ of Jacobians of smooth curves of genus g is closed in $\mathcal{A}_g^{\mathrm{irr}}$ and equals it for $g \leq 3$.

We also make extensive use of the following closed subsets of $\mathcal{A}_g$, introduced by Andreotti and Mayer in [An–M]:

$$N_k^g = \{(A, \Theta) \in \mathcal{A}_g \mid \dim \mathrm{Sing}\, \Theta \geq k\}, \qquad 0 \leq k \leq g - 2.$$

3.13. Set

$$\mathfrak{H}_g = \{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau = {}^t\tau,\ \operatorname{Im}\tau > 0\}.$$

For a type $\delta = (d_1, \dots, d_k)$ (see 3.3), let Δ_δ be the $g \times g$ diagonal matrix with diagonal $(1, \dots, 1, d_1, \dots, d_k)$, padded with $g - k$ leading entries equal to 1.

Every complex abelian variety polarized of type δ is isomorphic to a complex torus $\mathbb{C}^g/(\Delta_\delta\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$ for some $\tau \in \mathfrak{H}_g$. The group

$$\Gamma_g(\delta) = \left\{ M \in \mathrm{GL}_{2g}(\mathbb{Z}) \;\middle|\; {}^tM \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix} \right\}$$

acts on $\mathfrak{H}_g$, and the quotient is the moduli space $\mathcal{A}_{g,\delta}$.

3.14. For $a, b \in \mathbb{R}^g$, $z \in \mathbb{C}^g$, and $\tau \in \mathfrak{H}_g$, define

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}^g} \exp\bigl(\pi i\bigl[{}^t(n+a)\tau(n+a) + 2{}^t(n+a)(z+b)\bigr]\bigr).$$

These functions satisfy the heat equations

$$\frac{\partial^2\theta[\,{}^a_b\,]}{\partial z_j \partial z_k}(z,\tau) = 2\pi i(1+\delta_{jk})\frac{\partial\theta[\,{}^a_b\,]}{\partial\tau_{jk}}(z,\tau), \quad 1 \le j,k \le g.$$

The function $\theta[\,{}^0_0\,](z,\tau)$ is simply denoted by $\theta(z,\tau)$; it is the theta function. If (X,M) is the abelian variety polarized of type δ corresponding to $\tau \in \mathfrak{H}_g$, a basis of $H^0(X,M)$ is

$$\left\{ \theta\begin{bmatrix} r \\ 0 \end{bmatrix}(\cdot,\tau) \;\middle|\; r \in \Delta_\delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g \right\}.$$

I.4 Prym varieties

4.1. Let $\pi\colon \widetilde{C} \to C$ be an étale double cover of smooth connected projective curves. The kernel of the norm morphism $\mathrm{Nm}_\pi\colon J\widetilde{C} \to JC$ (see 2.6) has two connected components. Its identity component P is the Prym variety associated with π; the other component is denoted by P^-. The principal polarization of $J\widetilde{C}$ induces twice a principal polarization on P, generally denoted by Ξ.

The moduli space $\mathcal{R}_{g+1}$ of smooth curves C of genus $g+1$ equipped with such a cover π is irreducible of dimension $3g$ (for $g \ge 1$), and there is a Prym map $\mathrm{Pr}\colon \mathcal{R}_{g+1} \to \mathcal{A}_g$. The closure of its image is denoted by $\mathcal{P}_g$.

In [Be 1], Beauville introduced the notion of an admissible cover of a stable curve (see IV.A.4). This allows him to extend the Prym map to a proper morphism $\overline{\mathcal{R}}_{g+1} \to \mathcal{A}_g$, where $\overline{\mathcal{R}}_{g+1}$ is the moduli space of these covers. Its image is $\mathcal{P}_g$, which contains $\mathcal{J}_g$. The open subset of irreducible varieties in $\mathcal{P}_g$ is denoted by $\mathcal{P}_g^{\mathrm{irr}}$ (see 3.10).

CHAPTER II

The trisecant conjecture, the loci N^g_{g-4}, and Prym varieties

II.A Introduction

II.A.1 The trisecant conjecture

Let (A, Θ) be a principally polarized abelian variety of dimension $g \geq 2$. We assume that the divisor Θ is symmetric (see I.3.9). The linear system $|2\Theta|$ is base-point-free and induces a morphism $\psi\colon A \longrightarrow |2\Theta|^\vee \simeq \mathbb{P}^N$, where $N = 2^g - 1$ (see I.3.5). By [Mu 2], p. 335, there is a nondegenerate bilinear form on $H^0(A, 2\Theta)$ inducing an isomorphism $B\colon |2\Theta|^\vee \xrightarrow{\sim} |2\Theta|$ for which the following diagram commutes:

$$\begin{array}{ccc} & & |2\Theta|^\vee \\ & \overset{\psi}{\nearrow} & \big\downarrow B \\ A & \xrightarrow[\varphi]{} & |2\Theta| \end{array} \qquad \varphi(x) = \Theta_x + \Theta_{-x}. \tag{1.1}$$

When (A, Θ) is irreducible (see I.3.10), the morphism ψ has degree 2, and its image $K(A)$, the Kummer variety of A, is isomorphic to $A/\{\pm 1\}$. This variety is smooth away from the images of the points of order 2 of A.

We are interested in the existence of trisecants to $K(A)$, that is, in the most elementary sense, lines in $\mathbb{P}^N$ meeting $K(A)$ in three points.

The problem originates as follows. For the Jacobian JC of a smooth curve, there is a four-dimensional family of trisecants to its Kummer variety $K(JC)$. Following Mumford in [Mu 8], it is therefore tempting to conjecture that the existence of sufficiently many trisecants to the Kummer variety characterizes Jacobians among irreducible principally polarized abelian varieties. Before describing recent progress in this direction, it is useful to give a classical interpretation of the existence of a trisecant in terms of intersections of the theta divisor with its translates.

Thus let (A, Θ) be a principally polarized abelian variety, assumed irreducible, meaning that the divisor Θ is irreducible. Write θ for a generator of $H^0(A, \Theta)$ and, for each $x \in A$, write Θ_x for the translate of Θ by x (see I.3.2) and θ_x for a generator of $H^0(A, \Theta_x)$.

Let a, b, c, d be nonzero elements of A such that $a + b = c + d = x$ and $\{a, b\} \cap \{c, d\} = \emptyset$. The following three properties are equivalent:

(1.2) For every $y \in A$ such that $2y = x$, the three points $\varphi(y)$, $\varphi(y - a)$, and $\varphi(y - c)$ of $K(A)$ are collinear.

(1.3) The three elements $\theta\theta_x$, $\theta_a\theta_b$, and $\theta_c\theta_d$ of $H^0(A, \Theta + \Theta_x)$ are linearly dependent, so there are nonzero complex numbers λ, μ, ν such that

$$\lambda\theta\theta_x + \mu\theta_a\theta_b + \nu\theta_c\theta_d = 0.$$

(1.4) There is a scheme-theoretic inclusion

$$\Theta \cdot \Theta_a \subset \Theta_c \cup \Theta_d.$$

We briefly indicate the proofs.

(1.2) $\iff$ (1.3). By (1.1), the points $\psi(y)$, $\psi(y-a)$, and $\psi(y-c)$ of $\mathbb{P}^N$, which are distinct by assumption, are collinear if and only if the elements $\theta_y\theta_{-y}$, $\theta_{y-a}\theta_{a-y}$, and $\theta_{y-c}\theta_{c-y}$ of $H^0(A, 2\Theta)$ are linearly dependent. Translating by y gives the result. The complex numbers λ, μ, ν are nonzero because Θ is irreducible.

(1.3) $\implies$ (1.4). This is immediate.

(1.4) $\implies$ (1.3). The restriction of $\theta_c\theta_d$ to $\overline{\Theta}_a = \Theta \cdot \Theta_a$ therefore vanishes. There is an exact sequence

$$0 \longrightarrow H^0(\Theta, O(\Theta_b)) \xrightarrow{\cdot\theta_a} H^0(\Theta, O(\Theta_c + \Theta_d)) \longrightarrow H^0(\overline{\Theta}_a, O(\Theta_c + \Theta_d)).$$

Moreover, since $b \neq 0$, we have

$$H^0(A, O(\Theta_b)) \xrightarrow{\sim} H^0(\Theta, O(\Theta_b)).$$

It follows that there is a $\mu \in \mathbb{C}$ such that the restriction of $\mu\theta_a\theta_b + \theta_c\theta_d$ to Θ vanishes. We then use the exact sequence

$$0 \longrightarrow H^0(A, O(\Theta_x)) \xrightarrow{\cdot\theta} H^0(A, O(\Theta_x + \Theta)) \longrightarrow H^0(\Theta, O(\Theta_x + \Theta))$$

to conclude.

In particular, the existence of a trisecant implies that some intersection $\Theta \cdot \Theta_a$, with $a \neq 0$, is not integral. This is a very restrictive condition on the abelian variety (A, Θ) (see II.C).

We can now easily describe the trisecants to the Kummer variety of the Jacobian (JC, Θ) of a smooth curve C. For two points $x, y \in C$, write $x - y$ for the point of JC corresponding to the line bundle $O_C(x - y)$. It is easy to see (see II.B) that the intersections $\Theta \cdot \Theta_{x-y}$ are not integral and that

$$\forall x, y, z, t \in C, \qquad \Theta \cdot \Theta_{x-y} \subset \Theta_{x-t} \cup \Theta_{z-y}.$$

By (1.2), it follows that, for all points $x, y, z, t \in C$ and every $\zeta \in J^{-1}C$ with $2\zeta \equiv t - x - y - z$, the points $\psi(\zeta + x)$, $\psi(\zeta + y)$, and $\psi(\zeta + z)$ of $\mathbb{P}^N$ are collinear.

In II.B.4 we shall show that this construction gives all trisecants to $K(JC) \subset \mathbb{P}^N$.

Observe that the three points $\psi(\zeta + x)$, $\psi(\zeta + y)$, and $\psi(\zeta + z)$ are distinct if and only if the four points x, y, z, t of C are pairwise distinct.

It is interesting to examine what becomes of the trisecant when some of x, y, z, t coincide. We obtain lines in $\mathbb{P}^N$ tangent to the Kummer variety at one point and meeting it again at another, lines having contact of order 3 with $K(JC)$ (inflectional lines), or other degenerate cases, which are listed in II.C.2.8.

In particular, it is easy to see that there is a two-dimensional family of inflectional lines (the case $x = y = z$).

Finally, when all four points x, y, z, t coincide, we obtain a "trisecant" at the singular point $\psi(0)$, whose geometric description is unclear. However, this property is equivalent to the fact that the function $\theta(z, \tau)$ associated with JC (see I.3.14) satisfies a certain nonlinear partial differential equation, the KP equation (see II.C.2.10). Welters gives a geometric interpretation

of this equation in [We 8]. In II.C.2.11 we derive a consequence of the form (1.4) (this observation was communicated to me by E. Arbarello, whom I thank). Each of these properties of Jacobians of curves gives rise to a conjecture seeking to characterize them among principally polarized abelian varieties. These questions have prompted extensive work, notably by E. Arbarello, C. De Concini, B. A. Dubrovin, R. C. Gunning, Z. Ran, T. Shiota, and G. Welters.

Pride of place naturally goes to Shiota's theorem (formerly Novikov's conjecture), which states that an irreducible principally polarized abelian variety whose theta function satisfies the KP equation is the Jacobian of a curve. The original proof in [Sh] was simplified in [Ar-D 2] (see also [Ar]).

Next come Welters's results, of which we single out the following: a principally polarized abelian variety (A, Θ) with a one-dimensional family of elements a satisfying (1.4), and satisfying the additional condition $\dim \operatorname{Sing} \Theta \leq \dim A - 4$, is the Jacobian of a curve. Despite appearances, this last condition is substantial, as Theorem II.C.2.9 shows. This result is proved in [We 6]. One may also consult [We 7], which reaches the same conclusion without this condition, but assumes a one-dimensional family of relations of the form (1.3) with a, c fixed. Degenerate cases are treated as well.

It is therefore tempting to conjecture, as Welters did in [We 7], that the existence of a trisecant characterizes Jacobians.

Leaving aside degenerate cases, we state:

The trisecant conjecture. *Every irreducible principally polarized abelian variety whose Kummer variety admits a proper trisecant (that is, satisfies one of the equivalent conditions (1.2), (1.3), (1.4), for nonzero elements a, b, c, d with $\{a, b\} \cap \{c, d\} = \emptyset$) is the Jacobian of a curve.*

II.A.2 Methods and results

After examining in detail the reducibility of the intersections $\Theta \cdot \Theta_a$ on Jacobians in II.B, we approach the conjecture itself in II.C by showing, in collaboration with A. Beauville, that the existence of a trisecant, or equivalently a nontrivial inclusion $\Theta \cdot \Theta_a \subset \Theta_c \cup \Theta_d$ (see (1.4)), implies

$$\dim(\Theta_a \cap \operatorname{Sing} \Theta) \geq \dim A - 4.$$

By [An-M], this gives a weak form of the conjecture: the family of Jacobians is an irreducible component of the locus of irreducible abelian varieties whose Kummer variety admits a trisecant.

The remainder of this section concerns Prym varieties (see I.4) whose Kummer variety admits a trisecant. In II.D and II.E we describe the Prym varieties (P, Ξ) satisfying the condition $\dim \operatorname{Sing} \Xi \geq \dim P - 4$, which is necessary by II.C. At the same time we obtain irreducible components of the Andreotti–Mayer loci N^g_{g-4} (see I.3.12) other than $\overline{J_g}$ (see also Appendix I).

In II.F, either by studying the elements $a \in P$ for which $\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq \dim P - 4$, or by starting from identity (1.4) in certain cases, we prove the trisecant conjecture for generalized Prym varieties, hence in particular for all abelian varieties of dimension ≤ 5.

II.B Reducibility of the intersections $\Theta \cdot \Theta_a$ for Jacobians

A. Weil was the first to study systematically, in [W], the reducibility of the intersections $\Theta \cdot \Theta_a$ for a Jacobian.[1]

He proved that, in the Jacobian of a smooth curve C of genus ≥ 5, the intersection $\Theta \cdot \Theta_a$ is integral except when $a \in C - C$, that is, $a = O_C(p - q)$ for $p, q \in C$, and in one special case (case 2 of our theorem). This characterization of the surface $C - C$ in the Jacobian then enabled him to prove Torelli's theorem.

This section gives an elementary geometric proof of Weil's results, valid in every genus ≥ 3, and also describes the components of $\Theta \cdot \Theta_a$ in every case.

Recall (see Chapter I for the remaining notation) that "$\cdot$" denotes scheme-theoretic intersection. For a smooth curve C of genus g, we write "$-$" for the involution $L \mapsto \omega_C \otimes L^{-1}$ of $J^{g-1}C$, W^d for the image of $C^{(d)} \longrightarrow J^dC$, and, for $p \in C$, W^d_p for the image of W^d under $L \mapsto L \otimes O_C(p)$.

Our result is:

Theorem II.B.1. *Let C be a smooth projective curve of genus $g \geq 3$, let Θ be the canonical theta divisor of $J^{g-1}C$, and let a be a divisor of degree 0 on C that is not equivalent to zero. Set $\Theta_a = \Theta + a$. Then $\Theta \cdot \Theta_a$ is integral except in the following cases:*

1. *There are two points $p, q \in C$ such that $a \equiv p - q$.*
2. *There are a smooth elliptic curve E, a finite morphism $\pi\colon C \longrightarrow E$ of degree two, and a divisor e on E such that $a \equiv \pi^* e$.*

Remark II.B.2. *In the first case, $\Theta \cdot \Theta_a$ is the union of the irreducible subvarieties W^{g-2}_p and $-(W^{g-2}_q)$. In particular, $\Theta \cdot \Theta_a$ is reduced unless C is hyperelliptic and (p, q) is a conjugate pair.*

In the second case, $\Theta \cdot \Theta_a$ is reduced and generally has two explicitly describable components (Remarks 8 and 9), except in one special genus-three case (Remark 9), where there are three components.

Torelli's theorem follows from this theorem by an argument similar to that of [Ci].

Corollary II.B.3. *Let C, C' be smooth projective curves of the same genus $g \geq 3$. Suppose there is an isomorphism of principally polarized abelian varieties $v\colon JC \longrightarrow JC'$. Then there is an isomorphism $u\colon C' \longrightarrow C$ such that $v = \pm u^*$.*

Proof. The set $Z_C = \{a \in JC \mid \Theta \cdot \Theta_a \text{ is not integral}\}$ is independent of the theta divisor Θ chosen in JC to define it. In particular,

$$\begin{aligned} v(Z_C) &= \{v(a) \in JC' \mid \Theta \cdot \Theta_a \text{ is not integral}\} \\ &= \{v(a) \in JC' \mid v(\Theta) \cdot v(\Theta)_{v(a)} \text{ is not integral}\} = Z_{C'}. \end{aligned}$$

The theorem says that Z_C is the union of $C - C$ and, possibly, one or more elliptic curves $\pi^* JE \subset JC$. Since $C - C$ has dimension 2 in JC, it follows in every case that $v(C - C) = C' - C'$. For $L \in J^{g-1}C$, define the theta divisor $\Theta_L \subset JC$ by $\Theta_L = \{M \in JC \mid H^0(M \otimes L) \neq 0\}$. For $H^0(L) \neq 0$, this gives all theta divisors through 0. If $h^0(L) > 1$, then $h^0(L \otimes O_C(p - q)) \geq 1$ for all $p, q \in C$, so $\Theta_L \supset C - C$. If instead $h^0(L) = 1$, then $h^0(\omega_C \otimes L^{-1}) = 1$; writing

[1] This section was published, with some modifications, as "Sur la démonstration de A. Weil du théorème de Torelli pour les courbes" in *Compositio Mathematica* (reference [D 1]).

$|L| = \{x_1 + \cdots + x_{g-1}\}$ and $|\omega_C \otimes L^{-1}| = \{y_1 + \cdots + y_{g-1}\}$, we have

$$\Theta_L \cap (C - C) = \bigcup_{i=1}^{g-1} [(C - x_i) \cup (y_i - C)].$$

Choose $L \in J^{g-1}C$ with $h^0(L) = 1$. Then $v(\Theta_L \cap (C - C)) = \Theta_{L'} \cap (C' - C')$ is reducible and hence, by the preceding argument, is the union of $(g-1)$ translates of C' and $(g-1)$ translates of $(-C')$. Consequently,

$$\exists x \in C, \quad \exists x' \in C', \qquad v(C - x) = \pm(C' - x'),$$

which proves the corollary. □

Another consequence is the characterization of trisecants to the Kummer variety of the Jacobian of a curve announced in II.A.1.

Corollary II.B.4. *Let C be a smooth curve of genus $g \geq 3$. Suppose there are three nonzero elements $a, c, d \in JC$ such that*

$$\{0, a\} \cap \{c, d\} = \emptyset, \qquad \Theta \cdot \Theta_a \subset \Theta_c \cup \Theta_d.$$

Then there are four points $x, y, z, t \in C$ such that

$$a = O_C(x - y), \qquad \{c, d\} = \{O_C(x - t), O_C(z - y)\}.$$

Proof. The intersections $\Theta \cdot \Theta_a$, $\Theta \cdot \Theta_c$, and $\Theta \cdot \Theta_d$ are all reducible. Using Remarks 2, 8, and 9, we compare their possible components and show that case 2 of the theorem cannot occur for any of $a, c, d \in JC$.

First assume $g \geq 4$. In case 2 of the theorem, let σ denote the involution of C associated with π, and let Z^a_σ, Z_σ be the two components of $\Theta \cap \Theta_a$, for $a \in \pi^* \operatorname{Pic}^0 E$ (see Remark 8).

By Lemma 5, a generic element L of either Z_σ or Z^a_σ satisfies $h^0(L) = 1$, so Z_σ, Z^a_σ differ from every W^{g-2}_p, $p \in C$. We have $-Z_\sigma = Z_\sigma$, hence $-Z^a_\sigma = Z^{-a}_\sigma$; therefore Z_σ, Z^a_σ also differ from every $-(W^{g-2}_q)$, $q \in C$.

Moreover, if $Z_\tau = Z^a_\sigma$ for another involution τ, then for generic $(x_1, \ldots, x_{g-2}) \in C^{g-2}$ we would have $O_C(x_1 + \cdots + x_{g-2} + \sigma x_{g-2}) \in Z_\tau$, which is impossible. Similarly, Z_σ differs from Z_τ when $\sigma \neq \tau$. We have shown that W^{g-2}_p, $-(W^{g-2}_q)$, Z_σ, Z_τ, Z^a_σ, and Z^b_σ are distinct for $\sigma \neq \tau$, $a \neq b$, $g \geq 3$. This proves the assertion for $g \geq 4$.

The case $g = 3$ is similar: the components are of type Z_σ (cases 1 and 2 of Remark 9) or Z^a_σ (case 3). If a falls under case 1 of that remark, so that $\Theta \cdot \Theta_a = Z_{\sigma_1} \cup Z_{\sigma_2} \cup Z_{\sigma_3}$, two of these components lie, for example, in $\Theta \cdot \Theta_c$. Thus c also falls under case 1, and $\Theta \cdot \Theta_c = Z_{\sigma_1} \cup Z_{\sigma_2} \cup Z_\tau$. The Galois groups of $\phi_{|K+a|}$ and $\phi_{|K+c|}$, both generated by σ_1, σ_2, are consequently equal, and $K + a \equiv K + c$. Case 2 is treated identically; if a falls under case 3 of the remark, then Z^a_σ is a component of $\Theta \cdot \Theta_c$, so c also falls under that case for the same involution σ, and $a = c$. □

Proof of Theorem II.B.1. Weil's basic idea is to consider the linear system $|K_C + a|$. If $a \notin C - C$, it is base-point-free and therefore defines a morphism $\phi\colon C \longrightarrow |K + a|^\vee \simeq \mathbb{P}^{g-2}$. Let $p\colon \Theta \cap \Theta_a \dashrightarrow |K + a|$ be the rational map defined as follows. For any $L \in \Theta \cap \Theta_a$ with $h^0(L) = h^0(L - a) = 1$, we have $h^0(K - L + a) = 1$; if $|L| = \{D\}$ and $|K - L + a| = \{D'\}$, set $p(L) = D + D'$.

Lemma II.B.5. *The map p is defined on a dense open subset of $\Theta \cap \Theta_a$, and its restriction to each component is dominant. The intersection $\Theta \cdot \Theta_a$ is reduced.*

Proof. Since $\Theta \cap \Theta_a$ is locally defined by two equations in $J^{g-1}C$, all its components have dimension $g-2$. Moreover, $\dim(\operatorname{Sing}\Theta \cup \operatorname{Sing}\Theta_a) \leq g-3$. Thus, for each component Z of $\Theta \cap \Theta_a$ and generic $L \in Z$, the point L is smooth on both Θ and Θ_a, that is, $h^0(L) = h^0(L-a) = 1$. The map p is generically finite, so $p|_Z$ is dominant. Hence there is an $L \in Z$ such that $p(L)$ consists of $2g-2$ distinct points. The tangent space $T_L\Theta$ corresponds to the point $D + D^*$ of $|K| \simeq \mathbb{P}T_L^\vee(JC)$, where $D^* \in |K-L|$, and $T_L\Theta_a$ corresponds to $D' + D'^*$, where $D'^* \in |L-a|$. Since $a \neq 0$, we have $D \neq D'^*$, and by construction D, D' have no common point. The spaces $T_L\Theta, T_L\Theta_a$ are distinct, and Z is reduced. □

We may therefore restrict attention to the set $\Theta \cap \Theta_a$, which we describe geometrically. By Riemann–Roch, its inverse image under the natural map $C^{(g-1)} \longrightarrow \Theta \subset J^{g-1}C$ is the divisor

$$\overline{W} = \{D \in C^{(g-1)} \mid H^0(K_C + a - D) \neq 0\}.$$

Thus $\overline{W}$ consists of those $x_1 + \cdots + x_{g-1} \in C^{(g-1)}$ for which the $(g-1)$ points $\phi x_1, \ldots, \phi x_{g-1}$ lie on a common hyperplane. More precisely, let $U \subset |K+a|$ be the open set of hyperplanes in $|K+a|^\vee$ meeting $\phi(C) = C'$ transversely at points over which ϕ is étale. By the preceding lemma, studying the components of $\Theta \cap \Theta_a$ reduces to studying

$$W = \{x_1 + \cdots + x_{g-1} \in C^{(g-1)} \mid x_i \text{ pairwise distinct, } \phi \text{ smooth over the } \phi x_i, \\ \exists H \in U,\ \phi x_i \in H \text{ for every } i\}.$$

Continue to write $\phi\colon C \longrightarrow C'$ for the induced morphism from C onto its image. The formulas $2g-2 = \deg\phi \cdot \deg C'$ and $\deg C' \geq g-2$ give the following possibilities, which we consider in this order:

- ϕ is birational;
- ϕ has degree 2;
- ϕ has degree 3 and $g = 4$;
- $g = 3$.

(a) ϕ is birational. The space W, and hence $\Theta \cap \Theta_a$, is irreducible by the following theorem from [A-C-G-H 1], whose proof we reproduce.

Theorem II.B.6. *Let $C \subset \mathbb{P}^r$ be an irreducible curve of degree d, and let $U \subset (\mathbb{P}^r)^\vee$ be the open set of hyperplanes meeting C transversely. For every integer m with $1 \leq m \leq d$,*

$$I(m) = \{(x_1, \ldots, x_m; H) \in C^m \times U \mid x_i \text{ pairwise distinct and } x_i \in H\}$$

is irreducible.

Proof. The projection $\mathrm{pr}^m\colon I(m) \longrightarrow U$ is an étale covering. Choose a base point $H_0 \in U$, and let $F_m = (\mathrm{pr}^m)^{-1}(H_0)$ be its fibre. The irreducibility of $I(m)$ is equivalent to the transitivity of the monodromy action of $\pi_1(U, H_0)$ on F_m, that is, to its m-fold transitivity on F_1 through the monodromy of $\mathrm{pr}^1\colon I(1) \longrightarrow U$. It is enough to prove that the Galois group G of pr^1 is the symmetric group $\mathfrak{S}_d$, where d is the degree of C. For $d = 1$ this is immediate; otherwise it follows from two observations:

- G is doubly transitive. By the preceding discussion, this is equivalent to the irreducibility of $I(2)$. The projection $\mathrm{pr}_1 : I(2) \longrightarrow C \times C$ is dominant, with fibres that are dense open subsets of projective spaces of dimension $r-2$, so $I(2)$ is irreducible.
- G contains a transposition. Let H_1 be a hyperplane simply tangent to C at one point, and let $\{H_t\}_{t\in\mathbb{C},\,|t-1|<\varepsilon}$ be a one-parameter family transverse to the discriminant at H_1, with $H_t \in U$ for $t \neq 1$. Then $H_t \cap C$ contains two points that coalesce at the point of tangency of H_1 with C as t tends to 1. These two points are interchanged when t goes around 1.

□

It now suffices to observe that $W \simeq \mathrm{pr}(I(g-1))$, where

$$\mathrm{pr} : C^{g-1} \times U \longrightarrow C^{g-1} \longrightarrow C^{(g-1)}.$$

(b) ϕ has degree 2. **Lemma II.B.7.** *If ϕ has degree 2, or more generally if there is an involution σ on C such that ϕ factors through $\pi : C \longrightarrow C/\sigma$, then C/σ is a smooth elliptic curve E, and there is a divisor e on E such that $a \equiv \pi^* e$.*

Proof. The branched covering $\pi : C \longrightarrow C/\sigma = E$ has an associated element $\delta \in \mathrm{Pic}\, E$, defined by $\pi_* O_C = O_E \oplus O_E(-\delta)$. The morphism ϕ factors through π if and only if there is a degree-zero divisor e on E such that

$$a \equiv \pi^* e, \qquad H^0(E, K_E + e) = 0 \quad \text{or} \quad H^0(E, K_E + \delta + e) = 0.$$

Since δ has nonnegative degree, Riemann–Roch gives $g(E) = 0$ or 1. As a is not equivalent to 0, neither is e, so E is not rational. □

Remark II.B.8. *When ϕ has degree 2, the components of $\Theta \cap \Theta_a$ are*

$$\overline{\{O_C(x_1 + \cdots + x_{g-1}) \mid \phi x_1, \ldots, \phi x_{g-1} \text{ distinct, on a hyperplane in } U\}}$$

and

$$\{O_C(x_1 + \cdots + x_{g-2} + \sigma x_{g-2}) \mid x_i \in C \text{ arbitrary}\}.$$

Proof. It suffices to show that the first set is irreducible. As in A3.1.2, introduce the Galois group G of the étale projection

$$\mathrm{pr} : I = \{(x, H) \in C \times U \mid \phi x \in H\} \longrightarrow U.$$

It acts on a fibre $\{a_1^+, \ldots, a_{g-1}^+, a_1^-, \ldots, a_{g-1}^-\}$, where $a_i^+ = \sigma a_i^-$. By A3.1.2, the action of G on the set of $(a^{\varepsilon_1}_{\tau(1)}, \ldots, a^{\varepsilon_{g-1}}_{\tau(g-1)})$, with $\varepsilon_j \in \{-1, +1\}$ and $\tau \in \mathfrak{S}_{g-1}$, has at most two orbits: that of $(a_1^+, \ldots, a_{g-1}^+)$ and that of $(a_1^+, \ldots, a_{g-2}^+, a_{g-1}^-)$. As in the proof of Theorem 6, consider a family of hyperplanes $\{H_t\}_{t\in\mathbb{C},\,|t-1|<\varepsilon}$ with $H_t \in U$ for $t \neq 1$, while H_1 passes through a point of C' over which ϕ is ramified. For $t \neq 1$, the set $\phi^{-1}(H_t \cap C')$ contains two points that coalesce at the ramification point as t tends to 1. Thus G contains a transposition (a_i^+, a_i^-), proving the remark. □

(c) The case $g = 4$ and $\deg \phi = 3$. The image of ϕ is a conic $Q \subset \mathbb{P}^2$. Recall that

$$W = \{x_1 + x_2 + x_3 \in C^{(3)} \mid x_i \text{ pairwise distinct, } \phi \text{ étale on } \phi^{-1}(\phi x_i), \\ \phi x_1, \phi x_2, \phi x_3 \text{ collinear}\}.$$

The morphism $f : C \times C \longrightarrow C^{(3)}$ sending (x, y) to $(\phi^{-1}(\phi x) - x) + y$ induces a surjection from an irreducible dense subset of $C \times C$ onto W. This proves that W, and hence $\Theta \cap \Theta_a$, is irreducible.

(d) The case $g = 3$. The morphism ϕ has degree 4 onto $\mathbb{P}^1$, with branch locus $\Delta \subset \mathbb{P}^1$. Choose a base point $p_0 \in \mathbb{P}^1 \setminus \Delta$, with fibre $F = \phi^{-1}(p_0)$.

The Galois group G of ϕ is a subgroup of $\operatorname{Aut} F \simeq \mathfrak{S}_4$. It acts transitively on F because $C \setminus \phi^{-1}(\Delta)$ is connected. Recall that

$$W = \{x_1 + x_2 \in C^{(2)} \mid x_1 \neq x_2,\ \phi x_1 = \phi x_2 \notin \Delta\}.$$

The irreducible components of W correspond bijectively to the G-orbits in $F^{(2)} \setminus \operatorname{diag}$. In particular, W is irreducible if G is doubly transitive.

We briefly review the transitive subgroups G of $\mathfrak{S}_4$. Note that $|G|$ is divisible by 4:

1. $G \simeq (\mathbb{Z}/2)^2$. Clearly G contains no transposition, since otherwise it would not be transitive. Thus G is the Klein group
$$\{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}.$$
2. $G \simeq \mathbb{Z}/4$. Then G is conjugate to the subgroup generated by $(1\,2\,3\,4)$.
3. G has order 8. These groups are the Sylow 2-subgroups of $\mathfrak{S}_4$, hence are conjugate, for instance, to D_4, generated by $(1\,2\,3\,4)$ and $(1\,3)$.
4. G has order 12. It has index 2, so is normal and equals A_4.

Finally, if $x \in F$ and G_x is its stabilizer in G, then

$$\operatorname{Aut}(C/\mathbb{P}^1) \simeq N_G(G_x)/G_x.$$

The results needed here are summarized in the table below. They show that either W is irreducible or $\operatorname{Aut}(C/\mathbb{P}^1)$ contains an element of order 2. Together with Lemma 7, this completes the proof of the theorem.

Remark II.B.9. *The preceding analysis gives a more precise description of the irreducible components of $\Theta \cdot \Theta_a$ in case 2 of the theorem when $g = 3$. The double cover $\pi\colon C \longrightarrow C/\sigma = E$ corresponds to $\delta \in \operatorname{Pic}^2 E$ and a divisor $\Delta \in |2\delta|$ without multiple points. Let σ' be the involution of E associated with the degree-two morphism $\phi_{|\delta+e|}\colon E \longrightarrow \mathbb{P}^1$, where $a = \pi^* e$, and let R be its ramification divisor on E. Recall that $\phi = \phi_{|\delta+e|} \circ \pi$ (Lemma 7). Exactly one of the following cases occurs.*

G	*Stabilizer of* 1	$\operatorname{Aut}(C/\mathbb{P}^1)$	*Orbits in* $F^{(2)} \setminus \operatorname{diag}$
$(\mathbb{Z}/2)^2$	$\{\mathrm{id}\}$	$(\mathbb{Z}/2)^2$	3
$\mathbb{Z}/4$	$\{\mathrm{id}\}$	$\mathbb{Z}/4$	2
D_4	$\langle(2\,4)\rangle$	$\mathbb{Z}/2$	2
A_4	A_3	$\{\mathrm{id}\}$	1
$\mathfrak{S}_4$	$\mathfrak{S}_3$	$\{\mathrm{id}\}$	1

The last two actions are doubly transitive.

1. *$\Delta = x + y + \sigma' x + \sigma' y$, where $x, y \notin \operatorname{Supp} R$. Then a has order 2, and the Galois group G of ϕ is isomorphic to $(\mathbb{Z}/2)^2$. Writing $G = \{\mathrm{id}, \sigma_1, \sigma_2, \sigma_3\}$, the three components of $\Theta \cdot \Theta_a$ are*
$$\{O_C(x + \sigma_i x) \mid x \in C\}, \qquad i = 1, 2, 3.$$
2. *$\Delta = R$. Then a has order 2, and the Galois group G of ϕ is isomorphic to $\mathbb{Z}/4$. The two components of $\Theta \cdot \Theta_a$ are $\{O_C(x + \sigma x) \mid x \in C\}$ and $\{O_C(x + \sigma^2 x) \mid x \in C\}$, where σ generates G.*

3. *If Δ is of neither type above, the Galois group of ϕ is isomorphic to D_4. The automorphism group of C over $\mathbb{P}^1$ is generated by an involution σ, and*

$$\Theta \cdot \Theta_a = \{O_C(x + \sigma x) \mid x \in C\} \cup \{O_C(x_1 + x_2) \mid \pi x_2 = \sigma' \pi x_1\}.$$

II.C A relation between two approaches to the Schottky problem: a necessary condition for reducibility of $\Theta \cdot \Theta_a$ in an abelian variety

In collaboration with A. Beauville.[2]

The Schottky problem asks for a characterization of Jacobians among all abelian varieties. More precisely, let A_g be the moduli space of principally polarized abelian varieties of dimension g. Jacobians form a subvariety $J_g \subset A_g$, and the problem is to find equations for J_g, or its closure $\overline{J_g}$, in A_g.

Among the geometric approaches to this problem, two methods have proved particularly fruitful:

1. The Andreotti–Mayer approach uses the singularities of the divisor Θ. These authors prove that $\overline{J_g}$ is an irreducible component of the subvariety $N_{g-4} \subset A_g$ consisting of principally polarized abelian varieties (A, Θ) with $\dim \mathrm{Sing}(\Theta) \geq g - 4$ [An-M].
2. The approach based on the reducibility of $\Theta \cap \Theta_a$. It relies on an observation already used by Weil: for the Jacobian (JC, Θ) of a curve C, the intersection $\Theta \cap \Theta_a$ is reducible when $a = p - q$ for $p, q \in C$. More precisely, for distinct $p, q, r, s \in C$, one has $\Theta \cap \Theta_{p-q} \subset \Theta_{p-r} \cup \Theta_{s-q}$.

This leads us to consider several conditions on a principally polarized abelian variety (A, Θ), all satisfied when it is a Jacobian:

(i) There is a nonzero $a \in A$ such that $\Theta \cap \Theta_a$ is reducible.

(ii) There are three distinct nonzero elements $a, x, y \in A$ such that $\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y$.

(iii) The Kummer variety of (A, Θ) admits a trisecant.

(iv) The theta function associated with (A, Θ) satisfies a certain nonlinear partial differential equation, the KP equation (see (2.10) for a precise formulation).

These conditions and their mutual relations have recently been studied extensively, notably in [We 6], [We 7], and [Ar-D]. More precisely, these authors strengthen (ii) and (iii) by requiring one-dimensional families of trisecants or of triples (a, x, y) satisfying (ii).

The purpose of this article is to establish a link between the two approaches just described. We prove that each of (ii), (iii), and (iv) implies the Andreotti–Mayer condition $\dim \mathrm{Sing}(\Theta) \geq g - 4$. Moreover, a principally polarized abelian variety (A, Θ) satisfying (i) either satisfies the Andreotti–Mayer condition or contains an elliptic curve E with $(\Theta \cdot E) = 2$. These results and the Andreotti–Mayer theorem immediately imply that $\overline{J_g}$ is a component of the locus of principally polarized abelian varieties satisfying any one of (i)–(iv).

The proof starts from the observation that $\dim \mathrm{Sing}(\Theta) < g - 4$ implies that Θ is locally factorial. The reducibility of $\Theta \cap \Theta_a$ then gives a decomposition of $\Theta_a|_\Theta$ as a sum of effective Cartier divisors on Θ. Because of the isomorphism $\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\Theta)$, such a decomposition

[2] This section was published as "Une relation entre deux approches du problème de Schottky" in *Inventiones Mathematicae* (reference [Be-D 1]).

is very restrictive: in §1 we show that it is equivalent to A containing an elliptic curve E with $(\Theta \cdot E) = 2$ and $a \in E$. In §2 we deduce the results stated above.

We thank E. Arbarello for pointing out that condition (iv) implies a form of (i) and can therefore be treated by our method.

Notation and conventions. The varieties considered here are defined over an algebraically closed field k of arbitrary characteristic.

(0) Let V be an algebraic variety, $Z \subset V$ a subvariety, and L a line bundle on V. Suppose we are given a vector field X on V and a section $s \in H^0(V, L)$ vanishing on Z. There is a unique section $Xs \in H^0(Z, L|_Z)$ with the following property: for every open subset $U \subset V$ and every isomorphism $\lambda\colon O_U \longrightarrow L|_U$, we have

$$Xs = \lambda(X\lambda^{-1}(s))|_Z \qquad \text{on } Z \cap U.$$

Now let (A, Θ) be a principally polarized abelian variety, and let $\theta \in H^0(A, \Theta)$ have divisor Θ. The preceding construction gives a homomorphism

$$\tau\colon H^0(A, T_A) \longrightarrow H^0(\Theta, O_\Theta(\Theta)), \qquad \tau(X) = X\theta.$$

Through the identification $T_0(A) = H^0(A, T_A)$, each nonzero vector $a \in T_0(A)$ thus determines a section of $H^0(\Theta, O_\Theta(\Theta))$. By an abuse of notation, write $\Theta \cap \Theta_a$ for its zero scheme in Θ.

If a is a nonzero element of either A or $T_0(A)$, the subvariety $\Theta \cap \Theta_a$ defines a Cartier divisor on Θ, denoted $\Theta \cdot \Theta_a$.

II.C.1 Reducibility of $\Theta \cap \Theta_a$ in $\operatorname{Pic}\Theta$

(1.0) Let (A, Θ) be a principally polarized abelian variety, and $E \subset A$ an abelian subvariety. Let K be the kernel of the composite homomorphism $A \xrightarrow{\varphi_\Theta} \widehat{A} \longrightarrow \hat{E}$. It is an abelian subvariety of A, identified with $\widehat{A/E}$. Let $\pi\colon E \times K \longrightarrow A$ be the isogeny defined by $\pi(e, k) = e + k$. Denote the restrictions of Θ to E, K by Θ_E, Θ_K. The theorem of the square gives

$$\pi^*\Theta \equiv \operatorname{pr}_1^*\Theta_E + \operatorname{pr}_2^*\Theta_K;$$

thus the pullback of the polarization of A to $E \times K$ is the product polarization. We have

$$H(\Theta_E) = H(\Theta_K) = E \cap K,$$

whereas $\ker\pi$ consists of the elements $(e, -e)$ for $e \in E \cap K$. The induced polarizations Θ_E, Θ_K therefore have the same degree d. The next lemma, proved in A1.1.1, follows readily from Mumford's theory of theta groups:

Lemma II.C.1.1. *Let θ be a nonzero section of $H^0(A, \Theta)$. There are bases $(s_1, \dots, s_d)$ and $(t_1, \dots, t_d)$ of $H^0(E, \Theta_E)$ and $H^0(K, \Theta_K)$, respectively, such that*

$$\pi^*\theta = \sum_{i=1}^{d} s_i \otimes t_i.$$

(1.2) The case $\dim E = 1$ will play a special role. Set $(\Theta \cdot E) = d$; the induced polarization on E has degree d. The group $E \cap K$ is the d-torsion group $E[d]$, and $\ker\pi$ is consequently

isomorphic to $E[d]$, interpreted as a finite group scheme in positive characteristic. If $\operatorname{char} k \nmid d$, it is isomorphic to $(\mathbb{Z}/d)^2$.

If, for example, (A, Θ) is the Jacobian of a curve C, the embedding $C \hookrightarrow JC$ and the homomorphism $A \longrightarrow \hat{E} = E$ give a morphism $r\colon C \longrightarrow E$, with $(C \cdot K) = \deg r = d$. Conversely, a morphism $r\colon C \longrightarrow E$ of degree d to an elliptic curve induces a homomorphism $\bar{r}\colon E \longrightarrow JC$, not necessarily injective, and $\deg(\bar{r}^*\Theta) = d$.

Proposition II.C.1.3. *Assume* $\operatorname{char} k \neq 2$. *Let* (A, Θ) *be a principally polarized abelian variety and* $E \subset A$ *an abelian subvariety whose induced polarization* Θ_E *has degree* 2. *Let* e *be a nonzero element of* E *or* $T_0(E)$. *There are nonzero effective Weil divisors* C, C' *on* Θ *such that* $\Theta \cdot \Theta_e = C + C'$. *If* $\dim E = 1$, *then* C, C' *are Cartier divisors on* Θ.

The notation $\Theta \cdot \Theta_e$ was explained in (0) at the start of this section.

Proof. First consider $e \in E$, using the notation of (1.0). By Lemma 1.1 there are bases (s_1, s_2) of $H^0(E, \Theta_E)$ and (t_1, t_2) of $H^0(K, \Theta_K)$ such that $\pi^*\Theta$ is the divisor of $(x, y) \mapsto s_1(x)t_1(y) + s_2(x)t_2(y)$. The subvariety $\pi^{-1}(\Theta \cap \Theta_e) \subset E \times K$ is defined by

$$\begin{cases} s_1(x)t_1(y) + s_2(x)t_2(y) = 0, \\ s_1(x-e)t_1(y) + s_2(x-e)t_2(y) = 0. \end{cases} \tag{*}$$

It is therefore the union of the codimension-two subvarieties

- $\widetilde{C} = E \times B$, where B is the base locus of $|\Theta_K|$, defined by $t_1 = t_2 = 0$;
- $\widetilde{C}'$, defined by (*) and $s_1(x)s_2(x-e) - s_2(x)s_1(x-e) = 0$.

These subvarieties are stable under $\ker \pi$. Hence there are effective Weil divisors C, C' on Θ with $\widetilde{C} = \pi^*C$, $\widetilde{C}' = \pi^*C'$, and $\Theta \cdot \Theta_e = C + C'$.

Now suppose $\dim E = 1$. The equation $s_1(x)s_2(x-e) - s_2(x)s_1(x-e) = 0$ defines a divisor $\sum_i \varepsilon_i$ on E, linearly equivalent to $\Theta_E + (\Theta_E)_e$, and hence of degree 4. We have $\widetilde{C}' = \sum_{i=1}^4 \{\varepsilon_i\} \times \Delta_i$, where Δ_i is the divisor of $s_1(\varepsilon_i)t_1 + s_2(\varepsilon_i)t_2 \in H^0(K, \Theta_K)$. Thus $\widetilde{C}'$ is the inverse image of $\sum_i \varepsilon_i$ under the projection $\pi^{-1}(\Theta) \longrightarrow E$, so is a Cartier divisor on $\pi^{-1}(\Theta)$. Consequently C' is a Cartier divisor on Θ, as is $C = \Theta \cdot \Theta_e - C'$.

The case of a tangent vector e to E is identical: replace the second equation of (*) by

$$s_1'(x)t_1(y) + s_2'(x)t_2(y) = 0,$$

where $'$ denotes differentiation with respect to a local parameter. Then $\pi^{-1}(\Theta \cap \Theta_e)$ is the union of $\widetilde{C} = E \times B$ and $\widetilde{C}'$, defined by the modified equations (*) and $s_1(x)s_2'(x) - s_2(x)s_1'(x) = 0$. As above, this gives $\Theta \cdot \Theta_e = C + C'$. If $\dim E = 1$, the preceding case shows that C is Cartier on Θ, and hence so is C'. □

(1.4) We describe $\Theta \cdot \Theta_e$ more precisely when $\dim E = 1$.[3] Let $s\colon E \longrightarrow \mathbb{P}^1$ be the morphism $x \mapsto (s_1(x), s_2(x))$. The equation $s_1(x)s_2(x-e) - s_2(x)s_1(x-e) = 0$, respectively $s_1(x)s_2'(x) - s_2(x)s_1'(x) = 0$, says $s(x) = s(x-e)$, respectively $s'(x) = 0$, or equivalently $[x] + [x-e] \equiv \Theta_E$, respectively $2[x] \equiv \Theta_E$. If $\Theta_E \equiv [0] + [f]$ for $f \in E$, this is equivalent to $2x = e + f$, respectively

[3] Editorial note: the following description by permutation of four geometric points assumes $\operatorname{char} k \neq 2$. In characteristic 2, the fibres of $[2]$ have length 4 but are not étale; justification using group schemes and descent is required and is not supplied by this argument on points.

$2x = f$. Thus E_2 permutes $\varepsilon_1, \dots, \varepsilon_4$, and $\ker \pi$ permutes the subvarieties $\{\varepsilon_i\} \times \Delta_i$ of $\widetilde{C}'$. Setting $\varepsilon = \varepsilon_1$ and $\Delta = \Delta_1$, we obtain

$$C' = \pi(\{\varepsilon\} \times \Delta) = \Delta_\varepsilon, \qquad \Theta \cdot \Theta_e = \pi(E \times B) + \Delta_\varepsilon.$$

(1.5) Before stating the converse of Proposition 1.3, recall the results of Mumford and Kempf [Mu 9] on the cohomology of line bundles on an abelian variety A. A line bundle L on A is *nondegenerate* if the group $H(L)$ (I.3.3) is finite. There is then a unique integer i such that $H^i(A, L) \neq 0$, called its index and denoted $i(L)$.

For an arbitrary line bundle L on A, let K be the identity component of $H(L)$, and $p\colon A \longrightarrow A/K$ the canonical projection. There is a nondegenerate line bundle M on A/K, determined up to translation, such that L is algebraically equivalent to p^*M. Set

$$i_-(L) = i(M), \qquad i_+(L) = i(M) + \dim K.$$

If the cohomology of L is nonzero, we can choose M so that $L = p^*M$. For every integer i there is then a canonical isomorphism

$$H^i(A, L) \xrightarrow{\sim} H^{i(M)}(A/K, M) \otimes H^{i-i(M)}(K, \mathcal{O}_K).$$

In particular,

$$H^i(A, L) \neq 0 \quad \Longleftrightarrow \quad i_-(L) \leq i \leq i_+(L).$$

By definition, $i_-(L), i_+(L)$ depend only on the algebraic equivalence class of L. Moreover, for two line bundles L, L' on A, one has $i_-(L \otimes L') \leq i_-(L) + i_-(L')$. Indeed, the argument in [Mu 6], p. 159, step C, which proves this inequality for nondegenerate L, L', extends immediately to the general case.

(1.6) Proposition 1.3 has the following converse. We exclude product varieties, for which the divisor Θ itself is reducible. We call a polarized abelian variety *irreducible* if it is not isomorphic to a product of two nontrivial polarized abelian varieties.

Proposition II.C.1.7. *Assume* $\operatorname{char} k \neq 2$. *Let* (A, Θ) *be an irreducible principally polarized abelian variety of dimension* ≥ 4. *Suppose there are a nonzero element* a *of* A (*respectively* $T_0(A)$) *and nonzero effective Cartier divisors* C, C' *on* Θ *such that* $\Theta \cdot \Theta_a = C + C'$. *Then* A *contains an elliptic curve* E *with* $(\Theta \cdot E) = 2$, *and* $a \in E$ (*respectively* $a \in T_0(E)$).

Proof. First consider $a \in A$. Grothendieck's version of the Lefschetz theorem [Gro 3, Exp. XII, Cor. 3.6][4] gives divisors D, D' on A with $D|_\Theta \equiv C$, $D'|_\Theta \equiv C'$, and $D + D' \equiv \Theta_a$. Consider the cohomology sequence associated with

$$0 \longrightarrow \mathcal{O}_A(D - \Theta) \longrightarrow \mathcal{O}_A(D) \longrightarrow \mathcal{O}_\Theta(C) \longrightarrow 0.$$

Since C is effective, at least one of $H^0(A, D)$ and $H^1(A, D-\Theta)$ is nonzero. The same alternative holds for D'.

If both $H^0(A, D)$ and $H^0(A, D')$ are nonzero, Θ_a is reducible, contrary to our assumption. If both vanish, (1.5) gives $i_-(D-\Theta) \leq 1$ and $i_-(D'-\Theta) \leq 1$. But $\mathcal{O}_A(D+D'-2\Theta) = \mathcal{O}_A(\Theta_{-a})^{-1}$ is nondegenerate of index $g \geq 4$, contradicting $i_-(D + D' - 2\Theta) \leq i_-(D - \Theta) + i_-(D' - \Theta)$ (1.5).

[4] The vanishing of $H^i(\Theta, -n\Theta)$ for $i = 1, 2$ and $n \geq 1$, needed to apply this result, follows immediately from $H^i(A, -n\Theta) = 0$ for $1 \leq i \leq 3$, since $\dim A \geq 4$.

We may therefore assume that $H^1(A, D' - \Theta) \neq 0$ and $H^1(A, D - \Theta) = 0$, and take D effective with restriction C to Θ. By duality, $H^{g-1}(A, \Theta - D') \neq 0$. Writing $\hat{a}$ for the algebraically trivial divisor $\Theta_a - \Theta$, this says $H^{g-1}(A, D - \hat{a}) \neq 0$. By (1.5),

$$i_-(D) = i_-(D - \hat{a}) = 0, \qquad i_+(D) = i_+(D - \hat{a}) = g - 1.$$

Set $K = H(D)^0 = H(D - \hat{a})^0$, $E = A/K$, and let $p\colon A \longrightarrow E$ be the canonical homomorphism. We have $\dim K = i_+(D) - i_-(D) = g - 1$ and $\dim E = 1$. There are divisors $\mathfrak{d}, \mathfrak{e}$ on E with $D = p^*\mathfrak{d}$ and $\hat{a} = p^*\mathfrak{e}$. Identify E with an abelian subvariety of A through $\hat{p}$ and the principal polarizations of A, E. The degree-zero divisor $\mathfrak{e}$ corresponds to a point $e \in E$, and $\hat{a} = p^*\mathfrak{e}$ simply becomes $a = e$, so $a \in E$. We are now in the situation of (1.0), whose notation we retain. We determine the degree d of the induced polarizations Θ_E, Θ_K. Let θ be a nonzero element of $H^0(A, \Theta)$. On $E \times K$ we have the exact sequence

$$0 \longrightarrow \pi^*O_A(D' - \Theta) \xrightarrow{\pi^*\theta} \pi^*O_A(D') \longrightarrow \pi^*O_\Theta(C') \longrightarrow 0.$$

No divisor algebraically equivalent to D' is effective, since otherwise Θ would be reducible; hence $H^0(E \times K, \pi^*O_A(D')) = 0$. Since C' is effective, the map

$$H^1(\pi^*\theta)\colon H^1(E \times K, \pi^*O_A(D' - \Theta)) \longrightarrow H^1(E \times K, \pi^*O_A(D'))$$

is not injective.

Let φ be the restriction to E of $p\colon A \longrightarrow E$; it is multiplication by d. Set $\Delta = \varphi^*(\mathfrak{d} - \mathfrak{e})$. We have

$$\pi^*(D' - \Theta) \equiv \pi^*(\hat{a} - D) \equiv \pi^*p^*(\mathfrak{e} - \mathfrak{d}) = -\operatorname{pr}_1^*\Delta.$$

We obtain isomorphisms

$$\begin{aligned} H^1(E \times K, \pi^*O_A(D' - \Theta)) &= H^1(E \times K, -\operatorname{pr}_1^*\Delta) \simeq H^1(E, -\Delta), \\ H^1(E \times K, \pi^*O_A(D')) &= H^1(E \times K, \operatorname{pr}_1^*(\Theta_E - \Delta) + \operatorname{pr}_2^*\Theta_K) \\ &\simeq H^1(E, \Theta_E - \Delta) \otimes H^0(K, \Theta_K). \end{aligned}$$

Choose bases (s_i) and (t_i) as in (1.1), so that $\pi^*\theta = \sum_i s_i \otimes t_i$. The map $H^1(\pi^*\theta)$ is identified with

$$\begin{aligned} u\colon H^1(E, -\Delta) &\longrightarrow H^1(E, \Theta_E - \Delta) \otimes H^0(K, \Theta_K), \\ x &\longmapsto \sum_i (s_i \cdot x) \otimes t_i. \end{aligned}$$

Thus $u(x) = 0$ if and only if $s \cdot x = 0$ for every $s \in H^0(E, \Theta_E)$. By duality, this means that the image of the multiplication map

$$v\colon H^0(E, \Delta - \Theta_E) \otimes H^0(E, \Theta_E) \longrightarrow H^0(E, \Delta)$$

is contained in the kernel of x, viewed as a linear form on $H^0(E, \Delta)$. Consequently, u is injective if and only if v is surjective. Now

$$\deg \Theta_E = d, \qquad \deg \Delta = d^2 \deg \mathfrak{d} \geq d^2, \qquad \deg(\Delta - \Theta_E) \geq d^2 - d.$$

Since (A, Θ) is irreducible, $d \geq 2$. For two line bundles L, M of degrees at least 3 on an elliptic curve, multiplication $H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M)$ is surjective [Mu 9, Thm. 6]. It follows that $d = 2$, proving the first case.

We now consider $a \in T_0(A)$. The preceding argument, with a replaced by 0, yields an elliptic curve E with $\deg \Theta_E = 2$, and an isogeny $\pi\colon E \times K \longrightarrow A$. It remains to show that a is tangent to E. Extend a to an invariant vector field, still denoted by a. Consider the maps

$$H^0(A, T_A) \xrightarrow{\tau} H^0(\Theta, O_\Theta(\Theta)) \xrightarrow{\partial} H^1(A, O_A),$$

where τ was defined in (0) and ∂ is the boundary map of the exact sequence

$$0 \longrightarrow O_A \longrightarrow O_A(\Theta) \longrightarrow O_\Theta(\Theta) \longrightarrow 0.$$

By [Gr 1, 2.10], $\partial\tau$ is cup product with $c_1(\Theta) \in H^1(A, \Omega^1_A)$. Since $\pi^*\Theta \equiv \mathrm{pr}_1^*\,\Theta_E + \mathrm{pr}_2^*\,\Theta_K$, there is a commutative diagram

$$\begin{array}{ccc} H^0(A, T_A) & \xrightarrow{c_1(\Theta)} & H^1(A, O_A) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle H^1(\pi)} \\ H^0(E, T_E) \oplus H^0(K, T_K) & \xrightarrow[c_1(\Theta_E)\oplus c_1(\Theta_K)]{} & H^1(E, O_E) \oplus H^1(K, O_K). \end{array}$$

It therefore suffices to show that the image of $\partial\tau(a)$ under the canonical isomorphism $H^1(\pi)$ belongs to $H^1(E, O_E)$.

Let $t \in H^0(A, D)$ be a section with divisor D, let $\bar{t}$ be its restriction to Θ, and let $t' \in H^0(\Theta, C')$ be a section with divisor C'. After rescaling t, we have $\bar{t}\,t' = \tau(a)$. There is an exact diagram

$$\begin{array}{ccccccccc} 0 & \to & O_A(-D) & \to & O_A(\Theta - D) & \to & O_\Theta(C') & \to & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{t}} & & \\ 0 & \longrightarrow & O_A & \longrightarrow & O_A(\Theta) & \longrightarrow & O_\Theta(\Theta) & \to & 0. \end{array}$$

This diagram induces the commutative square

$$\begin{array}{ccc} H^0(\Theta, O_\Theta(C')) & \longrightarrow & H^1(A, O_A(-D)) \\ \downarrow{\scriptstyle \bar{t}} & & \downarrow{\scriptstyle t} \\ H^0(\Theta, O_\Theta(\Theta)) & \xrightarrow[\partial]{} & H^1(A, O_A). \end{array}$$

Hence $\partial\tau(a)$ lies in the image of

$$H^1(A, O_A(-D)) \xrightarrow{t} H^1(A, O_A).$$

Now $D \equiv p^*\mathfrak{d}$, and the maps

$$p^*\colon H^0(E, \mathfrak{d}) \longrightarrow H^0(A, D), \qquad p^*\colon H^1(E, -\mathfrak{d}) \longrightarrow H^1(A, -D)$$

are isomorphisms by (1.5). Let $u \in H^0(E, \mathfrak{d})$ satisfy $p^*u = t$. The preceding map is then identified with

$$H^1(E, O_E(-\mathfrak{d})) \xrightarrow{(2u,0)} H^1(E, O_E) \oplus H^1(K, O_K),$$

which proves the assertion. □

II.C.2 Application to the Schottky problem

Theorem II.C.2.1. *Assume* $\operatorname{char} k \neq 2$. *Let* (A, Θ) *be a principally polarized abelian variety of dimension* g. *If there is a nonzero element* a *of* A *(respectively of* $T_0(A)$*) such that* $\Theta \cdot \Theta_a$ *is not integral, then one of the following conclusions holds:*

1. $\dim \operatorname{Sing} \Theta \geq g - 4$;
2. *A contains an elliptic curve* E *such that* $(\Theta \cdot E) = 2$ *and* $a \in E$ *(respectively* $a \in T_0(E)$*).*

Proof. Write $\Theta \cdot \Theta_a = C + C'$, with C, C' nonzero effective Weil divisors. Suppose $\dim \operatorname{Sing} \Theta < g - 4$. The local rings of Θ are then regular in codimension at most 3 and, by Grothendieck's theorem (the "Samuel conjecture," [Gro 3, Exp. XI, Cor. 3.14]), factorial. Thus C, C' are Cartier divisors, and the theorem follows from Proposition 1.7. □

Remarks II.C.2.2.

1. *Conclusion* (i) *can be replaced by the following stronger assertion: for every decomposition* $\Theta \cdot \Theta_a = C + C'$ *into nonzero effective cycles,*

$$\dim(C \cap C' \cap \operatorname{Sing} \Theta) \geq g - 4.$$

   As with (i)*, this assertion is the alternative to case* (ii)*. Indeed, set* $Z = C \cap C' \cap \operatorname{Sing} \Theta$*. On* $\Theta \setminus Z$*, the divisors* C, C' *are Cartier; to extend them, it suffices that the local rings of* Θ *at points of* Z *be parafactorial [Gro 3, Exp. XI, no. 3]. If* $\dim Z \leq g - 5$*, this follows from Theorem 3.13* (ii) *of loc. cit. In particular, if* $\dim \operatorname{Sing} \Theta = g - 4$ *and case* (ii) *is excluded,* $C \cap C'$ *contains a component of* $\operatorname{Sing} \Theta$*.*

2. *The converse of* (ii) *follows from Proposition 1.3. Part II.D describes the components of* N_{g-4} *known at the time of writing: apart from* $\overline{J_g}$*, all but one correspond to principally polarized abelian varieties containing an abelian subvariety on which the induced polarization has degree* 2*. Such intersections are therefore reducible.*

3. *Appendix A1 shows that the theta divisor of a generic principally polarized abelian variety satisfying* (ii) *is smooth. Thus* (ii) *cannot be omitted. However, if there is an abelian subvariety* E *with* $1 < \dim E < g - 1$ *on which the induced polarization has degree* 2*, then* $\dim \operatorname{Sing} \Theta \geq g - 4$ *(A1.1.5).*

Theorem II.C.2.3. *Assume* $\operatorname{char} k \neq 2$. *Let* (A, Θ) *be a principally polarized abelian variety of dimension* g. *If there are distinct nonzero elements* a, x, y *of* A *such that* $\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y$*, then* $\dim(\Theta_a \cap \operatorname{Sing} \Theta) \geq g - 4$.

Proof. The hypothesis implies that $\Theta \cdot \Theta_a$ is not integral. By Remark 2.2.1, it suffices to exclude case (ii) of Theorem 2.1. Assume therefore that we are in that case and that $\dim(\Theta_a \cap \operatorname{Sing} \Theta) \leq g - 5$. There is an isogeny $\pi\colon E \times K \longrightarrow A$ of degree 4 and, with the notation of (1.4),

$$\Theta \cdot \Theta_a = \pi(E \times B) + \Delta_\varepsilon,$$

where B is the base locus of $|\Theta_K|$, $\Delta \in |\Theta_K|$, and $\varepsilon \in E$. If $\pi^*\Theta$ has equation $s_1(x)t_1(y) + s_2(x)t_2(y) = 0$, then

$$B\colon\ t_1 = t_2 = 0, \qquad \Delta\colon\ s_1(\varepsilon)t_1 + s_2(\varepsilon)t_2 = 0,$$

and $s_1(\varepsilon)s_2(\varepsilon - a) - s_2(\varepsilon)s_1(\varepsilon - a) = 0$.

Lemma II.C.2.4. *The subvarieties B and Δ are irreducible.*

Proof. It suffices to prove that they are smooth in codimension 1. Let Δ' be the divisor of the section $s_1'(\varepsilon)t_1 + s_2'(\varepsilon)t_2$ of $H^0(K, \Theta_K)$. For $y \in \Delta' \cap \operatorname{Sing}\Delta$, the Jacobian criterion shows that $\varepsilon + y$ is a singular point of Θ also lying in Θ_a. Hence

$$\dim \operatorname{Sing}\Delta - 1 \leq \dim(\Delta' \cap \operatorname{Sing}\Delta) \leq \dim(\Theta_a \cap \operatorname{Sing}\Theta) \leq g - 5.$$

The argument for B is the same. □

(2.5) We resume the proof of Theorem 2.3. Since Δ_ε is irreducible, we may assume $\Delta_\varepsilon \subset \Theta_x$. Write $x = \pi(e, k)$. Then

$$s_1(\varepsilon - e)t_1(y - k) + s_2(\varepsilon - e)t_2(y - k) = 0 \qquad (y \in \Delta).$$

This equation is nontrivial because s_1, s_2 have no common zero. It implies $\Delta_k \equiv \Delta$, so $k \in H(\Theta_K) = E \cap K$, and hence $x \in E$. Consequently,

$$(s_1(\varepsilon - x) : s_2(\varepsilon - x)) = (s_1(\varepsilon) : s_2(\varepsilon)) \in \mathbb{P}^1.$$

The morphism $s\colon E \longrightarrow \mathbb{P}^1$ has degree 2, and we already know that $s(\varepsilon - a) = s(\varepsilon)$. Thus $x = 0$ or $x = a$, contrary to the hypothesis. □ Theorem 2.3 extends to the case where some of a, x, y are infinitely near 0. To state it conveniently, let $\widetilde{A}$ be the blow-up of A at 0: its points are the nonzero points of A and the tangent directions at the origin.

Proposition II.C.2.6. *Assume* $\operatorname{char} k \neq 2$, *and let a, x, y be three distinct points of $\widetilde{A}$. The relation*

$$\Theta \cap \Theta_a \subset (\Theta \cap \Theta_x) \cup (\Theta \cap \Theta_y)$$

implies $\dim(\Theta_a \cap \operatorname{Sing}\Theta) \geq g - 4$.

Proof. As before, we reach $\Delta_\varepsilon \subset \Theta \cap \Theta_x$. If x is a point of A and a a tangent direction at the origin, then $s(\varepsilon - x) = s(\varepsilon)$, whereas ε is a ramification point of s, a contradiction.

Thus suppose x is a tangent direction at the origin, represented by $e + k$, with $e \in T_0(E)$ and $k \in T_0(K)$. Denote the associated vector fields by $\partial/\partial e$ and $\partial/\partial k$. The condition $\Delta_\varepsilon \subset \Theta \cap \Theta_x$ means

$$\frac{\partial s_1}{\partial e}(\varepsilon)t_1(y) + \frac{\partial s_2}{\partial e}(\varepsilon)t_2(y) + s_1(\varepsilon)\frac{\partial t_1}{\partial k}(y) + s_2(\varepsilon)\frac{\partial t_2}{\partial k}(y) = 0 \quad (y \in \Delta).$$

Set $t = s_1(\varepsilon)t_1 + s_2(\varepsilon)t_2 \in H^0(K, \Theta_K)$. Choose a section t' not proportional to t, and a basis $(\partial_1, \dots, \partial_{g-1})$ of $H^0(K, T_K)$. The argument of [Gr, pp. 92–93] shows that the images of $t', \partial_1 t, \dots, \partial_{g-1} t$ form a basis of $H^0(\Delta, O_\Delta(\Delta))$. The preceding equation gives $k = 0$ and

$$\left(\frac{\partial s_1}{\partial e}(\varepsilon) : \frac{\partial s_2}{\partial e}(\varepsilon)\right) = (s_1(\varepsilon) : s_2(\varepsilon)).$$

Thus ε is a ramification point of s, which is possible only if a is the tangent direction to E at 0. Hence $a = x$, contrary to the hypothesis. □

(2.7) Variants of Theorem 2.3 follow in the same way. For example, let $a \in A$ and $v \in T_0(A)$ be nonzero. Then

$$\Theta \cap \Theta_a \subset (\Theta \cap \Theta_v) \cup (\Theta \cap \Theta_v)_a \quad \Longrightarrow \quad \dim \operatorname{Sing} \Theta \geq g - 4.$$

Indeed, if the conclusion fails, we first obtain an elliptic curve $E \subset A$ with $(\Theta \cdot E) = 2$ and $a \in E$, and then $\Delta_\varepsilon \subset \Theta \cap \Theta_v$ or $\Delta_{\varepsilon - a} \subset \Theta \cap \Theta_v$. The argument of (2.6) yields a contradiction.

(2.8) Let (A, Θ) be an irreducible principally polarized abelian variety. The morphism $\psi\colon A \longrightarrow \mathbb{P}^N$ associated with $|2\Theta|$ induces an isomorphism from $A/\{\pm 1\}$ onto a subvariety W of $\mathbb{P}^N$, the *Kummer variety* of (A, Θ). We call a line ℓ in $\mathbb{P}^N$ a *trisecant* of W if it satisfies one of the following conditions:

1. ℓ contains at least three distinct points of W;
2. ℓ is tangent to W at a smooth point and contains another point of W;
3. ℓ is tangent to W at a singular point, meaning that it lies in the tangent cone of W there, and contains another point of W;
4. ℓ has contact of order 3 with W at a smooth point.

Theorem II.C.2.9. *Assume* $\operatorname{char} k \neq 2$. *Let* (A, Θ) *be an irreducible principally polarized abelian variety of dimension* g *whose Kummer variety admits a trisecant. Then* $\dim \operatorname{Sing} \Theta \geq g - 4$.

Proof. The existence of a trisecant implies conditions of the form (2.3); see [We 7] or [Mu 10]. More precisely:

1. If ℓ contains the distinct points $\psi(a), \psi(b), \psi(c)$, then
$$\Theta \cap \Theta_{b-a} \subset \Theta_{c-a} \cup \Theta_{-c-a}.$$
2. If ℓ is tangent to W at $\psi(a)$, with $2a \neq 0$, and passes through $\psi(b)$, then $\Theta \cap \Theta_{2a} \subset \Theta_{a+b} \cup \Theta_{a-b}$.
3. If ℓ is tangent to W at $\psi(0)$ in the direction $v \in T_0(A)$ and passes through $\psi(b)$, then $\Theta \cap \Theta_v \subset \Theta_b \cup \Theta_{-b}$.
4. If ℓ has contact of order 3 at $\psi(a)$ in the direction $v \in T_a(A)$, then
$$\Theta \cap \Theta_{2a} \subset (\Theta \cap \Theta_v) \cup (\Theta \cap \Theta_v)_{2a},$$
using the identification $T_a(A) = T_0(A)$.

The conclusion follows from Theorem 2.3, Proposition 2.6, and (2.7). □

(2.10) Although this is not strictly necessary, we assume here that $k = \mathbb{C}$. Let (A, Θ) be a principally polarized abelian variety, with Θ symmetric. Let $r\colon V \longrightarrow A$ be its universal cover and θ the theta function on V with divisor $r^*\Theta$. We say that (A, Θ) satisfies the *Novikov property* if there are constant vector fields x, y, t on V, with $x \neq 0$, and a constant $c \in \mathbb{C}$, such that

$$u = \frac{\partial^2}{\partial x^2} \log \theta + c$$

satisfies the Kadomtsev–Petviashvili equation

$$\frac{\partial}{\partial x}(u_{xxx} + 12uu_x - 4u_t) + 3u_{yy} = 0. \tag{KP}$$

As usual, $u_x = \partial u/\partial x$, and similarly for other derivatives. A direct calculation [Du] shows that, when (A, Θ) is irreducible, this equation is equivalent to the following one, where d is another constant:

$$\begin{aligned}\theta_{xxxx}\theta - 4\theta_{xxx}\theta_x + 3\theta_{xx}^2 + 4\theta_x\theta_t - 4\theta_{xt}\theta + 3\theta_{yy}\theta - 3\theta_y^2 \\ +12c(\theta_{xx}\theta - \theta_x^2) + d\theta^2 = 0.\end{aligned} \tag{KP$'$}$$

Novikov conjectured that Jacobians are the only irreducible principally polarized abelian varieties satisfying the Novikov property. This conjecture has just been proved by T. Shiota [Sh]. Theorem 2.1 yields the following weaker result.

Theorem II.C.2.11. *Let (A, Θ) be an irreducible principally polarized abelian variety of dimension g satisfying the Novikov property. Then* $\dim \operatorname{Sing} \Theta \geq g - 4$.

Proof. On $\Theta \cap \Theta_x$, equation (KP$'$) reduces to

$$0 = \theta_{xx}^2 - \theta_y^2 = (\theta_{xx} + \theta_y)(\theta_{xx} - \theta_y).$$

Set $X = \Theta \cap \Theta_x$. The functions $\theta_{xx} \pm \theta_y$ on V define two sections of $H^0(X, O_X(\Theta))$ whose product is zero. Lemma 2.12 below shows that they are not identically zero; hence X is the union of two subvarieties exchanged by $a \mapsto -a$. In particular, X is not integral. If $\dim \operatorname{Sing} \Theta < g - 4$, we are in case (ii) of Theorem 2.1; but (1.4) and Lemma 2.4 then show that X has two irreducible components preserved by $a \mapsto -a$, contradicting the preceding description.

The theorem thus follows from the next lemma.

Lemma II.C.2.12. *The section $\theta_{xx} + \theta_y$ of $H^0(X, O_X(\Theta))$ is not identically zero.*

Proof. The exact sequence

$$0 \longrightarrow O_\Theta \xrightarrow{\theta_x} O_\Theta(\Theta) \longrightarrow O_X(\Theta) \longrightarrow 0$$

gives the cohomology sequence

$$H^0(\Theta, O_\Theta(\Theta)) \longrightarrow H^0(X, O_X(\Theta)) \xrightarrow{\partial} H^1(\Theta, O_\Theta).$$

Since θ_y defines a section of $H^0(\Theta, O_\Theta(\Theta))$, we have $\partial(\theta_y) = 0$. We prove that $\partial(\theta_{xx}) \neq 0$. Let D denote differentiation in the direction x.

Let $(U_\alpha)_{\alpha \in I}$ be an open cover of A trivializing $O_A(\Theta)$, with transition functions $(\xi_{\alpha\beta})$. The section θ of $H^0(A, O_A(\Theta))$ is represented by functions θ_α on U_α satisfying $\theta_\alpha = \xi_{\alpha\beta}\theta_\beta$ on $U_\alpha \cap U_\beta$. There we have

$$\begin{aligned}D\theta_\alpha &= D\xi_{\alpha\beta}\,\theta_\beta + \xi_{\alpha\beta}D\theta_\beta, \\ D^2\theta_\alpha &= D^2\xi_{\alpha\beta}\,\theta_\beta + 2D\xi_{\alpha\beta}D\theta_\beta + \xi_{\alpha\beta}D^2\theta_\beta.\end{aligned}$$

The section θ_{xx} of $H^0(X, O_X(\Theta))$ is represented by $D^2\theta_\alpha$. By construction, its image in $H^1(\Theta, O_\Theta)$ is the class of the cocycle

$$(\alpha, \beta) \longmapsto c_{\alpha\beta} = \frac{D^2\theta_\alpha - \xi_{\alpha\beta}D^2\theta_\beta}{D\theta_\alpha}.$$

On $\Theta \cap U_\alpha \cap U_\beta$, the preceding formulas give

$$c_{\alpha\beta} = 2\frac{D\xi_{\alpha\beta}}{\xi_{\alpha\beta}} = 2\left\langle D, \frac{d\xi_{\alpha\beta}}{\xi_{\alpha\beta}} \right\rangle.$$

Thus $\partial(\theta_{xx}) = 4\pi i\, r(D \lrcorner c_1(\Theta))$, where $r\colon H^1(A, O_A) \longrightarrow H^1(\Theta, O_\Theta)$ is restriction. It is injective for $g \geq 2$, and cup product

$$H^0(A, T_A) \xrightarrow{c_1(\Theta)} H^1(A, O_A)$$

is bijective, proving the lemma and the theorem. □

We finish by applying these results to the Schottky problem (see the Introduction). Return to an arbitrary algebraically closed field k of characteristic different from 2. As in (2.6), $\widetilde{A}$ denotes the nonzero points of A together with tangent directions at 0.

Theorem II.C.2.13. *In the moduli space A_g of principally polarized abelian varieties of dimension g, the Jacobian locus J_g is an irreducible component of the set of irreducible varieties (A, Θ) satisfying any one of the following properties:*

1. *There exists $a \in \widetilde{A}$ such that $\Theta \cap \Theta_a$ is not integral.*
2. *There are three distinct elements a, x, y of $\widetilde{A}$ such that $\Theta \cap \Theta_a \subset (\Theta \cap \Theta_x) \cup (\Theta \cap \Theta_y)$.*
3. *The Kummer variety of (A, Θ) admits a trisecant.*
4. *(A, Θ) satisfies the Novikov property, when $k = \mathbb{C}$.*

Proof. It suffices to treat (i), since each other condition implies (i). Let N_{g-4} (respectively E_g) be the subvariety of A_g parametrizing irreducible (A, Θ) with $\dim \operatorname{Sing} \Theta \geq g-4$ (respectively containing an elliptic curve E with $(\Theta \cdot E) = 2$). Theorem 2.1 implies that the varieties satisfying (i) belong to $N_{g-4} \cup E_g$. By [An-M] and, in positive characteristic, [We 5], J_g is an irreducible component of N_{g-4}. Since J_g is not contained in E_g (1.2), the theorem follows. □

Condition (i) seems only slightly stronger than the Andreotti–Mayer condition (see Remark 2.2.2). However, no example is known to us of an irreducible principally polarized abelian variety satisfying (ii) or (iii) that is not a Jacobian; it is tempting to conjecture that each condition characterizes Jacobians.

II.D Prym varieties belonging to N_{g-4}^g for $g \geq 5$

[5] As its title indicates, this section describes Prym varieties (P, Ξ) of dimension $g \geq 5$ satisfying $\dim \operatorname{Sing} \Xi \geq g-4$.

It is based on results of Beauville who, extending results of Mumford ([Mu 2]), gives in [Be 1] the list of admissible covers whose Prym variety belongs to N_{g-4}^g and which also satisfy two conditions: condition $(*)$ on page 157 of that work and the stability of the curve C (see [De–M], p. 76). It follows from 4.11.3 and 5.4 of [Be 1] that the Prym varieties associated with

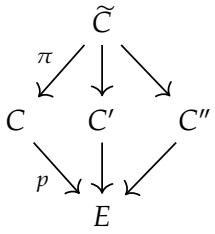
[5] This section will be published, with some modifications and together with Appendix 1, under the title "Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension 3" in Duke Mathematical Journal (reference [D 2]).

the covers on this list cover $(N^g_{g-4} - \overline{\mathcal{J}}_g) \cap \mathcal{P}^{\mathrm{irr}}_g$, where $\mathcal{P}^{\mathrm{irr}}_g$ denotes the set of indecomposable Prym varieties.

This list consists of several families, whose generic members we now describe.

1. The curve C is smooth and bielliptic: there is a degree-two morphism $p\colon C \to E$ from C onto an elliptic curve. The Galois group of the Galois closure of $p\pi\colon \widetilde{C} \to E$ is the dihedral group D_4. The resulting family $\mathcal{S}_{g+1}$ has dimension $2g$.

2. The curve C is bielliptic, but the Galois group of $p\pi\colon \widetilde{C} \to E$ is $(\mathbb{Z}/2)^2$. There is a commutative diagram:



   The family $\mathcal{S}_{g+1,t}$ consists of covers for which $g - t + 1 = g(C'') \geq g(C') = t + 1$. It has dimension $2g$, and the integer t ranges over $0, 1, \ldots, \lfloor g/2 \rfloor$. The corresponding family $\mathrm{Pr}(\mathcal{S}_{g+1,t})$ of Prym varieties will be denoted by $\mathcal{E}^2_{t,g-t}$ (we also write $\mathcal{E}^2_g = \mathrm{Pr}(\mathcal{S}_{g+1,0})$).

3. The curve C is obtained from a smooth hyperelliptic curve of genus $g - 1$ by identifying two pairs of points. The resulting family $\mathcal{H}_{g+1,0}$ has dimension $2g + 1$.

4. The curve C is the union of a smooth rational curve and a smooth hyperelliptic curve of genus $g - 2$, meeting at four points. The resulting family $\mathcal{H}_{g+1,1}$ has dimension $2g$.

5. The curve C is the union of a smooth curve of genus $t - 1 \geq 1$ and a smooth curve of genus $g - t - 1 \geq t - 1$, meeting at four points. The resulting families, for $2 \leq t \leq g/2$, are denoted by $\mathcal{H}_{g+1,t}$. They have dimension $3g - 4$.

6. When $g = 5$, C is the union of a smooth rational curve and a smooth curve of genus 3, meeting at the four points of a canonical divisor. The resulting family $\mathcal{T}_6$ has dimension 9.

7. In all the other cases listed by Beauville, the resulting Prym variety is a Jacobian. (The only remaining case, namely case e) of "even" covers of plane quintics, is treated in [Be 2], [Mas], [Tj 1], [Tj 2].)

Remark II.D. *Using arguments of the type found in [H–M], p. 29, one readily sees that the closures of the above families in the moduli space of admissible covers contain the families on Beauville's list.*

Furthermore, all these families are irreducible. We first make the following observation. Let $C \to S$ be an irreducible family of smooth curves. For any integer $d \geq 1$, write $C^d_S = C \times_S \cdots \times_S C$ (d factors) and let $J^d = \mathrm{Pic}^d(C/S)$ be the relative Picard scheme in degree d. There is a canonical morphism $C^d_S \to J^d$. Let $m\colon J^d \to J^{2d}$ be the morphism induced by squaring. Then the S-scheme $J^d \times_{J^{2d}} C^{2d}_S$ is irreducible, since the fibre over every point of S is irreducible of dimension $2d$ (Exercise G.8, p. 202 of [A–C–G–H 1]). This immediately gives the irreducibility of $\mathcal{S}_{g+1,0}$ (irreducibility of the family of elliptic curves with level-two structure and the preceding observation for elliptic curves with $d = g$), that of $\mathcal{S}_{g+1,t}$, $0 < t \leq g/2$ (the same observation for elliptic curves with $d = t, g - t$), that of $\mathcal{H}_{g+1,t}$, $t = 0, 1$ (hyperelliptic curves of genus $g - 1$, $g - 2$ and $d = 2$), and that of $\mathcal{H}_{g+1,t}$, $2 \leq t \leq g/2$ (curves of genus $t - 1$ and $g - t - 1$ with $d = 2$). The irreducibility of $\mathcal{T}_6$ is proved similarly; that of $\mathcal{S}_{g+1}$

follows from the existence of a dominant map $\mathcal{H}_{g+1,0} \dashrightarrow \mathcal{S}_{g+1}$ supplied by the tetragonal construction (3.2.4).

The following theorem collects the results of this section and also indicates where their proofs may be found.

Theorem II.D. *For $g \geq 5$:*

- $\overline{\mathcal{E}}^2_g = \mathrm{Pr}(\overline{\mathcal{S}}_{g+1,0}) = \mathrm{Pr}(\overline{\mathcal{H}}_{g+1,0}) = \mathrm{Pr}(\overline{\mathcal{S}}_{g+1})$ *(see 3.2.3, 3.2.4) has dimension $2g$ in $\mathcal{A}_g$ (see 1.4.2). It is an irreducible component of N^g_{g-4} (see 1.4.5) and contains the hyperelliptic Jacobians (see 3.1.2).*
- $\overline{\mathcal{E}}^2_{1,g-1} = \mathrm{Pr}(\overline{\mathcal{S}}_{g+1,1}) = \mathrm{Pr}(\overline{\mathcal{H}}_{g+1,1})$ *(see 3.2.3) has dimension $2g-1$ in $\mathcal{A}_g$ (see 1.4.2). It is an irreducible component of N^g_{g-4} (see 1.6.1, 4.2) and contains the bielliptic Jacobians (see 1.5.3, 3.1.2). We also have $\mathrm{Pr}(\mathcal{T}_6) \subset \mathcal{E}^2_{1,4}$ (see 2.2).*
- *For $2 \leq t \leq g/2$, $\mathcal{E}^2_{t,g-t} = \mathrm{Pr}(\overline{\mathcal{H}}_{g+1,t}) = \mathcal{D}^2_{t,g-t} \subset \mathcal{A}^2_{t,g-t}$ (see 3.2.3, 2.1 and A1.1.3, A2.3.5 for the definitions of $\mathcal{A}^2_{t,g-t}$ and $\mathcal{D}^2_{t,g-t}$).*

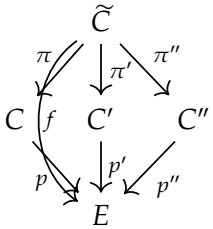
Taking account of the results of Appendix 1 (in particular A1.4.5), we have thus exhibited $2 + \lfloor g/2 \rfloor$ distinct irreducible components of N^g_{g-4} for $g \geq 5$, namely $\overline{\mathcal{J}}_g$, $\overline{\mathcal{E}}^2_g$, $\overline{\mathcal{E}}^2_{1,g-1}$ and the $\mathcal{A}^2_{t,g-t}$ for $2 \leq t \leq g/2$. Each of these components, except $\overline{\mathcal{J}}_g$, is contained in the divisor θ_{null} of $\mathcal{A}_g$, consisting of principally polarized abelian varieties (A, Θ) with Θ symmetric and $A[2] \cap \mathrm{Sing}\,\Theta \neq \emptyset$. It follows immediately (see 4.2) that N^5_1 has five irreducible components, namely $\overline{\mathcal{J}}_5$, $\mathcal{A}_{1,4}$, $\overline{\mathcal{E}}^2_5$, $\overline{\mathcal{E}}^2_{1,4}$ and $\mathcal{A}^2_{2,3}$, of respective dimensions 12, 11, 10, 9 and 9 (here $\mathcal{A}_{1,4}$ denotes the set of abelian varieties which are products of an elliptic curve and an abelian variety of dimension 4).

We point out that Donagi announced in [Do] results on the same subject which differ from ours: he gives the wrong dimension for $\mathcal{E}^2_5$ (9 instead of 10) and omits $\mathcal{A}_{1,4}$. He appears not to have noticed the inclusion $\mathrm{Pr}(\mathcal{T}_6) \subset \mathcal{E}^2_{1,4}$.

II.D.1 Study of the families $\mathcal{S}_{g+1,t}$

1.1. Notation

We are in the following situation. The curve C is smooth of genus $g+1 \geq 6$. There is a smooth elliptic curve E and a degree-two morphism $p\colon C \to E$, with associated branch divisor $\Delta = P_1 + \cdots + P_{2g}$ on E, a class $\delta \in \mathrm{Pic}^g E$ such that $\Delta \equiv 2\delta$, and an involution τ of C (I.2.6). There is a commutative diagram:



Write $g(C'') = g - t + 1 \geq g(C') = t + 1$. Associated with p' are its branch divisor $\Delta' = P_1 + \cdots + P_{2t}$, a class $\delta' \in \mathrm{Pic}^t E$ such that $\Delta' \equiv 2\delta'$, and an involution τ' of C'. Associated with p'' are $\Delta'' = \Delta - \Delta'$, $\delta'' = \delta - \delta'$ and an involution τ'' of C''.

Let $Q_1, \dots, Q_{2g}$ be the ramification points of p on C. The étale morphism π is associated with the following element of order two in $\operatorname{Pic}^0 C$:

$$\eta \equiv Q_1 + \cdots + Q_{2t} - p^*\delta' \equiv Q_{2t+1} + \cdots + Q_{2g} - p^*\delta''.$$

Denote by σ the fixed-point-free involution of $\widetilde{C}$ induced by π. The morphism π' (respectively π'') is branched over $p'^*(\Delta'')$ (respectively $p''^*(\Delta')$) and induces an involution σ' (respectively $\sigma'' = \sigma\sigma'$) of $\widetilde{C}$.

1.2. Study of the singular locus of the theta divisor

The Prym variety (P, Ξ) associated with π can be defined by ([Mu 2], proposition on p. 242):

$$P = \{L \in \operatorname{Pic}^{2g} \widetilde{C} \mid \operatorname{Nm} L = \omega_C,\ h^0(L) \text{ even}\}, \qquad \Xi = \{L \in P \mid h^0(L) \geq 2\}.$$

Proposition II.D.1.2.1. *We have*

$$\Xi = \left\{ \pi'^*L' \otimes \pi''^*L'' \;\middle|\; \begin{array}{c} L' \in \operatorname{Pic}^t C', \quad L'' \in \operatorname{Pic}^{g-t} C'', \\ h^0(L') \geq 1, \quad h^0(L'') \geq 1, \\ \operatorname{Nm}_{p'} L' \otimes \operatorname{Nm}_{p''} L'' = O_E(\delta) \end{array} \right\}.$$

Proof. Let $L \in \Xi$. Consider $\pi_* \colon |L| \to |\omega_C|$. Since $\dim |L| \geq 1$, its image meets the hyperplane $p^*|\delta|$ in $|\omega_C|$, and

$$\exists D \in |L|, \qquad \pi_* D = p^*M \quad \text{with } M \in |\delta|.$$

We can then write $D = \pi'^*D' + \pi''^*D''$ with D' and D'' effective. Thus Ξ is covered by the closed subsets

$$Z_a = \left\{ \pi'^*O_{C'}(D') \otimes \pi''^*O_{C''}(D'') \;\middle|\; \begin{array}{l} D' \text{ (respectively } D'') \text{ effective of degree } t+a \\ \text{(respectively } g-t-a), \quad p'_*D' + p''_*D'' \equiv \delta \end{array} \right\},$$

for $-t \leq a \leq g-t$. For $L \in Z_{g-t}$, we have

$$\begin{aligned} h^0(L) &= h^0(\pi'^*O_{C'}(D')) = h^0(D') + h^0(D' - p'^*\delta'') \\ &= h^0(D') + h^0(K_{C'} - D' + p'^*\delta'') + t - g \\ &= h^0(D') + h^0(p'^*\delta - D') + t - g \\ &= h^0(D') + h^0(\tau'^*D') + t - g = 2h^0(D') + t - g. \end{aligned}$$

By [Mu 3], p. 187, $Z_a \subset P \iff a$ is even. For $|a| \geq 2$, Riemann–Roch gives $h^0(D') \geq 2$ or $h^0(D'') \geq 2$. Now

$$\dim Z_a \leq \dim |\delta| - \min_{L \in Z_a} (\dim |D'| + \dim |D''|),$$

so $\dim Z_a < g - 1 = \dim \Xi$ for $|a| \geq 2$. The divisor Ξ therefore equals Z_0. □

The singular locus of the divisor Ξ is the union of the following two sets ([Mu 2], § 6):

$$\{L \in \Xi \mid h^0(L) \geq 4\}$$

and

$$\{L \in \Xi \mid L = \pi^*M \otimes N,\ M \in \operatorname{Pic} C,\ N \in \operatorname{Pic} \widetilde{C},\ h^0(M) \geq 2,\ h^0(N) \geq 1\}.$$

Points of the first set will be called stable singularities (they exist on every Prym variety of dimension at least 6 by 1.2.5); those of the second, exceptional singularities (they do not exist on a general Prym variety). A singularity can of course be both stable and exceptional.

Proposition II.D.1.2.2. *Let (P,Ξ) be the Prym variety associated with a cover belonging to $\mathcal{S}_{g+1,t}$. For $g \geq 5$:*

- *The singular locus of Ξ is the union of V, W_{-2}, W_0, W_2, where*

$$V = \left\{ \pi'^*L' \otimes \pi''^*L'' \;\middle|\; \begin{array}{c} \mathrm{Nm}_{p'} L' = O_E(\delta'), \quad \mathrm{Nm}_{p''} L'' = O_E(\delta''), \\ h^0(L') \geq 2, \quad h^0(L'') \geq 2 \end{array} \right\},$$

$$W_a = \left\{ \pi'^*L' \otimes \pi''^*L'' \otimes f^*M \;\middle|\; \begin{array}{c} L' \in \mathrm{Pic}^{t-2+a} C', \quad L'' \in \mathrm{Pic}^{g-t-2-a} C'', \quad M \in \mathrm{Pic}^2 E, \\ h^0(L') \geq 1, \quad h^0(L'') \geq 1, \\ \mathrm{Nm}_{p'} L' \otimes \mathrm{Nm}_{p''} L'' \otimes M^{\otimes 2} \simeq O_E(\delta) \end{array} \right\},$$

  and possibly finitely many points of order two in P.

- *The set V always has dimension at most $g-5$. For a generic member of $\mathcal{S}_{g+1,t}$, it is empty for $t \leq 2$, and is nonempty of pure dimension $g-6$ and is not contained in the W_a for $t \geq 3$. It has four irreducible components for $t = 3, g = 6$, two for $t = 3, g > 6$, and one for $t > 3$.*
- *The set W_2 is empty for $(t,g) = (2,5)$ or $(3,6)$, and is irreducible of dimension $g-4$ otherwise.*
- *The set W_0 is empty for $t \leq 1$, and is irreducible of dimension $g-4$ for $t \geq 2$.*
- *The set W_{-2} is empty for $t \leq 3$, and is irreducible of dimension $g-4$ for $t \geq 4$.*

Remark II.D.1.2.3. *With the notation of 1.1, if we make the additional assumption, satisfied generically on $\mathcal{S}_{g+1,t}$, that Δ is not the sum of two members of $|\delta|$, the singular locus of Ξ equals $V \cup W_{-2} \cup W_0 \cup W_2$ (see 1.2.9).*

The following two results follow from the proposition.

Corollary II.D.1.2.4. *For $g \geq 5$, the singular locus of the theta divisor of a generic member of $\mathrm{Pr}(\mathcal{S}_{g+1,t})$ is irreducible precisely in the following cases:*

(i) $t = 0$ or 1. Then $\mathrm{Sing}\,\Xi = W_2$.

(ii) $t = 2$ and $g = 5$. Then $\mathrm{Sing}\,\Xi = W_0$.

Corollary II.D.1.2.5. *For $g \geq 6$ and every Prym variety (P,Ξ) of dimension g, we have $\dim \mathrm{Sing}\,\Xi \geq g-6$.*

Proof. It follows from the proposition that, for $g \geq 6$ and (P,Ξ) belonging to $\mathrm{Pr}(\mathcal{S}_{g+1,3})$, $\mathrm{Sing}\,\Xi$ has an irreducible component of dimension $g-6$ consisting of stable singularities, namely V.

It is known (IV.A.4.5) that for any family of double covers $\widetilde{\mathcal{C}} \to \mathcal{C} \to T$, with corresponding family of Prym varieties $\mathcal{P} \to T$, the set

$$\bigcup_{t \in T} \{\text{stable singularities of } \Xi_t\}$$

is either empty or has codimension at most six in $\mathcal{P}$. If we take such a family containing our example, this set has codimension six at a point of one fibre, and therefore maps surjectively onto T. □

Remark II.D.1.2.6. *It follows from [We 1] and 1.2.5 that, for a generic Prym variety (P, Ξ) of dimension $g \geq 6$, $\operatorname{Sing}\Xi$ has pure dimension $g-6$ and its connected components are its irreducible components. A generic Jacobian is simple, and hence so is a generic Prym variety. By [Ba], for $g \geq 12$, $\operatorname{Sing}\Xi$ is generically irreducible of dimension $g-6$.*

Proof of Proposition 1.2.2

We begin with the following definition. Let $p\colon C \to B$ be a double cover of smooth curves, and let D be an effective divisor on C. We call D p-simple if there is no nonzero effective divisor E on B such that $D - p^*E$ is effective. We then have ([Mu 2], p. 338):

Proposition II.D.1.2.7. *Let $p\colon C \to B$ be a double cover of smooth curves and $O_B(-\delta) = \bigwedge^2 p_* O_C$; let L be a line bundle on B and D a p-simple effective divisor on C. Put $M = p^*L \otimes O_C(D)$. There is an exact sequence of O_B-modules:*

$$0 \longrightarrow L \longrightarrow p_*M \longrightarrow L \otimes O_B(p_*D - \delta) \longrightarrow 0.$$

Corollary II.D.1.2.8. *In our situation, consider $L = \pi'^* O_{C'}(D') \otimes \pi''^* O_{C''}(D'')$, where D' and D'' are effective, D' is p'-simple, and $\operatorname{Nm}_\pi L = \omega_C$. Then*

$$h^0(\widetilde{C}, L) \leq 2h^0(C'', D'') + g - t - \deg D''.$$

Proof. The divisor π'^*D' is π''-simple, so Proposition 1.2.7 gives

$$h^0(L) \leq h^0(D'') + h^0(D'' + \pi''_*\pi'^*D' - p''^*\delta').$$

Since $p'_*D' + p''_*D'' \equiv \delta$,

$$\begin{aligned} D'' + \pi''_*\pi'^*D' - p''^*\delta' &\equiv D'' + \tau''^*D'' + p''^*p'_*D' - \tau''^*D'' - p''^*\delta' \\ &\equiv p''^*(\delta - \delta') - \tau''^*D'' \equiv \tau''^*(K_{C''} - D''). \end{aligned}$$

The desired inequality follows by Riemann–Roch. □

Proof of Proposition 1.2.2. Let L be an element of Ξ corresponding to a singular point of Ξ. By 1.2.1, we can write

$$L = \pi'^*[O_{C'}(D') \otimes p'^* O_E(F')] \otimes \pi''^*[O_{C''}(D'') \otimes p''^* O_E(F'')]$$

with D', D'', F', F'' effective, D' p'-simple, D'' p''-simple, and

$$\deg D' + 2\deg F' = t = g(C') - 1, \qquad \deg D'' + 2\deg F'' = g - t = g(C'') - 1.$$

First suppose this singularity is exceptional but not stable. Every member of $|L|$ can then be written $\pi^*D + G$, with $h^0(D) \geq 2$ and G π-simple. Applying 1.2.7 to D and p, we are in one of the following cases:

(i) There is a divisor F of degree two on E such that $p^*F \leq D$. Then $F \leq F' + F''$, and we belong to W_{-2}, W_0 or W_2.

(ii) D is p-simple, $G = 0$ and $p_*D \equiv \delta$. Then $\pi^*D = \pi'^*D' + \pi''^*D''$, which is possible only if $D' \leq \operatorname{Ram} p'$ and $D'' \leq \operatorname{Ram} p''$. Hence

$$|\delta| \ni p_*D = p'_*D' + p''_*D'' \leq \Delta' + \Delta'' = \Delta.$$

(1.2.9) In particular, Δ is the sum of two members of $|\delta|$, which does not happen for a generic member of $\Delta_{g+1,t}$. We also have $L = \pi^* O_C(D)$, hence $L \simeq \sigma^* L$ and $L^{\otimes 2} \simeq \omega_{\widetilde{C}}$. It is therefore a point of order two in P.

Now suppose $h^0(L) \geq 4$. If $\deg F' \geq 1$, Corollary 1.2.8 gives

$$4 \leq h^0(L) \leq 2h^0(D'' + p''^* F'' + p''^* F') - 2\deg F',$$

hence $h^0(D'' + p''^* F'' + p''^* F') \geq 3$. Applying 1.2.7,

$$\begin{aligned} 3 &\leq h^0(F' + F'') + h^0(p''_* D'' + F' + F'' - \delta'') \\ &\leq \deg(F' + F'') + \max(\deg(F' - F''), 0) \\ &\leq 2\max(\deg F', \deg F''), \end{aligned}$$

so either F' or F'' has degree at least two, and we belong either to W_2 or to W_{-2}. The case $\deg F'' \geq 1$ is treated in the same way.

If $\deg F' = \deg F'' = 0$, Corollary 1.2.8 gives $h^0(D') \geq 2$ and $h^0(D'') \geq 2$. By ([We 2], 3.4), we belong to V. This proves the first part of the proposition.

The set V is clearly empty for $t = \deg L' \leq 1$. For $t = 2$, it is empty if C' is not hyperelliptic, as happens generically. The assertions about the number of irreducible components follow from [We 2], Proposition 3.6 for $t \geq 5$, from [Te] for $t = 4$ and, for $t = 3$, from the fact that the two g^1_3's on C' (assumed nonhyperelliptic), interchanged by τ'^*, satisfy $\mathrm{Nm}_{p'} g^1_3 = O_E(\delta')$ (use 1.2.7). Finally, V is not contained in any W_a: a generic element can be written

$$L = \pi'^* L' \otimes \pi''^* L'', \quad \mathrm{Nm}\, L' = O_E(\delta'), \quad \mathrm{Nm}\, L'' = O_E(\delta''), \quad h^0(L') = h^0(L'') = 2.$$

Moreover, all divisors in $|L'|$ (respectively $|L''|$) are p'-simple (respectively p''-simple). Such a bundle cannot belong to a W_a, $a \in \mathbb{Z}$.

Finally, the irreducibility and dimension assertions for the W_a follow from: □

Lemma II.D.1.2.10. *Let $p\colon C \to E$ be a bielliptic curve and fix $G \in \mathrm{Pic}^d E$. For $5 \leq d \leq g(C) + 1$, the set*

$$\{p^* M \otimes O_C(D) \mid M \in \mathrm{Pic}^2 E,\ D \in C^{(d-4)},\ O_E(p_* D) \otimes M^{\otimes 2} \simeq G\}$$

is irreducible of dimension $d - 4$.

Proof of Lemma 1.2.10. Put

$$Z(G) = \left\{ (x_1, \dots, x_{d-4}, M) \in C^{d-4} \times \mathrm{Pic}^2 E \;\middle|\; O_E\left(\sum_{i=1}^{d-4} p x_i\right) \otimes M^{\otimes 2} \simeq G \right\}.$$

First observe that $Z(G)$ is irreducible of dimension $d - 4$ for $d \geq 5$: this is clear for $d = 5$, and the first projection $\mathrm{pr}_1\colon Z(G) \to C$ has fibre $Z(G \otimes O_E(-px_1))$. For a generic element $(x_1, \dots, x_{d-4}, M)$ of $Z(G)$, the divisor $D = \sum_i x_i$ is p-simple (see 1.2.7), and there is an exact sequence

$$0 \to H^0(E, M) \to H^0(C, O_C(D) \otimes p^* M) \to H^0(E, O_E(p_* D - \delta) \otimes M) \to 0.$$

The last space vanishes when

$$0 \geq \deg[O_E(p_* D - \delta) \otimes M] = d - 1 - g(C),$$

except if

$$O_E(p_*D - \delta) \otimes M \simeq O_E \quad \Longleftrightarrow \quad M \simeq G \otimes O_E(-\delta).$$

Under the assumption $d \leq g(C) + 1$, it follows that, for a generic element $(x_1, \dots, x_{d-4}, M)$ of $Z(G)$, $D = \sum_i x_i$ is fixed in $|O_C(D) \otimes p^*M|$. Thus the map

$$Z(G) \longrightarrow \mathrm{Pic}^d C,$$
$$(x_1, \dots, x_{d-4}, M) \longmapsto O_C\left(\sum_{i=1}^{d-4} x_i\right) \otimes p^*M$$

is generically finite. Its image is therefore irreducible of dimension $d - 4$. □

1.3. Quadrics associated with double points of the theta divisor

We remain in the situation of 1.1, but consider only the cases $t = 0$ and $t = 1$.

We begin by recalling some facts about stable singularities of multiplicity two on Ξ ([Mu 2], [Tj 3]). Such a singularity corresponds to a line bundle L on $\widetilde{C}$ such that $h^0(L) = 4$ and $\mathrm{Nm}\, L = \omega_C$. There is a map

$$\phi\colon \bigwedge^2 H^0(L) \longrightarrow H^0(C, K_C + \eta),$$
$$s \wedge t \longmapsto s\sigma t - t\sigma s.$$

For a basis $\{s_1, \dots, s_4\}$ of $H^0(L)$, put

$$Q_L = \phi(s_1 \wedge s_2)\phi(s_3 \wedge s_4) - \phi(s_1 \wedge s_3)\phi(s_2 \wedge s_4) + \phi(s_1 \wedge s_4)\phi(s_2 \wedge s_3) \in S^2H^0(C, K_C + \eta).$$

When Q_L is nonzero, it is an equation of the tangent cone to Ξ at L in P (recall that $T_0P \simeq H^0(C, K_C + \eta)^\vee$). Moreover, the quadric defined by Q_L in $\mathbb{P}H^0(C, K_C + \eta)^\vee$ contains the image X of C under the morphism associated with the linear system $|K_C + \eta|$ (called the semicanonical curve).

It follows readily, for example from 1.4.1, that:

- For $t = 0$, X is the elliptic curve $\phi_{|\delta''|}(E)$.
- For $t = 1$, X lies on the cone Σ with vertex a point S and base $\phi_{|\delta''|}(E)$. It has a double point at S, and every quadric containing X contains Σ.

Proposition II.D.1.3.1. *Let (P, Ξ) be a generic member of $\mathcal{E}^2_{t,g-t}$, with $g \geq 5$. Then:*

1. *If $t = 0$, the intersection of the quadrics corresponding to the quadratic singularities of Ξ is the semicanonical curve X. Their linear span in $S^2T_0^\vee P$ has codimension $2g$.*
2. *If $t = 1$, the intersection of the quadrics is the cone Σ. Their linear span has codimension $3g - 2$.*

Proof. By 1.2.4(i), it suffices to compute the quadrics Q_L for elements L of W_2, which can be written

$$L = \pi'^*L' \otimes \pi''^*L'', \qquad \begin{cases} L' = O_{C'}(D') \otimes p'^*M,\ D' \geq 0,\ \deg L' = t + 4,\ \deg M = 2, \\ L'' = O_{C''}(D''),\ D'' \geq 0. \end{cases}$$

If D' is p'-simple, 1.2.7 gives the exact sequence

$$0 \to H^0(E,M) \xrightarrow{\alpha} H^0(C',L') \xrightarrow{\beta} H^0(E, M \otimes O_E(p'_*D' - \delta')) \to 0.$$

The morphisms α and β can be defined as follows:

$$\forall v \in H^0(E,M), \qquad \alpha(v) = p'^*(v)u', \quad \operatorname{div}(u') = D',$$
$$\forall u \in H^0(C',L'), \qquad u\tau'(u') - \tau'(u)u' = t'p'^*(\beta(u)),$$

where $\operatorname{div}(t')$ is the ramification divisor of p' on C'.

Choose a basis $\{v_1, v_2\}$ of $H^0(E,M)$ and a basis $\{v_3, v_4\}$ of $H^0(E, M \otimes O_E(p'_*D' - \delta'))$. There is then a basis $\{u'_1, \dots, u'_4\}$ of $H^0(C',L')$ such that

$$u'_1 = \alpha(v_1), \quad u'_2 = \alpha(v_2), \quad \beta(u'_3) = v_3, \quad \beta(u'_4) = v_4.$$

A basis of $H^0(L)$ is then $\{s_j = \pi'^*(u'_j)\pi''^*(u'')\}_{1\le j\le 4}$, where $\operatorname{div}(u'') = D''$. We have

$$\begin{aligned}\pi^*(\phi(s_j \wedge s_k)) &= \pi'^*(u'_j)\pi''^*(u'')\pi'^*(\tau'^*u'_k)\pi''^*(\tau''^*u'') \\ &\quad - \pi'^*(\tau'^*u'_j)\pi''^*(\tau''^*u'')\pi'^*(u'_k)\pi''^*(u'') \\ &= f^*(v'')\pi'^*(\phi(u'_j \wedge u'_k)),\end{aligned}$$

where $\operatorname{div}(v'') = p''_*D''$. Thus $\phi(s_1 \wedge s_2) = 0$ and

$$\begin{aligned}\pi^*(\phi(s_1 \wedge s_3)) &= f^*(v'')\pi'^*\left[u'p'^*(v_1)\tau'^*u'_3 - \tau'^*u'\, p'^*(v_1)u'_3\right] \\ &= f^*(v''v_1v_3)\pi'^*(t').\end{aligned}$$

Under the identification of 1.4.1, this reads $\phi(s_1 \wedge s_3) = p^*(v''v_1v_3)s'$. Hence

$$Q_L = -(v_1v_3v'')s' \cdot (v_2v_4v'')s' + (v_1v_4v'')s' \cdot (v_2v_3v'')s'.$$

We wish to show that these quadrics span the space of quadrics in $\mathbb{P}^{g-t-1}$ containing $\phi_{|\delta''|}(E)$. This is an easy exercise, whose proof we leave to the reader (it can also be found in [D 2]). □

1.4. Rank of the Prym map on $\mathcal{S}_{g+1,t}$. The Andreotti–Mayer theorem

Recall ([Be 2], Proposition 7.5) that the codifferential of the map $\Pr\colon \mathcal{R}_{g+1} \to \mathcal{A}_g$ at a point $(C,\eta) \in \mathcal{R}^0_{g+1}$ is

$$T^\vee_{(C,\eta)} \Pr\colon S^2H^0(C, K_C + \eta) \longrightarrow H^0(C, 2K_C).$$

We consider the differential of the restriction $\Pr_t$ of Pr to $\mathcal{S}_{g+1,t} \subset \mathcal{R}^0_{g+1}$. Let s' and s'' be two sections such that:

$$\operatorname{div}(s') = Q_1 + \cdots + Q_{2t}, \qquad \operatorname{div}(s'') = Q_{2t+1} + \cdots + Q_{2g} \quad \text{(notation of 1.1)},$$
$$s = s's'', \qquad \operatorname{div}(s) = Q_1 + \cdots + Q_{2g}.$$

We then have the decompositions

$$T^\vee_{(C,\eta)}\mathcal{R}_{g+1} \simeq H^0(C, 2K_C) \simeq \underbrace{p^*H^0(E, 2\delta)}_{\tau=\mathrm{id}} \oplus \underbrace{p^*H^0(E,\delta)s}_{\tau=-\mathrm{id}}.$$

The τ-invariant part corresponds to the locus in the deformation space of C on which the involution τ extends, that is, to $T^\vee_{(C,\eta)}\mathcal{S}_{g+1,t}$. We also have:

Lemma II.D.1.4.1. $H^0(C, K_C + \eta) \simeq p^*H^0(E,\delta'')s' \oplus p^*H^0(E,\delta')s''$.

Proof. We have (see 1.1)

$$\eta \equiv Q_1 + \cdots + Q_{2t} - p^*\delta' \equiv Q_{2t+1} + \cdots + Q_{2g} - p^*\delta'',$$

$$K_C + \eta \equiv \operatorname{div}(s') + p^*\delta'' \equiv \operatorname{div}(s'') + p^*\delta'.$$

Thus there is a morphism

$$\begin{aligned} H^0(E,\delta'') \oplus H^0(E,\delta') &\longrightarrow H^0(C, K_C + \eta), \\ (u'', u') &\longmapsto p^*u''\, s' + p^*u'\, s'', \end{aligned}$$

whose kernel is isomorphic to

$$H^0(p^*\delta'' - \operatorname{div}(s'')) = H^0(p^*\delta' - \operatorname{div}(s')) = H^0(C,\eta) = 0.$$

□

We may therefore write

$$\begin{aligned} S^2H^0(C, K_C + \eta) \simeq\ & [p^*S^2H^0(E,\delta'')s'^2 \oplus p^*S^2H^0(E,\delta')s''^2] \\ & \oplus [p^*H^0(E,\delta'') \otimes p^*H^0(E,\delta')]s's''. \end{aligned}$$

If $s_{\Delta'}$ (respectively $s_{\Delta''}$) is a section with divisor Δ' (respectively Δ'') such that $p^*s_{\Delta'} = s'^2$ (respectively $p^*s_{\Delta''} = s''^2$), we can write $T^\vee \Pr$ as the sum of

$$\begin{aligned} T^\vee \Pr_t \colon S^2H^0(E,\delta') \oplus S^2H^0(E,\delta'') &\longrightarrow H^0(E, 2\delta), \\ (u'v', u''v'') &\longmapsto u'v's_{\Delta''} + u''v''s_{\Delta'} \end{aligned}$$

and

$$\begin{aligned} H^0(E,\delta') \otimes H^0(E,\delta'') &\longrightarrow H^0(E,\delta), \\ u' \otimes u'' &\longmapsto u'u''. \end{aligned}$$

Proposition II.D.1.4.2. *For $g \geq 4$, $(g,t) \neq (4,2)$ and (C,η) generic in $\mathcal{S}_{g+1,t}$,*

$$\dim \ker T_{(C,\eta)} \Pr_t = \begin{cases} 0 & \text{for } t = 0, \\ 1 & \text{for } 0 < t \leq g/2. \end{cases}$$

In particular,

$$\dim \mathcal{E}^2_{t,g-t} = \begin{cases} 2g & \text{for } t = 0, \\ 2g-1 & \text{otherwise.} \end{cases}$$

Proof. Recall that $t = \deg \delta'$. If $t = 0$, $\dim \ker T \Pr_0$ is the corank of $S^2H^0(E,\delta'') \to H^0(E,2\delta'') = H^0(E,2\delta)$, which is surjective for $g = \deg \delta'' \geq 3$ ([Mu 4], p. 55). Suppose $t > 0$. We can write $T^\vee \Pr_t$ as the composite of

$$\phi' \oplus \phi'' \colon S^2H^0(E,\delta') \oplus S^2H^0(E,\delta'') \longrightarrow H^0(E,2\delta') \oplus H^0(E,2\delta'')$$

and

$$\begin{aligned} \phi \colon H^0(E,2\delta') \oplus H^0(E,2\delta'') &\longrightarrow H^0(E,2\delta), \\ (u', u'') &\longmapsto u's_{\Delta''} + u''s_{\Delta'}. \end{aligned}$$

Since Δ' and Δ'' have disjoint supports, the kernel of ϕ is spanned by $(s_{\Delta'}, -s_{\Delta''})$. If $t = 1$,

$$\operatorname{Im}(\phi' \oplus \phi'') \cap \ker \phi = [\mathbb{C}s^2_{\delta'} \oplus H^0(E,2\delta'')] \cap \mathbb{C}(s_{\Delta'}, -s_{\Delta''}) = 0,$$

because Δ' is the sum of two distinct points (ϕ'' is surjective since $\deg \delta'' = g-1 \geq 3$). Thus

$$\operatorname{rank} T^\vee \operatorname{Pr}_1 = \operatorname{rank}(\phi' \oplus \phi'') = 1 + 2(g-1) = 2g-1.$$

If $t = 2$, the same argument applies unless Δ' is the sum of two members of $|\delta'|$, which does not happen generically, or $\deg \delta'' = g-2 < 3$, the excluded case. If $t \geq 3$, ϕ' and ϕ'' are surjective and

$$\operatorname{rank} T^\vee \operatorname{Pr}_t = \operatorname{rank} \phi = 2g-1.$$

□

We can now show that $\overline{\mathcal{E}}^2_g = \operatorname{Pr}(\overline{\mathcal{J}}_{g+1,0})$ is an irreducible component of N^g_{g-4}. We recall the Andreotti–Mayer theorem, whose proof, if not its statement, can be found in [An–M].

Theorem II.D.1.4.3 (Andreotti–Mayer). *Let (A, Θ) be a principally polarized abelian variety of dimension g. Suppose that:*

- $\operatorname{Sing}\Theta$ *has pure dimension k.*
- *The set $\operatorname{Sing}_2\Theta$ of points of multiplicity two on Θ is dense in $\operatorname{Sing}\Theta$, and the image of each component of $\operatorname{Sing}_2\Theta$ under the map*

$$\begin{aligned} \operatorname{Sing}_2\Theta &\longrightarrow \mathbb{P}S^2T_0^\vee A, \\ x &\longmapsto \tau_x^*(\text{tangent cone to } \Theta \text{ at } x) \end{aligned} \tag{1.4.4}$$

  spans a vector space of dimension at least N.

Then the codimension of N^g_k at (A, Θ) is at least N.

Corollary II.D.1.4.5. *For $g \geq 5$, the set $\overline{\mathcal{E}}^2_g$ is an irreducible component of N^g_{g-4} of dimension $2g$.*

Proof. For a generic member (P, Ξ) of $\mathcal{E}^2_g$, $\operatorname{Sing}\Xi$ is irreducible for $g \geq 5$ by 1.2.4. The image of 1.4.4 spans a vector space of dimension $\binom{g+1}{2} - 2g$ (1.3.1), and $\dim \mathcal{E}^2_g = 2g$ (1.4.2). Theorem 1.4.3 gives the conclusion. □

Remark II.D.1.4.6. *Although for a generic member (P, Ξ) of $\mathcal{E}^2_{1,g-1}$ the locus $\operatorname{Sing}\Xi$ is irreducible (1.2.4), the image of 1.4.4 spans a vector space of dimension $\binom{g+1}{2} - (3g-2)$ (1.3.1), and $\dim \mathcal{E}^2_{1,g-1} = 2g-1$ (1.4.2). We cannot apply 1.4.3. Nevertheless, $\overline{\mathcal{E}}^2_{1,g-1}$ is indeed a component of N^g_{g-4} (1.6.1 and 4.2).*

1.5. Some remarks. The family $\overline{\mathcal{E}}^2_{1,g-1}$ contains the bielliptic Jacobians

We retain the same notation and add

$$P' = \ker(JC' \xrightarrow{\operatorname{Nm}} JE), \qquad P'' = \ker(JC'' \xrightarrow{\operatorname{Nm}} JE).$$

The polarization of JC' (respectively JC'') induces on P' (respectively P'') a polarization $L_{P'}$ (respectively $L_{P''}$) such that ([Mu 2], Corollary 1, p. 332; and I.3) $H(L_{P'}) \simeq p'^*JE[2]$ (respectively $H(L_{P''}) \simeq p''^*JE[2]$). This polarization has type (2), and $(P', L_{P'}) \in \mathcal{A}_{t,(2)}$ (respectively $(P'', L_{P''}) \in \mathcal{A}_{g-t,(2)}$).

Proposition II.D.1.5.1.

- *For $t = 0$, there is an isogeny $g\colon P'' \xrightarrow{\pi''^*} P = \mathrm{Prym}(\widetilde{C}/C)$ with kernel $\{0, p''^* O_E(\delta')\}$ such that $g^* L_P \sim L_{P''}$.*
- *For $0 < t \leq g/2$, there is an isogeny*

$$\begin{aligned} g\colon P' \times P'' &\longrightarrow P, \\ (x', x'') &\longmapsto \pi'^* x' - \pi''^* x'' \end{aligned}$$

with kernel $(p'^*, p''^*)(JE[2])$, such that $g^* L_P \sim L_{P'} \boxtimes L_{P''}$ (notation of I.3.10). In particular, $\mathcal{E}^2_{t,g-t} \subset \mathcal{A}^2_{t,g-t}$ (see Definition A1.1.3).

Proof. It is easy to check that g does take values in P and that $\ker g \subset P'[2] \times P''[2]$, so that g is an isogeny. For the polarizations, we must show

$$\widehat{\pi'^*}\, \phi_{L_P} \pi'^* = \phi_{L_{P'}}, \qquad \widehat{\pi''^*}\, \phi_{L_P} \pi'^* = 0.$$

We prove the first equality, for example. Put

$$i\colon P \hookrightarrow J\widetilde{C}, \quad i'\colon P' \hookrightarrow JC', \qquad \phi\colon J\widetilde{C} \xrightarrow{\sim} \widehat{J\widetilde{C}}, \quad \phi'\colon JC' \xrightarrow{\sim} \widehat{JC'}.$$

For every $p' \in P'$,

$$\begin{aligned} \widehat{\pi'^*}\, \phi_{L_P} \pi'^*(2p') &= \widehat{\pi'^*}\, \phi_{L_P^{\otimes 2}} \pi'^*(p') \\ &= \widehat{\pi'^*}\, \hat{i}\, \phi\, i \pi'^*(p') = \hat{i'}\, \widehat{\pi'^*}\, \phi\, \pi'^* i'(p') \\ &= \hat{i'}\, \phi' i'(2p') = \phi_{L_{P'}}(2p'). \end{aligned}$$

□

Observe that this result does not depend on the genus of E. If E has genus $b \geq 2$, one obtains Prym varieties belonging to $\mathcal{A}^{\delta}_{t,g-t}$, where δ is the polarization $(2, \dots, 2)$ of degree 2^b, and $b - 1 \leq t \leq g - (b-1)$.

Let us examine the cases $t = 0$ and $t = 1$ more closely. Let $\mathcal{E}_{g,(2)}$ be the set of Prym varieties (in the sense of [Mu 2]) associated with smooth bielliptic curves of genus $g + 1$. It is an irreducible subset of $\mathcal{A}_{g,(2)}$.

Proposition II.D.1.5.2.

1. *For every $(P, M) \in \overline{\mathcal{E}}_{g,(2)}$ and $a \in H(M)$, $a \neq 0$, the principally polarized abelian variety $P/\{0, a\}$ belongs to $\overline{\mathcal{E}}^2_g$.*
2. *For $g \geq 4$, every $(P, M) \in \overline{\mathcal{E}}_{g-1,(2)}$, every elliptic curve F and every isomorphism $\psi\colon H(M) \xrightarrow{\sim} F[2]$, the principally polarized abelian variety*

$$(P \times F)/\{(a, \psi a) \mid a \in H(M)\},$$

which belongs to $\mathcal{A}^2_{1,g-1}$ (see A1.1.3), belongs to $\overline{\mathcal{E}}^2_{1,g-1}$.

Proof. The first assertion follows immediately from 1.5.1. For the second, first observe that the family $\mathcal{F}^2_{1,g-1}$ so constructed is irreducible, since the family

$$\{p\colon C \to E \text{ a bielliptic structure, } g(C) = g, \text{ with an isomorphism } E[2] \simeq H(2)\}$$

is irreducible. Furthermore, $\dim \mathcal{F}^2_{1,g-1} \leq 2g - 2 + 1$. By 1.5.1 and A1.1.1, $\overline{\mathcal{E}}^2_{1,g-1} \subset \mathcal{F}^2_{1,g-1}$, and by 1.4.2, $\dim \mathcal{E}^2_{1,g-1} = 2g - 1$ for $g \geq 4$. Thus $\overline{\mathcal{E}}^2_{1,g-1} = \mathcal{F}^2_{1,g-1}$. □

Corollary II.D.1.5.3. *For $g \geq 4$, $\overline{\mathcal{E}}^2_{1,g-1}$ contains the bielliptic Jacobians.*

Proof. Indeed ([Mu 2], Corollary 1(b)), for a bielliptic structure $p\colon C \to E$, there is an isogeny $P \times JE \to JC$ with kernel $(p^*, \mathrm{id})(JE[2])$. □

1.6. The space $\overline{\mathcal{E}}^2_{1,g-1}$ is a component of N^g_{g-4}

We consider here only $g \geq 6$. The result is also true for $g = 5$ and will be proved in 4.2.

Theorem II.D.1.6.1. *For $g \geq 6$, $\overline{\mathcal{E}}^2_{1,g-1}$ is an irreducible component of N^g_{g-4} of dimension $2g-1$.*

Proof. Recall that the method of Andreotti and Mayer does not apply to $\overline{\mathcal{E}}^2_{1,g-1}$ (1.4.6). We shall use quasi-abelian varieties of rank one (called "rank 1 degenerations of abelian varieties" in [Mu 1], p. 350). We briefly explain what these are.

Quasi-abelian varieties of rank one and dimension g correspond bijectively to commutative group extensions of principally polarized abelian varieties of dimension $g-1$ by a one-dimensional torus. By [Mu 6], p. 227, all these extensions are of the form

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{j} \mathcal{G}(L_a \otimes L^{-1}) \longrightarrow A \longrightarrow 0,$$

where $(A, L) \in \mathcal{A}_{g-1}$ and $a \in A$ ([Mu 5], p. 290). Replacing a by $-a$ changes j to $-j$, and the resulting groups are isomorphic. There exists ([Ig 1], [Mu 1], p. 351, [Mu 7], [Na]) a coarse moduli space $\mathcal{A}^{(1)}_g$ for principally polarized abelian varieties or rank-one quasi-abelian varieties of dimension g, the disjoint union of $\mathcal{A}_g$ and a divisor $\partial\mathcal{A}_g$. The latter is fibred over $\mathcal{A}_{g-1}$ by $\rho_g\colon \partial\mathcal{A}_g \to \mathcal{A}_{g-1}$. The fibre over $(A, L) \in \mathcal{A}_{g-1}$ is isomorphic to $A/\operatorname{Aut}(A, L)$ under the correspondence above. For a closed subset Z of $\mathcal{A}_g$, write ∂Z for the intersection of its closure in $\mathcal{A}^{(1)}_g$ with $\partial\mathcal{A}_g$.

We shall need an extension calculation from [Fr–S 2], Theorem 5.2, concerning the following situation. Let $\pi_0\colon \widetilde{N} \to N$ be a connected étale double cover, with N smooth of genus g. Denote the induced involution on $\widetilde{N}$ by σ, and its Prym variety by $(P_0, L_0) \in \mathcal{A}_{g-1}$. Let C (respectively $\widetilde{C}$) be obtained from N (respectively $\widetilde{N}$) by identifying p with q (respectively $\tilde{p}$ with $\tilde{q}$, and $\sigma\tilde{p}$ with $\sigma\tilde{q}$, where $\pi_0(\tilde{p}) = p$ and $\pi_0(\tilde{q}) = q$). The induced étale cover $\pi\colon \widetilde{C} \to C$ is not admissible. Its Prym variety $P = \ker(J\widetilde{C} \xrightarrow{\mathrm{Nm}} JC)^0$ is an extension of P_0 by a one-dimensional torus. More precisely:

Lemma II.D.1.6.2 (Friedman–Smith). *The canonical extension $1 \to \mathbb{C}^* \to P \to P_0 \to 0$ corresponds to the element*

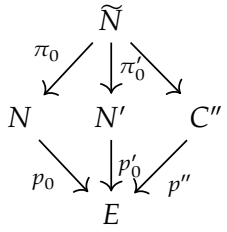
$$\mathcal{O}_{\widetilde{N}}(\pm(\tilde{q} - \tilde{p} - \sigma\tilde{q} + \sigma\tilde{p}))$$

of P_0.

Now consider a cover $\pi\colon \widetilde{C} \to C$ in $\mathcal{S}_{g,0}$, associated to $\eta = p^*\delta'$, where $\mathcal{O}(\delta')$ has order two in $\mathrm{Pic}^0 E$. If $f = p\pi\colon \widetilde{C} \to E$, write $f^*(JE[2]) = \{0, \varepsilon\}$, since $f^*\delta' = 0$.

Lemma II.D.1.6.3. *Let $(P, L) \in \Pr(\mathcal{S}_{g,0}) = \mathcal{E}^2_{g-1}$, and let ε be the unique nonzero element of $f^*(JE[2])$. Then the extension of P by ε belongs to $\partial\overline{\mathcal{E}}^2_{1,g-1}$.*

Proof. Consider a member of $\delta_{g+1,1}$, with the notation of 1.1. As P_1 tends to P_2 on E, the curve C' degenerates to $N'/(Q' \sim \tau_0' Q')$, where N' is smooth and $p_0' \colon N' \to E$ is an étale double cover, with $p_0'(Q') = P_1 = P_2$. There is a diagram



and the cover π_0 belongs to $\delta_{g,0}$. We have:

$$C = N/(Q \sim \tau_0 Q), \qquad p_0(Q) = P_1 = P_2,$$
$$\widetilde{C} = \widetilde{N}/(\widetilde{Q} \sim \sigma_0'' \widetilde{Q},\ \sigma_0 \widetilde{Q} \sim \sigma_0' \widetilde{Q}), \qquad \pi_0(\widetilde{Q}) = Q.$$

Let P (respectively P_0) be the Prym variety of $\pi \colon \widetilde{C} \to C$ (respectively $\pi_0 \colon \widetilde{N} \to N$). We are in the situation of 1.6.2, so there is an extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow P \longrightarrow P_0 \longrightarrow 0$$

corresponding to $a = O_{\widetilde{N}}(\pm(\widetilde{Q} - \sigma_0'' \widetilde{Q} - \sigma_0 \widetilde{Q} + \sigma_0' \widetilde{Q}))$. But $a = \pi_0'^* O_{N'}(\pm(Q' - \tau_0' Q'))$, where $Q' = \pi_0'(\widetilde{Q})$, and $O_{N'}(x' - \tau_0' x')$ is, for every $x' \in N'$, the unique nonzero element of $\ker(JN' \xrightarrow{\mathrm{Nm}} JE)$, namely $p_0'^* \alpha$, with $\alpha \in JE[2] \setminus \{0, O(\delta')\}$. Since $P_0 \in \mathrm{Pr}(\delta_{g,0})$ and $P \in \partial \overline{\mathcal{E}}^2_{1,g-1}$, this proves the lemma. □

We have thus exhibited an irreducible component $\partial' \overline{\mathcal{E}}^2_{1,g-1}$ of $\partial \overline{\mathcal{E}}^2_{1,g-1}$, of dimension $2g-2$, such that ρ_g induces a morphism

$$\phi_g \colon \partial' \overline{\mathcal{E}}^2_{1,g-1} \longrightarrow \overline{\mathcal{E}}^2_{g-1}, \qquad \phi_g^{-1}((P_0, L_0)) = \{\varepsilon\}$$

for $(P_0, L_0) \in \mathrm{Pr}(\delta_{g,0}) \subset \mathcal{E}^2_{g-1}$.

Mumford studied the spaces ∂N^g_k in [Mu 1]. His results ([Mu 1], p. 364) imply that ∂N^g_k is contained in the union of

$$\rho_g^{-1}(N^{g-1}_k),$$
$$\partial N^g_{k,1} = \{((A, \Theta), a) \in \partial \mathcal{A}_g \mid (A, \Theta) \in \mathcal{A}_{g-1},\ a \in A,\ \dim \mathrm{Sing}(\Theta \cdot \Theta_a) \geq k\},$$
$$\partial N^g_{k,2} = \{((A, \Theta), a) \in \partial \mathcal{A}_g \mid \dim(\mathrm{Sing}\,\Theta \cdot \mathrm{Sing}\,\Theta_a) \geq k-1\}.$$

Let C_g be an irreducible component of N^g_{g-4} containing $\overline{\mathcal{E}}^2_{1,g-1}$, and $\partial' C_g$ an irreducible component of ∂C_g containing $\partial' \overline{\mathcal{E}}^2_{1,g-1}$. Then

$$\dim \partial' C_g = \dim C_g - 1,$$
$$\partial' \overline{\mathcal{E}}^2_{1,g-1} \subset \partial' C_g \subset \rho_g^{-1}(N^{g-1}_{g-4}) \cup \partial N^g_{g-4,1} \cup \partial N^g_{g-4,2}.$$

For the moment, assume the following result:

Lemma II.D.1.6.4. *Let $p\colon C \to E$ be a bielliptic curve of genus $g \geq 6$, and let $\pi\colon \widetilde{C} \to C$ be an étale cover belonging to $\mathcal{S}_{g,0}$. Denote its Prym variety by (P_0, Ξ), and the unique nonzero element of $(p\pi)^*(JE[2]) \subset P_0$ by ε. Then:*

$$\operatorname{Sing}(\Xi \cdot \Xi_\varepsilon) = \operatorname{Sing}\Xi = \operatorname{Sing}\Xi_\varepsilon.$$

In particular, since $\dim \operatorname{Sing}\Xi = g-5$ (1.2.4(i)), the extension $P \in \partial'\overline{\mathcal{E}}^2_{1,g-1}$ of P_0 corresponding to ε belongs neither to $\rho_g^{-1}(N^{g-1}_{g-4})$ nor to $\partial N^g_{g-4,1}$. Hence

$$\partial'\overline{\mathcal{E}}^2_{1,g-1} \subset \partial' C_g \subset \partial N^g_{g-4,2}.$$

Taking images under ρ_g gives

$$\overline{\mathcal{E}}^2_{g-1} \subset \rho_g(\partial' C_g) \subset \rho_g(\partial N^g_{g-4,2}) \subset N^{g-1}_{g-5}.$$

For $g-1 \geq 5$, $\overline{\mathcal{E}}^2_{g-1}$ is an irreducible component of N^{g-1}_{g-5} (1.4.5). Therefore

$$\overline{\mathcal{E}}^2_{g-1} = \rho_g(\partial' C_g).$$

To show that $C_g = \overline{\mathcal{E}}^2_{1,g-1}$, it suffices to show that $\dim C_g = 2g-1$, or that $\dim \partial' C_g = 2g-2 = \dim \mathcal{E}^2_{g-1}$. This follows from the next lemma, which therefore completes the proof of the theorem. □

Lemma II.D.1.6.5. *Under the assumptions and notation of 1.6.4,*

$$\{a \in P_0 \mid \dim(\operatorname{Sing}\Xi \cdot \operatorname{Sing}\Xi_a) \geq g-5\} = \{0, \varepsilon\}.$$

Proof of 1.6.4 and 1.6.5. Write P for P_0, and replace g by $g+1 \geq 6$ to return to our usual setting. Recall (1.2.1, 1.2.2, 1.2.4) that

$$\begin{aligned}\Xi &= \{\pi''^*L'' \mid L'' \in \operatorname{Pic}^g C'',\ h^0(L'') \geq 1,\ \operatorname{Nm}_{p''} L'' = O(\delta)\},\\ \operatorname{Sing}\Xi &= \{\pi''^*(L'' \otimes p''^*M) \mid L'' \in \operatorname{Pic}^{g-4} C'',\ h^0(L'') \geq 1,\\ &\qquad\qquad M \in \operatorname{Pic}^2 E,\ \operatorname{Nm}_{p''} L'' \otimes M^{\otimes 2} = O(\delta)\}.\end{aligned}$$

Write $\varepsilon = f^*\alpha$, with $\alpha \in JE[2] \setminus \{0, O(\delta')\}$. If $L = \pi''^*L''$ belongs to Ξ, then

$$h^0(L \otimes \varepsilon) = h^0(L'' \otimes p''^*\alpha) + h^0(L'' \otimes p''^*\alpha \otimes p''^*O(\delta')).$$

If $L'' = O(D'')$ with D'' effective and p''-simple, 1.2.7 gives

$$\begin{aligned}h^0(L'' \otimes p''^*\alpha) &\leq h^0(E, \alpha) + h^0(\operatorname{Nm} L'' \otimes \alpha \otimes O(-\delta''))\\ &= h^0(O(\delta') \otimes \alpha) = 0.\end{aligned}$$

Thus $L \notin \Xi_\varepsilon$. It follows that

$$\begin{aligned}\Xi \cdot \Xi_\varepsilon = \{\pi''^*(L'' \otimes p''^*M) \mid &L'' \in \operatorname{Pic}^{g-2} C'',\ h^0(L'') \geq 1,\\ &M \in \operatorname{Pic}^1 E,\ \operatorname{Nm}_{p''} L'' \otimes M^{\otimes 2} = O(\delta)\}.\end{aligned}$$

For $L \in (\Xi \cdot \Xi_\varepsilon) \setminus \operatorname{Sing}\Xi$ (that is, with $L'' = O(D'')$, D'' effective and p''-simple), 1.2.7 gives

$$0 \longrightarrow H^0(E, M) \longrightarrow H^0(L'' \otimes p''^* M) \longrightarrow H^0(M \otimes \operatorname{Nm} L'' \otimes O(-\delta'')) = 0.$$

Let $s_{D''}$ be a section with divisor D'', and let s_N be a basis of $H^0(E, N)$ for every $N \in \operatorname{Pic}^1 E$. Then

$$H^0(\widetilde{C}, L) = \langle \pi''^*(s_{D''}\, p''^* s_M),\ \pi''^*(s_{D''}\, p''^* s_{M\otimes\delta'}) \rangle.$$

Similarly,

$$H^0(\widetilde{C}, L \otimes \varepsilon) = \langle \pi''^*(s_{D''}\, p''^* s_{M\otimes\alpha}),\ \pi''^*(s_{D''}\, p''^* s_{M\otimes\delta'\otimes\alpha}) \rangle.$$

If $G\colon \Xi_{\mathrm{sm}} \to \mathbb{P}T_0^\vee P \simeq \mathbb{P}H^0(K_C + \eta) \simeq \mathbb{P}H^0(\delta'')$ is the Gauss map, we obtain

$$G(L) = p''_* D'' + \operatorname{div}(s_M) + \operatorname{div}(s_{M\otimes\delta'}) \in |\delta''|,$$
$$G(L_\varepsilon) = p''_* D'' + \operatorname{div}(s_{M\otimes\alpha}) + \operatorname{div}(s_{M\otimes\delta'\otimes\alpha}) \in |\delta''|.$$

In particular, $G(L) \neq G(L_\varepsilon)$, and $\Xi \cdot \Xi_\varepsilon$ is smooth at L. This proves 1.6.4.

It remains to prove 1.6.5. Let $a \in P$ satisfy $\dim(\operatorname{Sing}\Xi \cdot \operatorname{Sing}\Xi_a) \geq g - 4$. Since $\operatorname{Sing}\Xi$ is irreducible of dimension $g - 4$ (1.2.4), we have $\operatorname{Sing}\Xi_a = \operatorname{Sing}\Xi$. Write $a = \pi''^* a''$ with $a'' \in P''$ (1.5.1), and denote by T''_2 the set

$$\{p''^* M \otimes O_{C''}(D'') \mid M \in \operatorname{Pic}^2 E,\ D'' \in C''^{(g-4)}\}.$$

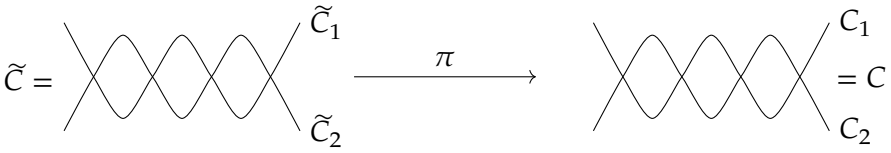
Since

$$\{L'' \in T''_2 \mid \operatorname{Nm} L'' = O(\delta)\} = (\pi''^*)^{-1}(\operatorname{Sing}\Xi),$$

we obtain $T''_2 + a'' = T''_2$. By A4.2, $a'' \in P'' \cap p''^* JE = p''^* JE[2]$, so $a \in \{0, \varepsilon\}$. □

II.D.2 The families $\mathcal{H}_{g+1,t}$ and the family $\mathcal{T}_6$

In this part we consider admissible covers of the form

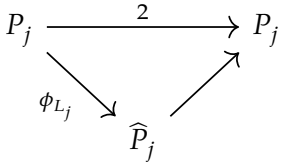


where $g(C_1) = t - 1$, $g(C_2) = g - t - 1$, the curves C_1, C_2 are smooth, and $1 \leq t \leq \lfloor g/2 \rfloor$.

The covers $\pi_j\colon \widetilde{C}_j \to C_j$ are branched at four points. Set $P_j = \ker(J\widetilde{C}_j \xrightarrow{\mathrm{Nm}} JC_j)$. The polarization of $J\widetilde{C}_j$ induces a polarization L_j on P_j, and ([Mu 2], Corollary 1, p. 332, and I.3.3)

$$H(L_j) \simeq \pi_j^*(JC_j[2]).$$

Thus these polarizations have type $(1, 2, \ldots, 2)$, with $\dim P_j - 1$ entries equal to two. The factorizations



induce polarizations $\hat{L}_j$ of type (2) on $\widehat{P}_j$.

Let (P, L) be the principally polarized Prym variety associated to the admissible cover π.

Proposition II.D.2.1. *There is an isogeny of degree four, $g\colon \widehat{P}_1 \times \widehat{P}_2 \to P$, such that $g^*L \simeq \hat{L}_1 \boxtimes \hat{L}_2$.*

In particular, $\Pr(\mathcal{H}_{g+1,t}) = \mathcal{P}^2_{t,g-t} \subset \mathcal{A}^2_{t,g-t}$ for $1 \le t \le \lfloor g/2 \rfloor$ (see definitions A1.1.3 and A2.3.5).

Proof. By [Be 1], p. 159, there is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{T}[2] & \longrightarrow & P \times \mathbb{Z}/2 & \longrightarrow & P_1 \times P_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{C}^*)^3 \simeq \widetilde{T} & \longrightarrow & J\widetilde{C} & \longrightarrow & J\widetilde{C}_1 \times J\widetilde{C}_2 & \to & 0 \\
 & & \downarrow & & \downarrow \mathrm{Nm} & & \downarrow \mathrm{Nm} & & \\
0 & \to & (\mathbb{C}^*)^3 \simeq T & \longrightarrow & JC & \longrightarrow & JC_1 \times JC_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus there is an isogeny of degree four, $f\colon P \to P_1 \times P_2$, satisfying $f^*(L_1 \boxtimes L_2) \simeq L^{\otimes 2}$.

Set $g = \phi_L^{-1} \hat{f}\colon \widehat{P}_1 \times \widehat{P}_2 \to \widehat{P} \xrightarrow{\sim} P$. One readily checks that $\hat{g}\phi_L g = \phi_{\hat{L}_1} \times \phi_{\hat{L}_2}$, proving the proposition. □

2.2. We now consider $\mathcal{J}_6$, for which $C = C_1 \cup C_2$, C_1 is smooth rational, $C_1 \cap C_2 = \{u_1, \dots, u_4\}$, and $\omega_{C_2} \simeq O_{C_2}(u_1 + \cdots + u_4)$. Assume C_2 is smooth nonhyperelliptic of genus three.

By 2.1, there is an isogeny

$$f\colon F \times \widehat{P}_2 \longrightarrow P = \mathrm{Prym}(\widetilde{C}/C), \tag{2.3}$$

where $F \simeq J\widetilde{C}_1$ is elliptic. It satisfies $f^*L \simeq L_F \boxtimes \hat{L}_2$, where L_F is a polarization of degree two on F.

Recall that we want to prove $\Pr(\mathcal{J}_6) \subset \overline{\mathcal{E}}^2_5$. Using A1.1.1 and the notation of 1.5.2, it suffices to show that $(\widehat{P}_2, \hat{L}_2) \in \overline{\mathcal{E}}_{4,(2)} \subset \mathcal{A}_{4,(2)}$. The curve C_1 no longer plays a role: C_2 is embedded in $\mathbb{P}^2$ by $|\omega_{C_2}|$, the cover $\pi_2\colon \widetilde{C}_2 \to C_2$ is branched along its section by a line D, and we want to show that the dual Prym variety $(\widehat{P}_2, \hat{L}_2)$ belongs to $\overline{\mathcal{E}}_{4,(2)}$.

Consider the special member of $\mathcal{J}_6$ with $C = D \cup C_2$. This is a plane quintic. The cover $\pi\colon \widetilde{C} \to C$ is even in the sense of [Ma], since its Prym variety (P, L) belongs to N^5_1. The latter is therefore isomorphic to the Jacobian (JN, Θ) of a smooth curve N of genus five ([Be 2], [Ma], [Tj 1], [Tj 2]). The singular locus of Θ is isomorphic to $\widetilde{C}$, hence reducible. The analysis in [Te] shows that there is a morphism $p\colon N \to E$ of degree two, where E is the elliptic double cover of D branched at $D \cap C_2$. Let $(Q, L_Q) \in \overline{\mathcal{E}}_{4,(2)}$ be the Prym variety associated to p. There are two isogenies:

$$\begin{aligned} f\colon E \times \widehat{P}_2 &\longrightarrow P && \text{(by 2.3)}, \\ g\colon E \times Q &\longrightarrow JN \simeq P && \text{(see 1.5.3)}, \end{aligned}$$

with $f^*L \simeq L_E \boxtimes \hat{L}_2$ and $g^*L \simeq L_E \boxtimes L_Q$.

We now show that $(\widehat{P}_2, \hat{L}_2)$ is isomorphic to (Q, L_Q), completing the argument. Use A1.1.1 and its notation. The injections $j_E(f)\colon E \hookrightarrow JN$ induced by f and $j_E(g)\colon E \hookrightarrow JN$ induced by g both induce degree-two morphisms from N onto E. Since C_2 is irreducible, [Te] implies that N has a unique bielliptic structure. After modifying f by automorphisms of E and N, we may assume $j_E(f) = j_E(g)$. The conclusion now follows directly from the uniqueness of X'' in A1.1.1.

II.D.3 Recillas' construction and Donagi's tetragonal construction

3.1. Recillas' construction

Recall the following construction, described in [Re] and made more precise in [Do–S], pp. 47–49.

Let X be a smooth curve of genus g, equipped with a $g^1_4, f\colon X \to \mathbb{P}^1$. The curve $\widetilde{C} = S^2_{\mathbb{P}^1} X$ has at most double points as singularities and admits a degree-six morphism $f^{(2)}\colon \widetilde{C} \to \mathbb{P}^1$. It has a natural involution ι sending $a + b$ to $f^{-1}f(a) - (a + b)$. Write $\pi\colon \widetilde{C} \to \widetilde{C}/\iota = C$ for the induced morphism through which $f^{(2)}$ factors. The curve C is trigonal.

Theorem II.D.3.1.1. *The cover $\pi\colon \widetilde{C} \to C$ is admissible, and its Prym variety is isomorphic to the Jacobian of X.*

Corollary II.D.3.1.2.

1. *The Prym varieties associated to admissible covers of curves obtained by identifying two points of a hyperelliptic curve are hyperelliptic Jacobians (compare [Be 1], Theorem 4.10, case b).*
2. *The locus of hyperelliptic Jacobians of genus g is contained in $\mathrm{Pr}(\overline{\mathcal{H}}_{g+1,0})$.*
3. *The locus of bielliptic Jacobians of genus g is contained in $\mathrm{Pr}(\overline{\mathcal{H}}_{g+1,1})$.*

Proof. Let X be hyperelliptic or bielliptic of genus g. Thus there is a degree-two morphism $p\colon X \to B$ with involution σ on X, branch divisor Δ_p on B, and $g(B) = 0$ or 1. Choose a g^1_2, $q\colon B \to \mathbb{P}^1$, with branch divisor Δ_q on $\mathbb{P}^1$ and involution τ. If $\tau(\Delta_p) \neq \Delta_p$, the Galois group of the Galois closure of $f = qp\colon X \to \mathbb{P}^1$ is D_4. The curve $\widetilde{C}$ obtained by Recillas' construction is

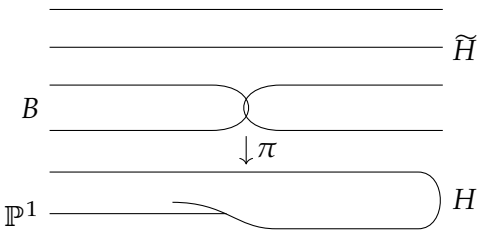
$$\widetilde{C} = \{x + \sigma x \mid x \in X\} \cup \{x + y \mid px = \tau py\} \simeq B \cup \widetilde{H},$$
$$C = \widetilde{C}/\iota \simeq \mathbb{P}^1 \cup H,$$

and H is hyperelliptic.

Over $Q \in \Delta_q \subset \mathbb{P}^1$, the fibre of $f^{(2)}$ consists of

- on B: $x + \sigma x$ twice, where $f(x) = Q$;
- on $\widetilde{H}$: $x + x$, $x + \sigma x$, $\sigma x + x$, $\sigma x + \sigma x$.

Thus the local picture is

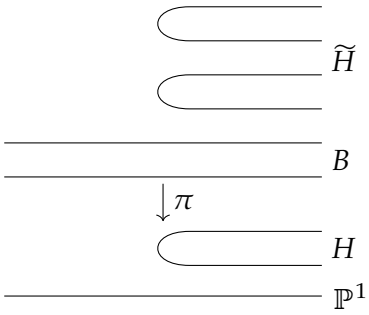


Hence H meets $\mathbb{P}^1$ at one point above Q.

Over $q(P)$, where $P \in \Delta_p \subset B$, the fibre consists of

- on B: $2x$, $y + \sigma y$, where $p(x) = P$, $p(y) = \tau P$;
- on $\widetilde{H}$: $x + y$ twice and $x + \sigma y$ twice.

The local picture is:



Thus H does not meet $\mathbb{P}^1$ here, but $\phi\colon H \to \mathbb{P}^1$ is ramified. Therefore:

- $H \cap \mathbb{P}^1$ maps under ϕ onto Δ_q and is Δ_q on $\mathbb{P}^1$;
- ϕ is branched over $q(\Delta_p)$.

If $g(B) = 0$, H meets $\mathbb{P}^1$ at two arbitrary points. It is easy to see that all hyperelliptic curves H of genus g arise this way. Inverting Recillas' construction ([Be 2], p. 389) gives the converse, proving 1) (use [Be 1], 4.11.3). If $\Delta_p = P_1 + \cdots + P_{2g+2}$, choose q so that $\tau P_i = P_j$ if and only if $\{i,j\} = \{1,2\}$. The curve H acquires a singular point above $q(P_1)$. This proves 2).

If $g(B) = 1$, H meets $\mathbb{P}^1$ at four points, so C belongs to $\mathcal{H}_{g+1,1}$. □

3.2. Donagi's tetragonal construction

Start with a connected étale double cover $\pi\colon \widetilde{C} \to C$ of a smooth curve C admitting a g^1_4, $h\colon C \to \mathbb{P}^1$. This defines a morphism $\mathbb{P}^1 \to C^{(4)}$ sending $P \in \mathbb{P}^1$ to the fibre $h^{-1}(P)$. Set

$$\begin{array}{ccc} \tilde{S} = \mathbb{P}^1 \times_{C^{(4)}} \widetilde{C}^{(4)} & \longrightarrow & \widetilde{C}^{(4)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & C^{(4)}. \end{array}$$

The involution σ induces an involution γ on $\tilde{S}$.

Theorem II.D.3.2.1 (Donagi; [Do] and [Be 3], §3(c)). *The curve $\tilde{S}$ is the disjoint union of two curves $\widetilde{C}_1, \widetilde{C}_2$ preserved by γ. The covers $\widetilde{C}_j \to \widetilde{C}_j/\gamma$, $j = 1,2$, are admissible, and their Prym varieties are isomorphic to the Prym variety of $\pi\colon \widetilde{C} \to C$.*

We now apply this construction to the covers in $\mathcal{S}_{g+1,t}$ and $\mathcal{S}_{g+1}$, for which C admits g^1_4's.

3.2.2. The families $\mathcal{S}_{g+1,t}$. Use the notation of 1.1 and choose a g^1_2, $\phi\colon E \to \mathbb{P}^1$, with involution ι. Set $h = \phi p\colon C \to \mathbb{P}^1$. The fibre of $P \in \mathbb{P}^1$ under $h\pi = \phi f\colon \widetilde{C} \to \mathbb{P}^1$ is $\{x, \sigma x, \sigma' x, \sigma'' x, y, \sigma y, \sigma' y, \sigma'' y\}$, where $f(x) = \iota f(y)$ and $\phi f(x) = P$.

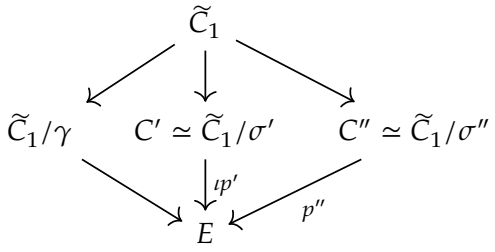
With the notation above,

$$\begin{aligned} \widetilde{C}_1 = \{&x + \sigma'' x + y + \sigma' y,\ \sigma x + \sigma'' \sigma x + y + \sigma' y, \\ &x + \sigma'' x + \sigma y + \sigma' \sigma y,\ x + \sigma' x + y + \sigma'' y, \text{ and their images under } \sigma\}, \\ \widetilde{C}_2 = \widetilde{C}'_2 \cup \widetilde{C}''_2, \end{aligned}$$

where

$$\begin{aligned}\widetilde{C}'_2 &= \{x+\sigma' x+y+\sigma' y,\ \sigma x+\sigma'\sigma x+y+\sigma' y, \text{ and their images under } \sigma\}\\ &\simeq \{a'+b' \in C'^{(2)} \mid p'a' = \iota p' b'\},\\ \widetilde{C}''_2 &\simeq \{a''+b'' \in C''^{(2)} \mid p''a'' = \iota p'' b''\}.\end{aligned}$$

1. Suppose $\iota P_i \neq P_j$ for $1 \leq i \leq 2t < j \leq 2g$. It is easy to see (using [Be 3], Proposition 3, a result of Welters [We 3]) that $\widetilde{C}_1$ is smooth and γ acts without fixed points. The group $\{1, \sigma, \sigma', \sigma''\}$ acts on $\widetilde{C}_1$ and induces a commutative diagram



   Thus the étale cover $\widetilde{C}_1 \to \widetilde{C}_1/\gamma$ belongs to $\mathcal{S}_{g+1,t}$. For $t = 0$ (that is, $2\delta' = 0$), one always has $\iota\delta' \equiv \delta'$, and the resulting cover is isomorphic to π. This is also the only case in which the fibres of $\mathrm{Pr}|_{\mathcal{S}_{g+1,t}}$ have dimension zero (1.4.2).

2. The curves $\widetilde{C}'_2$ and $\widetilde{C}''_2$ meet at four points above the branch locus of ϕ on $\mathbb{P}^1$. If $\iota P_i \neq P_j$ for $1 \leq i, j \leq 2t$ (respectively $2t < i, j \leq 2g$), then $\widetilde{C}'_2$ (respectively $\widetilde{C}''_2$) is smooth. There are degree-two morphisms

$$\begin{aligned}\widetilde{C}'_2/\gamma &\longrightarrow \mathbb{P}^1 \quad \text{branched at } \phi P_1, \dots, \phi P_{2t},\\ \widetilde{C}''_2/\gamma &\longrightarrow \mathbb{P}^1 \quad \text{branched at } \phi P_{2t+1}, \dots, \phi P_{2g}.\end{aligned}$$

   More precisely, distinguish the following cases.

   i) $t = 0$. The involution ι lifts to C', and $\widetilde{C}'_2$ is the disjoint union of two smooth rational curves preserved by γ. The curve $C_2 = \widetilde{C}_2/\gamma$ is the union of the hyperelliptic curve $\widetilde{C}''_2/\gamma$ of genus $g-1$ and two rational curves, each meeting it at two points. Its stable reduction belongs to $\overline{\mathcal{H}}_{g+1,0}$. All curves in $\mathcal{H}_{g+1,0}$ arise this way.

   ii) $t = 1$. The curve C_2 is the union of the hyperelliptic curve $\widetilde{C}''_2/\gamma$ of genus $g-2$ and a rational curve meeting it at four points. Thus it belongs to $\overline{\mathcal{H}}_{g+1,1}$. Every smooth hyperelliptic curve of genus $g-2$ can arise this way, and the four intersection points on it are arbitrary. They are also arbitrary on $\mathbb{P}^1$: for any $R_1, \dots, R_4, S_1, \dots, S_4 \in \mathbb{P}^1$ with $[R_1 : R_2 : R_3 : R_4] \neq [S_1 : S_2 : S_3 : S_4]$, there is a degree-two morphism $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\psi(S_j) = R_j$, $1 \leq j \leq 4$.

   iii) $t \geq 2$. We obtain two hyperelliptic curves meeting at four points, hence a member of $\overline{\mathcal{H}}_{g+1,t}$.

We have proved:

Theorem II.D.3.2.3. *For every g and $0 \leq t \leq \lfloor g/2 \rfloor$, $\overline{\mathcal{E}}^2_{t,g-t} \subset \mathrm{Pr}(\overline{\mathcal{H}}_{g+1,t})$, with equality for $t = 0$ and 1.*

3.2.4. The family $\mathcal{S}_{g+1}$. Recall that here C is smooth and has a bielliptic structure $p\colon C \to E$, with branch divisor $\Delta_p = P_1 + \cdots + P_{2g}$ on E and involution τ, while the Galois closure of $f = p\pi\colon \widetilde{C} \to E$ has group D_4. Choose a g^1_2, $\phi\colon E \to \mathbb{P}^1$, with branch divisor $\Delta_\phi = R_1 + \cdots + R_4$ on $\mathbb{P}^1$ and involution ι. We obtain a g^1_4, $h = \phi p\colon C \to \mathbb{P}^1$, whose monodromy group we assume is D_4 (for example if $\iota(\Delta_p) \neq \Delta_p$), and apply the tetragonal construction to it.

Let $P \in \mathbb{P}^1$. Write its fibre under $h\pi\colon \widetilde{C} \to \mathbb{P}^1$ as follows:

$$\begin{array}{ccccccc} \widetilde{C} & \xrightarrow{\pi} & C & \xrightarrow{p} & E & \xrightarrow{\phi} & \mathbb{P}^1 \\ x, \sigma x & \mapsto & a & \mapsto & e & \mapsto & P \\ y, \sigma y & \mapsto & \tau a & \mapsto & e & \mapsto & P \\ \bar{x}, \sigma\bar{x} & \mapsto & b & \mapsto & \iota e & \mapsto & P \\ \bar{y}, \sigma\bar{y} & \mapsto & \tau b & \mapsto & \iota e & \mapsto & P. \end{array}$$

With the preceding notation, the fibres of $\widetilde{C}_1, \widetilde{C}_2$ over P are respectively

$$\begin{aligned} (\widetilde{C}_1)_P = \{&x + \bar{x} + y + \bar{y},\ \sigma x + \sigma\bar{x} + y + \bar{y}, \\ &\sigma x + \bar{x} + \sigma y + \bar{y},\ \sigma x + \bar{x} + y + \sigma\bar{y}, \text{ and their images under } \sigma\}, \\ (\widetilde{C}_2)_P = \{&\sigma x + \bar{x} + y + \bar{y},\ x + \sigma\bar{x} + y + \bar{y}, \\ &x + \bar{x} + \sigma y + \bar{y},\ x + \bar{x} + y + \sigma\bar{y}, \text{ and their images under } \sigma\}. \end{aligned}$$

A group-action calculation shows that $\widetilde{C}_1, \widetilde{C}_2$ are irreducible. Also, τ induces an involution on C_1 (and on C_2), sending $[t + u + v + w] \in C_1$, with $f(t) = f(v)$, to $[\sigma t + u + \sigma v + w]$. Again, this can be proved rigorously by a group-action calculation that we omit. For example, $[x + \bar{x} + y + \bar{y}]$ maps to $[\sigma x + \bar{x} + \sigma y + \bar{y}] = [x + \sigma\bar{x} + y + \sigma\bar{y}]$. Set $D_j = C_j/\tau$. Then

$$\begin{aligned} (D_1)_P = \{&[x + \bar{x} + y + \bar{y}] = [\sigma x + \bar{x} + \sigma y + \bar{y}] = \text{images under } \sigma, \\ &[\sigma x + \sigma\bar{x} + y + \bar{y}] = [\sigma x + \bar{x} + y + \sigma\bar{y}] = \text{images under } \sigma\}, \\ (D_2)_P = \{&[\sigma x + \bar{x} + y + \bar{y}] = [x + \bar{x} + \sigma y + \bar{y}] = \text{images under } \sigma, \\ &[x + \sigma\bar{x} + y + \bar{y}] = [x + \bar{x} + y + \sigma\bar{y}] = \text{images under } \sigma\}. \end{aligned}$$

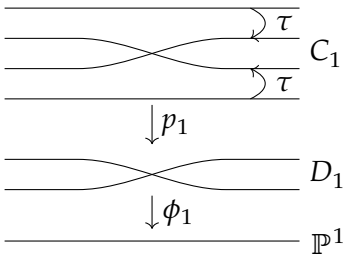
Thus there are morphisms

$$\widetilde{C}_j \xrightarrow{\pi_j} C_j \xrightarrow{p_j} D_j \xrightarrow{\phi_j} \mathbb{P}^1, \qquad j = 1, 2.$$

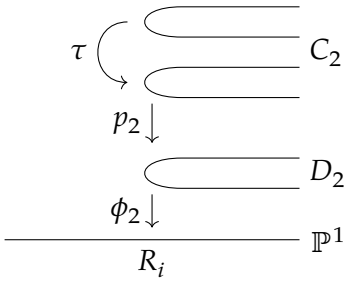
There is only one point on each of D_1, D_2 above R_i, $i = 1, \ldots, 4$. Hence either ϕ_j is ramified there, or D_j is singular there. What happens on C_1 above R_i? Here $x + y = \bar{x} + \bar{y}$, and

- either $x = \bar{x}$, $y = \bar{y}$, and $(C_1)_{R_i} = \{[2x + 2y], [2\sigma x + 2y], [x + \sigma x + y + \sigma y]\}$.

By Welters' criterion ([We 3], [Be 3], Proposition 3), C_1 is smooth at the first two of these points, but not the third. The curve D_1 is singular above R_i, and the local picture is:



We have $(C_2)_{R_i} = \{[x+\sigma x+2y], [2x+y+\sigma y]\}$, and Welters' criterion shows that C_2 is smooth at these points. The curve D_2 is smooth above R_i, and ϕ_2 is branched at R_i. The local picture is:



- or $x = \bar{y}$, $y = \bar{x}$, and the situation is reversed.

It follows that $\Delta_\phi = R_1 + \cdots + R_4$ is the disjoint union of the branch divisors of ϕ_1 and ϕ_2. Since the D_j are irreducible, each is obtained from $\mathbb{P}^1$ by identifying two pairs of points. Hence the C_j are obtained by identifying two pairs of points on hyperelliptic curves NC_j. We therefore belong to $\overline{\mathcal{H}}_{g+1,0}$. Finally, the branch divisor Δ_j of $p_j\colon NC_j \to ND_j \simeq \mathbb{P}^1$ satisfies $\phi_j(\Delta_j) = \phi(\Delta_p)$, showing that every cover in $\mathcal{H}_{g+1,0}$ arises this way. Thus

$$\Pr(\overline{\mathcal{S}}_{g+1}) = \Pr(\overline{\mathcal{H}}_{g+1,0}).$$

II.D.4 Closing remarks

In dimension $g \geq 6$, we now know that $\overline{\mathcal{J}}_g$, $\overline{\mathcal{E}}^2_g$, $\overline{\mathcal{E}}^2_{1,g-1}$ and the $\mathcal{A}^2_{t,g-t}$ for $2 \leq t \leq \lfloor g/2 \rfloor$ are irreducible components of N^g_{g-4} ([An–M], 1.4.5, 1.6.1, A1.4.5).

Proposition II.D.4.1. *For $g \geq 6$, these components are pairwise distinct.*

Proof. The closed subset $\overline{\mathcal{J}}_g$ differs from $\overline{\mathcal{E}}^2_{1,g-1}$ and $\mathcal{A}^2_{t,g-t}$ because a generic Jacobian is simple, and from $\overline{\mathcal{E}}^2_g$ because $3g-3 \neq 2g$. The only remaining equality of dimensions is $\dim \mathcal{E}^2_6 = \dim \mathcal{A}^2_{3,3} = 12$. For a generic member of $\mathcal{E}^2_6$, the intersection of the tangent cones at the double points of the theta divisor is an elliptic curve of degree six in $\mathbb{P}^5$ (see 1.3.1(1)). For a generic member of $\mathcal{A}^2_{3,3}$, it is the disjoint union of two projective planes (A1.4.4). They are therefore nonisomorphic. □

Proposition II.D.4.2. *The space N^5_1 has five irreducible components of respective dimensions $12, 10, 9, 9, 11$: $\overline{\mathcal{J}}_5$, $\overline{\mathcal{E}}^2_5$, $\overline{\mathcal{E}}^2_{1,4}$, $\mathcal{A}^2_{2,3}$ and $\mathcal{A}_{1,4}$ (defined below).*

Proof. Since $\Pr\colon \mathcal{R}_6 \to \mathcal{A}_5$ is surjective ([Be 1]), Beauville's list, its preceding commentary, and 2.1, 2.2, 3.2.3, 3.2.4 give

$$N^5_1 = \overline{\mathcal{J}}_5 \cup \overline{\mathcal{E}}^2_5 \cup \overline{\mathcal{E}}^2_{1,4} \cup \mathcal{A}^2_{2,3} \cup \mathcal{A}_{1,4} \cup \mathcal{A}_{2,3},$$

where $\mathcal{A}_{t,g-t} \subset \mathcal{A}_g$ denotes the locus of products of two abelian varieties of respective dimensions $t, g-t$. Their respective dimensions are $12, 10, 9, 9, 11, 9$ (1.4.2, A1.1.3). Observe immediately that $\mathcal{A}_{2,3} \subset \overline{\mathcal{J}}_5$. These sets are all irreducible, and $\overline{\mathcal{J}}_5$, $\overline{\mathcal{E}}^2_5$ and $\mathcal{A}^2_{2,3}$ are components of N^5_1 ([An–M], 1.4.5, A1.4.5). Since $\mathcal{A}_{1,4}$ is not contained in $\overline{\mathcal{J}}_5$, it too is a component

of N^5_1. It remains to check that $\overline{\mathcal{E}}^2_{1,4}$ is not contained in $\overline{\mathcal{J}}_5 \cup \overline{\mathcal{E}}^2_5 \cup \mathcal{A}^2_{2,3} \cup \mathcal{A}_{1,4}$. This is clear for $\mathcal{A}_{1,4}$, since $\mathcal{E}^2_{1,4} \not\subset N^5_2 \supset \mathcal{A}_{1,4}$.

If a generic member (P, Ξ) of $\mathcal{E}^2_{1,4}$ is the Jacobian of a genus-five curve C, then C is smooth nonhyperelliptic because $\dim \operatorname{Sing} \Xi = 1$ (1.2.4). The vector space spanned by the equations of the tangent cones at double points of Ξ has dimension $\binom{5+1}{2} - (3 \cdot 5 - 3)$ because $P \simeq JC$ ([Gr 1]), and $\binom{5+1}{2} - (3 \cdot 5 - 2)$ by 1.3.1(2), a contradiction.

If it belongs to $\mathcal{A}^2_{2,3}$, it is generic there too, since this locus has the same dimension as $\mathcal{E}^2_{1,4}$. The intersection of the tangent cones at double points of Ξ would have to be both a cone over an elliptic curve in $\mathbb{P}^3$ (1.3.1) and the disjoint union $\mathbb{P}^1 \sqcup \mathbb{P}^2$ (A1.4.4). If it belongs to $\overline{\mathcal{E}}^2_5$, it is the Prym variety of an admissible cover $\pi\colon \widetilde{C} \to C$ in $\overline{\mathcal{S}}_{6,0}$. Since $\operatorname{Sing} \Xi$ is irreducible, the exceptional case 1.2.9 does not occur. If C were smooth, 1.3.1 would therefore give a contradiction. In view of the dimension of $\mathcal{E}^2_{1,4}$, the curve C with its involution τ is one of the following.

1. $C = H/(x \sim y,\ \tau x \sim \tau y)$, where H is hyperelliptic of genus four with hyperelliptic involution τ. The variety P is isogenous to $R = \ker(J\widetilde{H} \xrightarrow{\mathrm{Nm}} JH)$ ([Be 1], Remark 3.6), where $\widetilde{H}$ is the normalization of $\widetilde{C}$. Likewise, the Prym variety $\overline{P}$ of the induced admissible cover of $\overline{C} = H/(x \sim \tau x,\ y \sim \tau y)$ is isogenous to R. But $\overline{C}$ is hyperelliptic, so $\overline{P}$ is a hyperelliptic Jacobian ([Be 1], Theorem 4.10). This is impossible because $P \in \mathcal{E}^2_{1,4}$ is not simple, whereas a generic hyperelliptic Jacobian is simple.
2. $C = N/(x \sim \tau x)$, where N is smooth bielliptic of genus five with involution τ. The cover $\pi\colon \widetilde{C} \to C$ cannot be the Wirtinger cover: then $P \simeq JN$, whereas bielliptic Jacobians of genus five form a family of dimension eight and $\dim \mathcal{E}^2_{1,4} = 9$. Thus $\widetilde{C}$ is irreducible, with normalization $\widetilde{N}$, and $\widetilde{N} \to N$ is branched at $x, \tau x$. The curve $\widetilde{C}$ has a double point O fixed by σ. Since $p_a(\widetilde{C}/\sigma') = 1$ (we are in $\overline{\mathcal{S}}_{6,0}$), and there is a morphism from $\widetilde{C}/\sigma' = C'$ onto a smooth elliptic curve, C' is smooth and $p'\colon C' \to E$ is étale. This contradicts the fact that $\pi'(O)$ is fixed by τ'. Hence this cover does not belong to $\overline{\mathcal{S}}_{6,0}$.

□

II.E Further results on Prym varieties belonging to N^g_{g-4}

The main theorem of II.D gives the following decomposition of the family of irreducible Prym varieties (P, Ξ) of dimension $g \geq 5$ satisfying $\dim \operatorname{Sing} \Xi \geq g - 4$:

$$N^g_{g-4} \cap \mathcal{P}^{\mathrm{irr}}_g = \mathcal{J}_g \cup \overline{\mathcal{E}^2_g} \cup \overline{\mathcal{E}^2_{1,g-1}} \cup \bigcup_{2 \leq t \leq g/2} \mathcal{P}^{2,\mathrm{irr}}_{t,g-t}.$$

The closures are taken in $\mathcal{A}^{\mathrm{irr}}_g$ (see I.3.12); $\mathcal{P}^{2,\mathrm{irr}}_{t,g-t}$ is the set of irreducible members of $\mathcal{P}^2_{t,g-t}$, defined in A2.3; and

$$\mathcal{E}^2_g = \Pr(\mathcal{S}_{g+1,0}), \qquad \mathcal{E}^2_{1,g-1} = \Pr(\mathcal{S}_{g+1,1})$$

(the definition of $\mathcal{S}_{g+1,t}$ is recalled below).

The results of II.D also give a precise geometric description of the members of $\mathcal{E}^2_g$ and $\mathcal{E}^2_{1,g-1}$.

This section describes the members of the families $\overline{\mathcal{S}}_{g+1,t}$ and their associated Prym varieties, namely $\overline{\mathcal{E}^2_{t,g-t}}$, and studies the singularities of the theta divisor of certain members of $\overline{\mathcal{E}^2_g}$.

We also briefly examine the situation in dimension 4.

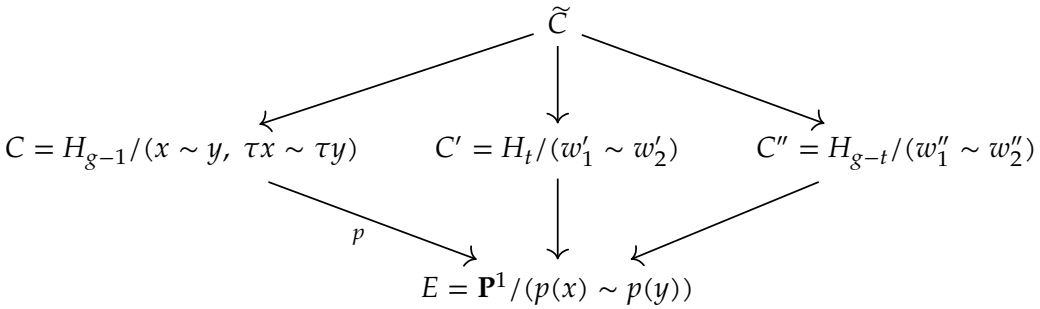
II.E.1 The families $\overline{\mathcal{E}^2_{t,g-t}}$

For $0 \leq t \leq g/2$, the family $\mathcal{S}_{g+1,t}$ is the subset of $\mathcal{R}_{g+1}$ consisting of étale double covers $\pi : \widetilde{C} \to C$ of smooth connected curves C of genus $g+1$ admitting a superelliptic structure $p : C \to E$, such that the Galois group of $p\pi$ is $\{1, \sigma, \sigma', \sigma''\}$, isomorphic to $(\mathbf{Z}/2)^2$, with $\widetilde{C}/\sigma = C$, $C' = \widetilde{C}/\sigma'$ of genus $t+1$, and $C'' = \widetilde{C}/\sigma''$ of genus $g-t+1$. We use the notation of II.D.1.1; in particular, $\mathcal{E}^2_{t,g-t} = \Pr(\mathcal{S}_{g+1,t})$. We study the degenerations of such covers for which π remains admissible.

If E remains smooth, II.D.1.5.1 and the fact that P' and P'' remain abelian varieties imply that JC' and JC'' remain abelian varieties. Consequently, one of the following cases occurs:

- C' and C'' are smooth and $\Delta' \cap \Delta'' = \emptyset$. We are still in $\mathcal{S}_{g+1,t}$.
- C' and C'' are smooth and $\Delta' \cap \Delta'' \neq \emptyset$. By II.D.3.2.2, a suitably chosen tetragonal construction reduces this to the first case without changing the Prym variety.
- C' is the union of two copies of E meeting at one point, so that $t = 1$. The Prym variety P is then the Jacobian of C''.

If E is irreducible and singular and $t > 0$, the situation is as follows:

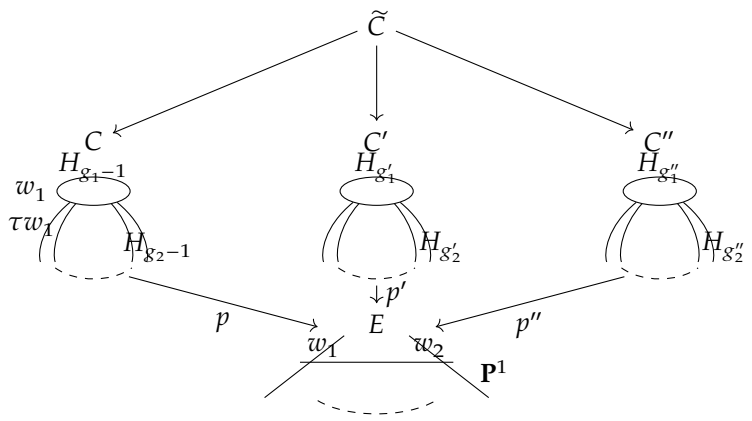


Here H_r denotes a smooth hyperelliptic curve of genus r, and w_1 and w_2 are Weierstrass points.

With the notation of A2.4, the associated Prym variety belongs to $\mathcal{H}^2_{t,g-t}$ $(0 < t \leq g/2)$.

If $t = 0$, we must add to the preceding case the case where $C' \xrightarrow{p'} E$ is the Wirtinger cover. The Prym variety associated with π is then the Jacobian of C''.

If E is reducible, the situation is as follows:



The curve E is the union of r copies of $\mathbf{P}^1$. On the jth copy there are $2g'_j$ points of Δ' and $2g''_j$ points of Δ'', hence $2g_j = 2g'_j + 2g''_j$ points of Δ, with $g_j > 0$. Thus $\sum g'_j = t$ and $\sum g''_j = g - t$.

The associated Prym variety then belongs to $\mathcal{H}^2_{g'_1,\dots,g'_r,g''_1,\dots,g''_s}$, with zero indices omitted (notation of A2.4).

We have therefore proved:

Theorem II.E.1.1. *For every g, the following equalities hold:*

$$\overline{\mathcal{E}^2_g} = \mathcal{E}^2_g \cup \mathcal{H}_g \cup \bigcup_{\substack{1\le g_1\le\cdots\le g_r\\ \sum g_i = g}} \mathcal{H}^2_{g_1,\dots,g_r},$$

$$\overline{\mathcal{E}^2_{1,g-1}} = \mathcal{E}^2_{1,g-1} \cup \mathcal{S}up_g \cup \bigcup_{\substack{1\le g_1\le\cdots\le g_r\\ \sum g_i = g-1}} \mathcal{H}^2_{1,g_1,\dots,g_r},$$

$$\overline{\mathcal{E}^2_{t,g-t}} = \mathcal{E}^2_{t,g-t} \cup \bigcup_{\substack{1\le g'_1\le\cdots\le g'_r,\ \sum g'_i = t\\ 1\le g''_1\le\cdots\le g''_s,\ \sum g''_i = g-t}} \mathcal{H}^2_{g'_1,\dots,g'_r,g''_1,\dots,g''_s},$$

where the closures are taken in the set of irreducible abelian varieties.

The families $\mathcal{E}^2_{t,g-t} = \mathrm{Pr}(\mathcal{S}_{g+1,t})$ were defined above, the families $\mathcal{H}^2_{g_1,\dots,g_r}$ in A2.4; the family $\mathcal{H}_g$ (respectively $\mathcal{S}up_g$) is the family of Jacobians of smooth hyperelliptic (respectively superelliptic) curves.

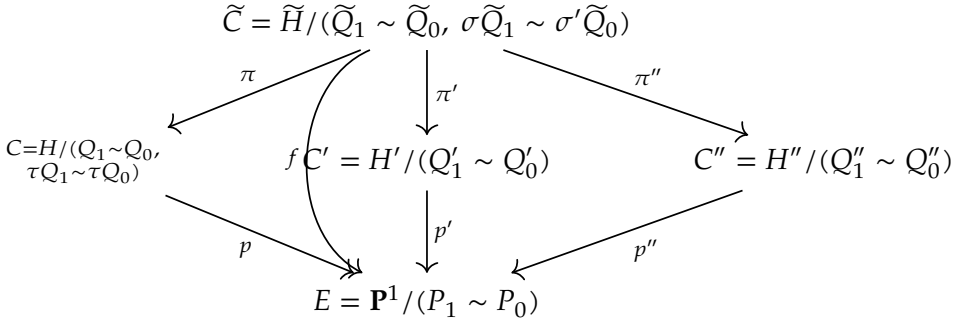
The decomposition of $N^g_{g-4} \cap \mathcal{P}^{\mathrm{irr}}_g$ at the beginning of this section, together with A2.4.4, A2.3.8, and A2.3.2, yields the following corollary.

Corollary II.E.1.2. *For $g \ge 5$, we have*

$$N^g_{g-4} \cap \mathcal{P}^{\mathrm{irr}}_g = \mathcal{J}_g \cup \mathcal{E}^2_g \cup \mathcal{H}^2_g \cup \mathcal{E}^2_{1,g-1} \cup \mathcal{H}^2_{1,g-1} \cup \bigcup_{2\le t\le g/2} \mathcal{P}^{2,\mathrm{irr}}_{t,g-t}.$$

II.E.2 Singularities of the theta divisor of members of $\mathcal{H}^2_g$ for $g \ge 5$

Recall that the members of $\mathcal{H}^2_g$ are the Prym varieties (P, Ξ) associated with covers $\pi : \widetilde{C} \to C$ that are degenerations of members of $\mathcal{S}_{g+1,0}$ of the following form:



Here $H' \simeq \mathbf{P}^1$; H, H'', and $\widetilde{H}$ are hyperelliptic of respective genera $g-1$, g, and $2g-1$; and Q''_0 and Q''_1 are Weierstrass points of H''.

Write $\tilde{n} : \widetilde{H} \to \widetilde{C}$ for the normalization of $\widetilde{C}$. We have the following exact sequences (see

[Be 1], 3.5 and II.D.1.5.1):

$$\begin{array}{ccccccccc} & & & & & & JH'' & & \\ & & & & & & \downarrow{\scriptstyle \pi''^*} & & \\ 0 & \to & \mathbf{Z}/2 & \to & P & \xrightarrow{\tilde{n}^*} & K = \ker(J\widetilde{H} \xrightarrow{\mathrm{Nm}} JH) & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{II.E.2.1}$$

Proposition II.E.2.2. *For $g \geq 5$, the singular locus of the theta divisor of a member of $\mathcal{H}^2_g$ has a unique irreducible component of dimension $\geq g-4$, namely*

$$W_2 = (\tilde{n}^*)^{-1}\{\pi''^* L'' \otimes f^* O_{\mathbf{P}^1}(2) \mid L'' \in \mathrm{Pic}^{g-4} H'',\ h^0(L'') \geq 1\}.$$

Proof. It follows from [Be 1], §4, that a line bundle L on $\widetilde{C}$ corresponding to a general point of a component of $\mathrm{Sing}\,\Xi$ of dimension $\geq g-4$ belongs to one of the following categories.

1) *Case c) in the list of Theorem 4.10 of* [Be 1]. We may write $L = (f^*M)(D)$, where M has degree 2 on E and D is nonsingular on $\widetilde{C}$. Then

$$\tilde{n}^* L = f^* O_{\mathbf{P}^1}(2) \otimes O(D), \qquad \pi_* D \in |\omega_H \otimes p^* O_{\mathbf{P}^1}(-2)|.$$

But

$$|\omega_H \otimes p^* O_{\mathbf{P}^1}(-2)| = p^* |O_{\mathbf{P}^1}(g-4)|.$$

Since D is nonsingular, we may write it (as in II.D.1.2.1) as

$$D = \pi'^* D' + \pi''^* D'', \qquad D', D'' \geq 0, \qquad \deg D' + \deg D'' = g-4.$$

For the associated family of line bundles to have dimension $g-4$, we must also have

$$h^0(H', D') = h^0(H'', D'') = 1$$

for a general member L. Since H' is smooth and rational, $D' = 0$. We therefore obtain the family in the statement.

2) *Case f) in the list of Theorem 4.10 of* [Be 1]. A general member L then satisfies (see 4.10.2 of [Be 1])

$$\tilde{n}^* L = f^* O_{\mathbf{P}^1}(1)(D + \widetilde{Q}_0 + \widetilde{Q}_1 + \sigma'\widetilde{Q}_0 + \sigma'\widetilde{Q}_1),$$

where $\pi_* D \in p^* |O_{\mathbf{P}^1}(g-4)|$. For every point $\tilde{x}$ of the hyperelliptic curve $\widetilde{H}$, however, we have

$$O_{\widetilde{H}}(\tilde{x} + \sigma'\tilde{x}) = \pi'^* O_{H'}(1),$$

so that $\tilde{n}^* L = f^* O_{\mathbf{P}^1}(2) \otimes O(D)$. As above, we conclude that we are in the family W_2.

It remains to prove that W_2 is irreducible. Diagram (II.E.2.1) extends to

$$\begin{array}{ccc} X & \xrightarrow{h} & JH'' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi''^*} \\ P & \xrightarrow[\tilde{n}^*]{} & K, \end{array} \tag{II.E.2.3}$$

where $X = \widehat{K}$ has a polarization of degree 2, $h = \widehat{\pi''^*}$, and $q = \widehat{\tilde{n}^*}$.

The isogeny h is associated with the element $\varepsilon = O_{H''}(Q''_0 - Q''_1)$ of $JH''[2]$. The variety $q^{-1}(W_2)$ is isomorphic to the inverse image under h of the subvariety W^{g-4} of $J^{g-4}H''$. It is irreducible because the restriction $\mathrm{Pic}^0 JH'' \to \mathrm{Pic}^0 W^{g-4}$ is an isomorphism when $g - 4 > 0$. □

Another description of W_2 will be given in II.F.3.4.

II.E.3 Singularities of the theta divisor of members of $\mathcal{H}^2_{1,g-1}$ for $g \geq 5$

The situation is the same as in the preceding subsection, except that now $g(H') = 1$, $g(H'') = g - 1$, and $\widetilde{H}$ is no longer hyperelliptic.

Diagram (II.E.2.3) becomes

$$\begin{array}{ccc} X' \times X'' & \xrightarrow{h=(h',h'')} & JH' \times JH'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (\pi'^*, -\pi''^*)} \\ P & \xrightarrow[\tilde{n}^*]{} & K. \end{array} \tag{II.E.3.1}$$

The kernel of the isogeny $(\pi'^*, -\pi''^*)$ has order two and is generated by

$$(O_{H'}(Q'_1 - Q'_0), O_{H''}(Q''_1 - Q''_0)) = (\varepsilon', \varepsilon'').$$

The isogenies q and h have degree 4. The isogeny q makes P a member of $\mathcal{H}^2_{1,g-1}$ (defined in A2.4).

Proposition II.E.3.2. *For $g \geq 5$, the singular locus of the theta divisor of a member of $\mathcal{H}^2_{1,g-1}$ has two irreducible components of dimension $\geq g - 4$, namely, writing $g'' = g(H'') = g - 1$,*

$$W_2 = (\tilde{n}^*)^{-1}\{\pi'^* L' \otimes \pi''^* L'' \otimes f^* O_{\mathbf{P}^1}(2) \mid L' \in \mathrm{Pic}^1 H',$$
$$L'' \in \mathrm{Pic}^{g''-4} H'',\ h^0(L')h^0(L'') > 0\},$$

and

$$W_1 = \pi'^*\varepsilon' + (\tilde{n}^*)^{-1}\{\pi''^* L'' \otimes f^* O_{\mathbf{P}^1}(2) \mid L'' \in \mathrm{Pic}^{g''-3} H'',\ h^0(L'') > 0\}.$$

Proof. As in the proof of II.E.2.2, the only relevant cases in Beauville's list are c) and f).

A general member in case c) satisfies, as in II.E.2.2,

$$\tilde{n}^* L = f^* O_{\mathbf{P}^1}(2) \otimes O(\pi'^* D' + \pi''^* D''), \qquad h^0(H', D') = h^0(H'', D'') = 1.$$

If $\deg D' = 0$, it is easy to check that $h^0(\widetilde{C}, L) = 3$, so we are not in P.

Thus $\deg D' = 1$, and we are in the family W_2.

A general member in case f) satisfies

$$\begin{aligned} \tilde{n}^* L &= f^* O_{\mathbf{P}^1}(1) \otimes O(D + \widetilde{Q}_0 + \widetilde{Q}_1 + \sigma'\widetilde{Q}_0 + \sigma'\widetilde{Q}_1) \\ &= f^* O_{\mathbf{P}^1}(2) \otimes O(D + \widetilde{Q}_0 - \widetilde{Q}_1 + \sigma'\widetilde{Q}_0 - \sigma'\widetilde{Q}_1) \\ &= f^* O_{\mathbf{P}^1}(2) \otimes \pi'^*\varepsilon' \otimes O(\pi'^* D' + \pi''^* D''). \end{aligned}$$

As above, either $\deg D' = 0$, in which case we are in W_1, or $\deg D' = 1$, in which case we are in W_2 (which is stable under translation by $\pi'^*\varepsilon'$).

We shall verify in II.F.5.1 that W_1 does indeed correspond to singularities of Ξ. The subvarieties W_1 and W_2 of P are isomorphic to the images under q of the inverse images under h of the subvarieties $\{0\} \times W^{g''-3}$ and $JH' \times W^{g''-4}$ of $JH' \times JH''$. They are therefore irreducible, since the natural maps $\mathrm{Pic}^0 J \to \mathrm{Pic}^0 W^d$ are injective for $d > 0$ (when $g'' = 4$, simply note that $W_2 = q(X' \times \ker h'') = q(X' \times \{0\})$). □

Another description of W_1 and W_2 will be given in II.F.5.

II.E.4 The case of dimension 4

For the families $\mathcal{S}_{5,t}$, $0 \le t \le 2$, the entire analysis of II.D.1 remains valid. In particular, assuming that the branch divisor Δ of $p : C \to E$ is not the sum of two members of $|\delta|$, we have

$$\mathrm{Sing}\,\Xi = \begin{cases} \emptyset, & t = 1, \\ \{f^*M \mid M^2 \simeq O(\delta)\}, & t = 0, 2. \end{cases}$$

When $t = 0$, we have $f^*\delta' = 0$ ($\delta' \in JE[2]$), so the singular locus of Ξ consists of two points of order 2. By II.D.1.4.2, $\mathcal{E}^2_4 = \mathrm{Pr}(\mathcal{S}_{5,0})$ has dimension 8. On $\mathcal{E}^2_{2,2} = \mathrm{Pr}(\mathcal{S}_{5,2})$ there are 4 singular points of order 2. The method of II.D.1.4.2 gives $\dim \mathcal{E}^2_{2,2} = 6$. Moreover, $\overline{\mathcal{E}^2_{2,2}} = \mathcal{A}^2_{2,2}$, recovering the fact that the theta divisor of a general member of $\mathcal{A}^2_{2,2}$ has 4 singular points, all of order 2 (A1.3.6). Finally, II.E.1.1 gives

$$\overline{\mathcal{E}^2_{2,2}} = \mathcal{A}^2_{2,2} = \overline{\mathcal{H}^2_{2,2}} \subset \overline{\mathcal{E}^2_4}.$$

We summarize these results in a proposition.

Proposition II.E.4.1.

1. *The subset $\mathcal{E}^2_4$ of $\mathcal{A}_4$ has dimension 8. Every member has precisely two points of order two as the singularities of its theta divisor.*
2. *The theta divisor of a general member of $\mathcal{E}^2_{1,3}$ is smooth.*
3. *The subset $\mathcal{E}^2_{2,2}$ of $\mathcal{A}_4$ has dimension 6. The theta divisor of a general member has precisely 4 points of order two as its singularities.*

We have $\overline{\mathcal{E}^2_{2,2}} = \mathcal{A}^2_{2,2} \subset \overline{\mathcal{E}^2_4}$.

Analogous results hold for $\mathcal{H}^2_{t,4-t}$, $0 \le t \le 2$.

Proposition II.E.4.2. *The subsets $\mathcal{H}^2_4$, $\mathcal{H}^2_{1,3}$, and $\mathcal{H}^2_{2,2}$ of $\mathcal{A}_4$ have respective dimensions 7, 6, and 6. The theta divisor of a general member has respectively 2, 2, and 4 points of order two as its only singularities.*

Proof. As in the proofs of II.E.2.2 and II.E.3.2, we examine the relevant cases in the list of Theorem 4.10 of [Be 1].

1) *Case c) in the list.* We obtain the two points of $W = (\tilde{n}^*)^{-1}\{f^*O_{\mathbf{P}^1}(2)\}$.

There is the additional condition that the corresponding line bundle $\tilde{L}$ on $\widetilde{C}$ must have even $h^0(\widetilde{C}, \tilde{L})$. We have

$$H^0(\widetilde{C}, \tilde{L}) \simeq \{s \in H^0(f^*O_{\mathbf{P}^1}(2)) \mid s(\widetilde{Q}_0) = s(\widetilde{Q}_1), \quad s(\sigma'\widetilde{Q}_0) = s(\sigma'\widetilde{Q}_1)\}.$$

For $t = 0$, $\widetilde{H}$ is hyperelliptic of genus 7, and

$$H^0(\widetilde{C}, \tilde{L}) \simeq \{s \in H^0(\widetilde{H}, 4g^1_2) \mid s(\widetilde{Q}_0) = s(\widetilde{Q}_1)\},$$

which has dimension 4.

For $t = 1$, H' is elliptic, and

$$H^0(\widetilde{C}, \tilde{L}) \simeq \{s \in H^0(H', p'^* O_{\mathbf{P}^1}(2)) \mid s(Q'_0) = s(Q'_1)\},$$

which has dimension 3.

For $t = 2$, $H^0(\widetilde{C}, \tilde{L})$ has dimension 2.

Thus W contributes singularities of Ξ only for $t = 0$ or $t = 2$.

2) *Case d) in the list.* We must determine whether C has line bundles N with $h^0(C, N) = 2$ and $N^2 \simeq \omega_C$. Let $n : H \to C$ be its normalization. The curve H is hyperelliptic of genus 3, with involution τ. We obtain C by identifying Q_0 with Q_1 and τQ_0 with τQ_1. We have $n^*N = 2g^1_2 + \gamma$, with $\gamma \in JH[2]$, and

$$2 = h^0(C, N) = \dim\{s \in H^0(n^*N) \mid s(Q_0) = s(Q_1), \quad s(\tau Q_0) = s(\tau Q_1)\}.$$

If $\gamma = 0$, we are in W_2. If $\gamma \neq 0$, the existence of a nonzero section of n^*N vanishing at Q_0 and Q_1 implies

$$\exists x, y \in H, \qquad 2g^1_2 + \gamma \equiv Q_0 + Q_1 + x + y.$$

There are then 4 Weierstrass points $w_1, \dots, w_4$ of H such that $\gamma \equiv w_1 + w_2 - w_3 - w_4$. The divisor $Q_0 + Q_1$ is therefore contained in a divisor of the base point free pencil $|w_1 + w_2 + w_3 + w_4|$, which does not occur for general Q_0 and Q_1 on H.

3) *Case f) in the list.* We obtain

$$(\tilde{n}^*)^{-1}\{f^* O_{\mathbf{P}^1}(1)(\widetilde{Q}_0 + \widetilde{Q}_1 + \sigma'\widetilde{Q}_0 + \sigma'\widetilde{Q}_1)\},$$

which equals W when $t = 0$ and is distinct otherwise. Thus, for $t = 1$ or 2, we obtain two additional singular points of order two.

4) *Case g) in the list.* This case does not in fact occur: the curve $H/(Q_1 \sim Q_0)$ has no thetanull, since the linear system $|\omega_H(Q_1 + Q_0)|$ contains no divisor of the form $2D$ with $h^0(D) \geq 2$.

We have therefore found the required number of singular points of order two for general members of $\mathcal{H}^2_4$, $\mathcal{H}^2_{1,3}$, and $\mathcal{H}^2_{2,2}$. To show that these points are indeed singularities of the theta divisor, it suffices to note that II.E.1.1 gives

$$\mathcal{H}^2_4 \cup \mathcal{H}^2_{1,3} \subset \overline{\mathcal{E}^2_4}, \qquad \mathcal{H}^2_{2,2} \subset \overline{\mathcal{E}^2_{2,2}},$$

and then apply II.E.4.1. The results for $\mathcal{H}^2_{2,2}$ also follow from the equality $\overline{\mathcal{H}^2_{2,2}} = \mathcal{A}^2_{2,2}$. □

Remark II.E.4.3. *The two preceding propositions, together with the fact that the theta divisor of a nonhyperelliptic Jacobian of dimension* 4 *cannot have two singular points of order two, show that no member of $\mathcal{E}^2_4$, $\mathcal{H}^2_4$, or $\mathcal{H}^2_{1,3}$ is a Jacobian. We also have $\mathcal{A}^2_{2,2} \cap \mathcal{J}_4 = \emptyset$, for example by II.C.1.3 and II.B.1.*

II.F The trisecant conjecture for Prym varieties

We can now search for possible trisecants to the Kummer varieties of irreducible Prym varieties as follows. The results of II.C show that the existence of an inclusion

$$\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d, \qquad a \neq 0, \qquad \{0, a\} \cap \{c, d\} = \emptyset,$$

on an irreducible Prym variety (P, Ξ) of dimension $g \geq 4$ implies $\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq g - 4$.

In particular, when $g \geq 5$, the variety (P, Ξ) belongs to one of the families studied in II.D and II.E, namely (II.E.1.2)

$$(P, \Xi) \in \mathcal{J}_g \cup \mathcal{E}^2_g \cup \mathcal{E}^2_{1,g-1} \cup \mathcal{H}^2_g \cup \mathcal{H}^2_{1,g-1} \cup \bigcup_{2 \leq t \leq g/2} \mathcal{P}^2_{t,g-t}.$$

When (P, Ξ) belongs to one of the families $\mathcal{E}^2_g$, $\mathcal{E}^2_{1,g-1}$, $\mathcal{H}^2_g$, or $\mathcal{H}^2_{1,g-1}$, we have a sufficiently precise description of $\operatorname{Sing} \Xi$ to determine, in dimension ≥ 5, the elements a of P satisfying $\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq g - 4$. For each of the first three families, we then prove that no such a admits a nontrivial inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$. This is the subject of subsections 1–3.

In subsection 4, we study the families $\mathcal{P}^2_{t,g-t}$ for $2 \leq t \leq g/2$, using an induction hypothesis. I should emphasize that the same techniques apply to members of $\mathcal{A}^2_{t,g-t}$, provided the trisecant conjecture holds in dimension $< g$. The sole obstacle to this approach, and a substantial one, is the absence of a complete description of N^g_{g-4}. If the components of N^g_{g-4} described in II.D and A1, namely $\overline{\mathcal{J}_g}$, $\overline{\mathcal{E}^2_g}$, $\overline{\mathcal{E}^2_{1,g-1}}$, and $\mathcal{A}^2_{t,g-t}$ for $2 \leq t \leq g/2$ (when $g \geq 5$), were to cover $N^g_{g-4} \cap \mathcal{A}^{\mathrm{irr}}_g$, the trisecant conjecture would follow. It would also suffice to show that an irreducible abelian variety whose Kummer variety has a trisecant, and which is not a Jacobian, necessarily belongs to some $\mathcal{A}^2_{t,g-t}$ with $2 \leq t \leq g/2$. This is exactly how we prove the conjecture in dimension 4: results of Ziv Ran ([Ra]) allow us to prove that $(P, \Xi) \in \mathcal{A}^2_{2,2} = \mathcal{P}^2_{2,2}$. The family $\mathcal{H}^2_{1,g-1}$ is studied at the end of this section, because its treatment uses the techniques of subsections 1–4. We also need the conjecture in dimension 4 to handle $\mathcal{H}^2_{1,4}$.

We thus prove the trisecant conjecture for (generalized) Prym varieties of arbitrary dimension, and in particular for all abelian varieties of dimension ≤ 5.

Theorem II.F. *Every irreducible Prym variety whose Kummer variety has a trisecant is the Jacobian of a curve.*

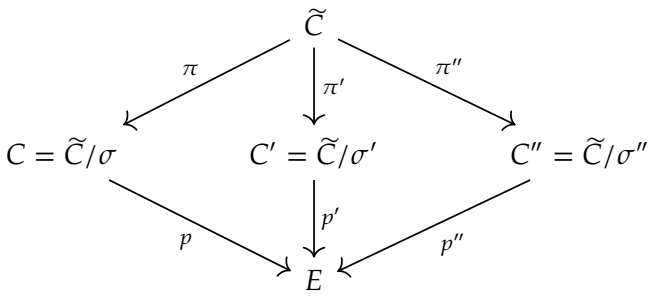
Proof. Let (P, Ξ) be the variety in question. We use induction on its dimension g. The first nontrivial case is $g = 4$, treated in II.F.4.1. We may therefore assume $g \geq 5$ and that the conjecture holds for Prym varieties of dimension $< g$. As explained above, either (P, Ξ) belongs to one of the families $\mathcal{E}^2_g$, $\mathcal{E}^2_{1,g-1}$, $\mathcal{H}^2_g$, $\mathcal{H}^2_{1,g-1}$, or $\mathcal{P}^2_{t,g-t}$ with $2 \leq t \leq g/2$, in which case its Kummer variety has no trisecant by II.F.1.1, 2.1, 3.1, 5.1, and 4.1, or (P, Ξ) is a Jacobian. □

II.F.1 The trisecant conjecture for members of $\mathcal{E}^2_g$, $g \geq 5$

We recall the definitions and notation used in II.D.1.

The family $\mathcal{S}_{g+1,0}$ consists of connected étale double covers $\pi : \widetilde{C} \to C$ of smooth curves C of genus $g+1$ admitting a superelliptic structure: a morphism p of degree 2 to a smooth elliptic curve E, for which the Galois group of $p\pi : \widetilde{C} \to E$ is isomorphic to $(\mathbf{Z}/2)^2$. Writing this group as $\{1, \sigma, \sigma', \sigma''\}$, with $C \simeq \widetilde{C}/\sigma$, we further require that $g(\widetilde{C}/\sigma') = 1$ and $g(\widetilde{C}/\sigma'') = g + 1$.

We use the notation of II.D.1.1:



$$\mathcal{O}(-\delta') = \bigwedge^2 p'_* \mathcal{O}_{C'}, \qquad \mathcal{O}(-\delta'') = \bigwedge^2 p''_* \mathcal{O}_{C''}, \qquad \mathcal{O}(-\delta) = \bigwedge^2 p_* \mathcal{O}_C = \mathcal{O}(-\delta' - \delta'').$$

The set $\mathcal{E}^2_g \subset \mathcal{P}_g$ consists of the Prym varieties corresponding to members of $\mathcal{S}_{g+1,0}$.

Theorem II.F.1.1. *Let (P, Ξ) be a member of $\mathcal{E}^2_g$ of dimension $g \geq 5$. Its associated Kummer variety has no trisecant.*

Proof. If the Kummer variety of (P, Ξ) has a trisecant, there exist (see II.A.1) nonzero elements a, c, and d of P with $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$ and $a \notin \{c, d\}$. By II.C.2.3, $\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq g - 4$. Recall (II.D.1.2.2) that $\operatorname{Sing} \Xi$ has dimension $g-4$ and a unique component of that dimension, namely

$$W_2 = \{\pi''^* \mathcal{O}(D'' + p''^* M) \mid D'' \in C''^{(g-4)},\ M \in E^{(2)},\ p''_* D'' + 2M \equiv \delta\}.$$

Thus $W_2 \subset \Xi_a$.

Recall also the isogeny $\pi''^* : P'' \to P$, where $P'' = \ker(JC'' \xrightarrow{\mathrm{Nm}} JE)$; its kernel is $\{0, p''^* \delta'\}$ (see II.D.1.5.1). The set

$$Z = \{\mathcal{O}(D'' + p''^* M) \mid D'' \in C''^{(g-4)},\ M \in E^{(2)},\ p''_* D'' + 2M \equiv \delta\}$$

is irreducible by A4.1. Let Θ'' denote the canonical theta divisor of $J^g C'' = J^{g''-1} C''$, where $g'' = g(C'') = g + 1$, and write $a = \pi''^* a''$ with $a'' \in P''$. We then have $Z \subset \Theta''_{a''}$. By A4.1, this implies $a'' = \mathcal{O}(-p''^* e + x + y)$ with $e \in E$ and $x, y \in C''$. Hence

$$\Xi \cdot \Xi_a = \pi''^* \{L'' \in \Theta'' \cdot \Theta''_{a''} \mid \operatorname{Nm} L'' = \mathcal{O}(\delta)\},$$

which is irreducible by A4.3(ii), except possibly in one of the following two cases (recall that $g(C'') = g + 1 \geq 6$).

First case: $a'' = \mathcal{O}(x'' - y'')$. Since $\operatorname{Nm} a'' = 0$, we have $y'' = \tau'' x''$. Then (see II.B.2)

$$\begin{aligned} \Xi \cdot \Xi_a = {} & \{\pi''^* \mathcal{O}(x'' + D'') \mid D'' \in C''^{(g''-2)},\ p''_* D'' \equiv \delta - p'' x''\} \\ & \cup \{\pi''^* \mathcal{O}(K_{C''} - \tau'' x'' - D'') \mid D'' \in C''^{(g''-2)}, \\ & \qquad p''_*(K_{C''} - \tau'' x'' - D'') \equiv \delta\}. \end{aligned}$$

For a member of the second set, we have

$$\begin{aligned} & \mathcal{O}(K_{C''} - \tau'' x'' - D'') \otimes \tau''^* \mathcal{O}(K_{C''} - \tau'' x'' - D'') \simeq p''^* \mathcal{O}(\delta), \\ \Longrightarrow \quad & K_{C''} - \tau'' x'' - D'' \equiv p''^* \delta' + x'' + \tau'' D'', \qquad p''_*(\tau'' D'') \equiv \delta - p'' x''. \end{aligned}$$

The two sets therefore differ by $\pi''^*p''^*\delta' = 0$. They are consequently equal and irreducible, and $\Xi \cdot \Xi_a$ is irreducible (although $(\pi''^*)^{-1}(\Xi \cdot \Xi_a)$ is not).

Second case: $a'' = O(-p''^*e + x + y)$ *with* $\delta \equiv p''x - e + \delta''$. Thus $\delta' \equiv p''x - e$. Since $\operatorname{Nm} a'' = 0$, we have $p''x \equiv p''y \equiv \delta' + e$, hence $p''x = p''y$. If $x = y$,

$$a'' = O(2x - p''^*(p''x - \delta')) = O(p''^*\delta' + x - \tau''x),$$

and $a = \pi''^*(x - \tau''x)$, the case already treated above. If $x = \tau''y$, then

$$a'' = O(-p''^*e + p''^*p''x) = O(p''^*\delta'),$$

and $a = 0$, which is impossible. □

Remark II.F.1.2. *Thus, for $g \geq 5$, no member of $\mathcal{E}^2_g$ is a Jacobian. This also holds for $g = 4$ by II.E.4.3 (these results also follow from [Sho 1]).*

Remark II.F.1.3. *The proof above and II.C.2.1 show that if $(P, \Xi) \in \mathcal{E}^2_g$, with $g \geq 5$, and $\Xi \cdot \Xi_a$ is reducible, then there is an elliptic curve $F \hookrightarrow P$ with $F \cdot \Xi = 2$ and $a \in F$. Since the superelliptic structure of C'' is unique, we have*

$$\begin{array}{ccccc} F'' = (\pi''^*)^{-1}(F) & \hookrightarrow & P'' & \hookrightarrow & JC'' \\ \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle 2:1} & & \\ F & \hookrightarrow & P & & \end{array}$$

and $F'' \cdot \Theta'' = 4$.

There is then a morphism of degree 4 from C'' to F''. For a general member of $\mathcal{E}^2_g$, all intersections $\Xi \cdot \Xi_a$ are irreducible.

II.F.2 The trisecant conjecture for members of $\mathcal{E}^2_{1,g-1}$, $g \geq 5$

The family $\mathcal{S}_{g+1,1}$ was defined in II.D.1.1. We have $\mathcal{E}^2_{1,g-1} = \Pr(\mathcal{S}_{g+1,1})$. We retain the notation of the preceding subsection, except that now $g(C') = g' = 2$ and $g(C'') = g'' = g$.

Theorem II.F.2.1. *Let (P, Ξ) be a member of $\mathcal{E}^2_{1,g-1}$ of dimension $g \geq 5$. Its associated Kummer variety has no trisecant.*

Proof. Suppose there is a nontrivial inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$. By II.D.1.2.2 and II.D.1.2.9, $\operatorname{Sing} \Xi$ has a unique irreducible component of dimension $g - 4$, namely

$$W_2 = \{O(\pi'^*x' + \pi''^*D'' + f^*M) \mid x' \in C',\ D'' \in C''^{(g''-5)},\ M \in E^{(2)},\ p'x' + p''_*D'' + 2M \equiv \delta\}.$$

As in II.F.1.1, $W_2 \subset \Xi_a$. Recall the isogeny $(\pi'^*, \pi''^*) : P' \times P'' \to P$, where $P' = \ker(JC' \to JE)$ and $P'' = \ker(JC'' \to JE)$ (see II.D.1.5.1). We therefore write $a = \pi'^*a' + \pi''^*a''$, with $\operatorname{Nm} a' = \operatorname{Nm} a'' = 0$.

Lemma II.F.2.2. *$W_2 \subset \Xi_a$ if and only if $a'' = O(x + y - p''^*e)$ with $x, y \in C''$ and $e \in E$.*

Proof. We begin with the following observation:

$$\begin{aligned} &\pi'^*L' \otimes \pi''^*L'' \simeq O_{\widetilde{C}}, \qquad \deg L' = \deg L'' = 0 \\ \Longrightarrow \quad &\exists N \in \operatorname{Pic}^0 E, \quad L' \simeq p'^*N, \quad L'' \simeq p''^*N^{-1}. \end{aligned} \tag{II.F.2.3}$$

Indeed, we then have

$$\pi'^*L' \simeq \pi''^*L''^{-1} \simeq \sigma''^*\pi'^*L' \simeq \pi'^*\tau'^*L'.$$

Since π' is ramified, π'^* is injective, so $L' \simeq \tau'^*L'$. Since p' is ramified at two points, there is a line bundle N on E with $L' \simeq p'^*N$ ([Mu 2], Corollary 2, p. 333). Thus $\pi''^*L'' \simeq \pi'^*L'^{-1} \simeq f^*N^{-1}$; injectivity of π''^* (since π'' is ramified) yields $L'' \simeq p''^*N^{-1}$.

Now take arbitrary $D'' \in C''^{(g''-5)}$ and $M \in E^{(2)}$. Choose $x' \in C'$ such that $p'x' + p''_*D'' + 2M \equiv \delta$. By hypothesis,

$$\exists \bar{x}' \in C', \quad \exists \bar{D}'' \in C''^{(g''-1)} : \quad \begin{cases} \pi'^*(\bar{x}' + a') + \pi''^*(\bar{D}'' + a'') \equiv \pi'^*x' + \pi''^*D'' + f^*M, \\ p'\bar{x}' + p''_*\bar{D}'' \equiv \delta. \end{cases}$$

By observation II.F.2.3, there is a divisor N of degree 0 on E such that

$$\begin{cases} x' - p'^*N \equiv \bar{x}' + a', \\ D'' + p''^*M + p''^*N \equiv \bar{D}'' + a''. \end{cases} \tag{II.F.2.4}$$

Keeping x' and D'' fixed but replacing M by $M_1 = M + \varepsilon$, with $\varepsilon \in \mathrm{Pic}^0 E[2]$, gives another divisor N_1 such that

$$\begin{cases} x' - p'^*N_1 \equiv \bar{x}'_1 + a', \\ D'' + p''^*M + p''^*\varepsilon + p''^*N_1 \equiv \bar{D}''_1 + a''. \end{cases}$$

Thus $\bar{x}'_1 \equiv \bar{x}' + p'^*(N - N_1)$.

By II.D.1.2.7, one of the following equivalences holds:

$$(\mathrm{A}') \quad N \equiv N_1, \qquad \text{or} \qquad (\mathrm{B}') \quad N - N_1 + p'\bar{x}' \equiv \delta'.$$

Similarly, $\bar{D}''_1 \equiv \bar{D}'' + p''^*(\varepsilon + N_1 - N)$ and, if $\bar{D}''$ is p''-simple (see II.D.1.2.7),

$$(\mathrm{A}'') \quad N \equiv N_1 + \varepsilon, \qquad \text{or} \qquad (\mathrm{B}'') \quad N_1 - N + \varepsilon + p''_*\bar{D}'' \equiv \delta''.$$

If (A′) holds, then (A″) fails, (B″) holds, and

$$\varepsilon \equiv \delta'' - p''_*\bar{D}''.$$

If (B′) holds, then $p''_*\bar{D}'' \equiv \delta - p'\bar{x}' \equiv N - N_1 + \delta''$, so (B″) fails, (A″) holds, and

$$\varepsilon \equiv N - N_1 \equiv \delta' - p'\bar{x}'.$$

Since we can always choose ε in

$$\mathrm{Pic}^0 E[2] \setminus \{0, \delta' - p'\bar{x}', \delta'' - p''_*\bar{D}''\},$$

we have shown that $\bar{D}''$ is not simple. By (II.F.2.4), this gives

$$\forall D'' \in C''^{(g''-5)}, \quad \forall M \in E^{(2)}, \quad \exists e \in E, \quad \exists G'' > 0, \quad \exists N \in \mathrm{Div}^0 E :$$

$$D'' + p''^*M - a'' \equiv p''^*(e - N) + G''.$$

In the notation of A4.2, this reads $(T_2)_{-a''} \subset \Theta$. Since $g'' \geq 5$, the conclusion follows. The converse is clear. □

Lemma II.F.2.5. *If there is a nontrivial inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$, then $a \in \pi'^* P' \cup \pi''^* P''$.*

Proof. We argue by contradiction, assuming $a' \notin p'^* \operatorname{Pic}^0 E$ and $a'' \notin p''^* \operatorname{Pic}^0 E$.

Set

$$\begin{aligned} U = \{(x', O(D''), \alpha) \mid x' \in C',\ D'' \in C''^{(g''-1)},\ \alpha \in \operatorname{Div}^0 E,\\ p'x' + p''_* D'' \equiv \delta,\\ H^0(x' - a' - p'^*\alpha) \neq 0,\\ H^0(D'' - a'' + p''^*\alpha) \neq 0\}. \end{aligned}$$

By observation II.F.2.3, the natural map $U \to \Xi \cap \Xi_a$ sending $(x', O(D''), \alpha)$ to $O(\pi'^* x' + \pi''^* D'')$ is surjective. Consider the projection

$$\operatorname{pr}_{1,3} : U \longrightarrow B' = \{(x', \alpha) \mid O(x') \in \Theta' \cdot \Theta'_{a' + p'^*\alpha}\}.$$

The curve B' is irreducible. Indeed, $\operatorname{pr}_2 : B' \to \operatorname{Pic}^0 E$ has degree 2, since $a' + p'^*\alpha$ never vanishes; and $\operatorname{pr}_1 : B' \to \Theta'$ also has degree 2 (the fibre of $x' \in \Theta'$ is $(p'^* \operatorname{Pic}^0 E) \cdot \Theta'_{a' - x'}$). If B' were reducible, it would have to be both the union of two copies of E and the union of two copies of $\Theta' \simeq C'$, which is absurd.

The projection $\operatorname{pr}_{1,3}$ is surjective: its fibre over $(x', \alpha) \in B'$ is

$$\{L'' \in \Theta'' \cdot \Theta''_{a'' - p''^*\alpha} \mid \operatorname{Nm} L'' \simeq O(\delta - p'x')\},$$

of dimension $g'' - 3$ by A4.3, since $a'' \notin p''^* \operatorname{Pic}^0 E$. Again by A4.3, this fibre is irreducible except in one of the following cases (recall from II.F.2.2 that $a'' = O(x + y - p''^* e)$):

- $\alpha = O(e - p''x)$ or $O(e - p''y)$;
- $\alpha = O(\delta' - p'x' + e - p''x)$ or $O(\delta' - p'x' + e - p''y)$;
- $g'' = 5$ and $O(\delta - p'x')$ belongs to a finite subset of $\operatorname{Pic}^4 E$.

Thus the fibres of $\operatorname{pr}_{1,3}$ have pure dimension $g'' - 3$ and are generically irreducible. Consequently, U has a unique irreducible component of dimension $\geq g'' - 3 + 1 = g - 2$. It follows that $\Xi \cdot \Xi_a$ is irreducible, a contradiction. □

We can apply II.F.2.5 to a, b, c, d. We claim that either $a, b, c, d \in P'$ or $a, b, c, d \in P''$, identifying these varieties with their images in P. Otherwise, for example, $a \in P' \setminus P''$ and $c \in P'' \setminus P'$. But we also have $\Xi \cdot \Xi_{a-c} \subset \Xi_{-c} \cup \Xi_d$, so $a - c \in P' \cup P''$, which is impossible.

1) If $a \in P'' \setminus P'$ and $b, c, d \in P''$, write

$$a = \pi''^* a'', \quad b = \pi''^* b'', \quad c = \pi''^* c'', \quad d = \pi''^* d''.$$

The members of $\Xi \cap \Xi_a$ are the $O(\pi'^* x' + \pi''^* D'')$ with

$$\begin{cases} p'x' + p''_* D'' \equiv \delta, \\ \exists \alpha \in \operatorname{Div}^0 E, \quad H^0(D'' - a'' - p''^*\alpha) \neq 0, \quad H^0(x' + p'^*\alpha) \neq 0. \end{cases}$$

By II.D.1.2.7, either $\alpha \equiv 0$ and $O(D'') \in \Theta'' \cdot \Theta''_{a''}$, or $\alpha + p'x' - \delta' \equiv 0$ and

$$\begin{aligned} D'' - a'' - p''^*\alpha &\equiv D'' - a'' - p''^*\delta' + p''^* p' x' \\ &\equiv D'' - a'' + p''^*\delta'' - p''^* p''_* D'' \\ &\equiv K_{C''} - a'' - \tau'' D'' \equiv K_{C''} - \tau''(D'' - a''), \end{aligned}$$

so again $O(D'') \in \Theta'' \cdot \Theta''_{a''}$.

The inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$ gives $\Theta'' \cdot \Theta''_{a''} \subset \Theta''_{c''} \cup \Theta''_{d''}$, and II.B.4 implies $a'' = O(m'' - \tau''m'')$ with $m'' \in C''$. It is then easy to verify that

$$\Xi \cdot \Xi_a = \{O(\pi'^*x' + \pi''^*m'' + \pi''^*D'') \mid x' \in C',\ D'' \in C''^{(g''-2)},\ p'x' + p''_*D'' \equiv \delta - p''m''\},$$

which is irreducible, a contradiction.

2) *If $a, b, c, d \in P'$*, write $a = \pi'^*a'$, and similarly for the other elements. As above, $\Xi \cdot \Xi_a$ is the union of

$$I' = \{O(\pi'^*x' + \pi''^*D'' + f^*e) \mid x' \in C',\ D'' \in C''^{(g''-3)},\ e \in E,\ p'x' + p''_*D'' + 2e \equiv \delta\},$$

which is irreducible and independent of a', and

$$I'(a') = \{O(\pi'^*x' + \pi''^*D'') \mid O(x') \in \Theta' \cdot \Theta'_{a'},\ D'' \in C''^{(g''-1)},\ p'x' + p''_*D'' \equiv \delta\}.$$

Since C' has genus 2, we can always write $a' = O_{C'}(m' - \tau'm')$ with $m' \in C'$. Then

$$I'(a') = \{O(\pi'^*m' + \pi''^*D'') \mid D'' \in C''^{(g''-1)},\ p''_*D'' \equiv \delta - p'm'\},$$

which is irreducible.

We have $I'(a') \subset \Xi_c \cup \Xi_d$, so, for example, $I'(a') \subset \Xi_c$. The preceding argument shows that $\Xi_c \supset I'$ because $c \in P'$, whence $\Xi_c \supset \Xi_a$, a contradiction. □

II.F.3 The trisecant conjecture for members of $\mathcal{H}^2_g$, $g \geq 5$

We recall the construction of the members of $\mathcal{H}^2_g$ (II.E.2 and A2.4).

Let H'' be a smooth hyperelliptic curve of genus $g \geq 5$, let Q''_0 and Q''_1 be two Weierstrass points of H'', and put $\varepsilon = O(Q''_1 - Q''_0) \in JH''[2]$. Let $h : X \to JH''$ be the étale double cover associated with ε. If Θ'' is a theta divisor of JH'', then $M = O(h^*\Theta'')$ defines a polarization of degree 2 on X. The group $H(M)$, isomorphic to $(\mathbf{Z}/2)^2$, contains $\ker h = \{0, \alpha\}$. For either of the two elements of $H(M) \setminus \ker h$, the quotient of (X, M) by that element is a principally polarized abelian variety (P, Ξ) belonging to $\mathcal{H}^2_g$.

Theorem II.F.3.1. *Let (P, Ξ) be a member of $\mathcal{H}^2_g$ of dimension $g \geq 5$. Its associated Kummer variety has no trisecant.*

Proof. As in subsection 1, suppose there is an inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$ with $a \neq 0$ and $\{0, a\} \cap \{c, d\} = \emptyset$. By II.C.2.3,

$$\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq g - 4. \tag{II.F.3.2}$$

We proved in II.E.2.2 that $\operatorname{Sing} \Xi$ has a unique component of dimension $\geq g-4$; we shall now give a different description of it. Recall diagram (II.E.2.3):

$$\begin{array}{ccc} X & \xrightarrow{h} & JH'' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi''^*} \\ P & \longrightarrow & K. \end{array} \tag{II.F.3.3}$$

In addition to the notation of II.E.2, we use the following:

- Θ'' is a theta divisor on JH'';
- Ξ is a theta divisor on P, with $\widetilde{\Xi} = q^*\Xi$, such that $M = O(\widetilde{\Xi}) = O(h^*\Theta'')$ is a polarization of degree 2 on X;
- $\ker \pi''^* = \{0, \varepsilon\}$, where ε is the element of JH'' associated with $O_{H''}(Q_0'' - Q_1'')$;
- $\ker h = \{0, \alpha\}$ and $\ker q = \{0, \beta\}$, so α and β generate $H(M)$;
- $s \in H^0(X, M)$ is an equation of $h^*\Theta''$ and $t \in H^0(X, M)$ is an equation of $h^*\Theta''_\varepsilon$, chosen so that $\widetilde{\Xi} = \operatorname{div}(s + t)$ (see A2.1.2).

On $J^{g-1}H''$, we have

$$\operatorname{Sing}\Theta'' \cdot \operatorname{Sing}\Theta''_\varepsilon = Z + Z_\varepsilon, \qquad Z = g^1_2 + Q_0'' + W^{g-4}.$$

The equation of $\widetilde{\Xi}$ shows that the points of $h^{-1}(\operatorname{Sing}\Theta'' \cap \operatorname{Sing}\Theta''_\varepsilon)$ are singular on $\widetilde{\Xi}$.

As noted just after II.E.2.3, the inverse image $\widetilde{Z}$ of Z under h is irreducible of dimension $g-4$ (for $g \geq 5$). Thus

$$\text{the component of } \operatorname{Sing}\Xi \text{ of dimension } g-4 \text{ is } q(\widetilde{Z}). \tag{II.F.3.4}$$

Finally, we have

$$\widetilde{\Xi} \cdot \widetilde{\Xi}_\alpha = \text{fixed part of } |M| = h^{-1}(\Theta'' \cdot \Theta''_\varepsilon) = \widetilde{T} + \widetilde{T}_\beta,$$

where $\widetilde{T} = h^{-1}(Q_0'' + W^{g-2})$ is an irreducible subvariety of X of dimension $g-2$. In particular, $\Xi \cdot \Xi_{q(\alpha)} = q(\widetilde{T})$ is irreducible. Return to our trisecant, and let $\tilde{a}$ be a lift of a under q. By (II.F.3.2) and (II.F.3.4), $\widetilde{\Xi}_{\tilde{a}} \supset \widetilde{Z} \cup \widetilde{Z}_\beta$. Since $\widetilde{Z} = \widetilde{Z}_\alpha$, we obtain

$$\widetilde{Z} \subset \widetilde{\Xi}_{\tilde{a}} \cap \widetilde{\Xi}_{\tilde{a}+\alpha} = \widetilde{T}_{\tilde{a}} \cup \widetilde{T}_{\tilde{a}+\beta}.$$

Replacing $\tilde{a}$ by $\tilde{a} + \beta$ if necessary, we may assume $\widetilde{Z} \subset \widetilde{T}_{\tilde{a}}$. Then $Z \subset T_{h(\tilde{a})}$, where $T = h(\widetilde{T})$, that is,

$$Q_0'' + g^1_2 + W^{g-4} \subset h(\tilde{a}) + Q_0'' + W^{g-2} \subset J^{g-1}H''.$$

Consequently, there are $x, y \in H''$ with $h(\tilde{a}) = O_{H''}(x - y)$. Using (II.F.3.3), this gives

$$\tilde{n}^*a = \pi''^* O_{H''}(x - y) = O_{\widetilde{H}}(\tilde{x} + \sigma''\tilde{x} - \tilde{y} - \sigma''\tilde{y}),$$

where $\pi''\tilde{x} = x$ and $\pi''\tilde{y} = y$.

Since $\widetilde{H}/\sigma' = H'$ is smooth and rational,

$$\tilde{y} + \sigma'\tilde{y} \equiv g^1_2 \equiv \sigma''\tilde{x} + \sigma'\sigma''\tilde{x},$$

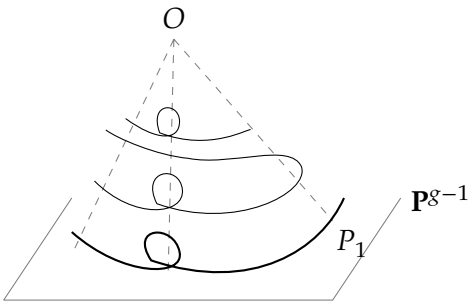
whence

$$\tilde{n}^*a = O_{\widetilde{H}}(\tilde{x} + \sigma\tilde{y} - \sigma\tilde{x} - \tilde{y}).$$

If both $\tilde{x}$ and $\tilde{y}$ are singular on $\widetilde{C}$, then $\tilde{n}^*a = 0$, so $a = q(\alpha)$. In that case, we saw that $\widetilde{\Xi} \cdot \widetilde{\Xi}_\alpha = \widetilde{T} + \widetilde{T}_\beta$, hence $\Xi \cdot \Xi_{q(\alpha)} = q(\widetilde{T})$ is irreducible.

If at most one of $\tilde{x}$ and $\tilde{y}$ is singular on $\widetilde{C}$, A3.2.2 shows that $\Xi \cdot \Xi_a$ is a generalized special subvariety, irreducible if the rational map $\varphi_{\tilde{x},\tilde{y}} : C \dashrightarrow \mathbf{P}^{g-2}$ obtained by composing the canonical map with projection from the chord joining $\tilde{x}$ to $\tilde{y}$ is birational. The canonical image of C in $\mathbf{P}^g$ is shown below. The nodal rational curve in $\mathbf{P}^{g-1}$ is the image of $\mathbf{P}^1$ under the linear system

$$\{s \in H^0(\mathbf{P}^1, O(g)) \mid s(P_1) = s(P_0)\}.$$



It is easy to see that $\varphi_{\tilde{x},\tilde{y}}$ is not birational if and only if $\pi\tilde{y} = \tau\pi\tilde{x}$. Under our hypotheses, $\tilde{x}$ and $\tilde{y}$ are then smooth points of $\widetilde{C}$ and, since $\tilde{n}^*a \neq 0$, we have $\tilde{y} = \sigma'\tilde{x}$ and

$$\begin{aligned}\tilde{n}^*a &= O_{\widetilde{H}}(\tilde{x} + \sigma''\tilde{x} - \sigma'\tilde{x} - \sigma\tilde{x}) \\ &= \pi''^*O_{H''}(x - \tau''x).\end{aligned}$$

Since the intersections $\Xi\cdot\Xi_a$, $\Xi\cdot\Xi_c$, and $\Xi\cdot\Xi_{a-c}$ are all reducible (II.A.1.3), there are $x, y, z \in H''$ such that

$$\begin{aligned}\tilde{n}^*a &= \pi''^*O_{H''}(x - \tau''x) \neq 0, \\ \tilde{n}^*c &= \pi''^*O_{H''}(y - \tau''y) \neq 0, \\ \tilde{n}^*(a - c) &= \pi''^*O_{H''}(z - \tau''z) \neq 0.\end{aligned}$$

This gives one of the two equivalences

$$\begin{aligned}& x - \tau''x - y + \tau''y \equiv z - \tau''z, \\ \text{or} \qquad & x - \tau''x - y + \tau''y \equiv z - \tau''z + Q''_1 - Q''_0.\end{aligned}$$

Each equivalence is possible only if one of x, y, or z is a Weierstrass point of H''. This is impossible, since $\tilde{n}^*a$, $\tilde{n}^*c$, and $\tilde{n}^*(a - c)$ are nonzero. □

Remark II.F.3.5. For $g \geq 5$, no member of $\mathcal{H}^2_g$ is therefore a Jacobian. The same holds for $g = 4$ by II.E.4.3; these results can also be deduced from [Sho 1].

II.F.4 The trisecant conjecture in dimension four and for members of $\mathcal{P}^2_{t,g-t}$, $g \geq 5$, $2 \leq t \leq g/2$

In this subsection we prove the trisecant conjecture in the following cases:

- The abelian variety has dimension 4. Here we make essential use of results of Ziv Ran [Ra].
- The abelian variety is a Prym variety of dimension $g \geq 5$ belonging to $\mathcal{P}^2_{t,g-t}$, with $2 \leq t \leq g/2$ (the loci defined in A2.3.5). We then assume the trisecant conjecture for Prym varieties of dimension $< g$.

Theorem II.F.4.1. *Let (A, Θ) be an irreducible Prym variety of dimension $g \geq 4$ whose Kummer variety has a trisecant. Assume that, if $g \geq 5$, (A, Θ) belongs to some $\mathcal{P}^2_{t,g-t}$ with $2 \leq t \leq g/2$, and that the trisecant conjecture holds for Prym varieties of dimension $< g$. Then (A, Θ) is the Jacobian of a curve.*

Preliminaries specific to dimension four

Start with an irreducible principally polarized abelian fourfold (A, Θ) whose Kummer variety has a trisecant. By II.A.1.3 there are nonzero elements a, b, c, d of A, with $a+b = c+d = x$ and $\{a, b\} \cap \{c, d\} = \emptyset$, and nonzero complex numbers λ, μ, ν satisfying

$$\forall \theta \in H^0(\Theta), \qquad \lambda\theta\theta_x + \mu\theta_a\theta_b + \nu\theta_c\theta_d = 0.$$

Assume that (A, Θ) is not a Jacobian. We then have:

Proposition II.F.4.2. *There are polarized abelian varieties X_1 and X_2 of degree 2 and dimension 2, and an isogeny $\pi : X_1 \times X_2 \to A$ making (A, Θ) a member of $\mathcal{A}^2_{2,2}$, with $\{a, b, c, d\} \cap \pi(X_1) \neq \emptyset$.*

Proof. We use the methods and results of [Ra]: the components of intersections of Θ with its translates are surfaces in a fourfold, so we have the intersection product at our disposal.

Lemma II.F.4.3. *One component of $\Theta \cap (\Theta_a \cup \Theta_c \cup \Theta_d)$ has zero self-intersection.*

Proof. Let $V_1, \dots, V_r$ be the components of $\Theta \cdot \Theta_a$. Assume $V_i^2 > 0$ for every i.

The values $V_i^2 = 2$ and $V_i^2 = 4$ are excluded by (3), page 474, and Lemma V.2 of [Ra], respectively.

If $V_iV_j = 0$, Lemma II.16 of [Ra] implies that, for $V = V_i$ or V_j, we have

$$\begin{cases} \exists E \subset T_0A, \quad \forall v \in V_{\mathrm{sm}}, \quad \tau_v^*(T_vV) \subset E, \quad \dim E = 3, \\ \text{or } \exists \ell \subset T_0A, \quad \forall v \in V_{\mathrm{sm}}, \quad \tau_v^*(T_vV) \supset \ell, \quad \dim \ell = 1. \end{cases}$$

This gives

$$\forall v, w \in V_{\mathrm{sm}}, \qquad \tau_v^*(T_vV) \cap \tau_w^*(T_wV) \neq 0,$$

and hence $V^2 = 0$ by Lemma II.14 of [Ra].

The value $V_iV_j = 1$ is also excluded by Lemma II.15 of [Ra]. Thus

$$V_i^2 \geq 6, \qquad V_iV_j \geq 2 \quad (i \neq j), \qquad \left(\sum_{i=1}^r V_i\right)^2 = 24.$$

In particular, $r = 2$. If $a - V_1 = V_2$, then $[V_1] = \frac{1}{2}[\Theta]^2$. Corollary II.2 of [Ra] then shows that V_1 is nondegenerate and $V_1^2 = 6$, contradicting Theorem 5 of [Ra].

Thus $a - V_1 = V_1$, $a - V_2 = V_2$, and

$$V_1^2 + V_2^2 = 24 - 2V_1V_2 \leq 20;$$

we may therefore assume, for example, that $V_1^2 \leq 10$.

This component V_1 is contained, for example, in Θ_c. If we also assume that no component of $\Theta \cdot \Theta_c$ has zero self-intersection, we similarly obtain $c - V_1 = V_1$. Hence V_1 is invariant under translation by $a - c$. Since $V_1^2 > 0$, $a - c$ has finite order $n \geq 2$. Let $p : A \to X$ be the quotient of A by the subgroup generated by $a - c$, and let W_1 be the image of V_1. Then $0 < V_1^2 = nW_1^2 \leq 10$. Recall that $W_1^2 = 2$ is excluded by [Ra]. The only remaining possibility is $n = 2$, $W_1^2 = 4$. By Lemma V.2 of [Ra], there are smooth connected curves C and D of genus 2 with

$$X \simeq JC \times JD, \qquad W_1 \simeq C \times D.$$

In the setting of that lemma, the abelian variety X is principally polarized, but this is not used in the proof.

The dual of p is an isogeny $\pi : JC \times JD \to A$ of degree 2, inducing polarizations M and N on JC and JD, respectively. Thus $2 = \deg \pi = \deg M \deg N$. One of M and N is therefore principal, and A is a product (see, for example, A1.1.1). □

We may therefore assume that $\Theta \cdot \Theta_a$, for example, has an irreducible component V with $V^2 = 0$.

By Lemma II.16 of [Ra], one of the following situations occurs:

- There is $v \in T_0 A$, $v \neq 0$, such that $v \in \tau_x^*(T_x V)$ for every $x \in V_{\mathrm{sm}}$.

  By Lemma II.13 of [Ra], we are in one of the following cases:

  1. There are an abelian subvariety X' of A and $V' \subset A' = A/X'$ such that, for $p : A \to A'$,
  $$V = p^{-1}(V'), \qquad \dim X' = 1, \qquad \dim V' = 1.$$
  2. The same holds with $\dim X' = 2$, $V' = \{x'\}$, so that $V = p^{-1}(x')$ is a translate of an abelian subvariety X' of A.

- There is $h \in T_0^* A$, $h \neq 0$, such that $\tau_x^*(T_x V) \subset \ker h$ for every $x \in V_{\mathrm{sm}}$.

  The abelian variety X' generated by V then has dimension ≤ 3. We are therefore in case 2), or in the case

  3. $\dim X' = 3$.

In every case there is an isogeny $\pi : X' \times X'' \to A$ (see A1.1.1) with

$$\begin{array}{lllll} \text{case 1)} & \dim X' = 1, & V = \pi(X' \times C''), & C'' \text{ a curve in } X'', \\ \text{case 2)} & \dim X' = 2, & V = \pi(X' \times \{x_0''\}), & x_0'' \in X'', \\ \text{case 3)} & \dim X' = 3, & V = \pi(S' \times \{x_0''\}), & S' \text{ a surface in } X'. \end{array}$$

Case 1). Let M' and M'' be the polarizations on X' and X'' induced by that of A. They have the same type, and there are bases $\{s_1', \dots, s_d'\}$ and $\{s_1'', \dots, s_d''\}$ of $H^0(X', M')$ and $H^0(X'', M'')$, respectively, such that $\pi^*\Theta$ has equation $\sum_{i=1}^d s_i' \otimes s_i''$ (A1.1.1.iv). Thus

$$\forall x'' \in C'', \qquad \sum_{i=1}^d s_i''(x'') s_i' = 0 \quad \text{in } H^0(X', M'),$$

so $C'' \subset \mathrm{Bs}\,|M''|$.

Lemma II.F.4.4. *Let (X, M) be a polarized abelian threefold which is not a product of two polarized abelian varieties, one of them principally polarized. Suppose that the base locus of $|M|$ contains an irreducible curve C. Then either $\deg M \leq 2$ or C is an elliptic curve.*

Proof. Let $p : (X, M) \to (A, \Theta)$ be an isogeny onto a principally polarized abelian variety. If A is irreducible, it is the Jacobian of a smooth curve D of genus 3. We have

$$p(C) \subset \bigcap_{g \in G} \Theta_g,$$

where $G = p(H(M))$ (notation I.3.3) has order $d = \deg M$.

If $d > 2$, it follows that $\Theta \cdot \Theta_g$ is reducible for every $g \in G \setminus \{0\}$. We now use II.B.1.

Let g and g' be distinct nonzero elements of G. Suppose $g = O_D(x-y)$ and $g' = O_D(x'-y')$. Then

$$\Theta \cap \Theta_g = D_x \cup (-D)_{-y}, \qquad \Theta \cap \Theta_{g'} = D_{x'} \cup (-D)_{-y'}.$$

Since $p(C)$ has dimension 1, one of the following cases occurs:

- $p(C) = D_x = D_{x'}$ and $x = x'$. Then $g - g' = O_D(y' - y)$ and $D_x \subset D_{y'} \cup (-D)_{-y}$. Since $x \neq y'$ (because $g' \neq 0$), D is hyperelliptic with involution τ, and $y = \tau x$. But then $\Theta \supset p(C) - 2g = D - x + 2\tau x$, which is impossible since $g \neq 0$.
- $p(C) = (-D)_{-y} = (-D)_{-y'}$ and $y = y'$. The argument is the same.
- $p(C) = D_x = (-D)_{-y'}$, so D is hyperelliptic and $y' = \tau x$. We have $g' - g = O_D(x' - \tau y)$ and $\Theta \cdot \Theta_{g'-g} = D_{x'} \cup D_y \supset p(C)$, hence $x = x'$. This reduces to the first case.

The curve D is therefore superelliptic. There are $E \in \mathcal{A}_{1,(2)}$, $P \in \mathcal{A}_{2,(2)}$, an isogeny $\pi : E \times P \to A$, and a nonzero element g of $E \cap G$.

By II.C.1.4 we then have

$$\Theta \cdot \Theta_g = \pi(E \times \{p\}) \cup \pi(\{e\} \times \Delta), \qquad e \in E, \quad p \in P.$$

The curve $p(C)$ is therefore:

- either $\pi(E \times \{p\})$, an elliptic curve;
- or $\pi(\{e\} \times \Delta)$. Then II.C.2.5 gives

  $$\forall g' \in A, \qquad \pi(\{e\} \times \Delta) \subset \Theta_{g'} \implies g' = 0 \text{ or } g,$$

  so $\deg M = \#G \leq 2$;
- or a component of $\pi(\{e\} \times \Delta)$. Thus Δ is reducible. By A2.1.1 there is an isogeny $E_1 \times E_2 \to P$, where E_1 and E_2 are elliptic curves, and the components of Δ are isomorphic to E_1 or E_2.

Now suppose $(A, \Theta) \simeq (A_1, \Theta_1) \times (A_2, \Theta_2)$, with $\dim A_1 = 1$, $\dim A_2 = 2$. The hypothesis on (X, M) implies that neither $G \subset A_1 \times \{0\}$ nor $G \subset \{0\} \times A_2$. If $g = (g_1, g_2)$ with $g_1 \neq 0$ and $g_2 \neq 0$, then

$$\Theta \cap \Theta_g = (\Theta_1 \times \Theta_{2,g_2}) \cup (\Theta_{1,g_1} \times \Theta_2) \cup (A_1 \times (\Theta_2 \cap \Theta_{2,g_2})).$$

If Θ_2 is reducible, all components are elliptic curves. If C is not elliptic, we therefore have $p(C) = \Theta_{1,g_1} \times \Theta_2$ or $p(C) = \Theta_1 \times \Theta_{2,g_2}$. For $h = (h_1, h_2) \in G$, this gives

$$\Theta_{1,g_1} \times \Theta_2 \subset (\Theta_{1,h_1} \times A_2) \cup (A_1 \times \Theta_{2,h_2}),$$

or

$$\Theta_1 \times \Theta_{2,g_2} \subset (\Theta_{1,h_1} \times A_2) \cup (A_1 \times \Theta_{2,h_2}),$$

so that

$$\begin{aligned} &\text{either } \forall h \in G, \quad g_1 = h_1 \text{ or } h_2 = 0, \\ &\text{or } \forall h \in G, \quad h_1 = 0 \text{ or } g_2 = h_2. \end{aligned}$$

In the first case, if $g_1 = h_1$, then $g_1 \neq g_1 + h_1$, so $g_2 + h_2 = 0$ and $h = (g_1, -g_2)$. If $h_2 = 0$, then $g_2 + h_2 \neq 0$, hence $g_1 = g_1 + h_1$ and $h = 0$. Thus $G = \{0, (g_1, -g_2)\}$ and $d = 2$. The second case is treated in the same way. □

Return to case 1). Since (A,Θ) is assumed irreducible, (X'',M'') satisfies the lemma's hypotheses. We may assume C'' is not elliptic; otherwise we reduce to case 2). The polarization M'' then has degree 2, and the isogeny $\pi : X' \times X'' \to A$ makes A a member of $\mathcal{A}^2_{1,3}$. Write F'' for the base locus of $|M''|$. We have

$$V \subset \Theta \implies \forall x'' \in C'', \quad s_1''(x'')s_1' + s_2''(x'')s_2' = 0 \implies C'' \subset F''.$$

Similarly, writing $a = \pi(a', a'')$, the inclusion $V \subset \Theta_a$ gives $C''_{-a''} \subset F''$.

First we show that $a \in X'$. Suppose $F'' = F''_{a''}$. Since X'' has dimension at least 3, we need only note that

$$\forall x'' \notin H(M''), \qquad H^0(X'', M''_{x''}) \hookrightarrow H^0(F'', M''_{x''}|_{F''}),$$

so $a'' \in H(M'')$ and $a \in X'$.

Thus suppose $F'' \neq F''_{a''}$; then F'' is reducible. By A2.2.4 we can choose a degree-2 isogeny from (X'',M'') onto an irreducible Jacobian (J,Θ). The image of F'' is $\Theta \cdot \Theta_\varepsilon$, where ε is the unique nonzero element of the image of $H(M'')$ in J. By II.B.1, one of the following cases occurs:

- $\Theta \cdot \Theta_\varepsilon$ is irreducible and F'' has two components differing by translation by an element of $H(M'')$. Since $F'' \supset C'' \cup C''_{a''}$, it is easy to see that $F'' \neq F''_{a''}$ is impossible.
- J is the Jacobian of a hyperelliptic curve. Again F'' has two components differing by translation by an element of $H(M'')$, giving a contradiction.
- J is the Jacobian of a superelliptic curve. There is then an isogeny $\pi'' : E \times P \to X''$, with $E \in \mathcal{A}_{1,(2)}$, $P \in \mathcal{A}_{2,(2)}$, so that $(A,\Theta) \in \mathcal{A}^2_{1,1,2}$. We have

  $$F'' = \pi''(E \times \{p\}) \cup \pi''(\{e\} \times \Delta), \qquad e \in E, \quad p \in P, \quad \Delta \in |M_P|.$$

  If Δ is reducible, its components are elliptic curves. Since C'' is not elliptic, Δ is irreducible and $C'' = C''_{-a''} = \pi''(\{e\} \times \Delta)$. Hence $a'' \in \pi''(H(M_P)) \subset H(M'')$ and $a \in X'$.

We have thus shown in every case that $a \in X'$, so II.C.1.4 gives

$$\Theta \cdot \Theta_a = \pi(X' \times F'') \cup \pi(\{x'\} \times \Delta''), \qquad x' \in X', \quad \Delta'' \in |M''|.$$

By II.C.2.5, the divisor Δ'' must be reducible. By A2.1.1, there is then a degree-2 isogeny $X''_1 \times X''_2 \to X''$ making A a member of $\mathcal{A}^2_{1,1,2}$. As in A2.2.1, we may view A as a member of $\mathcal{A}^2_{2,2}$. This proves the proposition in case 1).

Case 2). As in case 1), $x''_0 \in F''$.

Lemma II.F.4.5. *Let (X,M) be a polarized abelian surface which is not a product of two polarized abelian varieties, one of them principally polarized. If the base locus of $|M|$ is nonempty, then* $\deg M \leq 2$.

Proof. Let $p : (X,M) \to (A,\Theta)$ be an isogeny onto a principally polarized abelian variety, and let $G = p(H(M))$, of order $d = \deg M$. We have

$$p(\operatorname{Bs}|M|) = \bigcap_{g \in G} \Theta_g.$$

For an element a of this intersection,

$$\forall g \in G \setminus \{0\}, \qquad a + G \subset \Theta \cdot \Theta_g.$$

If $d > 2$, since $\Theta^2 = 2$, the intersection $\Theta \cdot \Theta_g$ has dimension 1. Thus Θ is reducible, $A \simeq E_1 \times E_2$, with $G \subset E_1 \times \{0\}$. This contradicts the hypothesis on (X, M). □

The isogeny $\pi : X' \times X'' \to A$ therefore makes (A, Θ) a member of $\mathcal{A}^2_{2,2}$. As in case 1), write $a = \pi(a', a'')$. Similarly,

$$\pi(X' \times \{x''_0\}) \subset \Theta_a \implies x''_0 - a'' \in F''.$$

Since $(M'')^2 = 4$ and $\dim F'' = 0$, we have $F'' = x''_0 + H(M'')$, hence $a'' \in H(M'')$ and $a \in X'$. This completes case 2).

Case 3). Use the same notation as in the two preceding cases. Then

$$\sum_{i=1}^{d} s''_i(x''_0) s'_i = 0 \quad \text{on } S'.$$

Let D' be the divisor in $|M'|$ defined by this equation. If $S' = D'$, then $D' \subset \Theta_a$. Writing $a = \pi(a', a'')$ gives

$$\sum_{i=1}^{d} s''_i(x''_0 - a'') s'_{i,-a'} = 0 \quad \text{on } D'.$$

As in II.C.2.5, one shows that $a' \in H(M')$. We may therefore assume $a' = 0$. Then $s''_i(x''_0) = s''_i(x''_0 - a'')$ for every i. Since $a'' \neq 0$, the linear system $|M''|$ is not very ample, and $d = \deg M'' \leq 2$.

Thus (A, Θ) belongs to $\mathcal{A}^2_{1,3}$. We finish by an argument already used: $a \in X''$, and by II.C.2.5 a trisecant can exist only if D' is reducible. There is then a degree-2 isogeny $\pi' : X'_1 \times X'_2 \to X'$, with $\dim X'_1 = 1$. This proves the proposition in this case.

If $S' \neq D'$, then D' is reducible and there is an isogeny $\pi' : X'_1 \times X'_2 \to X'$ as above, with

$$D' = \pi'(\{x'_1\} \times X'_2) \cup \pi'(X'_1 \times \Delta'_2), \qquad \Delta'_2 \in |M'_2|.$$

This reduces to case 2) or case 1). □

II.F.4.6. Proof of Theorem II.F.4.1

Proof. We now start with an irreducible principally polarized abelian variety (A, Θ) of dimension $g \geq 4$ satisfying the hypotheses of Theorem II.F.4.1. The preliminary argument and our hypotheses show that (A, Θ) belongs in particular to $\bigcup_{2 \leq t \leq g/2} \mathcal{A}^2_{t,g-t}$. By A2.2.2, there are abelian varieties $(X_1, M_1), \ldots, (X_r, M_r)$ polarized with degree 2, of dimensions $g_1, \ldots, g_r$, with $r \geq 2$ and $g = g_1 + \cdots + g_r$, generators α_i, β_i of $H(M_i)$, and an isogeny

$$\pi : X = X_1 \times \cdots \times X_r \longrightarrow A$$

whose kernel is generated by $(\alpha_1, 0, \ldots, 0, \alpha_i, 0, \ldots, 0)$ for $2 \leq i \leq r$, and $(\beta_1, \ldots, \beta_r)$.

Choose elements $\tilde{\alpha}_i, \tilde{\beta}_i$ of the theta group $\mathcal{G}(M_i)$, lifting α_i, β_i in $H(M_i)$, such that

$$\tilde{\alpha}_i^2 = \tilde{\beta}_i^2 = 1, \qquad \tilde{\alpha}_i \tilde{\beta}_i \tilde{\alpha}_i \tilde{\beta}_i = -1.$$

Let s^i be a nonzero section of M_i invariant under $\tilde{\alpha}_i$, and set $t^i = \tilde{\beta}_i s^i$; then $\{s^i, t^i\}$ is a basis of $H^0(M_i)$.

We may make the following assumptions:

(4.7) $(X_i, M_i) \in \mathcal{P}_{g_i,(2)}$ (see A2.3.6).

(4.8) For each i, either $g_i = 1$, or $\operatorname{div}(s^i)$ and $\operatorname{div}(t^i)$ are irreducible (see A2.3.6).

(4.9) For each i, $g_i \leq g - 2$.

(4.10) If $r = 2$, every member of $|M_1| \cup |M_2|$ is irreducible.

(4.11) If $g = 4$, either some i satisfies $g_i = 2$ and $a \in \pi(X_i)$, or some i, j satisfy $g_i = g_j = 1$ and $a \in \pi(X_i \times X_j)$. This follows from Proposition 4.2.

Remark II.F.4.12. *As explained in A2.2.1, and by A2.3.8, some of the X_i can be grouped together: if, for example, X'_1 is the quotient of $X_1 \times \cdots \times X_{r'}$ by the subgroup generated by $(\alpha_1, 0, \ldots, \alpha_i, 0, \ldots, 0)$ for $2 \leq i \leq r'$, then A is a quotient of $X'_1 \times X_{r'+1} \times \cdots \times X_r$ of the same type.*

Thus we may assume, for example, $r = 2$ to simplify certain calculations. Assumptions (4.8) and (4.10) are then lost.

Assume that there are elements

$$x = (x_1, \ldots, x_r), \quad a = (a_1, \ldots, a_r), \quad b = (b_1, \ldots, b_r),$$
$$c = (c_1, \ldots, c_r), \quad d = (d_1, \ldots, d_r)$$

of X, with $x = a + b = c + d$, satisfying

$$\begin{aligned} &\lambda \left(\prod_i s^i + \prod_i t^i \right) \left(\prod_i s^i_{x_i} + \prod_i t^i_{x_i} \right) \\ &+\mu \left(\prod_i s^i_{a_i} + \prod_i t^i_{a_i} \right) \left(\prod_i s^i_{b_i} + \prod_i t^i_{b_i} \right) \\ &+\nu \left(\prod_i s^i_{c_i} + \prod_i t^i_{c_i} \right) \left(\prod_i s^i_{d_i} + \prod_i t^i_{d_i} \right) = 0, \end{aligned} \tag{4.13}$$

where $\lambda\mu\nu \neq 0$. The three sets $\{0, x\} + \ker \pi$, $\{a, b\} + \ker \pi$, and $\{c, d\} + \ker \pi$ are assumed pairwise disjoint. All products without explicit limits in this section run over $i = 1, \ldots, r$.

As usual, $s^i_{x_i}$ is the image of s^i under

$$H^0(X_i, M_i) \xrightarrow{\tau^*_{-x_i}} H^0(X_i, M_{i,x_i}).$$

We choose once and for all isomorphisms $M_{i,a_i} \otimes M_{i,b_i} \simeq M_{i,c_i} \otimes M_{i,d_i} \simeq M_i \otimes M_{i,x_i}$, omitted from (4.13).

The group $\mathcal{G}(M_1 \boxtimes \cdots \boxtimes M_r)$ is a quotient of $\prod_i \mathcal{G}(M_i)$ [Mu 5, Lemma 1, p. 323]. Applying $(\tilde{\alpha}_1, 1, \ldots, 1)$ to (4.13) gives the two relations

$$\begin{aligned} &\lambda \left(\prod_i s^i s^i_{x_i} + \prod_i t^i t^i_{x_i} \right) + \mu \left(\prod_i s^i_{a_i} s^i_{b_i} + \prod_i t^i_{a_i} t^i_{b_i} \right) \\ &+\nu \left(\prod_i s^i_{c_i} s^i_{d_i} + \prod_i t^i_{c_i} t^i_{d_i} \right) = 0, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\lambda \left(\prod_i s^i t^i_{x_i} + \prod_i s^i_{x_i} t^i \right) + \mu \left(\prod_i s^i_{a_i} t^i_{b_i} + \prod_i s^i_{b_i} t^i_{a_i} \right) \\ &+\nu \left(\prod_i s^i_{c_i} t^i_{d_i} + \prod_i s^i_{d_i} t^i_{c_i} \right) = 0. \end{aligned} \tag{4.15}$$

Next apply $(\tilde{\beta}_1, 1, \dots, 1)$ to (4.14). For $\varepsilon = \pm 1$, this gives

$$\begin{aligned}
&\lambda(s^1 s^1_{x_1} + \varepsilon t^1 t^1_{x_1}) \left(\prod_{i=2}^r s^i s^i_{x_i} + \varepsilon \prod_{i=2}^r t^i t^i_{x_i} \right) \\
&+ \mu(s^1_{a_1} s^1_{b_1} + \varepsilon t^1_{a_1} t^1_{b_1}) \left(\prod_{i=2}^r s^i_{a_i} s^i_{b_i} + \varepsilon \prod_{i=2}^r t^i_{a_i} t^i_{b_i} \right) \\
&+ \nu(s^1_{c_1} s^1_{d_1} + \varepsilon t^1_{c_1} t^1_{d_1}) \left(\prod_{i=2}^r s^i_{c_i} s^i_{d_i} + \varepsilon \prod_{i=2}^r t^i_{c_i} t^i_{d_i} \right) = 0.
\end{aligned}$$

Continuing, and treating (4.15) in the same way, we obtain

$$\begin{cases} \lambda \bigotimes_{i=1}^r U_i^{\varepsilon_i} + \mu \bigotimes_{i=1}^r V_i^{\varepsilon_i} + \nu \bigotimes_{i=1}^r W_i^{\varepsilon_i} = 0, \\ \lambda \bigotimes_{i=1}^r \overline{U}_i^{\varepsilon_i} + \mu \bigotimes_{i=1}^r \overline{V}_i^{\varepsilon_i} + \nu \bigotimes_{i=1}^r \overline{W}_i^{\varepsilon_i} = 0, \end{cases} \qquad \varepsilon_i \in \{\pm 1\}, \quad \prod_i \varepsilon_i = 1, \tag{4.16}$$

where, for $\varepsilon \in \{\pm 1\}$ and $1 \le i \le r$,

$$\begin{aligned}
U_i^\varepsilon &= s^i s^i_{x_i} + \varepsilon t^i t^i_{x_i}, & V_i^\varepsilon &= s^i_{a_i} s^i_{b_i} + \varepsilon t^i_{a_i} t^i_{b_i}, \\
W_i^\varepsilon &= s^i_{c_i} s^i_{d_i} + \varepsilon t^i_{c_i} t^i_{d_i}, & \overline{U}_i^\varepsilon &= s^i t^i_{x_i} + \varepsilon s^i_{x_i} t^i, \\
\overline{V}_i^\varepsilon &= s^i_{a_i} t^i_{b_i} + \varepsilon s^i_{b_i} t^i_{a_i}, & \overline{W}_i^\varepsilon &= s^i_{c_i} t^i_{d_i} + \varepsilon s^i_{d_i} t^i_{c_i}
\end{aligned} \tag{4.17}$$

are elements of $H^0(X_i, M_i \otimes M_{i,x_i})$.

Remark II.F.4.18. *We have*

$$\begin{aligned}
&U_i^+ = 0 \iff x_i = \alpha_i + \beta_i, \quad V_i^+ = 0 \iff a_i - b_i = \alpha_i + \beta_i, \quad \dots, \\
&U_i^- = 0 \iff x_i = \beta_i, \quad \dots, \\
&\overline{U}_i^+ = 0 \iff x_i = \alpha_i, \quad \dots, \\
&\overline{U}_i^- = 0 \iff x_i = 0, \quad \dots.
\end{aligned}$$

We prove the first equivalence, omitting i. We have $s s_x = -t t_x$. Since s, t have no common component, $\operatorname{div}(s) = \operatorname{div}(t_x)$, so $x \in H(M)$ and, more precisely, $x \in \{\beta, \alpha + \beta\}$. If $x = \beta$, the element $\tilde{\beta} \in \mathcal{G}(M)$ corresponds to an isomorphism $u \colon M \longrightarrow M_\beta$, inducing $u^* \colon H^0(M_\beta) \longrightarrow H^0(M)$. Since $t = \tilde{\beta} s$, by definition $t = u^*(s_\beta)$. The isomorphism u induces $v \colon M \otimes M \longrightarrow M \otimes M_\beta$, and

$$v^*(s s_\beta + t t_\beta) = s\, u^*(s_\beta) + t\, u^*(t_\beta) = 2st \neq 0.$$

Likewise, one checks that $x = \alpha + \beta$ does annihilate U^+.

(4.19) First case: $a_1, c_1 \in H(M_1)$. Adding elements of $\ker \pi$ to a, c, and to b, d to preserve $x = a + b = c + d$, reduces us to $a_1 = c_1 = 0$.

First suppose $x_1 \notin H(M_1)$. The sections $s^1 s^1_{x_1}$, $t^1 t^1_{x_1}$, $s^1 t^1_{x_1}$, and $s^1_{x_1} t^1$ are then linearly independent. Grouping $X_2, \dots, X_r$ as in (4.12), assume $r = 2$. Relations (4.14)–(4.15) give

$$\begin{aligned}
&\lambda s^2 s^2_{x_2} + \mu s^2_{a_2} s^2_{b_2} + \nu s^2_{c_2} s^2_{d_2} \\
&= \lambda t^2 t^2_{x_2} + \mu t^2_{a_2} t^2_{b_2} + \nu t^2_{c_2} t^2_{d_2} \\
&= \lambda s^2 t^2_{x_2} + \mu s^2_{a_2} t^2_{b_2} + \nu s^2_{c_2} t^2_{d_2} \\
&= \lambda s^2_{x_2} t^2 + \mu s^2_{b_2} t^2_{a_2} + \nu s^2_{d_2} t^2_{c_2} = 0.
\end{aligned}$$

In other words, for all $\sigma, \sigma' \in H^0(M_2)$,

$$\lambda\sigma\sigma'_{x_2} + \mu\sigma_{a_2}\sigma'_{b_2} + \nu\sigma_{c_2}\sigma'_{d_2} = 0.$$

The following lemma gives $a_2 = c_2 = 0$, hence $a = c = 0$, contrary to the hypothesis.

Lemma II.F.4.20. *Let (X, M) be an irreducible abelian variety polarized with degree 2. Suppose there are $x, a, b, c, d \in X$ and nonzero complex numbers λ, μ, ν such that $x = a + b = c + d$ and*

$$\lambda\sigma\sigma'_x + \mu\sigma_a\sigma'_b + \nu\sigma_c\sigma'_d = 0 \qquad \text{for all } \sigma, \sigma' \in H^0(M).$$

Then $a = c = 0$.

Proof. Dimension 1 follows from Lemma 4.22 below. Assume $\dim X \geq 2$. By A2.1.2, the divisor of a generic section $\sigma \in H^0(M)$ is irreducible. Choose a basis $\{s, t\}$ as before.

First case: $c \notin H(M)$. A generic point $y \in \operatorname{div}(\sigma)$ does not lie in $\operatorname{div}(\sigma_c)$. There is $(p : q) \in \mathbb{P}^1$ such that $ps_b(y) + qt_b(y) = 0$. Taking $\sigma' = ps + qt$ also gives $ps_d(y) + qt_d(y) = 0$, so $(s_b t_d - s_d t_b)(y) = 0$. Since y is generic on $\operatorname{div}(\sigma)$ and σ generic in $H^0(M)$, we have $s_b t_d = s_d t_b$, hence $b = d$ by Remark 4.18. Thus

$$\lambda\sigma\sigma'_x + (\mu\sigma_a + \nu\sigma_c)\sigma'_b = 0 \qquad \text{for all } \sigma, \sigma'.$$

Since the base locus of $|M|$ has codimension 2, this implies $\operatorname{div}(\sigma'_b) \subset \operatorname{div}(\sigma'_x)$, hence $b = x$. This contradicts $c \notin H(M)$.

Second case: $a, c \in H(M)$. Two of s, s_a, s_c have the same divisor, say s, s_a. With $\gamma \neq 0$, we have

$$s(\lambda\sigma'_x + \gamma\mu\sigma'_b) + \nu s_c\sigma'_d = 0 \qquad (\sigma' \in H^0(M)).$$

It follows that $\operatorname{div}(s) = \operatorname{div}(s_a) = \operatorname{div}(s_c)$. Applying the same argument to t gives $a = c = 0$. □

We may therefore assume $x_1 = 0$. As above, in particular, without now assuming $r = 2$, we obtain

$$\lambda\prod_{i=2}^{r} s^i s^i_{x_i} + \mu\prod_{i=2}^{r} s^i_{a_i} s^i_{b_i} + \nu\prod_{i=2}^{r} s^i_{c_i} s^i_{d_i} = 0.$$

The triple $(s^i s^i_{x_i}, s^i_{a_i} s^i_{b_i}, s^i_{c_i} s^i_{d_i})$ in $H^0(M_i \otimes M_{i,x_i})$ therefore has rank at most 1 for every index except possibly one. When its rank is at most 1, one of a_i, b_i and one of c_i, d_i belong to $\{0, \alpha_i\}$. The preceding analysis then allows us to assume, for example, $a_i = b_i = c_i = d_i = 0$ for $i \neq r$. Thus

$$\lambda\sigma\sigma_{x_r} + \mu\sigma_{a_r}\sigma_{b_r} + \nu\sigma_{c_r}\sigma_{d_r} = 0 \qquad (\sigma \in H^0(M_r)). \tag{4.21}$$

Since $\dim X_r \leq g - 2$ by (4.9), the following lemma completes the argument.

Lemma II.F.4.22. *Let $(X, M) \in \mathcal{P}_{g,(2)}$ be an irreducible abelian variety polarized with degree 2. Suppose there are $x, a, b, c, d \in X$ and nonzero complex numbers λ, μ, ν such that $x = a + b = c + d$ and*

$$\lambda\sigma\sigma_x + \mu\sigma_a\sigma_b + \nu\sigma_c\sigma_d = 0 \qquad (\sigma \in H^0(M)).$$

If the trisecant conjecture holds for Prym varieties of dimension $g + 1$, then $\{a, b\} = \{c, d\} = \{0, x\}$.

Proof. Let E be an elliptic curve polarized with degree 2. We can construct an isogeny π of degree 4 from $E \times X$ onto a principally polarized abelian variety (A, Θ) of dimension $g+1$, belonging to $\mathcal{P}^2_{1,g}$ (A2.3.8), hence to $\mathcal{P}_{g+1}$. Regard X as a subvariety of A. If θ is an equation of Θ, then

$$\lambda\theta\theta_x + \mu\theta_a\theta_b + \nu\theta_c\theta_d = 0.$$

First assume $\dim X \geq 2$. By A2.2.3, (A, Θ) is irreducible and its Kummer variety has a trisecant if the lemma's conclusion fails. Hence it is the Jacobian of a curve C of genus $g+1 \geq 3$ admitting a degree-two morphism to E (see II.C.1.2).

By II.B.4, there are $x, y, z, t \in C$ such that, after interchanging c, d if necessary,

$$a \equiv x - y, \qquad b \equiv z - t, \qquad c \equiv x - t, \qquad d \equiv z - y.$$

But the images of a, b, c, d in $\mathrm{Pic}^0 E$ vanish since they belong to X. If τ is the involution of $C \longrightarrow E$, then

$$y = \tau x, \qquad t = \tau z, \qquad t = \tau x, \qquad z = \tau y,$$

so $a = b = c = d$, and then easily $a = b = c = d = 0$.

It remains to consider $\dim X = 1$. Choose $e, y \in X$ with $2[e] \in |M|$ and $2y = 2e + a$. If $\sigma \in H^0(M)$ satisfies $\sigma(y) = 0$, then $\sigma_a(y) = 0$, so either $\sigma_c(y) = 0$ or $\sigma_d(y) = 0$. Suppose, for example, $\sigma_c(y) = 0$. Then $y - c \in \mathrm{div}(\sigma) = [y] + [2e - y]$.

If $y - c = y$, then $c = 0$ and

$$(\lambda + \nu)\sigma\sigma_x + \mu\sigma_a\sigma_b = 0 \qquad (\sigma \in H^0(M)).$$

For every $z \in X$, this gives the equality of divisors

$$\begin{aligned}[z] + [2e - z] + [z + x] + [2e - z + x] = [z + a] + [2e - z + a] \\ + [z + b] + [2e - z + b].\end{aligned}$$

If $a, b \neq 0$, then for every z, either $z = 2e - z + a$ or $z = 2e - z + b$, which is impossible. Thus a or b vanishes, proving the lemma in this case.

If $y - c = 2e - y$, then $c = a$ and

$$\lambda\sigma\sigma_x + (\mu + \nu)\sigma_a\sigma_b = 0 \qquad (\sigma \in H^0(M)).$$

We finish as above. □

(4.23) Second case. For each i, assume that one cannot choose both an element of $\{a_i, b_i\} \cap H(M_i)$ and an element of $\{c_i, d_i\} \cap H(M_i)$; equivalently,

$$\{a_i, b_i\} \cap H(M_i) = \emptyset \quad \text{or} \quad \{c_i, d_i\} \cap H(M_i) = \emptyset.$$

We retain the previous notation, particularly (4.17), and assumptions (4.7)–(4.11).

Lemma II.F.4.24. *Let $i \in \{1, \dots, r\}$ satisfy $\dim X_i > 1$. At most one of the four triples*

$$(U_i^+, V_i^+, W_i^+), \quad (U_i^-, V_i^-, W_i^-), \quad (\overline{U}_i^+, \overline{V}_i^+, \overline{W}_i^+), \quad (\overline{U}_i^-, \overline{V}_i^-, \overline{W}_i^-)$$

in $H^0(M_i \otimes M_{i,x_i})$ has rank at most 1.

Proof. Take $i = 1$ and omit this index except in X_1 or when necessary.

First case: the triples (U^+, V^+, W^+) and (U^-, V^-, W^-) have rank at most 1.

1) $U^+ \neq 0$ and $U^- \neq 0$. We can write

$$s_a s_b = \gamma s s_x + \gamma' t t_x, \qquad s_c s_d = \delta s s_x + \delta' t t_x. \tag{4.25}$$

a) Assume $a \in H(M)$. We may assume $a = 0$. Grouping $X_2, \dots, X_r$ as in (4.12), relations (4.16) give

$$\overline{U}_1^{\varepsilon} \otimes (\lambda \overline{U}_2^{\varepsilon} + \mu \overline{V}_2^{\varepsilon}) + \nu \overline{W}_1^{\varepsilon} \otimes \overline{W}_2^{\varepsilon} = 0 \qquad (\varepsilon = \pm 1).$$

Since $2 s_{c_2}^2 t_{d_2}^2 = \overline{W}_2^{+} + \overline{W}_2^{-}$, these last two sections cannot both vanish. Assume, for example, $\overline{W}_2^{+} \neq 0$. Then

$$s_c t_d + s_d t_c = \rho(s t_x + s_x t).$$

Together with (4.25), for $p, q \in \mathbb{C}$ this gives

$$\begin{aligned}(p s_c + q t_c)(p s_d + q t_d) = s[(p^2\delta + q^2\delta')s_x + pq\rho t_x] \\ + t[pq\rho s_x + (p^2\delta' + q^2\delta)t_x].\end{aligned}$$

If $(p^2\delta + q^2\delta')(p^2\delta' + q^2\delta) - p^2q^2\rho^2 = 0$, then $\sigma_c \sigma_d = \sigma' \sigma''_x$, with $\sigma = ps + qt \in H^0(M)$ and $\sigma', \sigma'' \in H^0(M)$.

By (4.23), neither c nor d belongs to $H(M)$, since $a = 0$. Thus the divisor of σ is reducible. By A2.2.2, there is an isogeny $\pi' : X'_1 \times \cdots \times X'_n \longrightarrow X_1$, where the X'_j are polarized with degree 2, whose kernel is generated by $(\alpha'_1, 0, \dots, \alpha'_j, 0, \dots)$ for $2 \leq j \leq n$. If, for example, $\operatorname{div}(s + t)$ is reducible, then

$$\pi'^*(s + t) = s'^1 \cdots s'^n,$$

where $\operatorname{div}(s'^j)$ is irreducible when $\dim X'_j > 1$, and

$$\beta = \pi'(\alpha'_1, 0, \dots, 0), \qquad \alpha = \pi'(\beta'_1, \dots, \beta'_n).$$

If $c = \pi'(c'_1, \dots, c'_n)$ and $d = \pi'(d'_1, \dots, d'_n)$, then for every j, one of c'_j, d'_j belongs to $H(M'_j)$.

As in Remark 4.12, group $X_2, \dots, X_r$ to assume $r = 2$, then regard X as a quotient of $X'_1 \times \cdots \times X'_n \times X_2$. It therefore belongs to $\mathcal{A}^2_{g'_1, \dots, g'_n, g_2}$. We then have, for example, $a'_1 = c'_1 = 0$, reducing us to (4.19).

b) Assume $a, b, c, d \notin H(M)$ and $a - c \in H(M)$. We may assume $a = c$, $b = d$. Again,

$$\lambda \overline{U}_1^{\varepsilon} \otimes \overline{U}_2^{\varepsilon} + \overline{V}_1^{\varepsilon} \otimes (\mu \overline{V}_2^{\varepsilon} + \nu \overline{W}_2^{\varepsilon}) = 0 \qquad (\varepsilon = \pm 1),$$

which reduces us to (4.19) as above.

c) Assume $a, b, c, d, a - c, a - d \notin H(M)$. The principally polarized abelian variety $(X_1^{\alpha}, \Theta) = (X_1, M_1)/\{0, \alpha_1\}$, irreducible by (4.8), therefore admits a quadrisecant. Using the same letters for the images of elements of X_1 in X_1^{α}, (4.25) gives

$$\begin{aligned}\Theta \cdot \Theta_\beta \subset \Theta_a \cup \Theta_b, \quad \{a, b\} \cap \{0, \beta\} = \emptyset, \\ \Theta \cdot \Theta_\beta \subset \Theta_c \cup \Theta_d, \quad \{c, d\} \cap \{0, \beta\} = \emptyset.\end{aligned}$$

Since the trisecant conjecture is assumed for Prym varieties of dimension $\dim X_1$, X_1^{α} is a Jacobian JC. By II.B.4, we may write

$$\begin{aligned}P, Q, R, S, Q_0, Q_1 \in C, \qquad \beta \equiv Q_0 - Q_1, \\ a \equiv Q_0 - P, \quad b \equiv Q - Q_1, \quad c \equiv Q_0 - R, \quad d \equiv S - Q_1.\end{aligned}$$

Since β has order 2, X_1^α is a hyperelliptic Jacobian and Q_0, Q_1 are Weierstrass points. The equality $a+b=c+d$ gives $Q+R \equiv S+P$. There are three possibilities:

1. $P=Q$, $R=S$, so $x=\beta$. Then $s_a s_b$ is proportional to ss_x, hence $a \in H(M)$ or $b \in H(M)$, contrary to the assumption in case c).
2. $P=R$, $Q=S$, so $a=c$, $b=d$.
3. $R+Q \equiv S+P \equiv g^1_2$, so $c-b=d-a=\beta$.

The last two cases are excluded by $a-c, a-d \notin H(M)$. This completes case 1).

2) *$U^-=0$ and $U^+\neq 0$.* By Remark 4.18, $x=\beta \in H(M)$.

a) Assume $a \in H(M)$. We may assume $a=0$, giving $V^+=U^+=2st$. Thus

$$W^+ = s_c s_d + t_c t_d = \gamma st.$$

As in 1)a), we also have

$$s_c t_d + s_d t_c = \rho(st_x + s_x t) = \rho(s^2+t^2).$$

Hence

$$(s_c+t_c)(s_d+t_d) = \sigma'\sigma'', \qquad \sigma', \sigma'' \in H^0(M).$$

We finish as in 1)a), reducing to (4.19).

b) Assume $a,b,c,d \notin H(M)$ and $a-c \in H(M)$. Proceed as in 1)b).

c) Assume $a,b,c,d,a-c,a-d \notin H(M)$. The assumption $\operatorname{rank}(U^+,V^+,W^+)=1$ gives

$$\begin{cases} s_a s_b + t_a t_b = \gamma s s_x, \\ s_c s_d + t_c t_d = \delta s s_x, \end{cases} \qquad \begin{cases} \gamma s_{-a} s_b = s s_{b-a} + t t_{b-a}, \\ \delta s_{-c} s_d = s s_{d-c} + t t_{d-c}. \end{cases}$$

There are the following possibilities:

1. $\gamma\delta \neq 0$. As in 1)c), the trisecant conjecture for X_1^α yields a contradiction.
2. $\gamma = 0$. The assumption $\operatorname{rank}(V^-, W^-) \leq 1$ gives

$$s_c s_d - t_c t_d = \rho s_a s_b, \qquad \rho s_{a-c} s_{b-c} = s s_{d-c} - t t_{d-c}.$$

   If $\rho\delta \neq 0$, we obtain the same contradiction.
3. $\gamma = \delta = 0$. Then $s_c s_d = -t_c t_d$ and $2 s_c s_d = \rho s_a s_b$, contradicting c) by Remark 4.18.
4. $\gamma = \rho = 0$. Then $s_c s_d = t_c t_d$ and $2 s_c s_d = \delta s s_x$, again contradicting c).

Second case: $\operatorname{rank}(\overline{U}^+, \overline{V}^+, \overline{W}^+) \leq 1$ *and* $\operatorname{rank}(\overline{U}^-, \overline{V}^-, \overline{W}^-) \leq 1$. Proceed as in the first case, or reduce to it by replacing b, d by $b+\beta, d+\beta$.

Third case: $\operatorname{rank}(U^+, V^+, W^+) \leq 1$ *and* $\operatorname{rank}(\overline{U}^+, \overline{V}^+, \overline{W}^+) \leq 1$.

1) *$U^+ \neq 0$ and $\overline{U}^+ \neq 0$.* We can write

$$\begin{cases} u_a u_b = \gamma u u_x + \gamma' v v_x, \\ u_c u_d = \delta u u_x + \delta' v v_x, \end{cases} \qquad u = s+t, \quad v = s-t. \tag{4.26}$$

a) Assume $a \in H(M)$. As before, assume $a=0$. Grouping $X_2, \dots, X_r$ gives

$$\begin{cases} U_1^- \otimes (\lambda U_2^- + \mu V_2^-) + \nu W_1^- \otimes W_2^- = 0, \\ \overline{U}_1^- \otimes (\lambda \overline{U}_2^- + \mu \overline{V}_2^-) + \nu \overline{W}_1^- \otimes \overline{W}_2^- = 0. \end{cases}$$

Since $W_2^-, \overline{W}_2^-$ cannot both vanish, we have

$$\operatorname{rank}(U_1^-, W_1^-) \le 1 \quad \text{or} \quad \operatorname{rank}(\overline{U}_1^-, \overline{W}_1^-) \le 1.$$

Since $U_1^- = V_1^-$ and $\overline{U}_1^- = \overline{V}_1^-$, this reduces to one of the first two cases.

b) Assume $a, b, c, d \notin H(M)$ and $a - c \in H(M)$. Again, either $\operatorname{rank}(U_1^-, W_1^-) \le 1$ or $\operatorname{rank}(\overline{U}_1^-, \overline{W}_1^-) \le 1$, with $V_1^- = W_1^-$ and $\overline{V}_1^- = \overline{W}_1^-$. We return to one of the first two cases.

c) Assume $a, b, c, d, a - c, a - d \notin H(M)$ and $X_1^\beta = X_1/\{0, \beta_1\}$ irreducible. Proceed as in the first case, 1)c).

d) Assume $a, b, c, d, a - c, a - d \notin H(M)$ and X_1^β reducible. The divisor of $s + t$ is reducible. As in the first case, 1)a), there is an isogeny $\pi' : X_1' \times \cdots \times X_n' \longrightarrow X_1$ with

1. $\pi'^*(s + t) = s'^1 \cdots s'^n$, where $\operatorname{div}(s'^j)$ is irreducible for $\dim X_j' > 1$;
2. $a_1 = \pi'(a_1', \dots, a_n')$, and similarly for b_1, c_1, d_1, x_1.

The first equality (4.26) gives

$$\prod_j s'^j_{a_j'} s'^j_{b_j'} - \gamma \prod_j s'^j s'^j_{x_j'} - \gamma' \prod_j t'^j t'^j_{x_j'} = 0.$$

For every $j \in \{1, \dots, n\}$ except at most one, the sections $s'^j_{a_j'} s'^j_{b_j'}$, $s'^j s'^j_{x_j'}$, $t'^j t'^j_{x_j'}$ of $M_j' \otimes M'_{j,x_j'}$ are therefore proportional. Thus $a_j', b_j' \in H(M_j')$, reducing to (4.19).

2) $U^+ = 0$ and $\overline{U}^+ \neq 0$. By Remark 4.18, $x = \alpha + \beta$. The arguments of 1)a), 1)b), 1)d), which do not use the nonvanishing assumptions of 1), allow us to assume $a, b, c, d, a - c, a - d \notin H(M)$ and X_1^β irreducible. Proceed as in the first case, 2)c). □

(4.27) If $\dim X_i = 1$, all four triples have rank at most 1. Indeed, $\tilde{\alpha}_i, \tilde{\beta}_i$ generate a maximal level subgroup of $\mathcal{G}(M_i \otimes M_{i,x_i})$ [Mu 5, definition p. 291]. The sections

$$s^i s^i_{x_i} + t^i t^i_{x_i}, \quad s^i_{a_i} s^i_{b_i} + t^i_{a_i} t^i_{b_i}, \quad s^i_{c_i} s^i_{d_i} + t^i_{c_i} t^i_{d_i}$$

are invariant under this subgroup, hence proportional. For the other triples, use respectively the subgroups generated by $\{\tilde{\alpha}_i, -\tilde{\beta}_i\}$, $\{-\tilde{\alpha}_i, \tilde{\beta}_i\}$, and $\{-\tilde{\alpha}_i, -\tilde{\beta}_i\}$.

Lemma II.F.4.28. *At most one of the X_i has dimension greater than 1. In particular, $r \ge 3$.*

Proof. Suppose $\dim X_1 > 1, \dim X_2 > 1$. By the preceding lemma, at least two of the following four properties hold:

$$\begin{aligned}
&\operatorname{rank}(U_i^+, V_i^+, W_i^+) \ge 2 \quad (i = 1, 2),\\
&\operatorname{rank}(U_i^-, V_i^-, W_i^-) \ge 2 \quad (i = 1, 2),\\
&\operatorname{rank}(\overline{U}_i^+, \overline{V}_i^+, \overline{W}_i^+) \ge 2 \quad (i = 1, 2),\\
&\operatorname{rank}(\overline{U}_i^-, \overline{V}_i^-, \overline{W}_i^-) \ge 2 \quad (i = 1, 2).
\end{aligned}$$

If $r = 2$, equations (4.16) immediately give a contradiction. In view of (4.9), $r = 2$ is therefore excluded altogether.

If $r \ge 3$, group $X_3, \dots, X_r$ as in (4.12). Equations (4.16) force two of the four triples $(U_3^\varepsilon, V_3^\varepsilon, W_3^\varepsilon)$, $(\overline{U}_3^\varepsilon, \overline{V}_3^\varepsilon, \overline{W}_3^\varepsilon)$ to vanish identically, which is impossible. □

Henceforth assume $\dim X_1 = \cdots = \dim X_{r-1} = 1$.

Lemma II.F.4.29. *Assume $r \geq 4$. For each $i \in \{1, \dots, r-1\}$ and $\varepsilon \in \{\pm 1\}$, at most one of $U_i^\varepsilon, V_i^\varepsilon, W_i^\varepsilon$ (respectively $\overline{U}_i^\varepsilon, \overline{V}_i^\varepsilon, \overline{W}_i^\varepsilon$) vanishes.*

Proof. Take $i = 1$. If $U_1^+ = V_1^+ = 0$, Remark 4.18 gives $x_1 = \alpha_1 + \beta_1 = a_1 + b_1$ and $b_1 - a_1 = \alpha_1 + \beta_1$, so $a_1, b_1 \in X_1[2] = H(M_1)$. Similarly, $U_1^+ = W_1^+ = 0$ gives $c_1, d_1 \in H(M_1)$. If $V_1^+ = W_1^+ = 0$, then $b_1 - a_1 = d_1 - c_1 = \alpha_1 + \beta_1$; together with $a_1 + b_1 = c_1 + d_1$, this gives $c_1 - a_1 = d_1 - b_1 \in H(M_1)$ and $b_1 - c_1, a_1 - c_1 \in H(M_1)$. Translating II.A.1.3 by $-c$ gives

$$\lambda s_{-c} s_d + \mu s_{a-c} s_{b-c} + \nu s s_{d-c} = 0.$$

In each case it therefore suffices to exclude $a_1, b_1 \in H(M_1)$. Thus assume $a_1 = b_1 = x_1 = 0$.

Using (4.15) and $s^1_{c_1} t^1_{d_1} + s^1_{d_1} t^1_{c_1} = \gamma s^1 t^1$ (4.27), we obtain

$$\begin{aligned} s^1 t^1 \Big[\lambda \Big(\prod_{i=2}^r s^i t^i_{x_i} + \prod_{i=2}^r s^i_{x_i} t^i \Big) + \mu \Big(\prod_{i=2}^r s^i_{a_i} t^i_{b_i} + \prod_{i=2}^r s^i_{b_i} t^i_{a_i} \Big) \\ + \nu\gamma \prod_{i=2}^r s^i_{d_i} t^i_{c_i} \Big] + \nu s^1_{c_1} t^1_{d_1} \Big[\prod_{i=2}^r s^i_{c_i} t^i_{d_i} - \prod_{i=2}^r s^i_{d_i} t^i_{c_i} \Big] = 0. \end{aligned}$$

By (4.23), neither c_1 nor d_1 belongs to $H(M_1)$, so $s^1 t^1$ and $s^1_{c_1} t^1_{d_1}$ are linearly independent. Both brackets vanish. The equality $\prod_{i=2}^r s^i_{c_i} t^i_{d_i} = \prod_{i=2}^r s^i_{d_i} t^i_{c_i}$ gives $d_i - c_i \in \{0, \alpha_i\}$ for $i \geq 2$. Thus assume $c_i = d_i$ for $i \geq 2$.

As in (4.25), we can also write $s^1_{c_1} s^1_{d_1} = \delta (s^1)^2 + \delta' (t^1)^2$. Equation (4.14) gives

$$\begin{cases} \lambda \prod_{i=2}^r s^i s^i_{x_i} + \mu \prod_{i=2}^r s^i_{a_i} s^i_{b_i} + \nu\delta \prod_{i=2}^r (s^i_{c_i})^2 + \nu\delta' \prod_{i=2}^r (t^i_{c_i})^2 = 0, \\ \lambda \prod_{i=2}^r t^i t^i_{x_i} + \mu \prod_{i=2}^r t^i_{a_i} t^i_{b_i} + \nu\delta' \prod_{i=2}^r (s^i_{c_i})^2 + \nu\delta \prod_{i=2}^r (t^i_{c_i})^2 = 0. \end{cases} \tag{4.30}$$

Choose $y_3 \in X_3$, $y_4 \in X_4$ with $s^3_{c_3}(y_3) = t^4_{c_4}(y_4) = 0$. Substitution in (4.30) yields one of two alternatives:

- a linear dependence between $\prod_{i \in I} s^i s^i_{x_i}$ and $\prod_{i \in I} s^i_{a_i} s^i_{b_i}$, or between $\prod_{i \in I} t^i t^i_{x_i}$ and $\prod_{i \in I} t^i_{a_i} t^i_{b_i}$, where $I = \{2, 5, \dots, r\}$. Then for every $i \in I$, either a_i or b_i belongs to $\{0, \alpha_i\}$;
- the following four products vanish:

$$\begin{aligned} s^3(y_3) s^3_{x_3}(y_3) s^4(y_4) s^4_{x_4}(y_4) &= s^3_{a_3}(y_3) s^3_{b_3}(y_3) s^4_{a_4}(y_4) s^4_{b_4}(y_4) \\ &= t^3(y_3) t^3_{x_3}(y_3) t^4(y_4) t^4_{x_4}(y_4) \\ &= t^3_{a_3}(y_3) t^3_{b_3}(y_3) t^4_{a_4}(y_4) t^4_{b_4}(y_4) = 0. \end{aligned}$$

But in the second alternative, for example,

$$s^3(y_3) = 0 \implies c_3 = d_3 \in H(M_3), \quad x_3 = 0.$$

By (4.23), $a_3 - c_3, b_3 - c_3 \notin H(M_3)$, so

$$s^3_{a_3}(y_3) s^3_{b_3}(y_3) t^3_{a_3}(y_3) t^3_{b_3}(y_3) \neq 0.$$

It follows that

$$s^4_{a_4}(y_4)s^4_{b_4}(y_4) = t^4_{a_4}(y_4)t^4_{b_4}(y_4) = 0, \quad a_4 - c_4, b_4 - c_4 \in H(M_4).$$

Moreover,

$$t^3(y_3)t^3_{x_3}(y_3) \neq 0 \implies t^4(y_4)t^4_{x_4}(y_4) = 0 \implies c_4 \in H(M_4),$$

contradicting (4.23). Thus the first alternative holds. We infer that for every $i \geq 2$, either a_i or b_i belongs to $\{0, \alpha_i\}$. Substitution in (4.30) gives

$$(\lambda \pm \mu)\prod_{i=2}^{r} s^i s^i_{x_i} + \nu\delta\prod_{i=2}^{r}(s^i_{c_i})^2 + \nu\delta'\prod_{i=2}^{r}(t^i_{c_i})^2 = 0.$$

Choose $y_2 \in X_2$ with $s^2(y_2) = 0$. Since $\prod_{i=3}^r (s^i_{c_i})^2$ and $\prod_{i=3}^r (t^i_{c_i})^2$ are linearly independent, $\delta(s^2_{c_2}(y_2))^2 = \delta'(t^2_{c_2}(y_2))^2 = 0$. Hence $\delta = 0$ or $\delta' = 0$, so $c_1, d_1 \in H(M_1)$, contradicting (4.23). Every case therefore yields a contradiction. □

Lemma II.F.4.31. *For $i < r$, none of the four triples $(U^\varepsilon_i, V^\varepsilon_i, W^\varepsilon_i)$, $(\overline{U}^\varepsilon_i, \overline{V}^\varepsilon_i, \overline{W}^\varepsilon_i)$, with $\varepsilon = \pm 1$, vanishes identically.*

Proof. Omit i. If, for example, $U^+ = V^+ = W^+ = 0$, Remark 4.18 gives

$$\begin{aligned} U^+ = ss_x + tt_x = 0 &\Longrightarrow x = \alpha + \beta \in H(M), \\ V^+ = 0 &\Longrightarrow b - a = \alpha + \beta, \\ W^+ = 0 &\Longrightarrow d - c = \alpha + \beta. \end{aligned}$$

Thus $2a = 2b = 2c = 2d = 0$. Since $\dim X = 1, a, b, c, d \in X[2] = H(M)$, contrary to (4.23). □

Lemma II.F.4.32. *For every $i \in \{1, \dots, r\}$, $\dim X_i = 1$.*

Proof. Suppose $\dim X_r > 1$ and, by Lemma 4.24,

$$\operatorname{rank}(U^\varepsilon_r, V^\varepsilon_r, W^\varepsilon_r) \geq 2 \qquad (\varepsilon = \pm 1). \tag{4.33}$$

First show that none of $U^\pm_i, V^\pm_i, W^\pm_i$ for $i < r$ vanishes. Suppose to the contrary that $U^+_1 = 0$, initially with $r = 3$. Relations (4.16) give

$$\begin{aligned} \lambda U^-_1 U^{-\varepsilon}_2 U^\varepsilon_3 + \mu V^-_1 V^{-\varepsilon}_2 V^\varepsilon_3 + \nu W^-_1 W^{-\varepsilon}_2 W^\varepsilon_3 = 0, \\ \mu V^+_1 V^\varepsilon_2 V^\varepsilon_3 + \nu W^+_1 W^\varepsilon_2 W^\varepsilon_3 = 0 \end{aligned} \qquad (\varepsilon = \pm 1).$$

By (4.33), the two relations are proportional for each ε; hence one vanishes identically. Since $V^+_1 = W^+_1 = 0$ is impossible by Lemma 4.31, as is $U^+_3 = U^-_3 = 0$, and so on, the same relation cannot be trivial for both signs. Thus, for example,

$$\begin{cases} U^-_1 U^+_2 = V^-_1 V^+_2 = W^-_1 W^+_2 = 0, \\ V^+_1 V^+_2 = W^+_1 W^+_2 = 0, \end{cases} \qquad \Longrightarrow \quad U^+_2 = V^+_2 = W^+_2 = 0.$$

Lemma 4.31 therefore excludes $r = 3$. For $r > 3$, the same method gives

$$\prod_{i=2}^{r-1} U^{\varepsilon_i}_i = \prod_{i=2}^{r-1} V^{\varepsilon_i}_i = \prod_{i=2}^{r-1} W^{\varepsilon_i}_i = 0 \qquad \text{if } \prod_{i=2}^{r-1} \varepsilon_i = 1.$$

It follows that for every $i \in \{2, \dots, r-1\}$ there are $\varepsilon_i, \varepsilon_i', \varepsilon_i''$ such that $U_i^{\varepsilon_i} = V_i^{\varepsilon_i'} = W_i^{\varepsilon_i''} = 0$, contradicting Lemma 4.29.

Thus assume all $U_i^{\pm}$ nonzero for $i < r$. By (4.27), write

$$V_i^{\varepsilon} = A_i^{\varepsilon} U_i^{\varepsilon}, \qquad W_i^{\varepsilon} = B_i^{\varepsilon} U_i^{\varepsilon}, \quad A_i^{\varepsilon}, B_i^{\varepsilon} \in \mathbb{C}^*.$$

Relations (4.16) give

$$\lambda U_r^+ + \mu\Big(\prod_{i<r} A_i^{\varepsilon_i}\Big) V_r^+ + \nu\Big(\prod_{i<r} B_i^{\varepsilon_i}\Big) W_r^+ = 0 \quad \text{if } \prod_{i<r} \varepsilon_i = 1,$$
$$\lambda U_r^- + \mu\Big(\prod_{i<r} A_i^{\varepsilon_i}\Big) V_r^- + \nu\Big(\prod_{i<r} B_i^{\varepsilon_i}\Big) W_r^- = 0 \quad \text{if } \prod_{i<r} \varepsilon_i = -1.$$

By (4.33), these relations are proportional, so for $i < j < r$,

$$A_i^+ A_j^+ = A_i^- A_j^-, \qquad A_i^+ A_j^- = A_i^- A_j^+,$$
$$B_i^+ B_j^+ = B_i^- B_j^-, \qquad B_i^+ B_j^- = B_i^- B_j^+.$$

Thus some $\varepsilon, \varepsilon' \in \{\pm 1\}$ satisfy $A_i^- = \varepsilon A_i^+$ and $B_i^- = \varepsilon' B_i^+$ for every $i < r$. On each X_i, omitting indices,

$$s_a s_b + t_a t_b = A^+(s s_x + t t_x),$$
$$s_a s_b - t_a t_b = \varepsilon A^+(s s_x - t t_x).$$

Hence $s_a s_b$ is proportional to $s s_x$ or $t t_x$, giving $a \in H(M)$ or $b \in H(M)$. The same argument with c, d contradicts (4.23). □

Lemma II.F.4.34. *At most one of the sections $U_i^{\varepsilon}, V_i^{\varepsilon}, W_i^{\varepsilon}$, for $1 \le i \le r$ and $\varepsilon \in \{\pm 1\}$, vanishes.*

Proof. Since every X_i has dimension 1, $r \ge 4$ and Lemma 4.29 applies. Three situations could contradict the conclusion:

i) $U_1^+ = V_2^+ = 0$. Relations (4.16) give $\prod_{i=3}^r W_i^{\varepsilon_i} = 0$ for $\prod_{i=3}^r \varepsilon_i = 1$. Thus for every $i \ge 3$ there is $\varepsilon_i'' \in \{\pm 1\}$ with $W_i^{\varepsilon_i''} = 0$. Repeat the argument with $U_1^+ = W_3^{\varepsilon_3''} = 0$: for every $i \ge 4$, there is ε_i' with $V_i^{\varepsilon_i'} = 0$. Starting from $V_2^+ = W_3^{\varepsilon_3''} = 0$ likewise gives, for every $i \ge 4$, a sign ε_i with $U_i^{\varepsilon_i} = 0$. Two of $\varepsilon_4, \varepsilon_4', \varepsilon_4''$ agree, contradicting Lemma 4.29.

ii) $U_1^+ = V_1^+ = 0$. Relations (4.16) again give $\prod_{i=2}^r W_i^{\varepsilon_i} = 0$ for $\prod_{i=2}^r \varepsilon_i = 1$. Finish as above.

iii) $U_1^+ = V_1^- = 0$. Then

$$\lambda U_1^- \prod_{i=2}^r U_i^{\varepsilon_i} + \nu W_1^- \prod_{i=2}^r W_i^{\varepsilon_i} = 0 \quad \text{if } \prod_{i=2}^r \varepsilon_i = -1,$$
$$\mu V_1^+ \prod_{i=2}^r V_i^{\varepsilon_i} + \nu W_1^+ \prod_{i=2}^r W_i^{\varepsilon_i} = 0 \quad \text{if } \prod_{i=2}^r \varepsilon_i = 1.$$

For $i \ge 2$, write $W_i^{\varepsilon} = B_i^{\varepsilon} U_i^{\varepsilon} = C_i^{\varepsilon} V_i^{\varepsilon}$, with $B_i^{\varepsilon}, C_i^{\varepsilon} \in \mathbb{C}^*$. Since $r \ge 4$, for $2 \le i < j \le r$ we obtain

$$B_i^+ B_j^+ = B_i^- B_j^-, \qquad C_i^+ C_j^+ = C_i^- C_j^-.$$

Thus some $\varepsilon, \varepsilon' \in \{\pm 1\}$ satisfy $B_i^- = \varepsilon B_i^+$ and $C_i^- = \varepsilon' C_i^+$ for every $i \ge 2$. As at the end of 4.32, this implies

$$[c_i \in H(M_i) \text{ or } d_i \in H(M_i)]$$

and

$$[a_i - c_i = d_i - b_i \in H(M_i)] \quad \text{or} \quad [a_i - d_i = c_i - b_i \in H(M_i)],$$

contrary to (4.23). □

We can finally complete the proof. For $1 \le i \le r$, $\varepsilon \in \{\pm 1\}$, write

$$V_i^\varepsilon = A_i^\varepsilon U_i^\varepsilon, \qquad W_i^\varepsilon = B_i^\varepsilon U_i^\varepsilon.$$

We have the following two properties:

1. The $A_i^\varepsilon, B_i^\varepsilon$ are complex numbers, all nonzero except at most one (Lemma 4.34), and $r \ge 4$.
2. If $r = 4$, (4.11) gives two indices i with $a_i = 0$, hence $A_i^+ = A_i^- = 1$ for these indices.

Relations (4.16) become

$$\lambda + \mu \prod_{i=1}^r A_i^{\varepsilon_i} + \nu \prod_{i=1}^r B_i^{\varepsilon_i} = 0, \qquad \prod_{i=1}^r \varepsilon_i = 1. \tag{4.35}$$

We show that every solution also satisfying 1) and 2) has

$$\exists i \in \{1, \dots, r\}, \qquad A_i^- = \pm A_i^+, \quad B_i^- = \pm B_i^+. \tag{4.36}$$

As above, this contradicts (4.23) and completes the theorem's proof. We readily reduce to the case where

$$A = \prod_{i=1}^r A_i^-, \qquad B = \prod_{i=1}^r B_i^-$$

are nonzero. Set

$$\lambda' = \frac{\lambda}{AB}, \quad \mu' = \frac{\mu}{B}, \quad \nu' = \frac{\nu}{A}, \qquad \mu_i = \frac{A_i^+}{A_i^-}, \quad \nu_i = \frac{B_i^+}{B_i^-}.$$

For $r > 5$, reduce immediately to $r = 5$ by taking $\varepsilon_6 = \cdots = \varepsilon_r = 1$ in (4.35). The system becomes

$$r = 4 : \begin{cases} \lambda' + \mu' + \nu' = 0, \\ \lambda' + \mu'\mu_i\mu_j + \nu'\nu_i\nu_j = 0 & (1 \le i < j \le 4), \\ \lambda' + \mu'\mu_1\mu_2\mu_3\mu_4 + \nu'\nu_1\nu_2\nu_3\nu_4 = 0, \\ \mu_1 = \mu_2 = 1; \end{cases}$$

$$r = 5 : \begin{cases} \lambda' + \mu'\mu_i + \nu'\nu_i = 0 & (1 \le i \le 5), \\ \lambda' + \mu'\mu_i\mu_j\mu_k + \nu'\nu_i\nu_j\nu_k = 0 & (1 \le i < j < k \le 5), \\ \lambda' + \mu'\mu_1\mu_2\mu_3\mu_4\mu_5 + \nu'\nu_1\nu_2\nu_3\nu_4\nu_5 = 0. \end{cases}$$

In both cases, at most one of the μ_i, ν_i vanishes.

For $r = 4$, we have

$$\begin{aligned} \lambda' + \mu' + \nu' = \lambda' + \mu' + \nu'\nu_1\nu_2 = 0 \quad &\Longrightarrow \nu_1\nu_2 = 1, \\ \lambda' + \mu'\mu_3 + \nu'\nu_1\nu_3 = \lambda' + \mu'\mu_3 + \nu'\nu_2\nu_3 = 0 \quad &\Longrightarrow \nu_1\nu_3 = \nu_2\nu_3. \end{aligned}$$

Similarly, $\nu_1\nu_4 = \nu_2\nu_4$. Since ν_3, ν_4 cannot both vanish, $\nu_1 = \nu_2 = \pm 1$, proving (4.36).

Now assume $r = 5$.

1) One of the μ_i, ν_i vanishes. The others are nonzero. If, for example, $\mu_1 = 0$, then $1 = \nu_j\nu_k = \nu_2\nu_3$ for $k > j \geq 2$. Hence $\nu_2 = \nu_3 = \nu_4 = \nu_5 = \varepsilon \in \{\pm 1\}$. We have

$$\mu_i = \mu_i\mu_j\mu_k = \mu_2\mu_3\mu_4 \quad (k > j > i \geq 2),$$

so $\mu_2 = \mu_3 = \mu_4 = \mu_5 = \varepsilon' \in \{\pm 1\}$, proving (4.36).

2) None of the μ_i, ν_i vanishes. If $\mu_1\nu_2 - \nu_1\mu_2 \neq 0$, the system

$$\begin{cases} \lambda' + \mu'\mu_1 x + \nu'\nu_1 y = 0, \\ \lambda' + \mu'\mu_2 x + \nu'\nu_2 y = 0 \end{cases}$$

has the unique solution $x = y = 1$. Hence

$$\mu_i\mu_j = \nu_i\nu_j = 1 \qquad (j > i \geq 3),$$

which implies (4.36). We may therefore assume

$$\frac{\mu_1}{\nu_1} = \cdots = \frac{\mu_5}{\nu_5} = k.$$

We then have

$$\begin{cases} \lambda' + \nu_1(k\mu' + \nu') = 0, \\ \lambda' + \nu_1^3(k^3\mu' + \nu') = 0, \\ \lambda' + \nu_1^5(k^5\mu' + \nu') = 0. \end{cases}$$

Thus

$$(k\mu' + \nu')(k^5\mu' + \nu') = (k^3\mu' + \nu')^2 \implies k(k^2-1)^2 = 0 \implies k = \pm 1, \quad \nu_1 = \pm 1.$$

This proves (4.36) and completes the proof of the theorem. □

II.F.5 The trisecant conjecture for elements of $\mathcal{H}^2_{1,g-1}$, $g \geq 5$

The elements of $\mathcal{H}^2_{1,g-1}$ are defined in A2.4. The geometry of the singularities of their theta divisors is studied in II.E.3. The following notation is that of A2 and differs slightly from that of II.E.3.

Let H be a smooth hyperelliptic curve of genus $g-1$, let Q_0, Q_1 be Weierstrass points, let $\varepsilon \in JH[2]$ be the element associated with $O_H(Q_0 - Q_1)$, and let $h\colon (X_2, M_2) \to (JH, \Theta)$ be the degree-2 isogeny associated with ε. Also take $(X_1, M_1) \in \mathcal{A}_{1,(2)}$. Choose generators α_1, β_1 of $H(M_1)$ and α_2, β_2 of $H(M_2)$ with $h(\alpha_2) = 0$. The quotient (P, Ξ) of $X_1 \times X_2$ by the subgroup generated by (α_1, α_2) and (β_1, β_2) is principally polarized and belongs to $\mathcal{H}^2_{1,g-1}$.

Recall that the principally polarized abelian varieties obtained in this way are irreducible (A2.4.5) and are Prym varieties (II.E.1.1 or A2.4.2).

Theorem II.F.5.1. *Let $(P, \Xi) \in \mathcal{H}^2_{1,g-1}$ have dimension $g \geq 5$. If $g = 5$, assume in addition that the trisecant conjecture is true in dimension 4. Then the Kummer variety of (P, Ξ) has no trisecant.*

Proof. As usual, start from a nontrivial inclusion $\Xi \cdot \Xi_a \subset \Xi_c \cup \Xi_d$. It follows that

$$\dim(\Xi_a \cap \operatorname{Sing} \Xi) \geq g - 4.$$

We proved in II.E.3.2 that $\operatorname{Sing} \Xi$ has two components of dimension $\geq g-4$, which we now describe differently.

In addition to the notation of II.E.3, use that of A2.2.1. In particular, if $\pi\colon X_1 \times X_2 \to P$ is the isogeny, there are respective bases $\{s^1, t^1\}$ and $\{s^2, t^2\}$ of $H^0(X_1, M_1)$ and $H^0(X_2, M_2)$ such that $s^1s^2 + t^1t^2$ is an equation of $\widetilde{\Xi} = \pi^*\Xi$, and s^2 is an equation of $h^*\Theta$.

The two components in question are

$$\begin{aligned} W_2 &= \pi(X_1 \times Z_2), \quad Z_2 = h^{-1}(g^1_2 + Q_0 + W^{g-5}), \\ W_1 &= \pi(\{y_1\} \times S_2), \quad S_2 = h^{-1}(g^1_2 + W^{g-4}), \quad t^1(y_1) = 0. \end{aligned}$$

On Z_2, we have $s^2 = \partial s^2/\partial x_j^2 = t^2 = \partial t^2/\partial x_j^2 = 0$; on S_2, $s^2 = \partial s^2/\partial x_j^2 = t^2 = 0$. It is easy to check that W_1, W_2 do indeed correspond to singularities of Ξ.

Return to our trisecant.

Lemma II.F.5.2. *Under the preceding hypotheses and with the same notation,*

$$\dim(\Xi_a \cap \operatorname{Sing}\Xi) \geq g-4 \iff \begin{array}{c} \exists a_1 \in X_1,\ a_2 \in X_2,\ x, y \in H, \\ a = \pi(a_1, a_2), \qquad h(a_2) = O_H(x-y). \end{array}$$

Proof. Write $a = \pi(a_1, a_2)$. If $X_1 \times Z_2 \subset \widetilde{\Xi}_{(a_1,a_2)}$, then

$$\begin{aligned} &\forall x \in X_1,\ z \in Z_2, \quad s^1(x-a_1)s^2(z-a_2) + t^1(x-a_1)t^2(z-a_2) = 0 \\ &\iff \quad \forall z \in Z_2, \quad s^2(z-a_2) = t^2(z-a_2) = 0 \\ &\iff \quad g^1_2 + Q_0 + W^{g-5} - h(a_2) \subset (Q_0 + W^{g-3}) \cup (Q_1 + W^{g-3}). \end{aligned}$$

Thus either $h(a_2)$ or $h(a_2)+\varepsilon$ belongs to $H-H$. This proves the lemma, after adding $(\beta_1, \beta_2) \in \ker\pi$ to (a_1, a_2) if necessary.

If $\{y_1\} \times S_2 \subset \widetilde{\Xi}_{(a_1,a_2)}$, invariance of S_2 under translation by α_2 gives

$$\{y_1\} \times S_2 \subset \widetilde{\Xi}_{(a_1,a_2)} \cdot \widetilde{\Xi}_{(a_1,a_2+\alpha_2)}.$$

Hence, for every $w \in S_2$,

$$s^1(y_1 - a_1)s^2(w - a_2) = t^1(y_1 - a_1)t^2(w - a_2) = 0.$$

Projection to JH gives

$$g^1_2 + W^{g-4} - h(a_2) \subset W^{g-2} \cup W^{g-2}_\varepsilon,$$

which proves the lemma in this case as above. □

Lemma II.F.5.3. *None of the four elements a, b, c, d of P belongs to $\pi(X_1)$.*

Proof. Suppose, for example, that $a = \pi(a_1, 0)$. By II.C.1.4,

$$\Xi \cdot \Xi_a = \pi(X_1 \times T_2) \cup \pi(\{e\} \times D_2),$$

where

$$\begin{aligned} T_2 &= h^{-1}(Q_0 + W^{g-3}), \\ e &\in \operatorname{div}(s^1 t^1_{a_1} - s^1_{a_1} t^1), \\ D_2 &= \operatorname{div}(s^1(e)s^2 + t^1(e)t^2) \in |M_2|. \end{aligned}$$

If D_2 is reducible, A1 shows that one of the abelian varieties

$$X_2^\alpha = X_2/\{0, \alpha_2\} = JH, \quad X_2^\beta = X_2/\{0, \beta_2\}, \quad X_2^{\alpha+\beta} = X_2/\{0, \alpha_2 + \beta_2\}$$

is reducible, contradicting A2.4.2. Thus, for example, $\{e\} \times D_2 \subset \widetilde{\Xi}_{(c_1,c_2)}$. As in II.C.2.5, this allows us to assume $c_2 = 0$. But then $\Xi \cdot \Xi_a \subset \Xi \cdot \Xi_c$, a contradiction. □

By II.A.1.3, each of $a, b, c, d, a-c, a-d, b-c, b-d \in P$ satisfies the hypotheses of Lemma 5.2.

Lemma II.F.5.4. *We may write $a = \pi(a_1, a_2)$, $b = \pi(b_1, b_2)$, $c = \pi(c_1, c_2)$, $d = \pi(d_1, d_2)$ with*

$$h(a_2) = O_H(x-y), \quad h(b_2) = O_H(z-t),$$
$$h(c_2) = O_H(x-t), \quad h(d_2) = O_H(z-y),$$

where x, y, z, t are pairwise distinct points of H.

Proof. By 5.2, write $a = \pi(a_1, a_2)$, $c = \pi(c_1, c_2)$, $a-c = \pi(m_1, m_2)$, with

$$h(a_2) = O_H(x-y), \quad h(c_2) = O_H(z-t), \quad h(m_2) = O_H(u-v).$$

The points x, y (respectively z, t and u, v) are distinct by 5.3. To prove the lemma, it suffices to show that the following cannot hold simultaneously:

$$x \notin \{z, \tau t, u, \tau v\}, \qquad y \notin \{\tau z, t, \tau u, v\},$$
$$z \notin \{\tau u, v\}, \qquad t \notin \{u, \tau v\}.$$

Argue by contradiction. We are in one of the cases

$$1) \quad x - y - (z-t) \equiv u - v,$$
$$2) \quad x - y - (z-t) \equiv u - v + Q_0 - Q_1.$$

The points Q_0, Q_1 are Weierstrass points of H. In case 1), since $x \notin \{y, z, u\}$, $|x + t + v|$ is a g^1_3. But $x \notin \{\tau t, \tau v\}$ and $t \neq \tau v$, a contradiction.

In case 2), since $x \notin \{y, z, u\}$, $|x + t + Q_1 + v|$ is a g^1_4. It follows that $Q_1 \in \{x, t, v\}$ and $\#(\{x, t, v\} \cap \{u, y, z\}) = 2$, a contradiction. □

To proceed, use results from the proof (4.6) of Theorem 4.1. Hypothesis (4.23) holds here. Indeed, by Lemma 5.3, it suffices to check that $a, c \in \pi(X_2)$ is impossible. If this were the case, then, as in (4.19), $a_1 = b_1 = c_1 = d_1 = 0$, and (4.21) would hold with $r = 2$. The two principally polarized abelian varieties

$$X_2^{\beta} = X_2/\{0, \beta_2\}, \qquad X_2^{\alpha+\beta} = X_2/\{0, \alpha_2 + \beta_2\},$$

which belong to $\mathcal{H}^2_{g-1}$ by construction and are irreducible by A2.4.2, would each admit a trisecant. This contradicts 3.1 if $g-1 \geq 5$ and, by II.E.4.3, the trisecant conjecture in dimension 4 if $g - 1 = 4$.

(5.5) Thus we are in the second case (4.23) of the proof of 4.1.

With the notation of that proof, particularly (4.17), 5.3 implies that none of the sections $U_2^{\pm}, V_2^{\pm}, W_2^{\pm}, \overline{U}_2^{\pm}, \overline{V}_2^{\pm}, \overline{W}_2^{\pm}$ of $H^0(X_2, M_2 \otimes M_{2,x_2})$ vanishes.

By 5.4, we also have $\Theta \cdot \Theta_{h(a_2)} \subset \Theta_{h(c_2)} \cup \Theta_{h(d_2)}$ in JH, or, by II.A.1, a relation

$$\theta\theta_{h(a_2+b_2)} + \gamma\theta_{h(a_2)}\theta_{h(b_2)} + \delta\theta_{h(c_2)}\theta_{h(d_2)} = 0,$$

with $\gamma, \delta \in \mathbb{C}^*$, for every $\theta \in H^0(JH, O(\Theta)) \setminus \{0\}$. Since $h^*\theta$ is proportional to s^2,

$$s^2 s^2_{x_2} + \gamma s^2_{a_2} s^2_{b_2} + \delta s^2_{c_2} s^2_{d_2} = 0.$$

Applying $\tilde{\beta}_2 \in \mathcal{G}(M_2)$ as in (4.14), we obtain

$$\begin{cases} U_2^+ + \gamma V_2^+ + \delta W_2^+ = 0, \\ U_2^- + \gamma V_2^- + \delta W_2^- = 0. \end{cases} \tag{5.6}$$

We distinguish two cases.

i) $\operatorname{rank}(U_2^+, V_2^+, W_2^+) = \operatorname{rank}(U_2^-, V_2^-, W_2^-) = 2$. Relations (4.16) and (5.6) give

$$V_1^\pm = \gamma U_1^\pm, \qquad W_1^\pm = \delta U_1^\pm,$$

and hence

$$s^1_{a_1} s^1_{b_1} = \gamma s^1 s^1_{x_1}, \qquad s^1_{c_1} s^1_{d_1} = \delta s^1 s^1_{x_1}.$$

This forces both $\{a_1, b_1\} \cap H(M_1) \neq \emptyset$ and $\{c_1, d_1\} \cap H(M_1) \neq \emptyset$, contradicting (5.5).

ii) *By Lemma 4.24, it remains to consider the case*

$$\operatorname{rank}(U_2^+, V_2^+, W_2^+) = 1, \qquad \operatorname{rank}(\overline{U}_2^+, \overline{V}_2^+, \overline{W}_2^+) = 2.$$

Let $\theta \in H^0(JH, O(\Theta)) \setminus \{0\}$. By hypothesis, the four sections

$$\theta\theta_{x+z-y-t}, \quad \theta_\varepsilon \theta_{x+z-y-t+\varepsilon}, \quad \theta_{x-y}\theta_{z-t}, \quad \theta_{x-y+\varepsilon}\theta_{z-t+\varepsilon}$$

of $H^0(JH, O(\Theta + \Theta_{x+z-y-t}))$ are linearly dependent, where $\varepsilon = O_H(Q_0 - Q_1) \in JH[2]$. Since $\theta, \theta_\varepsilon, \theta_{x-y}$ vanish on $\operatorname{Sing}\Theta = g^1_2 + W^{g-4}$, we have, for example,

$$g^1_2 + W^{g-4} - x + y + Q_0 - Q_1 \subset W^{g-2},$$

and therefore $\{Q_0, Q_1\} \cap \{x, y\} \neq \emptyset$.

Suppose, for example, that $x = Q_0$. Then

$$\begin{aligned} h(a_2 + \beta_2) &= O_H(x - y + Q_0 - Q_1) = O_H(Q_1 - y), \\ h(c_2 + \beta_2) &= O_H(Q_1 - t), \\ h(d_2) &= O_H(z - y). \end{aligned}$$

By II.A.1, there is thus a relation

$$s^2 t^2_{x_2} + \gamma' s^2_{b_2} t^2_{a_2} + \delta' s^2_{d_2} t^2_{c_2} = 0,$$

which gives

$$\overline{U}_2^\pm + \gamma' \overline{V}_2^\pm + \delta' \overline{W}_2^\pm = 0.$$

This relation is analogous to (5.6). As in case i), it yields a contradiction with (5.5). □

CHAPTER III

Vanishing theta constants on abelian varieties of dimension four

Recall ([Be 1]) that in dimension four, the divisor N_0^4 in $\mathcal{A}_4$, consisting of principally polarized abelian varieties with singular theta divisor, has two irreducible components: $\overline{\mathcal{J}}_4$, the closure of the Jacobian locus, and θ_{null}, the locus of abelian varieties (A, Θ) with Θ symmetric for which $\operatorname{Sing}\Theta$ contains a point of order two. This property is independent of the choice of symmetric divisor Θ. It is equivalent to the vanishing, at the point $\tau \in \mathfrak{H}_4$ corresponding to A, of one of the functions $\theta[\,{}^a_b\,](0, 2\tau)$, where $a, b \in \frac{1}{2}\mathbb{Z}^4/\mathbb{Z}^4$ and $2\,{}^t a \cdot b \in \mathbb{Z}$. One therefore also says that a theta constant vanishes on A.

We study here the structure of the subsets $\theta_{\text{null}}^{(p)}$ of $\mathcal{A}_4$ consisting of abelian varieties on which at least p theta constants vanish.

Our main result is that there is exactly one irreducible abelian fourfold on which ten theta constants vanish and which is not a hyperelliptic Jacobian. Varley has already studied this variety in detail ([Va]).

More precisely, with the notation

$$\begin{aligned} \mathcal{A}_4^{\text{irr}} &= \{\text{irreducible members of } \mathcal{A}_4\}, \\ \theta_{\text{null}}^{(p),\text{irr}} &= \theta_{\text{null}}^{(p)} \cap \mathcal{A}_4^{\text{irr}}, \\ \mathcal{H}_4 &= \{\text{hyperelliptic Jacobians of dimension 4}\}, \\ \mathcal{A}_{r,s} &= \{A_1 \times A_2 \mid A_1 \in \mathcal{A}_r,\ A_2 \in \mathcal{A}_s\}, \end{aligned}$$

we have:

Theorem III.1.

i) $\theta_{\text{null}}^{(2)}$ *is irreducible of dimension eight.*

ii) $\theta_{\text{null}}^{(3)}$ *has three irreducible components, including* $\overline{\mathcal{H}}_4$ *and* $\mathcal{A}_{1,3}$*, each of dimension seven.*

iii) $\theta_{\text{null}}^{(9),\text{irr}} \setminus \mathcal{H}_4$ *is irreducible of dimension one. Its members belong to* $\mathcal{A}_{1,1,1,1}^2$ *and are quotients of a product* E^4*, with* E *elliptic.*

iv) $\theta_{\text{null}}^{(10),\text{irr}} \setminus \mathcal{H}_4$ *consists of a single member, corresponding to the case where* E *has complex multiplication by* i*.*

Proof. Let $\mathcal{A}_4(4,8)$ be the moduli space of principally polarized abelian varieties with a suitable level structure (see [Ig 2] or [V] for details). We use only that the functions $\theta[\,{}^a_b\,](0, 2\tau)$ (see I.3.14), for $a, b \in \frac{1}{2}\mathbb{Z}^4/\mathbb{Z}^4$ and $2\,{}^t a \cdot b \in \mathbb{Z}$, define ample Cartier divisors on $\mathcal{A}_4(4,8)$. Write p for the projection $\mathcal{A}_4(4,8) \to \mathcal{A}_4$, and, for two of these divisors D, D', write q for its restriction to $D \cap D'$. The locus $\theta_{\text{null}}^{(2)}$ has pure dimension eight, since θ_{null} is irreducible of dimension nine. It contains $\overline{\mathcal{E}}_4^2$ (see II.E.4.1; closure here is in $\mathcal{A}_4$), which is therefore a component, as it has dimension eight. We prove it is the only one. Arguing by contradiction, write

$$D \cap D' = M + M',$$

where M, M' are reduced Weil divisors on D, with no common component, and $M = q^{-1}(\overline{\mathcal{E}}_4^2)$.

Also write $\overline{\mathcal{A}_4(4,8)}$ for the Satake compactification, in which all closures will be taken.

We first show that $M \cap M'$, which is contained in the singular locus of $D \cap D'$, has dimension seven. Consider its trace on $\mathcal{J}_4'(4,8) = p^{-1}(\mathcal{J}_4')$, where $\mathcal{J}_4'$ is the closure of $\mathcal{J}_4$ in $\mathcal{A}_4$. We have

$$D \cap D' \cap \mathcal{J}_4'(4,8) \subset q^{-1}(\mathcal{J}_4' \cap \theta_{\text{null}}^{(2)}) = q^{-1}(\mathcal{H}_4') \cup q^{-1}(\mathcal{A}_{1,3}),$$

where $\mathcal{H}_4'$ is the closure in $\mathcal{A}_4$ of the family $\mathcal{H}_4$ of hyperelliptic Jacobians of dimension four. We also have $\mathcal{H}_4 \subset \overline{\mathcal{E}}_4^2$ by II.E.1.1, and $\mathcal{A}_{1,3} \subset \overline{\mathcal{E}}_4^2$. Indeed, consider the étale covers

$$\pi\colon \widetilde{C} = \widetilde{C}_0 \cup F_1 \cup F_2 \longrightarrow C = C_0 \cup F,$$

where $\pi_0\colon \widetilde{C}_0 \to C_0$ belongs to $\mathcal{S}_{4,0}$, the isomorphic elliptic curves F_1, F_2, F meet $\widetilde{C}_0, \widetilde{C}_0, C_0$, respectively, transversely at one point, and C_0 is bielliptic of genus four. This shows that $\overline{\mathcal{E}}_4^2$ contains $\mathcal{E}_3^2 \times \mathcal{A}_1$. The proof of II.D.1.4.2 shows that $\mathcal{E}_3^2$ has dimension six, hence is dense in $\mathcal{A}_3$. Therefore

$$M \supset q^{-1}(\mathcal{H}_4') \cup q^{-1}(\mathcal{A}_{1,3}).$$

Thus M contains every component of $D \cap D' \cap \mathcal{J}_4'(4,8)$.

Now $\overline{\mathcal{J}_4'(4,8)}$ is an irreducible ample Cartier divisor on $\overline{\mathcal{A}_4(4,8)}$ (see, for example, [Ig 3]). It therefore meets $\overline{M'}$, and $\dim(\overline{M'} \cap \overline{\mathcal{J}_4'(4,8)}) \geq 7$. Since the complement of $\mathcal{A}_4(4,8)$ in $\overline{\mathcal{A}_4(4,8)}$ has dimension six, $M' \cap \mathcal{J}_4'(4,8)$ is nonempty of dimension at least seven. But

$$\begin{aligned} M' \cap \mathcal{J}_4'(4,8) &\subset D \cap D' \cap \mathcal{J}_4'(4,8) = M \cap \mathcal{J}_4'(4,8) \\ &= q^{-1}(\mathcal{H}_4') \cup q^{-1}(\mathcal{A}_{1,3}). \end{aligned}$$

Hence some component Z of $q^{-1}(\mathcal{H}_4') \cup q^{-1}(\mathcal{A}_{1,3})$ is contained in both M and M', and therefore in the singular locus of $D \cap D'$. The Jacobian criterion shows that if (A, Θ) corresponds to a singular point of $D \cap D'$, then $\operatorname{Sing}\Theta$ contains at least two points of order two, and either the tangent cones at two of them coincide, or one has multiplicity at least four.

Since $q(Z)$ is either $\mathcal{H}_4'$ or $\mathcal{A}_{1,3}$, it suffices to check that neither condition holds at a generic point of either family to obtain a contradiction.

On a hyperelliptic Jacobian, the ten points of order two in $\operatorname{Sing}\Theta$ correspond to the linear systems $g_2^1 + w_j$, $1 \leq j \leq 10$, where w_j are the Weierstrass points. They have multiplicity two, and the tangent cones to Θ there have equations

$$(s_0^2 u_j)(s_1^2 u_j) - (s_0 s_1 u_j)^2 = 0,$$

where s_0, s_1, u_j are sections of g_2^1, with $\{s_0, s_1\}$ spanning $H^0(g_2^1)$ and $\operatorname{div}(u_j) = 2w_j$ (see I.2.4). These ten quadrics are pairwise distinct.

On a product $E \times JC$, with C smooth nonhyperelliptic of genus three, the 28 vanishing theta constants correspond to the points of order two on the theta divisor Θ of JC. The tangent cones are reducible, the union of $T_x JC$ and $E \times T_x\Theta$. The 28 hyperplanes $T_x\Theta$ correspond to the 28 bitangents of the canonical curve of C, so are pairwise distinct.

We have thus proved by contradiction that $D \cap D' = q^{-1}(\overline{\mathcal{E}}_4^2)$, and hence

$$\theta_{\text{null}}^{(2)} = \overline{\mathcal{E}}_4^2,$$

proving (i).

For the remaining assertions, use the description of $\overline{\mathcal{E}}_4^2$ in II.E.1.1 (closure in $\mathcal{A}_4$). With the notation there,

$$\overline{\mathcal{E}}_4^2 \setminus (\mathcal{A}_{1,3} \cup \mathcal{A}_{2,2}) = \mathcal{E}_4^2 \cup \mathcal{H}_4 \cup \mathcal{H}_4^2 \cup \mathcal{H}_{1,3}^2 \cup \mathcal{H}_{2,2}^2 \cup \mathcal{H}_{1,1,2}^2 \cup \mathcal{H}_{1,1,1,1}^2.$$

By (i), $\theta_{\text{null}}^{(3)}$ has pure dimension seven. Its generic member therefore belongs to $\mathcal{A}_{1,3}$, $\mathcal{E}_4^2$, $\mathcal{H}_4$ or $\mathcal{H}_4^2$. The last family is excluded by II.E.4.2. By II.E.4.1, a member of $\mathcal{E}_4^2 = \Pr(\mathcal{S}_{5,0})$ has three vanishing theta constants if, for the corresponding member of $\mathcal{S}_{5,0}$, Δ is the sum of two members of $|\delta|$. This family is determined by choosing an elliptic curve E, $\delta \in \operatorname{Pic}^4 E$, $\delta' \in (\operatorname{Pic}^0 E)[2] \setminus \{0\}$, and $P_1+P_2+P_3+P_4$, $P_5+P_6+P_7+P_8 \in |\delta|$. It is therefore irreducible, proving (ii). To prove (iii) and (iv), we examine the possible numbers of vanishing theta constants on the families above, first reducing the list. Recall that $\mathcal{E}_{2,(2)}$ is the family of Prym varieties, in the sense of [Mu 2], associated to bielliptic structures on smooth curves of genus three. It is an irreducible subset of $\mathcal{A}_{2,(2)}$.

Lemma III.2. *We have* $\mathcal{A}_{2,(2)}^{\text{irr}} = \mathcal{E}_{2,(2)}$.

Proof. Let $P \in \mathcal{A}_{2,(2)}^{\text{irr}}$ and let E be any elliptic curve. The member of $\mathcal{A}_{1,2}^2$ constructed from E, P is irreducible by A2.2.3. It is therefore the Jacobian of a smooth genus-three curve C admitting a bielliptic structure $C \to E$ (see, for example, II.C.1.2), whose Prym variety is P. □

The curve E in this proof is arbitrary. Given any $P', P'' \in \mathcal{A}_{2,(2)}^{\text{irr}}$, we can find two morphisms $C' \to E$, $C'' \to E$ to the same elliptic curve, with Prym varieties P', P'', respectively. It follows (II.D.1.5) that

$$\mathcal{A}_{2,2}^{2,\text{irr}} = \mathcal{E}_{2,2}^2$$

and

$$(\overline{\mathcal{E}}_4^2 \setminus \mathcal{H}_4) \cap \mathcal{A}_4^{\text{irr}} = \mathcal{E}_4^2 \cup \mathcal{H}_4^2 \cup \mathcal{H}_{1,3}^2 \cup \mathcal{E}_{2,2}^2.$$

We examine these families in turn in a series of lemmas.

Lemma III.3. *Members of* $\mathcal{E}_4^2$ *have at most eight vanishing theta constants.*

Proof. By II.E.4.1, the number of vanishing theta constants is two plus the number of decompositions of Δ as the sum of two members of $|\delta|$.

Suppose $\delta \equiv P_1+P_2+P_3+P_4 \equiv P_5+P_6+P_7+P_8$. For any other decomposition $\Delta = \Delta_1+\Delta_2$, we may assume $\#(\operatorname{Supp}\Delta_1 \cap \{P_1,P_2,P_3,P_4\}) = 2$. Once this intersection is fixed, Δ_1 has at most two distinct possibilities. Replacing Δ_1 by $\Delta - \Delta_1$ gives the same decomposition. There are therefore at most $\binom{4}{2} \cdot 2 \cdot \frac{1}{2} = 6$ other decompositions. If this bound is attained, then, after reordering the P_i, the situation is

$$\begin{aligned} \forall\, 1 \le i < j \le 4, \quad P_i + P_j &\equiv P_1 + P_2 + P_3 + P_4 - P_i - P_j \\ \equiv P_{4+i} + P_{4+j} &\equiv P_5 + P_6 + P_7 + P_8 - P_{4+i} - P_{4+j}, \\ \implies \quad 2P_i + 2P_j &\equiv 2P_{4+i} + 2P_{4+j} \equiv \delta, \\ \implies \quad P_j = P_1 + \varepsilon_j, &\qquad P_{4+j} = P_5 + \varepsilon_j', \end{aligned}$$

with $E[2] = \{\varepsilon_1 = 0, \varepsilon_2, \varepsilon_3, \varepsilon_4\} = \{\varepsilon_1' = 0, \varepsilon_2', \varepsilon_3', \varepsilon_4'\}$. Returning to the first equivalence gives

$$\begin{aligned} i = 1 &\implies 2P_1 + \varepsilon_j \equiv 2P_5 + \varepsilon_j', \\ 1 < i < j &\implies 2P_1 + \varepsilon_i + \varepsilon_j \equiv 2P_5 + \varepsilon_i' + \varepsilon_j'. \end{aligned}$$

Hence $\varepsilon_i = \varepsilon_i'$ and then $2P_1 \equiv 2P_5$. Thus P_5 equals one of P_1, P_2, P_3, P_4, a contradiction. There are therefore at most $2 + 1 + 5 = 8$ vanishing theta constants.

Taking

$$P_i = P_1 + \varepsilon_i, \qquad P_{4+i} = P_5 + \varepsilon_i \quad (1 \le i \le 4), \qquad E[2] = \{\varepsilon_1 = 0, \varepsilon_2, \varepsilon_3, \varepsilon_4\},$$

with the two translates of $E[2]$ disjoint and $\delta \equiv 2P_1 + 2P_5$, gives an abelian variety with eight vanishing theta constants. This is essentially the only way to obtain them. □

Lemma III.4. *Members of $\mathcal{H}_4^2 \cup \mathcal{H}_{1,3}^2$ have at most eight vanishing theta constants.*

Proof. These abelian varieties are the Prym varieties of certain covers of a curve C obtained from a hyperelliptic curve H of genus three, with hyperelliptic involution τ, by identifying Q_0 with Q_1 and τQ_0 with τQ_1 (II.E.1).

The number of vanishing theta constants is two plus the number of decompositions (II.E.4.2)

$$Q_0 + Q_1 + x + y \equiv w_i + w_j + w_k + w_\ell,$$

where w_i, w_j, w_k, w_ℓ are distinct Weierstrass points on H. Suppose one decomposition has $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, and seek the other possible index sets.

1. If $\{1, 2, 3, 5\}$ works, then $x = w_5$, and $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$ and $\{2, 3, 4, 5\}$ work as well. If a further index set works, we may replace it by its complement so that it meets $\{1, 2, 3, 4, 5\}$ in three points. Then y is another w_i, say w_6. But

   $$Q_0 + Q_1 \equiv w_1 + w_2 + w_3 + w_4 - w_5 - w_6 \equiv w_7 + w_8,$$

   which is impossible. Thus there are at most five decompositions in this case, giving seven vanishing theta constants.

2. Now suppose two index sets that work meet in two points. As for members of $\mathcal{S}_{5,0}$, there are seven possible decompositions. If all seven work, then

   $$O(Q_0 + Q_1) \in T = \Theta \cap \Theta_\varepsilon \cap \Theta_{\varepsilon'} \cap \Theta_{\varepsilon''} \cap \Theta_{\varepsilon+\varepsilon'} \cap \Theta_{\varepsilon+\varepsilon''} \cap \Theta_{\varepsilon'+\varepsilon''} \cap \Theta_{\varepsilon+\varepsilon'+\varepsilon''},$$

   where $\varepsilon = w_1 + w_2 - w_3 - w_4$, $\varepsilon' = w_1 + w_2 - w_5 - w_6$ and $\varepsilon'' = w_1 + w_3 - w_5 - w_7$. The set T then contains the eight distinct points

   $$Q_0 + Q_1 + \{0, \varepsilon, \varepsilon', \varepsilon'', \varepsilon + \varepsilon', \varepsilon + \varepsilon'', \varepsilon' + \varepsilon'', \varepsilon + \varepsilon' + \varepsilon''\}.$$

   Since $\Theta^3 = 6$, any three of these translates of Θ intersect in dimension one, so the intersection of two is reducible. This is possible only if there is a degree-two morphism $p\colon H \to E$ onto an elliptic curve (use II.B.1). Then

   $$\varepsilon, \varepsilon', \varepsilon'', \varepsilon + \varepsilon', \varepsilon + \varepsilon'', \varepsilon' + \varepsilon'', \varepsilon + \varepsilon' + \varepsilon'' \in p^*E[2],$$

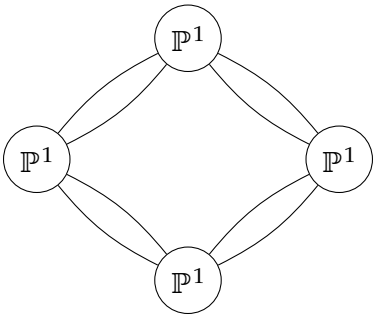
   which is absurd. Thus there are at most $2 + 6 = 8$ vanishing theta constants.

□

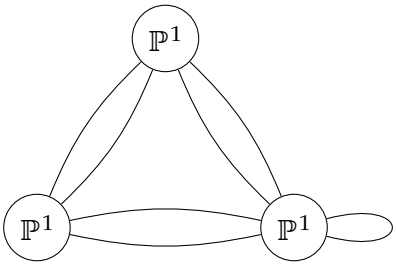
Before the next lemma, introduce a subset $\mathcal{B}_{g_1,\dots,g_4}^2$ of $\mathcal{A}_{g_1+\cdots+g_4}$ constructed like $\mathcal{A}_{g_1,\dots,g_4}^2$. As in A2.2.1, let $(X_i, M_i) \in \mathcal{A}_{g_i,(2)}$, $i = 1, \dots, 4$, and let α_i, β_i generate $H(M_i)$. The quotient of $X_1 \times X_2 \times X_3 \times X_4$ by the subgroup generated by

$$(\alpha_1, \alpha_2, 0, 0), \quad (0, 0, \alpha_3, \alpha_4), \quad (\alpha_1, 0, \beta_3, \beta_4), \quad (\beta_1, \beta_2, \alpha_3, 0)$$

is a principally polarized abelian variety. Denote the resulting locus by $\mathcal{B}^2_{g_1,\dots,g_4}$. It is contained in $\mathcal{A}^2_{g_1+g_2,g_3,g_4}$ and $\mathcal{A}^2_{g_1,g_2,g_3+g_4}$. For example, $\mathcal{A}^2_{1,1,1}$ is the locus of Prym varieties of admissible covers of curves of the type in A2.3.7. Every component is $\mathbb{P}^1$. The following dual graph represents these curves: vertices represent components, and edges represent double points.



The locus $\mathcal{B}^2_{1,1,1,1}$ corresponds to covers of curves with dual graph



again with every component equal to $\mathbb{P}^1$.

Lemma III.5. *Members of $\mathcal{E}^2_{2,2}$ have at most ten vanishing theta constants. Those with nine or ten belong to $\mathcal{A}^2_{1,1,1,1}$ or $\mathcal{B}^2_{1,1,1,1}$.*

Proof. The number of vanishing theta constants is now four plus the number of decompositions of Δ into two members of $|\delta|$ for which, if one is $P_i + P_j + P_k + P_\ell$, the number

$$h^0(Q_i + Q_j + Q_k + Q_\ell + \eta) = h^0(Q_i + Q_j + Q_k + Q_\ell + Q_1 + Q_2 + Q_3 + Q_4 - p^*\delta')$$

is even. This is equivalent to $\#(\{1,2,3,4\} \cap \{i,j,k,\ell\})$ being even. As in Lemma 3, this gives at most $4 + 6 = 10$ vanishing theta constants.

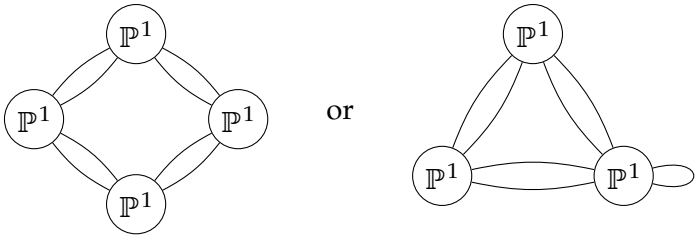
Recall the effect of the tetragonal construction on members of $\mathcal{S}_{5,2}$ (II.D.3.2.2). Let $\phi\colon E \to \mathbb{P}^1$ have degree two. One resulting curve C' is constructed as follows. Let C_1, C_2 be elliptic double covers of $\mathbb{P}^1$ branched at $\phi P_1 + \cdots + \phi P_4$ and $\phi P_5 + \cdots + \phi P_8$, respectively. Then C' is the union of C_1, C_2, meeting at four points.

If at least nine theta constants vanish, that is, there are at least five decompositions of Δ, one easily checks that there is a relation of the form

$$P_1 + P_2 \equiv P_3 + P_4 \equiv P_5 + P_6 \equiv P_7 + P_8.$$

Choose ϕ to be the morphism associated to $O_E(P_1 + P_2)$. Then C_1, C_2 become reducible, each consisting of two copies of $\mathbb{P}^1$ meeting at two points. The curve C' has one of the following

two forms, after stable reduction where necessary (shown as dual graphs):



Every component shown is $\mathbb{P}^1$. Thus the Prym variety belongs to $\mathcal{A}^2_{1,1,1,1}$ or $\mathcal{B}^2_{1,1,1,1}$. □

Briefly recall the construction of $\mathcal{A}^2_{1,1,1,1}$. Start with four elliptic curves E_1, E_2, E_3, E_4, and choose generators α_j, β_j of $E_j[2]$. The quotient of $E_1 \times E_2 \times E_3 \times E_4$ by the subgroup generated by

$$(\alpha_1, \alpha_2, 0, 0), \quad (\alpha_1, 0, \alpha_3, 0), \quad (\alpha_1, 0, 0, \alpha_4), \quad (\beta_1, \beta_2, \beta_3, \beta_4)$$

is a principally polarized abelian variety in $\mathcal{A}^2_{1,1,1,1}$.

Lemma III.6. *We have $\mathcal{A}^2_{1,1,1,1} \subset \theta^{(6),\mathrm{irr}}_{\mathrm{null}}$. The subvarieties $\mathcal{A}^2_{1,1,1,1} \cap \theta^{(p)}_{\mathrm{null}}$, $6 \le p \le 10$, are irreducible of dimension $10 - p$. In particular, exactly one member of $\mathcal{A}^2_{1,1,1,1}$ has ten vanishing theta constants.*

Proof. There is an isogeny $\pi\colon E_1 \times E_2 \times E_3 \times E_4 \to A$. Let o_j be the origin of E_j and $M_j = O(2o_j)$ the symmetric line bundle. Choose $\tilde{\alpha}_j, \tilde{\beta}_j \in \mathcal{G}(M_j)$ mapping to α_j, β_j in $H(M_j) = E_j[2]$, such that

$$\tilde{\alpha}_j^2 = \tilde{\beta}_j^2 = 1, \qquad \tilde{\alpha}_j \tilde{\beta}_j \tilde{\alpha}_j \tilde{\beta}_j = -1.$$

Let s_j be a section of M_j invariant under $\tilde{\alpha}_j$, and set $t_j = \tilde{\beta}_j \cdot s_j$. Then $\operatorname{div}(s_j) = x_j + (x_j + \alpha_j)$, with $2x_j = \alpha_j$. An equation for the inverse image in $E_1 \times E_2 \times E_3 \times E_4$ of the theta divisor of A is

$$s = s_1 s_2 s_3 s_4 + t_1 t_2 t_3 t_4.$$

Let $(y_1, \dots, y_4)$ be a singular point of $\operatorname{div}(s)$. Distinguish two cases.

i) If $2y_1 \neq 0$, then $s_1(y_1)\dot{t}_1(y_1) - \dot{s}_1(y_1)t_1(y_1) \neq 0$, where a dot denotes differentiation with respect to a local parameter on E_j. Therefore

$$s_2(y_2)s_3(y_3)s_4(y_4) = t_2(y_2)t_3(y_3)t_4(y_4) = 0.$$

If $s_2(y_2)t_2(y_2) \neq 0$, then $s_3(y_3)s_4(y_4) = t_3(y_3)t_4(y_4) = 0$. For example, $s_3(y_3) = t_4(y_4) = 0$. In particular, y_3, y_4 are not of order two. The same method therefore gives $s_1(y_1)s_3(y_3) = t_1(y_1)t_3(y_3) = s_1(y_1)s_4(y_4) = t_1(y_1)t_4(y_4) = 0$, which is impossible. Hence for every $j \in \{2, 3, 4\}$, either $s_j(y_j) = 0$ or $t_j(y_j) = 0$.

In particular, y_2 is not of order two, so $s_1(y_1) = 0$ or $t_1(y_1) = 0$ as well. More precisely, the necessary and sufficient condition is

$$\exists \sigma \in \mathfrak{S}_4, \qquad \begin{cases} s_{\sigma(1)}(y_{\sigma(1)}) = s_{\sigma(2)}(y_{\sigma(2)}) = 0, \\ t_{\sigma(3)}(y_{\sigma(3)}) = t_{\sigma(4)}(y_{\sigma(4)}) = 0. \end{cases}$$

This gives the following six distinct points on A, singular on Θ:

$$\pi(x_1 + x_2 + x_3 + x_4 + \beta_1 + \beta_j + \epsilon\alpha_1), \qquad j \in \{2, 3, 4\}, \quad \epsilon \in \{0, 1\}.$$

They have order two because $2\sum_j x_j = \sum_j \alpha_j$ lies in the kernel of π.

ii) If one y_j has order two, the first case shows that all do. The images of $o_j, \alpha_j, \beta_j, \alpha_j + \beta_j$ under $(s_j, t_j)\colon E_j \to \mathbb{P}^1$ are $\lambda_j, -\lambda_j, 1/\lambda_j, -1/\lambda_j$ ([Mu 5], p. 350), where

$$\lambda_j \in U = \mathbb{C} \setminus \{0, 1, -1, i, -i\}.$$

The complex number λ_j determines E_j, the points α_j, β_j, and a theta structure for the degree-two polarization on E_j (see [Ig 2] or [V]). Thus U is the moduli space $\mathcal{A}_1(2,4)$.

We can therefore define a surjective morphism $U^4 \to \mathcal{A}^2_{1,1,1,1}$.

The cross-ratio of $\lambda, -\lambda, 1/\lambda, -1/\lambda$ in $\mathbb{P}^1$ is

$$\left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right)^2.$$

It is invariant under the group G generated by $\lambda \mapsto -\lambda$ and $\lambda \mapsto -1/\lambda$.

Thus $U^4 \to \mathcal{A}^2_{1,1,1,1}$ is invariant under G^4 on U^4 and under the symmetric group $\mathfrak{S}_4$. The image of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in U^4$ belongs to $\theta^{(7)}_{\text{null}}$ if and only if there are $g_1, \dots, g_4 \in G$ such that

$$1 + g_1(\lambda_1) g_2(\lambda_2) g_3(\lambda_3) g_4(\lambda_4) = 0.$$

Thus $\theta^{(7)}_{\text{null}} \cap \mathcal{A}^2_{1,1,1,1}$ is the image of the irreducible hypersurface in U^4 with equation $1 + \lambda_1\lambda_2\lambda_3\lambda_4 = 0$.

Likewise, $\theta^{(8)}_{\text{null}} \cap \mathcal{A}^2_{1,1,1,1}$ is the image of the union of the subvarieties of U^4 defined by

$$\begin{cases} 1 + \lambda_1\lambda_2\lambda_3\lambda_4 = 0, \\ 1 + g_1(\lambda_1) g_2(\lambda_2) g_3(\lambda_3) g_4(\lambda_4) = 0, \end{cases}$$

where

$$g_1(\lambda_1) g_2(\lambda_2) g_3(\lambda_3) g_4(\lambda_4) \notin \{\lambda_1\lambda_2\lambda_3\lambda_4, (\lambda_1\lambda_2\lambda_3\lambda_4)^{-1}\}.$$

Using the actions of G^4 and $\mathfrak{S}_4$, we reduce to the four subvarieties

$$\lambda_1\lambda_3 = \lambda_2\lambda_4 = \pm i, \qquad \lambda_1\lambda_3 = -\lambda_2\lambda_4 = \pm 1,$$

and then to points of the form $(\lambda_1, \lambda_2, \lambda_1, \lambda_2)$ or $(\lambda_1, \lambda_2, i\lambda_1, i\lambda_2)$.

The transformation $\lambda \mapsto i\lambda$ takes (E, α, β) to $(E, \alpha, \alpha+\beta)$. Hence the morphism $U^4 \to \mathcal{A}^2_{1,1,1,1}$ is invariant under

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \longmapsto (i\lambda_1, i\lambda_2, \lambda_3, \lambda_4).$$

It follows that $\theta^{(8)}_{\text{null}} \cap \mathcal{A}^2_{1,1,1,1}$ is the image of the irreducible surface in U^4 consisting of the points $(\lambda_1, \lambda_2, \lambda_1, \lambda_2)$.

Similarly, $\theta^{(9)}_{\text{null}} \cap \mathcal{A}^2_{1,1,1,1}$ is the image of the diagonal of U^4, and $\theta^{(10)}_{\text{null}} \cap \mathcal{A}^2_{1,1,1,1}$ corresponds to the case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$, with $\lambda^4 = -1$. Then $E_1, \dots, E_4$ are isomorphic to the elliptic curve with complex multiplication by i, whose invariant is $j = 1728$. This recovers Varley's variety [Va]. □

The following lemma handles $\mathcal{B}^2_{1,1,1,1}$ and completes the proof of the last two assertions of Theorem III.1.

Lemma III.7. *We have*

$$\mathcal{B}^2_{1,1,1,1} \subset \theta^{(4)}_{\text{null}}, \qquad \mathcal{B}^2_{1,1,1,1} \cap \theta^{(7)}_{\text{null}} \subset \mathcal{A}^2_{1,1,1,1}.$$

Proof. In the notation of the preceding lemma, an equation for Θ is

$$s = s_1 s_2 (s_3 s_4 + t_3 t_4) + t_1 t_2 (s_3 s_4 - t_3 t_4).$$

Let (y_1, y_2, y_3, y_4) be a singular point.

i) Suppose $2y_1 \neq 0$. Then

$$\begin{aligned} s_2(y_2)(s_3(y_3)s_4(y_4) + t_3(y_3)t_4(y_4)) &= 0, \\ t_2(y_2)(s_3(y_3)s_4(y_4) - t_3(y_3)t_4(y_4)) &= 0. \end{aligned}$$

1) If $s_3(y_3)s_4(y_4) = t_3(y_3)t_4(y_4) = 0$, then y_3, y_4 do not have order two. Thus

$$\begin{aligned} s_4(y_4)(s_1(y_1)s_2(y_2) + t_1(y_1)t_2(y_2)) &= 0, \\ t_4(y_4)(s_1(y_1)s_2(y_2) - t_1(y_1)t_2(y_2)) &= 0, \\ s_3(y_3)(s_1(y_1)s_2(y_2) + t_1(y_1)t_2(y_2)) &= 0, \\ t_3(y_3)(s_1(y_1)s_2(y_2) - t_1(y_1)t_2(y_2)) &= 0. \end{aligned}$$

It follows that $s_1(y_1)s_2(y_2) = t_1(y_1)t_2(y_2) = 0$, so for every j either $s_j(y_j) = 0$ or $t_j(y_j) = 0$. This gives the following four distinct points of order two on A:

$$\begin{gathered} \pi(x_1, x_2 + \beta_2, x_3, x_4 + \beta_4), \\ \pi(x_1, x_2 + \beta_2, x_3 + \alpha_3, x_4 + \beta_4), \\ \pi(x_1, x_2 + \beta_2, x_3 + \beta_3, x_4), \\ \pi(x_1, x_2 + \beta_2, x_3 + \beta_3, x_4 + \alpha_4). \end{gathered}$$

2) If, for example, $s_3(y_3)s_4(y_4) + t_3(y_3)t_4(y_4) \neq 0$, then $s_2(y_2) = 0$, hence $2y_2 \neq 0$. Likewise, $s_1(y_1) = 0$. We then have

$$\begin{aligned} s_3(y_3)s_4(y_4) - t_3(y_3)t_4(y_4) &= 0, \\ \dot{s}_3(y_3)s_4(y_4) - \dot{t}_3(y_3)t_4(y_4) &= 0, \\ s_3(y_3)\dot{s}_4(y_4) - t_3(y_3)\dot{t}_4(y_4) &= 0. \end{aligned}$$

It follows that y_3, y_4 have order two. We may therefore assume

$$(y_3, y_4) \in \{(o_3, o_4), (o_3, \beta_4)\},$$

that is, $1 - \lambda_3\lambda_4 = 0$ or $\lambda_3 = \lambda_4$. There is then an isomorphism $E_3 \to E_4$ taking α_3 to α_4. We have the following diagram (A2.2.3):

$$\begin{array}{ccccc} E_3 \times E_3 & & & & E_3' \times E_3' \\ & \searrow^{p} & & \swarrow_{p'} & \\ & & (Y, M) & & \end{array} \qquad \begin{aligned} \operatorname{Ker} p &= \langle (\alpha_3, \alpha_3) \rangle, \\ \operatorname{Ker} p' &= \langle (\alpha_3', \alpha_3') \rangle. \end{aligned}$$

Here $E_3' = E_3/\{0, \alpha_3\}$ and α_3' is the image of β_3. The points

$$p(\beta_3, \beta_3) = p'(\alpha_3', 0), \qquad p(\alpha_3, 0) = p'(\beta_3', \beta_3')$$

generate $H(M)$. Thus (A, Θ) belongs to $\mathcal{A}^2_{1,1,1,1}$ and is a quotient of $E_1 \times E_2 \times E_3' \times E_3'$.

ii) If $(A,\Theta) \notin \mathcal{A}^2_{1,1,1,1}$, any remaining singular points of Θ therefore have the form $\pi(y_1,\dots,y_4)$, with $2y_j = 0$ for every j. Suppose one of them is $\pi(o_1,o_2,o_3,o_4)$. Then

$$1 + \lambda_3\lambda_4 + \lambda_1\lambda_2(1-\lambda_3\lambda_4) = 0.$$

By the preceding argument we may assume $\lambda_3\lambda_4 \neq \pm 1$. Any other singular point then belongs to

$$\{\pi(o_1,\beta_2,\alpha_3,o_4),\ \pi(o_1,\beta_2,\beta_3,o_4),\ \pi(o_1,\beta_2,\alpha_3+\beta_3,o_4),$$
$$\pi(o_1,\beta_2,\alpha_3,\beta_4),\ \pi(o_1,\beta_2,\beta_3,\beta_4),\ \pi(o_1,\beta_2,\alpha_3+\beta_3,\beta_4)\},$$

and there cannot be two such points.

□

□

CHAPTER IV

The Schottky problem for Prym varieties and the loci N^g_{g-6}

IV.A Prym varieties and the loci N^g_{g-6}

In II.D.1.2.5 we showed that, for $g \geq 6$, the family $\mathcal{P}_g$ of Prym varieties of dimension g is contained in the Andreotti–Mayer locus

$$N^g_{g-6} = \{(A, \Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing} \Theta \geq g - 6\}.$$

We shall refine this result by proving the following analogue of the main theorem of [An-M]:

Theorem IV.A. *For $g \geq 7$, $\mathcal{P}_g$ is an irreducible component of N^g_{g-6}.*

The corresponding assertion for $g = 6$ is false: $\mathcal{P}_6$ has dimension 18, $\mathcal{A}_6$ has dimension 21, and for every g, N^g_0 is a divisor in $\mathcal{A}_g$ ([Be 1], [Mu 1]).

The method is essentially that of [An-M]. Andreotti and Mayer exhibit a Jacobian (J, Θ) with $\dim \operatorname{Sing} \Theta = g - 4$ such that the tangent cones to Θ at its points of multiplicity 2 span a vector space of codimension $3g - 3$ in $S^2 T^*_0 J$. They then show that this implies their result.

One could use the same method for Prym varieties. Unfortunately, although a generic Prym variety satisfies $\dim \operatorname{Sing} \Xi = g - 6$, to my knowledge no explicit example with this property is known. Moreover, the tangent cones at the double points of the theta divisor seem difficult to study directly.

To overcome this difficulty, we use degenerations of Prym varieties, namely certain extensions of Prym varieties by $\mathbf{C}^*$. Under the isomorphism $\operatorname{Ext}^1(P, \mathbf{C}^*) \simeq P$, these are the extensions corresponding to $a = \mathcal{O}(r + s - \sigma r - \sigma s) \in P$, where $P = \operatorname{Prym}(\widetilde{C} \to \widetilde{C}/\sigma)$ and $r, s \in \widetilde{C}$.

In subsection 2 we show that the Andreotti–Mayer method applies on the boundary of $\mathcal{A}_g$. This reduces the problem to studying $\operatorname{Sing}(\Xi \cdot \Xi_a)$ for $(P, \Xi) \in \mathcal{P}_{g-1}$ and a as above. We then use a second degeneration, where (P, Ξ) is a Jacobian $(JN, \Theta) \in \mathcal{J}_{g-1}$ and $a = \mathcal{O}_N(p+q-r-s)$, with $p, q, r, s \in N$. In this case $\dim \operatorname{Sing}(\Theta \cdot \Theta_a) = g - 6$.

Using the results of subsections 4 and 5 on the behavior of $\operatorname{Sing} \Xi$ and $\operatorname{Sing}(\Xi \cdot \Xi_a)$ in families, subsection 6 identifies those components of $\operatorname{Sing}(\Theta \cdot \Theta_a)$ which arise by specialization from the generic situation. Calculations in this special case are then straightforward. We conclude in subsection 7.

Just as N^g_{g-4} always has components other than $\mathcal{J}_g$ for $g \geq 4$, N^g_{g-6} always has components other than $\mathcal{P}_g$ for $g \geq 7$ (A1.4.5).

There is another analogy with Jacobians. If C is not hyperelliptic, its canonical embedding is projectively normal (Noether's theorem). If C is also neither trigonal nor a smooth plane quintic, the ideal of its canonical curve is generated by quadrics (the Enriques–Babbage–Petri theorem). By [Gr 1], for $g \geq 5$ the intersection of the tangent cones at the double points of the theta divisor in $\mathbf{P}T_0 JC \simeq \mathbf{P}H^0(C, \omega_C)^*$ is therefore the canonical curve. This proves Torelli's theorem for these curves.

Now consider a Prym variety (P, Ξ) of dimension $g \geq 6$, associated with a generic curve C and an element η of order 2 in JC. The semicanonical curve

$$X = \varphi_{|\omega_C \otimes \eta|}(C) \subset \mathbf{P}H^0(C, \omega_C \otimes \eta)^* \simeq \mathbf{P}T_0P$$

is projectively normal [Gr-L]. Our results imply that the tangent cones at the double points of Ξ span the space of quadrics containing X.

In a paper currently in preparation, we show that, for $g \geq 7$, the ideal of X in $\mathbf{P}^{g-1}$ is generated by its degree-2 elements. In particular, X is the intersection of the quadrics containing it. This gives a constructive proof of the generic Torelli theorem for Prym varieties, proved by other methods in [Fr-S 1] for $g \geq 6$.

IV.A.1 Notation, conventions, and recollections

For a smooth connected curve C, write $C^{(d)}$ for the set of effective divisors of degree d on C, or equivalently the symmetric product of d copies of C (I.2), and put

$$C_d^r = \{D \in C^{(d)} \mid h^0(C, O(D)) \geq r+1\}, \qquad W_d^r = \{L \in J^dC \mid h^0(C, L) \geq r+1\}.$$

An element of W_d^r is called a g_d^r. We often write W for W_{g-3}^0. For $p, q \in C$, put

$$\begin{aligned} W_{p,q} &= \{L(p+q) \in J^{g-1}C \mid L \in W\} \\ &= \{L \in J^{g-1}C \mid h^0(L(-p-q)) \geq 1\}, \\ C_{p,q}^{(d)} &= \{D \in C^{(d+2)} \mid D \geq p+q\}. \end{aligned}$$

Write s_{pq} for a section of $O(p+q)$ with divisor $p+q$. Similarly, for $D \in C^{(d)}$, write s_D for a section of $O(D)$ with divisor D.

We also write "$-$" for the involution $L \mapsto \omega_C \otimes L^{-1}$ of $J^{g-1}C$, and $(-Z)$ for the image of a subset Z of $J^{g-1}C$ under this involution.

The loci W_d^r and C_d^r have natural scheme structures ([A-C-G-H 1], Chapter IV). Then:

- For $L \in W_d^r \setminus W_d^{r+1}$, the tangent space $T_L W_d^r \subset T_L J^dC \simeq H^0(C, \omega_C)^*$ is the annihilator of the image of the natural map

$$H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C, \omega_C).$$

- If C is a general curve of genus g, then W_d^r has dimension $\rho = g - (r+1)(g-d+r)$, is irreducible if $\rho > 0$, smooth outside W_d^{r+1}, and Cohen–Macaulay ([A-C-G-H 1], Chapter VII).

We have used, and shall continue to use, the convention that a set of negative dimension is empty.

IV.A.2 The Andreotti–Mayer theorem and its application to quasi-abelian varieties of rank one

We begin by recalling the results of §3 of [An-M] on higher ramification loci of an analytic map. The theorem below is not stated in this form in [An-M], but follows directly from its results.

IV.A.2.1. Let X and B be smooth connected analytic varieties, and let $p : X \to B$ be a proper smooth morphism. Let D be a divisor in X, let $S \subset D$ be the critical locus of $p|_D : D \to B$, and write $\pi : S \to B$ for the restriction of p to S.

Throughout, f denotes a local equation of D in X. For every $x \in X$, there is an exact sequence

$$0 \longrightarrow T^*_b B \xrightarrow{p^*} T^*_x X \xrightarrow{i^*} T^*_x X_b \longrightarrow 0,$$

where $b = p(x)$ and $X_b = p^{-1}(b)$. If $x \in S$, then by definition $i^* df(x) = 0$, so $(p^*)^{-1}(df(x))$ is well defined. Put

$$\forall x \in S, \qquad T_x = \ker[(p^*)^{-1}(df(x))] \subset T_b B.$$

This vector space does not depend on the chosen local equation f. We also write

$$\begin{aligned} S_k &= \{x \in S \mid \dim_x \pi^{-1}(\pi(x)) \geq k\}, \quad \text{a closed analytic subset of } S, \\ \pi_k &= \pi|_{S_k}, \\ N_k &= \pi(S_k), \qquad\qquad\qquad\quad \text{a closed analytic subset of } B. \end{aligned} \tag{IV.A.2.2}$$

Let $b \in B$, and assume $\dim \pi^{-1}(b) = k$, so $b \in N_k$. Let N be a local irreducible component of N_k at b, and let $S^1, \dots, S^r$ be the irreducible components of $\pi_k^{-1}(N)$ dominating N. Write

$$S(N) = \bigcup_{i=1}^{r} (S^i \cap \pi^{-1}(b)),$$

so $S(N)$ is a union of k-dimensional irreducible components of $S_b = \pi^{-1}(b) = \operatorname{Sing} D_b$. The result is as follows:

Theorem IV.A.2.3. *The reduced tangent cone to N at b is contained in $\bigcap_{x \in S(N)} T_x \subset T_b B$. In particular,*

$$\dim N \leq \operatorname{codim}_{T^*_b B} \left\langle (p^*)^{-1}(df(x)) \mid x \in S(N) \right\rangle.$$

Proof. The proof is given in [An-M], pages 214–217. We recall its main steps. There are points $x_1, \dots, x_m$ of $S(N)$ such that

$$\bigcap_{x \in S(N)} T_x = \bigcap_{\alpha=1}^{m} T_{x_\alpha} = T.$$

For $\alpha \in \{1, \dots, m\}$ there is $j(\alpha) \in \{1, \dots, r\}$ such that $x_\alpha \in S^{j(\alpha)}$. Let U_α be a connected neighborhood of x_α in $S^{j(\alpha)}$. By upper semicontinuity, we may assume that the fibers of all the maps $\pi^j : S^j \to N$ have dimension exactly k everywhere. As in [An-M], Remmert's theorem then implies that the maps π^j are open.

The set $\bigcap_{\alpha=1}^m \pi^{j(\alpha)}(U_\alpha)$ is open, and every smooth point y of N in it has a preimage in each U_α.

The open sets U_α can be chosen arbitrarily small. For every sequence $y^{(n)}$ of smooth points of N converging to b, we can choose sequences $x_\alpha^{(n)}$ in $\pi_k^{-1}(N)$ converging to x_α with $\pi_k(x_\alpha^{(n)}) = y^{(n)}$.

By Lemma 8 of [An-M],

$$T_{y^{(n)}} N \subset \bigcap_{\alpha=1}^{m} T_{x_\alpha^{(n)}}.$$

Assuming that the limit of the tangent spaces exists, we obtain

$$\lim_n T_{y^{(n)}} N \subset T.$$

The reduced tangent cone to N at b is contained in the union of all these limits, proving the result. □

IV.A.2.4. Throughout this article, a quasi-abelian variety of rank 1 and dimension $g+1$ means an extension of a principally polarized abelian variety of dimension g by the torus $\mathbf{C}^*$.

There is a coarse moduli space $\mathcal{A}_{g+1}^{(1)}$ for principally polarized abelian varieties and rank-1 quasi-abelian varieties of dimension $g+1$ ([Ig 1], [Mu 1], [Na]). It is the disjoint union of an open subset isomorphic to $\mathcal{A}_{g+1}$ and a divisor $\partial\mathcal{A}_{g+1}$. There is a morphism $\rho_{g+1} : \partial\mathcal{A}_{g+1} \to \mathcal{A}_g$ whose fiber over (A, Θ) is isomorphic to $A/\operatorname{Aut}(A, \Theta)$. It is induced on $\partial\mathcal{A}_{g+1}$ by the morphism from $\mathcal{A}_{g+1}^{(1)}$ to the Satake compactification $\mathcal{A}_{g+1}^* = \mathcal{A}_{g+1} \cup \mathcal{A}_g \cup \cdots \cup \mathcal{A}_0$ of $\mathcal{A}_{g+1}$.

There is a "universal family" of rank-1 quasi-abelian varieties over $B = \mathbf{C}^g \times \mathfrak{H}_g$, for which we now construct a compactification $p : X \to B$. Let

- $\widetilde{X} = (\mathbf{C}^2 \setminus \{(0,0)\}) \times \mathbf{C}^g \times B$;
- $\Gamma = \mathbf{C}^* \times \mathbf{Z}^g \times \mathbf{Z}^g$, acting on $\widetilde{X}$ by

$$(\alpha, p, q) \cdot (\lambda, \mu, z, a, \tau) = (\alpha\lambda, \alpha\mu e^{-2\pi i\, {}^t qa}, z + p + \tau q, a, \tau);$$

- $X = \widetilde{X}/\Gamma$;
- $p : X \to B$ the projection onto the last two factors.

The fiber over $(a, \tau) \in B$ is isomorphic to the projective bundle $\mathbf{P}(O_A(\Theta) \oplus O_A(\Theta_a))$ over the principally polarized abelian variety (A, Θ) associated with $\tau \in \mathfrak{H}_g$. Removing its two canonical sections gives an extension of A by $\mathbf{C}^*$.

Define a morphism $f : \widetilde{X} \to \mathbf{C}$ by

$$f(\lambda, \mu, z, a, \tau) = \lambda\theta(z, \tau) + \mu\theta(z - a, \tau).$$

Then

$$\begin{aligned} f((\alpha, p, q) \cdot (\lambda, \mu, z, a, \tau)) &= \alpha\lambda\theta(z + p + \tau q, \tau) + \alpha\mu e^{-2\pi i\, {}^t qa}\theta(z - a + p + \tau q, \tau) \\ &= \alpha\lambda e^{\pi i(-{}^t q\tau q - 2{}^t qz)}\theta(z, \tau) \\ &\quad + \alpha\mu e^{-2\pi i\, {}^t qa} e^{\pi i(-{}^t q\tau q - 2{}^t q(z-a))}\theta(z - a, \tau) \\ &= \alpha e^{\pi i(-{}^t q\tau q - 2{}^t qz)} f(\lambda, \mu, z, a, \tau). \end{aligned}$$

The divisor $\widetilde{D}$ of f is therefore the inverse image of a divisor D on X. We take f as a local equation of D. We use the notation of IV.A.2.1 to apply its results to this situation.

The fiber over $(a, \tau) \in B$ of $\widetilde{\pi} : \tilde{S} \to B$ is

$$\left\{ (\lambda, \mu, z) \;\middle|\; \begin{array}{c} \theta(z, \tau) = \theta(z - a, \tau) = 0, \\ \lambda \dfrac{\partial\theta}{\partial z_j}(z, \tau) + \mu \dfrac{\partial\theta}{\partial z_j}(z - a, \tau) = 0 \quad (1 \le j \le g) \end{array} \right\}.$$

The set N_k of IV.A.2.2 thus corresponds to

$$\left\{ (a, \tau) \in B \;\middle|\; \begin{array}{c} \dim \operatorname{Sing}(\Theta \cdot \Theta_a) \ge k \text{ or} \\ \dim(\operatorname{Sing}\Theta \cap \operatorname{Sing}\Theta_a) \ge k - 1 \end{array} \right\}.$$

To distinguish this from the Andreotti–Mayer loci, we denote it by Γ_k^g.

IV.A.2.5. By [Mu 1], (2.4), pages 363–364, the boundary of the Andreotti–Mayer locus

$$N_k^{g+1} = \{(A,\Theta) \in \mathcal{A}_{g+1} \mid \dim \operatorname{Sing}\Theta \geq k\}$$

in the partial compactification $\mathcal{A}^{(1)}_{g+1}$ is contained in

$$\rho_{g+1}^{-1}(N_k^g) \cup \partial' N_k^{g+1},$$

where $\partial' N_k^{g+1}$ is the image of Γ_k^g under $B = \mathbf{C}^g \times \mathfrak{H}_g \to \partial\mathcal{A}_{g+1}$.

We use Theorem IV.A.2.3 in the following form:

Proposition IV.A.2.6. *Let $f : (\mathcal{A},\Theta) \to T$ be a family of principally polarized abelian varieties of dimension g, where T is smooth and connected, equipped with a section a. This gives a family of rank-1 quasi-abelian varieties, and hence a morphism $\varphi : T \to \partial\mathcal{A}_{g+1}$. Put*

$$\mathcal{T} = \bigcup_{t\in T} \operatorname{Sing}(\Theta_t \cdot \Theta_{t,a_t}) \subset \mathcal{A}.$$

Assume:

(*i*) *For every $t \in T$, $\mathcal{T}_t$ has dimension k, and it is irreducible for generic t. Thus exactly one component $\mathcal{T}^0$ of $\mathcal{T}$ dominates T.*

(*ii*) *For generic $t \in T$, $\dim(\operatorname{Sing}\Theta_t \cap \operatorname{Sing}\Theta_{t,a_t}) \leq k-2$.*

(*iii*) *For every $t \in T$, the only automorphisms of (A_t,Θ_t) are $\pm\mathrm{id}$, and a_t does not have order 2.*

In particular, $\varphi(T) \subset \partial' N_k^{g+1}$. Let N be an irreducible component of $\partial' N_k^{g+1}$ containing $\varphi(T)$. For every $t \in T$, the dimension of N at $\varphi(t)$ is at most the codimension of the projective span of the image of $\mathcal{T}_t^0$ under the rational map

$$\mathcal{T}_t = \operatorname{Sing}(\Theta_t \cdot \Theta_{t,a_t}) \dashrightarrow \mathbf{P}^{g+\binom{g+1}{2}-1},$$

$$z \longmapsto \left[\left(\lambda\frac{\partial\theta}{\partial z_i}(z,\tau)\right)_{1\leq i\leq g},\quad \left(\lambda\frac{\partial^2\theta}{\partial z_i\partial z_j}(z,\tau) + \mu\frac{\partial^2\theta}{\partial z_i\partial z_j}(z-a_t,\tau)\right)_{1\leq i\leq j\leq g}\right],$$

where $\lambda\frac{\partial\theta}{\partial z_i}(z,\tau) + \mu\frac{\partial\theta}{\partial z_i}(z-a_t,\tau) = 0$ for every i.

Proof. Since the question is local, we may assume T simply connected. By (iii), the morphism $\varphi : T \to \partial\mathcal{A}_{g+1}$ lifts to $\psi : T \to B$. There is a Cartesian diagram

$$\begin{array}{ccc} P = \mathbf{P}(\mathcal{O}_{\mathcal{A}}(\Theta) \oplus \mathcal{O}_{\mathcal{A}}(\Theta_a)) & \xrightarrow{\Psi} & X \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow \\ T & \xrightarrow[\psi]{} & B. \end{array}$$

Consider the lift $\mathcal{T}'$ of $\mathcal{T}$ in the critical locus of P. The projection $h : \mathcal{T}' \to \mathcal{T}$ is birational except over points of $\operatorname{Sing}\Theta_t \cap \operatorname{Sing}\Theta_{t,a_t}$. By (ii) and (i), $\mathcal{T}_t'$ has dimension k and is irreducible for generic t. Thus exactly one component $\mathcal{T}'^0$ of $\mathcal{T}'$ dominates T. Its image under h is $\mathcal{T}^0$.

Let N' be a local irreducible component of Γ_k^g containing $\psi(T)$ whose image in $\partial\mathcal{A}_{g+1}$ contains N. As in IV.A.2.2, let $S^1,\dots,S^r$ be the irreducible components of $\pi_k^{-1}(N')$ dominating

N'. For every $t \in T$, the $S^j_{\psi(t)}$ are components of $\mathcal{J}_t{}'$ of dimension $\geq k$. Hence for generic t, $S^j_{\psi(t)} = \mathcal{J}'^0_t$ for every j, so $S^j \supset \Psi(\mathcal{J}'^0)$. We can therefore apply Theorem IV.A.2.3.

Choose $\tilde{x} = (\lambda, \mu, z, a, \tau) \in \widetilde{X}$ above $x \in S_t$. If $\mu \neq 0$, we may assume $\mu = 1$. With

$$f(\lambda, 1, z, a, \tau) = \lambda\theta(z, \tau) + \theta(z - a, \tau),$$

we have

$$\begin{aligned}
\frac{\partial f}{\partial \lambda} &= 0 = \theta(z, \tau), \\
\frac{\partial f}{\partial z_j} &= 0 = \lambda \frac{\partial \theta}{\partial z_j}(z, \tau) + \frac{\partial \theta}{\partial z_j}(z - a, \tau), && \tilde{x} \in \tilde{S}_t = \operatorname{Sing} \widetilde{D}_t, \\
\frac{\partial f}{\partial a_j} &= -\frac{\partial \theta}{\partial z_j}(z - a, \tau) = \lambda \frac{\partial \theta}{\partial z_j}(z, \tau), \\
\frac{\partial f}{\partial \tau_{jk}} &= \lambda \frac{\partial \theta}{\partial \tau_{jk}}(z, \tau) + \frac{\partial \theta}{\partial \tau_{jk}}(z - a, \tau) \\
&= \frac{1}{2\pi i(1 + \delta_{jk})} \left(\lambda \frac{\partial^2 \theta}{\partial z_j \partial z_k}(z, \tau) + \frac{\partial^2 \theta}{\partial z_j \partial z_k}(z - a, \tau) \right),
\end{aligned}$$

by the heat equation (I.3.14). Applying IV.A.2.3 proves the proposition. □

IV.A.3 Some results on linear systems on curves

We collect results on the subschemes W^r_d of a smooth curve's Jacobian which will be needed later.

For a smooth curve C of genus g, here and below put

$$W = W^0_{g-3} \subset J^{g-3}C.$$

In accordance with IV.A.1, for $p, q \in C$ write

$$\begin{aligned}
W_{p,q} &= W + p + q \subset J^{g-1}C, \\
C^{(d)}_p &= \{D \in C^{(d+1)} \mid D \geq p\}, \\
C^{(d)}_{p,q} &= \{D \in C^{(d+2)} \mid D \geq p + q\}.
\end{aligned}$$

Proposition IV.A.3.1. *Let C be a generic smooth curve of genus g. For generic $p, q \in C$,*

$$\forall L \in \operatorname{Pic}(C), \qquad h^0(L) \geq 2 \implies h^1(L^2(-p - q)) = 0.$$

Proof. On a generic curve, [Gi] gives

$$h^0(L) \geq 2 \implies h^1(L^2) = 0 \implies h^0(L^2) = 2d + 1 - g,$$

where $d = \deg L$. If p were a base point of $|L^2|$, it would also be a base point of $|L|$, giving

$$h^0(L(-p)) \geq 2 \implies h^0(L^2(-2p)) = 2d - 2 + 1 - g = h^0(L^2) - 2,$$

a contradiction. Thus

$$\forall p \in C, \qquad h^0(L^2(-p)) = h^0(L^2) - 1 = 2d - g \iff h^0(\omega_C \otimes L^{-2}(p)) = 0.$$

If $d > g$, then $h^1(L^2(-p - q)) = 0$ immediately, since $\deg(\omega_C \otimes L^{-2}(p + q)) < 0$. We also treat $d = g$. On the diagonal $p = q$, an exceptional line bundle would satisfy $L^2 = \omega_C(2p)$,

so $L = \kappa(p)$ with $\kappa^2 = \omega_C$. Gieseker's property recalled above gives $h^0(\kappa) \leq 1$: otherwise it would imply $h^1(\omega_C) = 0$. There are only finitely many theta characteristics κ. Choose p outside the divisors of the sections of the effective ones. Then $h^0(\kappa(-p)) = 0$ for every κ, and Riemann–Roch gives $h^0(\kappa(p)) = 1$. Thus there is no exceptional line bundle of degree g for this (p,p). Properness of the incidence locus below implies the same for generic (p,q).

It remains to consider $d < g$, for which $2g - 2d > 0$. Consider the closed set

$$F = \{(L,p,q) \in J^dC \times C^2 \mid h^0(L) \geq 2,\ h^1(L^2(-p-q)) \neq 0\}.$$

If its projection onto C^2 is surjective, there are a surface Σ, a finite covering $\pi = (p,q) : \Sigma \to C^2$, and a morphism $f : \Sigma \to J^dC$ such that, for every $x \in \Sigma$,

$$h^0(L_x) \geq 2, \qquad h^1(L_x^2(-p(x) - q(x))) = h^0(\omega_C \otimes L_x^{-2}(p(x) + q(x))) \geq 1,$$

where $L_x = f(x)$. The preceding argument gives $h^0(\omega_C \otimes L_x^{-2}(p(x))) = 0$, hence $h^0(\omega_C \otimes L_x^{-2}(p(x) + q(x))) = 1$.

Thus every $x \in \Sigma$ determines a unique divisor D_x in $|\omega_C \otimes L_x^{-2}(p(x) + q(x))|$. This defines a morphism

$$g : \Sigma \longrightarrow C^{(2g-2d)}, \qquad x \longmapsto D_x,$$

and $q(x) \notin \operatorname{Supp} D_x$. Fixing $q(x) = q$ gives a morphism $\Sigma_q = \pi^{-1}(C \times \{q\}) \to C^{(2g-2d)}$ whose image avoids the ample divisor $C_q^{(2g-2d-1)}$ in $C^{(2g-2d)}$ ([Fu-L], Lemma 2.7). It is therefore constant, equal to D_q, with $q \notin \operatorname{Supp} D_q$. We have

$$\forall x \in \Sigma, \qquad L_x^{-2}(p(x) + q(x)) = \omega_C^{-1}(D_{q(x)}),$$

and also $L_x^{-2}(p(x) + q(x)) = \omega_C^{-1}(D_{p(x)})$. It follows that $D_{p(x)}$ is constant, equal to D. But then $p \notin \operatorname{Supp} D$ for every p, a contradiction. Thus the projection of F onto C^2 is not surjective, proving the proposition. □

Proposition IV.A.3.2. *Let C be a generic smooth curve of genus $g \geq 6$. For generic $p, q \in C$, $W_{p,q} \cap \operatorname{Sing} \Theta$ is irreducible and reduced of dimension $g - 5$. Its cohomology class is $\theta^5/24$.*

Proof. Consider the inverse images of $\operatorname{Sing} \Theta$ and $W_{p,q}$ in $C^{(g-1)}$. The first is C^1_{g-1}, irreducible of dimension $g - 3$; the second is $C_{p,q}^{(g-3)}$, the zero locus of a section of an ample locally free sheaf of rank 2 ([Fu-L], Lemma 2.7). Their intersection T, of pure dimension $g-5$ for generic (p,q), is therefore connected in codimension 1 when $g - 3 > 2$ ([Fu-L] or [A-C-G-H 1], p. 314). If T is reducible, it is consequently singular in codimension 1.

The morphism $C_{p,q}^{(g-3)} \to \Theta$ is an isomorphism away from $(C^1_{g-3})_{p,q}$, which has dimension $g-7$ by IV.A.1. Its restriction to T is therefore an isomorphism in codimension 1, so the image of T, namely $W_{p,q} \cap \operatorname{Sing} \Theta$, is also singular in codimension 1.

Let L be a generic point of a component Z of $\operatorname{Sing}(W_{p,q} \cap \operatorname{Sing} \Theta)$ of dimension $\geq g - 6$. Since $\dim W^2_{g-1} = g - 9$ and $\dim W^1_{g-3} = g - 8$, we have $h^0(L) = 2$ and $h^0(L(-p-q)) = 1$. For $D \in |L(-p-q)|$, the notation of IV.A.1 gives

$$\begin{aligned}
H^0(L) &= \mathbf{C}s_D s_{pq} \oplus \mathbf{C}s_2, \\
H^0(\omega_C \otimes L^{-1}) &= \mathbf{C}t_1 \oplus \mathbf{C}t_2, \\
T_L \operatorname{Sing} \Theta &= \langle s_D s_{pq} t_1, s_D s_{pq} t_2, s_2 t_1, s_2 t_2 \rangle^{\perp}, \\
H^0(L(-p-q)) &= \mathbf{C}s_D, \\
H^0(\omega_C \otimes L^{-1}(p+q)) &= \mathbf{C}t_1 s_{pq} \oplus \mathbf{C}t_2 s_{pq} \oplus \mathbf{C}t, \\
T_L W_{p,q} &= \langle s_D t_1 s_{pq}, s_D t_2 s_{pq}, s_D t \rangle^{\perp}.
\end{aligned}$$

But L is singular on $W_{p,q} \cap \operatorname{Sing}\Theta$ and smooth on both $\operatorname{Sing}\Theta$ and $W_{p,q}$. Thus $T_L \operatorname{Sing}\Theta \subset T_L W_{p,q}$, and after modifying t and t_1 if necessary,

$$s_D t = s_2 t_1. \tag{IV.A.3.3}$$

Let F be the fixed part of L. First assume $p, q \notin \operatorname{Supp} F$. We can write

$$\begin{aligned} D &= F + E, \\ \operatorname{div} s_1 &= p + q + F + E, \\ \operatorname{div} s_2 &= F + M, \qquad \operatorname{Supp} M \cap \operatorname{Supp}(p + q + E) = \emptyset. \end{aligned}$$

Equation IV.A.3.3 becomes $s_E t = s_M t_1$. Thus t vanishes on M and can be written $t = s_M u$, with

$$\begin{aligned} u &\in H^0(\omega_C \otimes L^{-1}(p + q - M)) \\ &= H^0(K_C - 2M + p + q - F) \subset H^0(K_C - 2M + p + q). \end{aligned}$$

By IV.A.3.1, for generic p, q the last space vanishes for every M with $h^0(M) \geq 2$. This gives the contradiction $t = 0$.

We cannot have $p + q \leq F$, since $h^0(L(-p - q)) = 1$. Hence, for example, $p \in \operatorname{Supp} F$, $q \notin \operatorname{Supp} F$, and

$$\begin{aligned} F &= p + G, \qquad D = G + E, \\ \operatorname{div} s_1 &= p + q + G + E, \\ \operatorname{div} s_2 &= p + G + M, \qquad \operatorname{Supp} M \cap \operatorname{Supp}(q + E) = \emptyset. \end{aligned}$$

Then $s_E t = s_M s_p t_1$. As in the first case, $t = s_M u$, with

$$u \in H^0(\omega_C \otimes L^{-1}(p + q - M)) \subset H^0(K_C - 2M + p + q) = 0,$$

a contradiction. Thus $W_{p,q} \cap \operatorname{Sing}\Theta$ is nonsingular in codimension 1, hence irreducible. Its cohomology class is calculated in IV.A.3.7. □

Proposition IV.A.3.4. *Let C be a smooth connected curve of genus $g \geq 5$, neither hyperelliptic, trigonal, nor superelliptic, and let p, q be distinct points of C. Consider the rational map*

$$\begin{aligned} \phi : \operatorname{Sing}\Theta \dashrightarrow \mathbf{P}S^2 T_0^* JC \simeq \mathbf{P}S^2 H^0(C, \omega_C), \\ x \in \operatorname{Sing}_2 \Theta \longmapsto \tau_x^*(\textit{tangent cone to } \Theta \textit{ at } x). \end{aligned}$$

The projective span of the image of $W_{p,q} \cap \operatorname{Sing}\Theta$ has codimension $3g - 2$. It is the projective space of quadrics in $|\omega_C|^*$ containing the canonical curve and the chord pq.

Proof. Let $X \subset \mathbf{P}H^0(C, \omega_C)^* = \mathbf{P}^{g-1}$ be the canonical curve, and let $\mathcal{I}_X$ be its ideal in $\mathbf{P}^{g-1}$. By [Gr 1], the rational map

$$\psi : \operatorname{Sing}\Theta \dashrightarrow \mathbf{P}H^0(\mathcal{I}_X(2)) \subset \mathbf{P}S^2 H^0(C, \omega_C)$$

induced by ϕ is nondegenerate and associated with a vector subspace V of $H^0(\operatorname{Sing}\Theta, \mathcal{O}(\Theta))$. Let $\langle pq \rangle$ be the line through $p, q \in X$ in $\mathbf{P}^{g-1}$, and put

$$H_{p,q} = \mathbf{P}H^0(\mathcal{I}_{X \cup \langle pq \rangle}(2)) \subset \mathbf{P}H^0(\mathcal{I}_X(2)).$$

Under our hypotheses, the intersection of the quadrics containing X is X itself, except when C is a plane quintic ($g = 6$). In that case it is the Veronese surface in $\mathbf{P}^5$, which contains no line [SD]. Thus $H_{p,q}$ is a hyperplane of $\mathbf{P}H^0(\mathcal{I}_X(2))$. We shall show that

$$\psi^*(H_{p,q}) = (W_{p,q} \cap \operatorname{Sing}\Theta) + ((-W_{p,q}) \cap \operatorname{Sing}\Theta) = D_{p,q}. \tag{IV.A.3.5}$$

This gives the exact sequence

$$0 \longrightarrow H^0(O_{\operatorname{Sing}\Theta}) \longrightarrow H^0(O_{\operatorname{Sing}\Theta}(\Theta)) \xrightarrow{r} H^0(O_{D_{p,q}}(\Theta)),$$

showing that the span of $\psi(W_{p,q} \cap \operatorname{Sing}\Theta) = \psi(D_{p,q})$ is the hyperplane $H_{p,q}$, defined by the restriction $r(V)$. This concludes the proof.

Equation IV.A.3.5 follows from the next two lemmas and the fact that the cohomology class of $\operatorname{Sing}\Theta$ is $\theta^4/12$ ([Gri-H], p. 358), so that of $\psi^*(H_{p,q})$ is $\theta^5/12$. □

Lemma IV.A.3.6. *There is a set-theoretic equality*

$$\phi^{-1}(H_{p,q}) = [W_{p,q} \cup (-W_{p,q})] \cap \operatorname{Sing}\Theta.$$

Proof. First let $L \in W_{p,q} \cap \operatorname{Sing}_2\Theta$. Choose bases

$$H^0(L) = \mathbf{C}s_{pq}s \oplus \mathbf{C}t, \qquad H^0(\omega_C \otimes L^{-1}) = \mathbf{C}s' \oplus \mathbf{C}t'.$$

Then $\phi(L) = (s_{pq}ss')(tt') - (s_{pq}st')(s't)$ is a quadric containing the canonical curve X and the chord $\langle pq\rangle$.

Conversely, let $L \in \operatorname{Sing}_2\Theta$, and choose bases $\{s,t\}$ of $H^0(L)$ and $\{s',t'\}$ of $H^0(\omega_C \otimes L^{-1})$ such that $s(p) = s'(p) = 0$. If $L \notin W_{p,q}$, then $t(p)s(q) \neq 0$. If $\phi(L) = [(st')(ts') - (ss')(tt')]$ contains $\langle pq\rangle$, then

$$\begin{aligned}(st')(p)(ts')(q) + (st')(q)(ts')(p)\\ - (ss')(p)(tt')(q) - (ss')(q)(tt')(p) = 0.\end{aligned}$$

Hence $s'(q)t'(p) = 0$: either s' vanishes at p,q, or p is a base point of $\omega_C \otimes L^{-1}$. Thus $L \in (-W_{p,q})$. □

Lemma IV.A.3.7. *Under the hypotheses of IV.A.3.4, $W_{p,q} \cap \operatorname{Sing}\Theta$ has pure dimension $g-5$ and cohomology class $\theta^5/24$.*

Proof. By [Te], under our hypotheses $\operatorname{Sing}\Theta$ is irreducible of dimension $g-4$. If $\operatorname{Sing}\Theta \subset W_{p,q}$, then

$$\forall r,s \in C, \qquad \operatorname{Sing}\Theta \subset \Theta_{p+q-r-s}.$$

By [We 2], this would be possible only if

$$\exists x,y \in C, \qquad p+q-r-s \equiv x-y,$$

implying $\dim W_3^1 \geq 1$, hence that C is hyperelliptic by [Ma].

Thus $\dim(W_{p,q} \cap \operatorname{Sing}\Theta) \leq g-5$. The construction in [Ke-L] shows that $W_{p,q} \cap \operatorname{Sing}\Theta$ always has codimension ≤ 5 at every point, so in our situation it has pure dimension $g-5$. Its cohomology class w is calculated in the notation of [Gri-H], page 352:

$$\begin{aligned}w &= \alpha^*(\sigma_{3,2}) = \alpha^*(-\sigma_1\sigma_4 + \sigma_3\sigma_2)\\ &= -\theta\frac{\theta^4}{4!} + \frac{\theta^3}{3!}\frac{\theta^2}{2!} = \frac{\theta^5}{24}.\end{aligned}$$

□

Proposition IV.A.3.8. *Let C be a smooth curve of genus g, neither hyperelliptic, trigonal, nor superelliptic; let p,q,r,s be four distinct points of C, and let $a \in JC$ correspond to $L_a = O_C(p+q-r-s)$. There is an equality of cycles*

$$\Theta_a \cdot \operatorname{Sing}\Theta = (W_{p,q} \cap \operatorname{Sing}\Theta) + ((-W_{r,s}) \cap \operatorname{Sing}\Theta).$$

Proof. Let u be a section of $O_C(r+s)$ with divisor $r+s$, and v a section of $O_C(p+q)$ with divisor $p+q$. The kernel of

$$H^0(L) \oplus H^0(L \otimes L_a^{-1}) \xrightarrow{(\cdot u, \cdot v)} H^0(L(r+s))$$

is isomorphic to $H^0(L(-p-q))$. Hence

$$h^0(L(r+s)) + h^0(L(-p-q)) \geq h^0(L) + h^0(L \otimes L_a^{-1}),$$

and, by Riemann–Roch, if $\deg L = g-1$,

$$h^0(\omega_C \otimes L^{-1}(-r-s)) + h^0(L(-p-q)) \geq h^0(L) + h^0(L \otimes L_a^{-1}) - 2.$$

This gives the set-theoretic equality in the statement. Equality of cycles follows from IV.A.3.7 and the fact that $\operatorname{Sing}\Theta$ has class $\theta^4/12$. □

Lemma IV.A.3.9. *Let C be a generic smooth curve of genus $g \geq 6$, and let p,q be generic points of C. Then $W_{p,q} \cap (-W_{p,q}) \cap \operatorname{Sing}\Theta$ has dimension $\leq g-6$.*

Proof. By IV.A.3.2 it is enough to show that $W_{p,q} \cap \operatorname{Sing}\Theta \subset (-W_{p,q})$ cannot hold for generic p,q. Otherwise, for generic p,q,

$$\begin{aligned} \forall M \in W_{g-2}^1, \qquad M(p) &\in W_{p,q} \cap \operatorname{Sing}\Theta \\ &\Longrightarrow h^0(\omega_C \otimes M^{-1}(-2p-q)) \geq 1 \\ &\Longrightarrow h^0(\omega_C \otimes M^{-1}) \geq 4 \\ &\Longrightarrow h^0(M) \geq g-2+1-g+4 = 3. \end{aligned}$$

This is impossible, since W_{g-2}^1 is not contained in W_{g-2}^2; by IV.A.1, W_{g-2}^1 is nonempty for $g \geq 6$. □

Proposition IV.A.3.10. *Let C be a smooth nonhyperelliptic curve of genus $g \geq 6$. For generic $p,q,r,s \in C$, the locus $W_{p,q} \cap (-W_{r,s})$ is irreducible of dimension $g-5$, with cohomology class $6\theta^5/5!$. Its image under the Gauss map $\Theta \dashrightarrow \mathbf{P}T_0^*JC = |\omega_C|$ is $|\omega_C(-p-q-r-s)| \simeq \mathbf{P}^{g-5}$.*

Proof. Let $\varphi : C^{(g-1)} \to \Theta$ be the natural map. Put

$$Z = \{D \in C^{(g-3)} \mid \exists E \in |K_C - p - q - r - s|,\ E \geq D\}.$$

Then $W_{p,q} \cap (-W_{r,s}) = \varphi(Z_{p,q})$. For generic p,q,r,s, $|H| = |K_C - p - q - r - s|$ is a g^{g-5}_{2g-6}, and Z has dimension $g-5$ ([A-C-G-H 1], Lemma 3.2, p. 342) and is connected for $g \geq 6$ by [Fu-L]. If $|H|$ had a base point x for every $(p,q,r,s) \in C^4$, the line bundles $O(p+q+r+s+x)$ would describe a subvariety of dimension ≥ 3 in W_5^1, forcing C to be hyperelliptic by [Ma]. Thus we may assume $|H|$ base point free.

It is classical ([A-C-G-H 1], p. 112, or II.B.6) that Z is irreducible if $\varphi_{|H|} : C \to \mathbf{P}^{g-5}$ is birational. For $g \geq 7$, this follows because the canonical map is birational and a nondegenerate

curve in $\mathbf{P}^r$, $r \geq 3$, has at most a one-dimensional family of trisecants (Exercise L.1, p. 152 of [A-C-G-H 1]): projection from a generic point of the curve is therefore birational.

For $g = 6$ this never holds, so we use another approach. Since Z is connected, so is $W_{p,q} \cap (-W_{r,s})$. It suffices to show that this intersection is smooth.

Let $L \in W_{p,q} \cap (-W_{r,s})$. Then

$$\begin{aligned} H^0(L(-p-q)) &= \mathbf{C}u, \\ H^0(\omega_C \otimes L^{-1}(-r-s)) &= \mathbf{C}u', \\ H^0(\omega_C \otimes L^{-1}(p+q)) &= \mathbf{C}u's_{pqrs} \oplus \mathbf{C}u'_1 \oplus \mathbf{C}u'_2, \\ H^0(L(r+s)) &= \mathbf{C}us_{pqrs} \oplus \mathbf{C}u_1 \oplus \mathbf{C}u_2. \end{aligned}$$

Thus in $T_L JC \simeq H^0(C, \omega_C)^*$,

$$\begin{aligned} T_L W_{p,q} &= \langle uu's_{pqrs}, uu'_1, uu'_2 \rangle^\perp, \\ T_L(-W_{r,s}) &= \langle uu's_{pqrs}, u'u_1, u'u_2 \rangle^\perp. \end{aligned}$$

If the intersection is not smooth at L, there is a relation of the form $uu'_1 = u'u_1$. Write

$$\begin{aligned} \operatorname{div}(u) = E + A, \quad \operatorname{div}(u') = E + B, \quad A \cap B = \emptyset, \\ \operatorname{div}(u_1) = F + C, \quad \operatorname{div}(u'_1) = F + D, \quad C \cap D = \emptyset. \end{aligned}$$

Then $A = C$ and $B = D$, so

$$\begin{cases} F \equiv p + q + r + s + E, \\ 2E + A + B \equiv K_C - p - q - r - s. \end{cases}$$

We cannot have $F \geq p + q + r + s$, hence $h^0(F) \geq 2$.

If $\deg E = 1$, then F is a g^1_5. This gives $\dim W^1_5 \geq 3$, forcing C to be hyperelliptic [Ma].

If $\deg E \geq 2$, then $2E+A+B+s$ corresponds to a bitangent of $\varphi_{|K_C-p-q-r|}(C) \subset \mathbf{P}^2$, or to a line through a point of multiplicity ≥ 4. Since a plane curve has only finitely many bitangents, the second possibility holds generically: $h^0(K_C - p - q - r - 2E) \geq 2$. This contradicts $\dim W^1_3 \leq 0$. Thus $W_{p,q} \cap (-W_{r,s})$ is smooth and connected, hence irreducible.

The Gauss map induces on Z the rational map

$$\begin{aligned} &Z \dashrightarrow |\omega_C|, \\ &D \longmapsto (D + p + q) + (D' + r + s), \end{aligned} \qquad D' \in |\omega_C(-p-q-r-s-D)|,$$

whose image is $|H| \subset |\omega_C|$.

Finally, apply [M] or [A-C-G-H 1], Chapter VIII, to calculate the cohomology class. In their notation,

$$\begin{aligned} [Z_{p,q}] &= \eta^2 [t^2]((1+\eta t)^{-1} e^{t\varphi^*\theta}) \\ &= \eta^2 \left(\eta^2 - \eta\varphi^*\theta + \tfrac{1}{2}\varphi^*\theta^2\right), \end{aligned}$$

and

$$\begin{aligned} [W_{p,q} \cap (-W_{r,s})] &= \varphi_* \left(\eta^4 - \eta^3\varphi^*\theta + \tfrac{1}{2}\eta^2\varphi^*\theta^2\right) \\ &= \frac{\theta^5}{5!} - \frac{\theta^4}{4!}\theta + \frac{1}{2 \cdot 3!}\theta^3\theta^2 = 6\frac{\theta^5}{5!}. \end{aligned}$$

□

IV.A.4 Singularities of the theta divisor of a generalized Prym variety

In [Be 1], §3, Beauville extended Mumford's results on theta-divisor singularities for Prym varieties of smooth étale double covers to singular curves. However, he considers only covers satisfying property $(*)$ of [Be 1], page 157, with no components exchanged by the involution.

We show that these results extend to all covers satisfying condition $(**)$ below, which give every generalized Prym variety.

Let $\widetilde{C}$ be a connected curve of genus $2g+1$ with at most nodal singularities, equipped with an involution σ, and let C be its quotient. Assume:

$$\begin{gathered}\sigma \text{ is not the identity on any component of } \widetilde{C};\\ \text{the fixed points of } \sigma \text{ are nodes of } \widetilde{C}, \text{ with branches not exchanged by } \sigma;\\ \text{the number of components of } \widetilde{C} \text{ exchanged by } \sigma \text{ equals the number of exchanged nodes.}\end{gathered} \tag{**}$$

By Lemma 5.1 of [Be 1], C has genus $g+1$, and $P^0 = (\ker \mathrm{Nm})^0 \subset J\widetilde{C}$ is an abelian variety of dimension g.

Proposition IV.A.4.1. *Let M be a line bundle on $\widetilde{C}$ with $\mathrm{Nm}\, M = \omega_C$. The function $L \mapsto h^0(L \otimes M) \bmod 2$ is constant on P^0. Such an M exists with $h^0(M) = 0$.*

Proof. The first assertion has the same proof as Proposition 3.4 of [Be 1]. We prove the existence of M. For every multidegree $\mathbf{d}$ on C, there is a multidegree $\widetilde{\mathbf{d}}$ on $\widetilde{C}$ such that $\mathrm{Nm}(\mathrm{Pic}^{\widetilde{\mathbf{d}}}(\widetilde{C})) \subset \mathrm{Pic}^{\mathbf{d}}(C)$. Since $\mathrm{Nm} : J\widetilde{C} \to JC$ is surjective, a line bundle M on $\widetilde{C}$ with $\mathrm{Nm}\, M = \omega_C$ exists.

If $H^0(\widetilde{C}, M) \neq 0$, the argument of Step II, Lemma 3.2 of [Mu 3], gives a smooth point x of $\widetilde{C}$ such that

$$h^0(\widetilde{C}, M \otimes O_{\widetilde{C}}(x - \sigma x)) = h^0(\widetilde{C}, M) - 1.$$

Thus we can find M with $\mathrm{Nm}\, M = \omega_C$ and $H^0(\widetilde{C}, M) = 0$. □

It can happen that no M with these two properties has the form $\pi^* L_0$ (see the example in subsection 6 and compare Proposition 3.10 of [Be 1]).

Fix such an M, and define $\mathrm{Pic}^M(\widetilde{C})$ as the translate of $J\widetilde{C}$ by M: it is the connected component of $\mathrm{Pic}(\widetilde{C})$ consisting of line bundles with the same multidegree as M. Let P be the connected component of $\{L \in \mathrm{Pic}^M(\widetilde{C}) \mid \mathrm{Nm}\, L = \omega_C\}$ containing M, and put

$$\widetilde{\Theta} = \{L \in \mathrm{Pic}^M(\widetilde{C}) \mid h^0(L) \geq 1\}.$$

Then $\widetilde{\Theta}$ is a divisor in $\mathrm{Pic}^M(\widetilde{C})$ (Proposition 2.2 of [Be 1]).

Proposition IV.A.4.2. *We have*

$$P = \{L \in \mathrm{Pic}^M(\widetilde{C}) \mid \mathrm{Nm}\, L = \omega_C,\ h^0(L) \text{ even}\},$$

and $\widetilde{\Theta}|_P = 2\Xi$, where $\Xi = \{L \in P \mid h^0(L) \geq 2\}$.

Proof. The forward inclusion in the first assertion follows from IV.A.4.1. The proof of 5.4 in [Be 1] gives two possibilities:

- $\widetilde{C} = A \cup \sigma A$ with $\#(A \cap \sigma A) = 2$. Then $\mathrm{Nm} : J\widetilde{C} \to JC$ has connected kernel P^0, and $P = \{L \in \mathrm{Pic}^M(\widetilde{C}) \mid \mathrm{Nm}\, L = \omega_C\}$.

- Some component of $\widetilde{C}$ is invariant under σ, and $\ker \mathrm{Nm}$ has two connected components P^0 and P^1. Translating by M gives the two components of the norm fiber over ω_C. If L belongs to the first, and x is a smooth point on an invariant component of $\widetilde{C}$, then

$$h^0(L \otimes O_{\widetilde{C}}(x - \sigma x)) = h^0(L) \pm 1.$$

  The twisted bundle therefore belongs to the second component, where h^0 is odd. This proves the first assertion.

The second assertion has the same proof as 3.10 of [Be 1]. □

There are two types of singularities of Ξ: either $h^0(L) \geq 4$; or $h^0(L) = 2$ and, for every basis $\{s, t\}$ of $H^0(L)$,

$$\sigma^* s \otimes t = s \otimes \sigma^* t \quad \text{in } H^0(\widetilde{C}, L \otimes \sigma^* L) = H^0(\widetilde{C}, \omega_{\widetilde{C}}).$$

The first are called stable singularities, the second exceptional singularities. We now study stable singularities in families of double covers.

IV.A.4.3. Let $\widetilde{C} \to T$ be a stable curve of genus $2g+1$ with an involution σ satisfying $(**)$ on the fibers. Assume $\widetilde{C}$ smooth and T a polydisc; we may shrink T without always saying so.

Put $C = \widetilde{C}/\sigma$. It is a normal analytic space. By Theorem 3.1 of [Gro 2], the Picard spaces $\mathrm{Pic}(\widetilde{C}/T)$ and $\mathrm{Pic}(C/T)$ exist. These analytic spaces over T are generally nonseparated, with respective fibers $\mathrm{Pic}(\widetilde{C}_t)$ and $\mathrm{Pic}(C_t)$. There is a norm morphism $\mathrm{Nm} : \mathrm{Pic}(\widetilde{C}/T) \to \mathrm{Pic}(C/T)$.

We also use the analytic subspace $\mathrm{Pic}^0(\widetilde{C}/T)$, whose fiber at t is $\mathrm{Pic}^0(\widetilde{C}_t)$ and which is quasi-projective over T ([De], Proposition 4.3).

Lemma IV.A.4.4. *There is a line bundle M on $\widetilde{C}$ such that*

$$\mathrm{Nm}\, M = \omega_{C/T}, \qquad H^0(\widetilde{C}_t, M_t) = 0 \quad \textit{for every } t \in T.$$

Proof. Let $\mathcal{D}$ be an effective Cartier divisor on C with $O_C(\mathcal{D}) = \omega_{C/T}$ whose restriction to C_0 is a sum of $2g$ distinct smooth points. After shrinking T, $\mathcal{D}$ is a sum of $2g$ sections of $C \to T$, which lift to sections of $\widetilde{C} \to T$. This gives an effective Cartier divisor $\widetilde{\mathcal{D}}$ on $\widetilde{C}$, and $M = O(\widetilde{\mathcal{D}})$ satisfies the first condition.

As in the proof of IV.A.4.1, there are smooth points $x_1^0, \dots, x_m^0$ of $\widetilde{C}_0$ such that

$$H^0\left(\widetilde{C}_0, M_0\Big(\sum_{i=1}^m (x_i^0 - \sigma x_i^0)\Big)\right) = 0.$$

Choose local sections $x_1, \dots, x_m$ with these values at 0. If X_i are their images, then $M \otimes O(\sum_{i=1}^m (X_i - \sigma X_i))$ satisfies both conditions after shrinking T. □

Fix such an M and define $\mathrm{Pic}^M(\widetilde{C}/T)$ as the translate of $\mathrm{Pic}^0(\widetilde{C}/T)$ by M. The associated relative Prym variety is

$$\mathcal{P} = \{L \in \mathrm{Pic}^M(\widetilde{C}/T) \mid \mathrm{Nm}\, L = \omega_{C/T}\}^M,$$

whose fiber at t is $\{L_t \in \mathrm{Pic}^{M_t}(\widetilde{C}_t) \mid \mathrm{Nm}\, L_t = \omega_{C_t},\ h^0(\widetilde{C}_t, L_t) \text{ even}\}$; the superscript M denotes the component containing M. Following the usual determinantal construction ([Kl-L]), one can define an analytic subspace $\mathcal{W}_M^r$ of $\mathrm{Pic}^M(\widetilde{C}/T)$ whose underlying set is

$$\{L_t \in \mathrm{Pic}^{M_t}(\widetilde{C}_t) \mid h^0(\widetilde{C}_t, L_t) \geq r+1\}.$$

Set $\Xi = \mathcal{P} \cdot \mathcal{W}_M^1$ and $S = \mathcal{P} \cdot \mathcal{W}_M^3$. For $t \in T$, (P_t, Ξ_t) is then the Prym variety associated with the cover $\widetilde{C}_t \to C_t$, and S_t is the set of stable singularities of Ξ_t.

Proposition IV.A.4.5. *The subspace S of $\mathcal{P}$ is empty or has codimension at most 6 everywhere.*

Proof. Use the construction of [Mu 3], generalized in Theorem 1.1 of [Be 1], and apply Remark (1.6) of [H] (see the proof of IV.A.5.5). □

Section II.D.1.2.5 shows that, when $g \geq 6$, S_t is never empty if C_t is smooth. It is easy to deduce that, in the situation above, S is never empty for $g \geq 6$.

IV.A.5 The intersections $\Xi \cdot \Xi_{r+s-\sigma r-\sigma s}$ on a Prym variety

In many respects, the intersections $\Xi \cdot \Xi_{r+s-\sigma r-\sigma s}$ on Prym varieties play a role analogous to that of $\Theta \cdot \Theta_{p-q}$ on Jacobians (see IV.B). We study the former here, particularly their singular loci and the behavior of these loci in families.

Theorem IV.A.5.1. *Let $\pi : \widetilde{C} \to C$ be an étale double cover of smooth connected curves, with associated involution σ. Suppose that C is neither hyperelliptic nor trigonal and has genus $g+1$, and let (P, Ξ) be the Prym variety associated with π in $\operatorname{Pic}^{2g} \widetilde{C}$. Take two points $r, s \in \widetilde{C}$ with $s \neq \sigma r$, and let $a \in \operatorname{Pic}^0 \widetilde{C}$ correspond to the line bundle $L_a = \mathcal{O}_{\widetilde{C}}(r+s-\sigma r-\sigma s)$. Then the intersection $\Xi \cdot \Xi_a$ is reduced and*

$$\Xi \cdot \Xi_a = \{L \in P \mid h^0(\widetilde{C}, L(-r-s)) \geq 1\},$$
$$\operatorname{Sing}(\Xi \cdot \Xi_a) \supseteq \{L \in P \mid h^0(\widetilde{C}, L(-r-s)) \geq 2\},$$

with equality in the second formula for general C and general r, s on $\widetilde{C}$.

Proof. We begin with the following lemma.

Lemma IV.A.5.2. *For $L \in P$, one has*

$$h^0(L(-r-s)) \geq \tfrac{1}{2}[h^0(L) + h^0(L \otimes L_a^{-1})] - 1.$$

Proof. Let t be a section of $\mathcal{O}(r+s)$ with divisor $r+s$. The kernel of the map

$$H^0(L) \oplus H^0(L \otimes L_a^{-1}) \xrightarrow{(\sigma^* t, -t)} H^0(L(\sigma r + \sigma s))$$

is isomorphic to $H^0(L(-r-s))$. Hence

$$\begin{aligned} h^0(L) + h^0(L \otimes L_a^{-1}) - h^0(L(-r-s)) \\ &\leq h^0(L(\sigma r + \sigma s)) = h^0(\sigma^* L(r+s)) \\ &= h^0(\omega_{\widetilde{C}} \otimes (\sigma^* L)^{-1}(-r-s)) + 2 \\ &= h^0(L(-r-s)) + 2, \end{aligned}$$

by Riemann–Roch and the isomorphism $L \otimes \sigma^* L \simeq \omega_{\widetilde{C}}$. □

It follows that, set-theoretically,

$$\Xi \cap \Xi_a = \{L \in P \mid h^0(L(-r-s)) \geq 1\}. \tag{IV.A.5.3}$$

Since C is neither hyperelliptic nor trigonal, the linear system $|H| = |\omega_C(-\pi r - \pi s)|$ is a base-point-free g^{g-2}_{2g-2}. Following [Be 3], associate with it the reduced subvariety $S = (\pi^{(2g-2)})^{-1}(|H|)$ of $\widetilde{C}^{(2g-2)}$.

The image of $S + r + s \subset \widetilde{C}^{(2g)}$ in $\mathrm{Pic}^{2g}\,\widetilde{C}$ then has two connected components. Exactly one of them, V, lies in P, and by IV.A.5.3 it is $\Xi \cap \Xi_a$. Theorem 1 of [Be 3] shows that the class of V in $H^4(P, \mathbf{Q})$ is $[\Xi]^2$, so the intersection $\Xi \cdot \Xi_a$ is reduced.

Let $L \in P$ satisfy $h^0(L(-r-s)) \geq 2$. We show that L is singular on $\Xi \cdot \Xi_a$. If $h^0(L) \geq 4$ or $h^0(L \otimes L_a^{-1}) \geq 4$, then L is singular on Ξ or on Ξ_a, hence on $\Xi \cdot \Xi_a$. If $h^0(L) = h^0(L \otimes L_a^{-1}) = 2$, take a basis $\{u, v\}$ of $H^0(L(-r-s))$. Writing s_{rs} for the section with divisor $r+s$, we have

$$H^0(L) = \mathbf{C}us_{rs} \oplus \mathbf{C}vs_{rs},$$
$$H^0(L \otimes L_a^{-1}) = \mathbf{C}u\sigma^* s_{rs} \oplus \mathbf{C}v\sigma^* s_{rs}.$$

By [Mu 2], §6, the hyperplane $T_L\Xi$ in $T_L P \simeq H^0(\widetilde{C}, \omega_{\widetilde{C}})^{*,-}$ corresponds to

$$[us_{rs}\,\sigma^* v\,\sigma^* s_{rs} - \sigma^* u\,\sigma^* s_{rs}\,vs_{rs}],$$

and $T_L\Xi_a$ corresponds to

$$[u\sigma^* s_{rs}\,\sigma^* v\,s_{rs} - \sigma^* u\,s_{rs}\,v\sigma^* s_{rs}].$$

The two hyperplanes are equal, so L is singular on $\Xi \cdot \Xi_a$.

Now suppose that C is general and r, s are general on $\widetilde{C}$. Recall ([Be 3], §1) that there is a surjective morphism

$$f : S_0 \subset \widetilde{C}^{(2g-2)} \longrightarrow \Xi \cap \Xi_a, \qquad D \longmapsto O_{\widetilde{C}}(D + r + s).$$

The fiber over L is isomorphic to $|L(-r-s)|$. Thus it suffices to show that S_0 is smooth; by Proposition 3 of [Be 3], this means that, for every effective divisor A on C,

$$h^0(H - 2A) \neq 0 \quad \Longrightarrow \quad h^0(H - A) = h^0(H) - \deg A.$$

By Riemann–Roch, with $r' = \pi(r)$ and $s' = \pi(s)$, this becomes

$$h^0(A + r' + s') \geq 2 \quad \Longrightarrow \quad h^0(K_C - 2A - r' - s') = 0.$$

This property follows from Proposition IV.A.3.1. □

The points $L \in P$ such that $h^0(\widetilde{C}, L(-r-s)) \geq 2$ will be called stable singularities of $\Xi \cdot \Xi_a$.

This definition is justified by the analogy between their properties and those of the stable singularities of the divisor Ξ defined in IV.A.4. In particular, we shall show that they always exist for $g \geq 5$ (see IV.A.6.9), and that their locus has dimension at least $g - 5$ everywhere (see IV.A.5.5).

IV.A.5.4. We now study these singularities in families. Retain the assumptions and notation of IV.A.4.3, with a family of double covers $\widetilde{\mathcal{C}} \to \mathcal{C} = \widetilde{\mathcal{C}}/\sigma \to T$. We have also chosen a line bundle M on $\widetilde{\mathcal{C}}$ satisfying $\mathrm{Nm}\, M = \omega_{\mathcal{C}/T}$ and $H^0(\widetilde{\mathcal{C}}_t, M_t) = 0$ for every t.

Take two disjoint sections r, s and define the analytic subspace $S^{r,s}$ of $\mathcal{P}$ by the procedure of [Kl-L]. Using the Poincaré bundle on $\widetilde{\mathcal{C}} \times_T \mathrm{Pic}^M(\widetilde{\mathcal{C}}/T)$, construct locally free sheaves E, F on $\mathrm{Pic}^M(\widetilde{\mathcal{C}}/T)$ of ranks $m, m+2$, respectively, together with a morphism $f : E \to F$ such that, set-theoretically,

$$\{L_t \mid \mathrm{rk}\, f_{L_t} \leq m - 2\} = \{L_t \mid h^0(\widetilde{C}_t, L_t(-r(t) - s(t))) \geq 2\}.$$

The space $S^{r,s}$ is then defined as the determinantal variety

$$S^{r,s} = Z\left(\bigwedge^{m-1} f\Big|_{\mathcal{P}}\right).$$

When C_t is smooth, $S_t^{r,s}$ is thus the set of stable singularities of $\Xi_t \cdot \Xi_{t,a(t)}$, where $a(t) = r(t) + s(t) - \sigma r(t) - \sigma s(t)$.

Proposition IV.A.5.5. *The subspace $S^{r,s}$ of $\mathcal{P}$ is empty or has codimension at most 5 everywhere.*

Proof. Choose $N+1$ sections $x_0, \dots, x_N$ of $\widetilde{\mathcal{C}} \to T$ such that

$$r, s, \sigma x_0, \dots, \sigma x_N \quad \text{are pairwise disjoint,} \tag{IV.A.5.6}$$

and, for every t,

$$\text{all components of the multidegree of } M_t\Big(r(t) + s(t) - \sum_{i=1}^{N} x_i(t)\Big) \text{ are negative.} \tag{IV.A.5.7}$$

Use the following notation:

$$\begin{array}{ccccc} \mathcal{P} \times_T \widetilde{\mathcal{C}} & & & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \tilde{f}} & & & \\ \mathcal{P} \times_T C & \xrightarrow[f]{} & \mathcal{P} & \xrightarrow{p} & T. \end{array}$$

We again write $r, s, x_0, \dots, x_N$ for the induced sections of $\tilde{f}$, and $R, S, X_0, \dots, X_N$ for their images in $\mathcal{P} \times_T \widetilde{\mathcal{C}}$.

Let $\mathcal{F}$ be a Poincaré bundle on $\mathcal{P} \times_T \widetilde{\mathcal{C}}$, with $\mathcal{F}|_{\{L_t\}\times \widetilde{C}_t} \simeq L_t$ for $L_t \in \mathcal{P}$ and $\mathcal{F}|_{X_0} \simeq \mathcal{O}_{X_0}$.

The sheaf $E = \pi_* \mathcal{F}$ is locally free of rank 2 on $\mathcal{P} \times_T C$. It carries a nondegenerate quadratic form with values in $\omega_{\mathcal{P}\times_T C/\mathcal{P}}$, as one proves just as in Proposition 3.4 of [Be 1]. Following [Mu 3], set

$$\begin{aligned} A &= \pi_*\Big(\sum_{i=1}^{N} X_i\Big), && \text{a Cartier divisor on } \mathcal{P} \times_T C, \\ V &= f_*(E(A)/E(-A)), \\ W_1 &= f_*E(A), \qquad W_2 = f_*(E/E(-A)), && \text{subsheaves of } V. \end{aligned}$$

By IV.A.5.7, for every $L_t \in \mathcal{P}$,

$$H^0(C_t, E_{L_t}(-A_t)) \simeq H^0(\widetilde{C}_t, L_t(-\pi^* A_t)) = 0.$$

As in [Mu 3], it follows that W_1, W_2, V are locally free of ranks $2N, 2N, 4N$, respectively. There is a nondegenerate quadratic form on V for which W_1, W_2 are totally isotropic. Moreover,

$$W_{1,L_t} \cap W_{2,L_t} \simeq H^0(\widetilde{C}_t, L_t) \qquad (L_t \in \mathcal{P}).$$

We also need

$$W_0 = f_*(\pi_*(\mathcal{F}(-R-S)) \otimes \mathcal{O}(A)) \subset W_1.$$

This sheaf is locally free of rank $2N-2$ (apply IV.A.5.7), and

$$\begin{aligned} W_{0,L_t} &\simeq H^0(C_t, \pi_*(L_t(-r_t - s_t)) \otimes \mathcal{O}(A_t)) \\ &\simeq H^0(\widetilde{C}_t, L_t(-r_t - s_t + \pi^* A_t)). \end{aligned}$$

There is the following commutative diagram, analogous to the one in [Mu 3], p. 183. To simplify its notation, put $\widetilde{C} = \widetilde{C}_t$, $L = L_t$, $r = r_t$, $s = s_t$, and $B = \pi^* A_t$:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(L(-r-s)) & \longrightarrow & H^0(L) = (W_1 \cap W_2)_{L_t} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(L(-r-s+B)) & \longrightarrow & H^0(L(B)) = W_{1,L_t} \\
 & & \downarrow & & \downarrow \\
 & & H^0(L(-r-s+B)|_B) & = & H^0(L(B)|_B).
\end{array}$$

This proves that $(W_0 \cap W_2)_{L_t} \simeq H^0(\widetilde{C}_t, L_t(-r_t - s_t))$. Thus, set-theoretically,

$$S^{r,s} = \{L \in \mathcal{P} \mid \dim(W_{0,L} \cap W_{2,L}) \geq 2\}.$$

To prove the proposition, it therefore suffices to apply the following lemma.

Lemma IV.A.5.8. *Let P be an irreducible variety, $E \to P$ a vector bundle of rank $2n$ with a nondegenerate quadratic form Q, and W^n, W^k isotropic subbundles of ranks n and $k < n$, respectively. The locus*

$$\{x \in P \mid \dim(W^n_x \cap W^k_x) \geq r\}$$

is empty or has codimension at most $\frac{r(r-1)}{2} + r(n-k)$ at every point.

Proof. As in [H], 1.6, it suffices to prove the following. Let V be a vector space of dimension $2n$ with a nondegenerate quadratic form Q, let Λ_0 be an isotropic subspace of dimension n, and let Σ^k be the irreducible subvariety of $\mathrm{Gr}(k, V)$ formed by isotropic subspaces ([Gri-H], p. 739). Then

$$\Sigma^k(\Lambda_0) = \{\Lambda^k \in \Sigma^k \mid \dim(\Lambda^k \cap \Lambda_0) \geq r\}$$

is closed and irreducible of codimension $\frac{r(r-1)}{2} + r(n-k)$.

As in [H], 1.5, introduce

$$\Gamma = \{(\Lambda^k, \Delta) \in \Sigma^k \times \mathrm{Gr}(r, \Lambda_0) \mid \Delta \subset \Lambda^k\}.$$

By [Gri-H], p. 739, the fiber of $\mathrm{pr}_2 : \Gamma \to \mathrm{Gr}(r, \Lambda_0)$ is irreducible of dimension

$$(k-r)(2n-2r-k+r) - \frac{(k-r)(k-r+1)}{2}.$$

The projection $\mathrm{pr}_1 : \Gamma \to \Sigma^k$ is birational onto its image $\Sigma^k(\Lambda_0)$. Hence

$$\begin{aligned}
\mathrm{codim}_{\Sigma^k} \Sigma^k(\Lambda_0) &= \mathrm{codim}_{\Sigma^k}(\text{fiber of } \mathrm{pr}_2) - \dim \mathrm{Gr}(r, \Lambda_0) \\
&= k(2n-k) - \frac{k(k+1)}{2} \\
&\quad - (k-r)(2n-2r-k+r) + \frac{(k-r)(k-r+1)}{2} - r(n-r) \\
&= 2nr - r^2 - kr + \frac{r(r-1)}{2} - nr + r^2 \\
&= r(n-k) + \frac{r(r-1)}{2}.
\end{aligned}$$

□

Apply the lemma with $n = 2N$, $k = 2N-2$, and $r = 2$. The codimension of $S^{r,s}$ is therefore at most $1+4 = 5$. □

IV.A.6 The fundamental example

We now apply the results of IV.A.4 and IV.A.5 to the following situation. Let Δ be the unit disk in $\mathbf{C}$, and consider a family $\tilde{f} : (\widetilde{C}, \sigma) \to \Delta$ such that:

- $\widetilde{C}$ is a smooth surface;
- C_0 is obtained from a general curve N of genus $g \geq 6$ by identifying two general points p, q of N;
- for $t \neq 0$, C_t is a general smooth irreducible curve of genus $g+1$;
- $\pi_0 : \widetilde{C}_0 \to C_0$ is the Wirtinger cover of C_0: the curve $\widetilde{C}_0$ is obtained from the disjoint union $N_1 \sqcup N_2$, where $N_1 = N_2 = N$, by identifying p (respectively q) on N_1 with q (respectively p) on N_2.

If L_0 is a line bundle of degree g on C_0 with $L_0^{\otimes 2} \simeq \omega_{C_0}$, [H], Theorem 2.14, gives that $\pi_0^* L_0$ has multidegree (g, g) and

$$h^0(\pi_0^* L_0) = 2h^0(L_0) \pm 1 \quad \text{is odd.}$$

Applying Lemma 2.1(iii) of [Be 1], we conclude that any bundle M on $\widetilde{C}$ satisfying the conclusions of IV.A.4.4, namely

$$\operatorname{Nm} M = \omega_{C/\Delta}, \qquad h^0(\widetilde{C}_t, M_t) = 0,$$

has multidegree $(g-1, g+1)$ or $(g+1, g-1)$ on $\widetilde{C}_0$. The two cases are exchanged by replacing M with $\sigma^* M$.

The choice of M thus defines a family of Prym varieties $\mathcal{P}$. If, for example, $\deg M_0 = (g-1, g+1)$, then $P_0 \subset \operatorname{Pic}^{(g-1,g+1)} \widetilde{C}_0$. We also know ([Be 1], Theorem 5.4) that $P_0 \simeq JN$. This can be seen as follows. Let n be the normalization of $\widetilde{C}_0$ and let pr_1 be the projection $J(N_1 \sqcup N_2) \to JN_1 \xrightarrow{\sim} JN$. The morphism

$$\operatorname{pr}_1 n^* : P_0 \longrightarrow J^{g-1}N$$

is an isomorphism that sends

$$\Xi_0 = \{L \in P_0 \mid h^0(L) \geq 2\} \quad \text{to} \quad \Theta = \{L \in J^{g-1}N \mid h^0(L) \geq 1\}.$$

Proof. Let $\mathcal{L} \in P_0$. The equality $\mathcal{L} \otimes \sigma^* \mathcal{L} = \omega_{\widetilde{C}_0}$ gives

$$n^* \mathcal{L} = (L, \omega_N(p+q) \otimes L^{-1}), \qquad L \in J^{g-1}N.$$

There is an exact sequence

$$0 \longrightarrow H^0(\mathcal{L}) \xrightarrow{n^*} H^0(L) \oplus H^0(\omega_N(p+q) \otimes L^{-1}) \longrightarrow \mathbf{C} \oplus \mathbf{C}.$$

If $L \notin \Theta$, then $H^0(L) = H^0(\omega_N \otimes L^{-1}) = 0$ and

$$\begin{aligned} H^0(\mathcal{L}) &\simeq \{u \in H^0(\omega_N(p+q) \otimes L^{-1}) \mid u(p) = u(q) = 0\} \\ &\simeq H^0(\omega_N \otimes L^{-1}) = 0, \end{aligned}$$

so $\mathcal{L} \notin \Xi_0$. Thus $\operatorname{pr}_1 n^*(\Xi_0) \subset \Theta$; since Θ is irreducible, equality holds. □

We now determine which points of $\operatorname{Sing}\Theta$ correspond to stable singularities in the Prym sense. Use the notation of IV.A.3:

$$W = W^0_{g-3} \subset J^{g-3}N, \qquad W_{p,q} = W + p + q \subset J^{g-1}N.$$

Lemma IV.A.6.1. *The stable singularities of Ξ_0 are the points of $W_{p,q} \cap \operatorname{Sing}\Theta$.*

Proof. Retain the notation above. If $h^0(\mathcal{L}) \geq 4$, then $\mathcal{L}$ is singular on Ξ_0, so L is singular on Θ and $h^0(L) \geq 2$. If $L \notin W_{p,q}$, then $h^0(L(-p-q)) = 0$, $h^0(\omega_N(p+q) \otimes L^{-1}) = 2$, and $h^0(L) = 2$. All sections of $\omega_N(p+q) \otimes L^{-1}$ therefore vanish at p, q, and

$$H^0(\mathcal{L}) \simeq H^0(L(-p-q)) \oplus H^0(\omega_N \otimes L^{-1})$$

has dimension 2, a contradiction. Hence $L \in W_{p,q}$.

Conversely, if $L \in W_{p,q} \cap \operatorname{Sing}\Theta$, choose nonzero $s \in H^0(L(-p-q))$ and linearly independent $\bar{s}, \bar{t} \in H^0(\omega_N \otimes L^{-1})$. Then

$$n^*H^0(\mathcal{L}) \supset \langle (ss_{pq}, 0), (0, \bar{s}s_{pq}), (0, \bar{t}s_{pq}) \rangle,$$

where s_{pq} is a section with divisor $p+q$. Thus $h^0(\mathcal{L}) \geq 3$; since this dimension is even, it is at least 4. □

Remark IV.A.6.2. In IV.A.4.3 we introduced the following closed subset of $\mathcal{P}$:

$$S = \bigcup_{t \in \Delta} \{\text{stable singularities of } \Xi_t\}.$$

Since all singularities of Ξ are stable for a general cover, $S_t = \operatorname{Sing}\Xi_t$ for $t \neq 0$. By [We 4],

$$\lim_{t \to 0} S_t \subset W_{p,q} \cap (-W_{p,q}) \cap \operatorname{Sing}\Theta,$$

which has dimension at most $g-6$ by IV.A.3.9. Thus S is reducible and has codimension 6 everywhere in $\mathcal{P}$. One component is $W_{p,q} \cap \operatorname{Sing}\Theta$, over 0; the others dominate Δ.

Now consider two disjoint, general sections r, s of $\widetilde{C} \to \Delta$. Replacing them by their images under σ if necessary, we may assume that $r_0, s_0 \in N_2 \setminus N_1$; their images in N are chosen general and distinct from p, q.

The line bundle $\mathcal{O}_{\widetilde{C}}(r+s-\sigma r-\sigma s+N_2)$ has multidegree zero on every fiber. It corresponds to a section a of $\mathcal{P}^0 \to \Delta$ such that, for $t \neq 0$, a_t corresponds to $\mathcal{O}_{\widetilde{C}_t}(r_t + s_t - \sigma r_t - \sigma s_t)$, and a_0 corresponds in JN to

$$\operatorname{pr}_1 n^*\mathcal{O}_{\widetilde{C}_0}(r_0 + s_0 - \sigma r_0 - \sigma s_0 + N_2) = \mathcal{O}_N(-r_0 - s_0 + p + q).$$

As in IV.A.5.4, introduce the closed subspace $S^{r,s}$ of $\mathcal{P}$ with underlying set

$$\{L_t \in \mathcal{P} \mid h^0(\widetilde{C}_t, L_t(-r_t - s_t)) \geq 2\}.$$

By Theorem IV.A.5.1, for $t \neq 0$,

$$S^{r,s}_t = \operatorname{Sing}(\Xi_t \cdot \Xi_{t,a_t}).$$

Over 0, we have the following result, omitting the subscripts in r_0, s_0, a_0.

Proposition IV.A.6.3. *For $g \geq 6$, the analytic space $S^{r,s}_0$ is pure of dimension $g-5$ and has three irreducible components:*

$$W_{p,q} \cap \operatorname{Sing}\Theta, \qquad (-W_{r,s}) \cap \operatorname{Sing}\Theta_a, \qquad W_{p,q} \cap (-W_{r,s}).$$

It is smooth at every point of $(W_{p,q} \cap \operatorname{Sing}\Theta) \setminus (-W_{r,s})$.

Proof. We already know (IV.A.6.1) that the points of $W_{p,q} \cap \operatorname{Sing}\Theta$ correspond to line bundles $\mathcal{L}$ on $\widetilde{C}_0$ with $h^0(\mathcal{L}) \geq 4$, hence to elements of $S_0^{r,s}$.

The points of $-(W_{p,q} \cap \operatorname{Sing}\Theta) + a = (-W_{r,s}) \cap \operatorname{Sing}\Theta_a$ correspond to line bundles $\mathcal{L}$ on $\widetilde{C}_0$ with $h^0(\mathcal{L}(\sigma r + \sigma s - r - s)) \geq 4$, hence also to elements of $S_0^{r,s}$.

If $L \in W_{p,q} \cap (-W_{r,s})$, choose nonzero $u \in H^0(L(-p-q))$ and $\bar{u} \in H^0(\omega_N \otimes L^{-1}(-r-s))$. With the preceding notation,

$$n^* H^0(\mathcal{L}(-r-s)) \supset \langle (us_{pq}, 0), (0, \bar{u}s_{pqrs}) \rangle,$$

so L is indeed a point of $S_0^{r,s}$.

Conversely, if $\mathcal{L}$ corresponds to a point of $S_0^{r,s}$, distinguish two cases.

(i) If $h^0(L) \geq 2$ or $h^0(L(-p-q+r+s)) \geq 2$, we are in $\Theta_a \cap \operatorname{Sing}\Theta$ or $\Theta \cap \operatorname{Sing}\Theta_a$. If, for example, $L \in \Theta_a \cap \operatorname{Sing}\Theta$, Proposition IV.A.3.8 gives

$$L \in ((-W_{r,s}) \cap \operatorname{Sing}\Theta) \cup (W_{p,q} \cap \operatorname{Sing}\Theta).$$

If L lies in the first component but not the second, then

$$\begin{aligned} h^0(L) &= 2, & h^0(\omega_N \otimes L^{-1}(-r-s)) &= 1, \\ h^0(L(-p-q)) &= 0, & h^0(\omega_N \otimes L^{-1}(p+q-r-s)) &= 1. \end{aligned}$$

It follows that

$$n^* H^0(\mathcal{L}(-r-s)) \simeq H^0(L(-p-q)) \oplus H^0(\omega_N \otimes L^{-1}(-r-s)),$$

which is impossible, since the space on the right has dimension 1. Thus $L \in W_{p,q} \cap \operatorname{Sing}\Theta$. Similarly, if $L \in \Theta \cap \operatorname{Sing}\Theta_a$, then $L \in (-W_{r,s}) \cap \operatorname{Sing}\Theta_a$.

(ii) If $h^0(L) = h^0(L(-p-q+r+s)) = 1$, then

$$n^* H^0(\mathcal{L}(-r-s)) \simeq H^0(L) \oplus H^0(\omega_N \otimes L^{-1}(p+q-r-s)),$$

which is possible only if

$$\begin{aligned} H^0(L) &\simeq H^0(L(-p-q)), \\ H^0(\omega_N \otimes L^{-1}(p+q-r-s)) &\simeq H^0(\omega_N \otimes L^{-1}(-r-s)), \end{aligned}$$

that is, $L \in W_{p,q} \cap (-W_{r,s})$.

We now compute the tangent space to $S_0^{r,s}$ at $\mathcal{L}$. The construction of $S^{r,s}$ commutes with base change. In particular, its fiber $S_0^{r,s}$ is the space associated with the family $\widetilde{C}_0 \to \{0\}$ and the sections r, s, namely $P_0 \cdot (W^1_{(g-1,g-1)})_{r,s}$.

When $h^0(\widetilde{C}_0, \mathcal{L}(-r-s)) = 2$, take a basis $\{u_1, u_2\}$ of $H^0(\mathcal{L}(-r-s))$ and a basis

$$\{\bar{u}_1 = \sigma^* u_1 s_{rs} \sigma^* s_{rs}, \bar{u}_2 = \sigma^* u_2 s_{rs} \sigma^* s_{rs}, \bar{u}_3, \bar{u}_4\}$$

of $H^0(\omega_{\widetilde{C}_0} \otimes \mathcal{L}^{-1}(r+s))$. Then $T_{\mathcal{L}} S_0^{r,s}$ is the annihilator of the eight elements

$$\{u_i \bar{u}_j - \sigma^*(u_i \bar{u}_j)\}_{1 \leq i \leq 2,\, 1 \leq j \leq 4}$$

in the space of sections of $\omega_{\widetilde{C}_0}$ anti-invariant under σ^*, which is $T^*_{\mathcal{L}} P_0$. The special form of $\bar{u}_1, \bar{u}_2$ reduces these eight elements to the following five:

$$\{(u_1 \sigma^* u_2 - u_2 \sigma^* u_1) s_{rs} \sigma^* s_{rs}, \quad u_i \bar{u}_j - \sigma^*(u_i \bar{u}_j)\ (1 \leq i \leq 2 < j \leq 4)\}.$$

We compute this tangent space when $\mathcal{L}$ corresponds to a point $L \in W_{p,q} \cap \operatorname{Sing}\Theta$ with $L \notin (-W_{r,s})$, to show that its dimension is $g-5$. Then

$$h^0(L) \geq 2, \qquad h^0(L(-p-q)) \geq 1, \qquad h^0(L(r+s)) = 2.$$

It follows easily that

$$h^0(L) = h^0(L(r+s)) = 2,$$
$$h^0(L(-p-q)) = h^0(L(r+s-p-q)) = 1.$$

There are exact sequences

$$0 \longrightarrow H^0(\mathcal{L}(-r-s)) \xrightarrow{n^*} H^0(L) \oplus H^0(\omega_N \otimes L^{-1}(p+q-r-s)) \longrightarrow \mathbf{C}_1 \oplus \mathbf{C}_2,$$
$$0 \longrightarrow H^0(\omega_{\widetilde{C}_0} \otimes \mathcal{L}^{-1}(r+s)) \xrightarrow{n^*} H^0(\omega_N \otimes L^{-1}(p+q)) \oplus H^0(L(r+s)) \longrightarrow \mathbf{C}_1 \oplus \mathbf{C}_2.$$

Choose the following bases:

basis	space
$\{u_1 = us_{pq}, u_2\}$	$H^0(L)$
$\{\bar{u}\}$	$H^0(\omega_N \otimes L^{-1}(p+q-r-s))$
$\{\bar{u}_1, \bar{u}_2\}$	$H^0(\omega_N \otimes L^{-1})$
$\{\bar{u}_1 s_{pq}, \bar{u}_2 s_{pq}, \bar{u} s_{rs}\}$	$H^0(\omega_N \otimes L^{-1}(p+q))$
$\{us_{pqrs}, u_2 s_{rs}\}$	$H^0(L(r+s))$.

Since $\bar{u}$ does not vanish at p, q, the pair $(0, \bar{u})$ is not in $n^*H^0(\mathcal{L}(-r-s))$, which therefore has dimension at most 2, and exactly 2 by the preceding argument. We can choose u_2 so that a basis is

$$\{(us_{pq}, 0), (u_2, \bar{u})\}.$$

A basis of $n^*H^0(\omega_{\widetilde{C}_0} \otimes \mathcal{L}^{-1}(r+s))$ is then

$$\{(0, us_{pqrs}), (\bar{u}s_{rs}, u_2 s_{rs}), (\bar{u}_1 s_{pq}, 0), (\bar{u}_2 s_{pq}, 0)\}.$$

The tangent space $T_{\mathcal{L}} S_0^{r,s}$ is the annihilator, in $T_{\mathcal{L}} P_0 \simeq T_L JN \simeq H^0(N, \omega_N)^*$, of

$$\{u\bar{u}s_{rs}, u\bar{u}_1 s_{pq}, u\bar{u}_2 s_{pq}, u_2\bar{u}_1, u_2\bar{u}_2\}.$$

Comparing with the computations in the proof of IV.A.3.2, with $t = \bar{u}s_{rs}$, $s_D = u$, $t_i = \bar{u}_i$, and $s_i = u_i$, gives

$$T_{\mathcal{L}} S_0^{r,s} \simeq T_L(W_{p,q} \cdot \operatorname{Sing}\Theta).$$

The end of that proof shows that this tangent space has dimension $g-5$, completing the proof. □

Remark IV.A.6.4. The same method shows that $S_0^{r,s}$ is smooth at a general point of $W_{p,q} \cap (-W_{r,s})$. In particular, by IV.A.3.2 and IV.A.3.10, its cohomology class, and hence that of $S_t^{r,s}$ for every t, is, using IV.A.3.7 and IV.A.3.10,

$$\frac{\theta^5}{24} + \frac{\theta^5}{24} + \frac{\theta^5}{20} = \frac{2}{15}\theta^5.$$

We can now prove the main result of this paragraph.

Theorem IV.A.6.5. *For $g \geq 6$, the analytic subspace $S^{r,s}$ of $\mathcal{D}$ is irreducible, has codimension 5, and dominates Δ.*

Proof. It suffices to show that every irreducible component T of $S^{r,s}$ satisfies, after shrinking Δ if necessary:

(i) T has codimension 5 and dominates Δ;

(ii) $T_0 \supset W_{p,q} \cap \operatorname{Sing}\Theta$.

Indeed, IV.A.6.3 shows that $S^{r,s}$ is smooth at a general point of $W_{p,q} \cap \operatorname{Sing}\Theta$, so exactly one irreducible component of $S^{r,s}$ contains this locus. The theorem then follows immediately from (i) and (ii).

By IV.A.5.5, $S^{r,s}$ is empty or has codimension at most 5 everywhere in $\mathcal{D}$. Since its fiber over 0 has codimension 5 everywhere in P_0 (IV.A.6.3), (i) follows.

We may assume that, for every t in a dense subset Ω of Δ, P_t is simple and $a_t = r_t + s_t - \sigma r_t - \sigma s_t$ has infinite order in P_t. Section IV.B.2.1 then gives

$$\dim(T_t \cap \operatorname{Sing}\Xi_t) \geq g-6 \qquad (t \in \Omega).$$

IV.A.6.6. Observe that this also proves Theorem II.D.1.2.5 again: $\dim \operatorname{Sing}\Xi \geq g-6$ for every Prym variety (P,Ξ).

By [We 1], $\operatorname{Sing}\Xi_t = S_t$ is pure of dimension $g-6$. Thus T_t contains a component of $\operatorname{Sing}\Xi_t$ for $t \in \Omega$. We also know ([We 4], 3.4) that the limit of S_t as $t \to 0$ is contained in $W_{p,q} \cap (-W_{p,q}) \cap \operatorname{Sing}\Theta$, which has dimension at most $g-6$ by IV.A.3.9. Hence T_0 contains a component of dimension $g-6$ of this locus. By IV.A.6.3, the components of T_0 are among

$$W_{p,q} \cap \operatorname{Sing}\Theta, \qquad (-W_{r,s}) \cap \operatorname{Sing}\Theta_a, \qquad W_{p,q} \cap (-W_{r,s}).$$

Assertion (ii) now follows from the next lemma.

Lemma IV.A.6.7. *Fix p,q. For general r,s, no component of dimension $g-6$ of $W_{p,q} \cap (-W_{p,q}) \cap \operatorname{Sing}\Theta$ is contained in $-W_{r,s}$.*

Proof. Equivalently, no component Z is contained in $W_{r,s}$. If Z were contained in every $W_{r,s}$, then $Z \subset W^2_{g-1}$, which is impossible since $\dim Z = g-6$ whereas $\dim W^2_{g-1} = g-9$ (see IV.A.1). □

This proves (ii), completing the proof of Theorem IV.A.6.5. □

We also obtain the following additional result.

Corollary IV.A.6.8. *Let $\pi : \widetilde{C} \to C$ be an étale double cover of smooth connected curves, where C is general of genus $g+1 \geq 7$. Let r,s be two general points of $\widetilde{C}$, and let a be the element of the Prym variety (P,Ξ) of π associated with $O_{\widetilde{C}}(r+s-\sigma r-\sigma s)$. Then $\operatorname{Sing}(\Xi \cdot \Xi_a)$ is irreducible of dimension $g-5$, with cohomology class $\frac{2}{15}[\Xi]^5$.*

Remark IV.A.6.9. When $g=5$, the same argument shows that $\operatorname{Sing}(\Xi \cdot \Xi_a)$ is never empty and is generally a set of $\frac{2}{15}5! = 16$ points.

IV.A.7 The main theorem

It is now easy to prove the following theorem.

Theorem IV.A.7.1. *For every $g \geq 7$, the Prym locus $\mathcal{D}_g$ is an irreducible component of N^g_{g-6}.*

Proof. We show that, for $g \geq 6$, $\mathcal{P}_{g+1}$ is a component of N^{g+1}_{g-5}. First recall that $\mathcal{P}_{g+1} \subset N^{g+1}_{g-5}$ for $g \geq 5$ (IV.A.6.6 or II.D.1.2.5).

Consider the boundary $\partial\mathcal{P}_{g+1} \subset \partial\mathcal{A}_{g+1}$ of $\mathcal{P}_{g+1}$ in the partial compactification $\mathcal{A}^{(1)}_{g+1}$ of $\mathcal{A}_{g+1}$ (see IV.A.2.4). It is pure of dimension $3g+2$, and $\rho_{g+1}(\partial\mathcal{P}_{g+1}) = \mathcal{P}_g$ ([Fr-S 2], Theorem 3.6). Section IV.A.2.5 also gives

$$\mathcal{P}_{g+1} \subset N^{g+1}_{g-5} \quad \Longrightarrow \quad \partial\mathcal{P}_{g+1} \subset \partial N^{g+1}_{g-5} \subset \rho^{-1}_{g+1}(N^g_{g-5}) \cup \partial' N^{g+1}_{g-5}.$$

Consider the family $f : \mathcal{P} \to \Delta$ studied in IV.A.6, with the section $a : t \mapsto O(r(t)+s(t)-\sigma r(t)-\sigma s(t))$ on the smooth fibers, extended as above. By [Fr-S 2], Theorem 5.2, the associated family of quasi-abelian varieties is the family of Pryms associated with the double covers

$$\widetilde{C}/(r \sim \sigma s,\ s \sim \sigma r) \longrightarrow C/(\pi r \sim \pi s) \longrightarrow \Delta.$$

It therefore determines a morphism $\varphi : \Delta \to \partial\mathcal{P}_{g+1}$.

Let $\mathcal{C}_{g+1}$ be an irreducible component of N^{g+1}_{g-5} containing $\mathcal{P}_{g+1}$, with boundary $\partial\mathcal{C}_{g+1}$. Let $\partial\mathcal{P}^*_{g+1}$ be an irreducible component of $\partial\mathcal{P}_{g+1}$ containing $\varphi(\Delta)$, and let $\partial\mathcal{C}^*_{g+1}$ be an irreducible component of $\partial\mathcal{C}_{g+1}$ containing $\partial\mathcal{P}^*_{g+1}$. Then

$$\varphi(\Delta) \subset \partial\mathcal{P}^*_{g+1} \subset \partial\mathcal{C}^*_{g+1} \subset \rho^{-1}_{g+1}(N^g_{g-5}) \cup \partial' N^{g+1}_{g-5},$$
$$\varphi(\Delta) \not\subset \rho^{-1}_{g+1}(N^g_{g-5}) \quad \text{by IV.A.6.2, for example.}$$

Hence $\partial\mathcal{C}^*_{g+1} \subset \partial' N^{g+1}_{g-5}$. We now apply IV.A.2.6 to the family $\mathcal{P} \to \Delta$ with its section a. Hypothesis (i) follows from IV.A.6.8 with $\mathcal{T}^0 = S^{r,s}$, and (iii) from the fact that a general Jacobian has only the automorphisms $\pm$id.

If (ii) failed, then, for $t \neq 0$, there would be a component Z_t of $\operatorname{Sing}\Xi_t$ such that $Z_{t,a_t} \subset \operatorname{Sing}\Xi_t$. As in IV.A.6.5, we may assume that P_t is simple for t in a dense subset of Δ. The set of x_t such that

$$\dim(\operatorname{Sing}\Xi_t \cap (\operatorname{Sing}\Xi_t)_{x_t}) = g-6$$

is then finite. Thus (ii) holds for a general choice of r, s.

It follows that the dimension of $\partial\mathcal{C}^*_{g+1}$ at $\varphi(0)$ is at most the codimension of the projective span of the image of

$$(W_{p,q} \cap \operatorname{Sing}\Theta) \cup ((-W_{r,s}) \cap \operatorname{Sing}\Theta_a) \cup (W_{p,q} \cap (-W_{r,s}))$$

under the rational map

$$z \longmapsto \left[\left(\lambda\frac{\partial\theta}{\partial z_i}(z,\tau)\right)_{1\leq i\leq g}, \left(\lambda\frac{\partial^2\theta}{\partial z_i\partial z_j}(z,\tau) + \mu\frac{\partial^2\theta}{\partial z_i\partial z_j}(z-a,\tau)\right)_{1\leq i\leq j\leq g}\right],$$

where

$$\lambda\frac{\partial\theta}{\partial z_i}(z,\tau) + \mu\frac{\partial\theta}{\partial z_i}(z-a,\tau) = 0 \quad (1 \leq i \leq g),$$

and a corresponds to $O(p+q-r-s) \in JN$.

By IV.A.3.4, the image of the first component, for which generally $\mu = 0$, spans $\mathbf{P}(\{0\} \times V)$, where

$$\dim V = \binom{g+1}{2} - (3g-2).$$

The image of the second component is the same. By IV.A.3.10, the image of the third spans $\mathbf{P}(V')$, where the projection of V' onto the first g factors has dimension $g-4$. Thus

$$\dim \partial C^*_{g+1} \leq 4 + 3g - 2 = 3g + 2.$$

Since ∂C_{g+1} is pure of dimension $\dim C_{g+1} - 1$, it follows that

$$3g + 3 = \dim \mathcal{P}_{g+1} \leq \dim C_{g+1} \leq 3g + 2 + 1,$$

so $\mathcal{P}_{g+1} = C_{g+1}$. □

IV.B On the Schottky problem for Prym varieties

Joint work with A. Beauville.

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g, and $\mathcal{J}_g$ its subvariety of Jacobians. The usual Schottky problem seeks equations, or geometric characterizations, for $\mathcal{J}_g$ in $\mathcal{A}_g$. A recently fruitful approach uses the reducibility of certain intersections $\Theta \cap \Theta_a$ for a Jacobian (JC, Θ). This reducibility implies the existence of trisecants to the Kummer variety of JC. In II.C we proved that $\overline{\mathcal{J}}_g$ is a component of the locus of $(A, \Theta) \in \mathcal{A}_g$ for which an intersection $\Theta \cap \Theta_a$ is reducible, or whose Kummer variety admits a trisecant. The proof shows that these properties essentially imply $\dim \operatorname{Sing} \Theta \geq g - 4$; one then applies the Andreotti–Mayer theorem [An–M], which states that $\overline{\mathcal{J}}_g$ is a component of the locus N^g_{g-4} of pairs satisfying this condition.

Prym varieties form a closed subvariety $\mathcal{P}_g$ of $\mathcal{A}_g$, strictly containing $\overline{\mathcal{J}}_g$ in the dimensions considered here. In this section we seek a geometric characterization of $\mathcal{P}_g$, which one may call the Schottky problem for Prym varieties. There is a striking parallel between the two Schottky problems, at least in the approach described above. We prove that, for $g \geq 7$, $\mathcal{P}_g$ is an irreducible component of the locus of $(A, \Theta) \in \mathcal{A}_g$ satisfying any one of the following conditions:

(i) There are distinct nonzero elements $a, b \in A$ such that $\Theta \cap \Theta_a \cap \Theta_b$ is not integral.

(ii) There are distinct nonzero elements $a, b, x, y \in A$ such that $\Theta \cap \Theta_a \cap \Theta_b \subset \Theta_x \cup \Theta_y$.

(iii) The Kummer variety $K(A)$ admits a quadrisecant plane.

The proof is entirely analogous to that of II.C and relies essentially on the theorem of IV.A that $\mathcal{P}_g$ is a component of the Andreotti–Mayer locus N^g_{g-6}. Section 1 explains the proof and the relations between conditions (i)–(iii); two technical results used in the proof are established in Sections 2 and 3.

IV.B.1 Reducibility of $\Theta \cap \Theta_a \cap \Theta_b$ and quadrisecant planes

All varieties considered in this section are defined over the complex numbers.

Let $\widetilde{C}, C$ be smooth projective curves, $\pi\colon \widetilde{C} \to C$ an étale double cover, and σ the corresponding involution of $\widetilde{C}$. Recall that its Prym variety P is the abelian subvariety of $J\widetilde{C}$ which is the image of $1 - \sigma^*$. It carries a principal polarization, defined as follows. After translation, it is convenient to view P as the variety of divisor classes D on $\widetilde{C}$, modulo linear equivalence, satisfying $\pi_* D \equiv K_C$ and $h^0(\widetilde{C}, D)$ even. The subvariety of effective divisor classes is then a divisor Θ on P.

We study reducibility of the intersections $\Theta \cap \Theta_a \cap \Theta_b$, taken scheme-theoretically throughout this section. We exclude cases in which an intersection $\Theta \cap \Theta_a$ is already reducible. For $p, q \in \widetilde{C}$, write $[p, q]$ for the element $p + q - \sigma p - \sigma q$ of P.

Proposition IV.B.1.1. *Assume that C is neither hyperelliptic, nor trigonal, nor a double cover of an elliptic curve. Let p, q, r, s be four points of $\widetilde{C}$ such that $[p, q]$ and $[p, r]$ are distinct nonzero elements of P. Then $\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]}$ has codimension 3, is reducible, and is scheme-theoretically contained in $\Theta_{[p,s]} \cup \Theta_{[q,r]}$.*

Proof. By definition, $\Theta \cap \Theta_{[p,q]}$ is the set of effective divisor classes D on $\widetilde{C}$ satisfying

$$\pi_* D \equiv K_C, \qquad h^0(D) \text{ even} \geq 2, \qquad h^0(D - p - q + \sigma p + \sigma q) \geq 2.$$

Together with the first two conditions, the last is equivalent to $h^0(D - p - q) \geq 1$. Indeed, if $h^0(D - p - q) = 0$, Riemann–Roch and $K_{\widetilde{C}} - D \equiv \sigma D$ imply $h^0(D + \sigma p + \sigma q) = 2$. Thus σp and σq are base points of $|D + \sigma p + \sigma q|$, so that

$$|D - p - q + \sigma p + \sigma q| = |D - p - q| + \sigma p + \sigma q = \emptyset.$$

Conversely, if $h^0(D - p - q) \geq 1$, then $h^0(D - p - q + \sigma p + \sigma q)$ is even and nonzero, hence at least 2.

Consider the special subvariety S_{pq} consisting of effective divisor classes E on $\widetilde{C}$ such that $\pi_* E \in |K_C - \pi p - \pi q|$ and $h^0(E + p + q)$ is even [Be 3]. Identify S_{pq} with a subvariety of P by translation by $(p + q)$. The preceding argument gives $S_{pq} = \Theta \cap \Theta_{[p,q]}$ set-theoretically. This equality is in fact scheme-theoretic, since the two have the same cohomology class in P ([Be 3], Theorem 1). By the hypothesis on C, the system $|K_C - \pi p - \pi q|$ defines a morphism from C to projective space, birational onto its image. The corollary to Proposition 3 of [Be 3] therefore shows that $\Theta \cap \Theta_{[p,q]}$ is irreducible and normal.

Thus $\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]}$ is a Cartier divisor in $\Theta \cap \Theta_{[p,q]}$. Set-theoretically, it consists of effective divisor classes D on $\widetilde{C}$ satisfying

$$\pi_* D \equiv K_C, \quad h^0(D) \text{ even}, \quad h^0(D - p - q) \geq 1, \quad h^0(D - p - r) \geq 1.$$

It is therefore the union of the subvariety S_{pqr} of classes $D \in P$ with $h^0(D - p - q - r) \geq 1$, and the subvariety W_p of classes $D \in P$ with $h^0(D - p) \geq 2$. As above, the first is identified with the special subvariety associated with $|K_C - \pi p - \pi q - \pi r|$. It is reduced, irreducible for sufficiently general r, and its cohomology class in P is $4\Theta^3/6$. It follows that W_p is a reduced Weil divisor in $\Theta \cap \Theta_{[p,q]}$, and that there is an equality of cycles

$$\Theta \cdot \Theta_{[p,q]} \cdot \Theta_{[p,r]} = S_{pqr} + \widetilde{W}_p,$$

where $\widetilde{W}_p$ is a Weil divisor in $\Theta \cap \Theta_{[p,q]}$ with the same support as W_p. Varying r gives $\widetilde{W}_p \subset \Theta_{[p,s]}$ for every $s \in \widetilde{C}$, whereas $S_{pqr} \subset \Theta_{[q,r]}$. This proves the proposition. □

Remark. A tangent-space calculation shows that $\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]}$ is smooth at a sufficiently general point of W_p. Thus $\widetilde{W}_p = W_p$ in the preceding proof, and the class of W_p in P is $2\Theta^3/6$.

We shall see that 1.1 is essentially equivalent to the existence of quadrisecant planes to the Kummer variety of a Prym variety. Let (A, Θ) be a principally polarized abelian variety; unless otherwise stated, we always assume Θ symmetric. Let $\psi \colon A \to \mathbf{P}^N$ be the morphism defined by $|2\Theta|$, and $K(A)$ its image, the Kummer variety of A. Recall that when (A, Θ) is

irreducible, that is, not isomorphic to a product of two nonzero polarized abelian varieties, ψ identifies $K(A)$ with the quotient of A by the involution $x \mapsto -x$.

Proposition IV.B.1.2. *Let (A, Θ) be a principally polarized abelian variety of dimension at least 3, and let a, b, x, y be distinct nonzero elements of A. Assume that $s = x + y$ differs from $a, b, a + b$, and that $\Theta \cap \Theta_a \cap \Theta_b$ has codimension 3. Consider the following conditions:*

(i) The intersection $\Theta \cap \Theta_a \cap \Theta_b$ is not integral.

(ii) Scheme-theoretically, $\Theta \cap \Theta_a \cap \Theta_b \subset \Theta_x \cup \Theta_y$.

(iii) There are complex numbers λ, μ, ν, ρ, not all zero, such that the following identity in z holds:

$$\lambda\theta(z)\theta(z-s) + \mu\theta(z-a)\theta(z+a-s) + \nu\theta(z-b)\theta(z+b-s) + \rho\theta(z-x)\theta(z-y) = 0.$$

(iv) For every $\zeta \in A$ with $2\zeta = s$, the four distinct points $\psi(\zeta)$, $\psi(\zeta - a)$, $\psi(\zeta - b)$, $\psi(\zeta - x)$ of $\mathbf{P}^N$ are coplanar.

Then conditions (ii)–(iv) are equivalent and imply (i).

Proof. For $u \in A$, let θ_u be a nonzero section of $H^0(A, O_A(\Theta_u))$.

(iii) $\iff$ *(iv)*. There is a basis (ψ_α) of $H^0(A, O_A(2\Theta))$ satisfying

$$\theta_u\theta_{-u} = \sum_\alpha \psi_\alpha(u)\psi_\alpha \qquad (u \in A)$$

(Riemann's addition formula). It follows immediately that the formula (iii)′, obtained by replacing z with $z + \zeta$ in (iii), is equivalent to (iv).

(iii) $\Longrightarrow$ *(ii)*. The scalar ρ in (iii) cannot vanish, since otherwise $\Theta \cap \Theta_a \cap \Theta_b$ would have a component of codimension 2. Thus (iii) implies that the section $\theta_x\theta_y$ of $O_A(\Theta_x + \Theta_y)$ vanishes on $\Theta \cap \Theta_a \cap \Theta_b$, giving (ii).

(ii) $\Longrightarrow$ *(iii)*. Let $\mathcal{I}$ be the ideal of $\Theta \cap \Theta_a \cap \Theta_b$ in A. Tensoring the Koszul complex of this complete intersection by $O_A(\Theta_x + \Theta_y)$ gives an exact sequence

$$\begin{aligned} 0 &\longrightarrow O_A(-\Theta_{a+b-s}) \\ &\longrightarrow O_A(\Theta_s - \Theta_a) \oplus O_A(\Theta_s - \Theta_b) \oplus O_A(\Theta_s - \Theta_{a+b}) \\ &\longrightarrow O_A(\Theta_s) \oplus O_A(\Theta_{s-a}) \oplus O_A(\Theta_{s-b}) \\ &\xrightarrow{(\theta, \theta_a, \theta_b)} \mathcal{I}(\Theta_x + \Theta_y) \longrightarrow 0. \end{aligned}$$

By the hypothesis on s, $H^0(A, \mathcal{I}(\Theta_x + \Theta_y))$ is generated by $\theta\theta_s$, $\theta_a\theta_{s-a}$, and $\theta_b\theta_{s-b}$. Condition (ii) says that $\theta_x\theta_y$ lies in this space, giving (iii). *(ii)* $\Longrightarrow$ *(i)*. For $u \in A$, tensoring the Koszul sequence by $O_A(\Theta_u)$ gives

$$\begin{aligned} 0 &\longrightarrow O_A(-\Theta - \Theta_{a+b-u}) \\ &\longrightarrow O_A(-\Theta_{a-u}) \oplus O_A(-\Theta_{b-u}) \oplus O_A(-\Theta_{a+b-u}) \\ &\longrightarrow O_A(\Theta_u - \Theta) \oplus O_A(\Theta_u - \Theta_a) \oplus O_A(\Theta_u - \Theta_b) \\ &\longrightarrow \mathcal{I}(\Theta_u) \longrightarrow 0. \end{aligned}$$

For $u \notin \{0, a, b\}$, it follows that $H^0(A, \mathcal{I}(\Theta_u)) = 0$. Thus $\Theta \cap \Theta_a \cap \Theta_b$ is not contained in Θ_u. Condition (ii), without a hypothesis on s, therefore implies (i). □

Remarks. 1) The hypothesis $\operatorname{codim}(\Theta \cap \Theta_a \cap \Theta_b) = 3$ is essential. Suppose, for example, that A contains an elliptic curve E with $(\Theta \cdot E) = 2$. It follows readily from II.C.1.4 that, for distinct nonzero $a, b \in E$, the intersection $\Theta \cap \Theta_a \cap \Theta_b$ is integral of codimension 2 and is contained in Θ_x for every $x \in E$. For general $y \in A$, we therefore have $\Theta \cap \Theta_a \cap \Theta_b \subset \Theta_x \cup \Theta_y$, but (iii) is not satisfied. Nevertheless, $\psi(E)$ is a conic in $\mathbf{P}^N$, so any four points of $\psi(E)$ are coplanar.

2) The hypothesis $s \notin \{a, b, a+b\}$ merely simplifies the statements of (iii) and (iv). If, for instance, $s = a$, condition (iv) should say that there is a line tangent to $K(A)$ at $\psi(\zeta)$, coplanar with $\psi(\zeta - b)$ and $\psi(\zeta - x)$.

More generally, as in II.C, one can consider specializations of (ii)–(iv) in which some of a, b, x, y become infinitely close to 0; we leave this rather tedious exercise to the reader. For Jacobians, letting a, x, y tend to 0 in $\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y$ yields the KP equation. For a Prym variety of smooth curves, the points $[p, q]$, $[p, r]$, $[p, s]$, and $[q, r]$ cannot all tend to 0. This becomes possible when C is singular, and yields a nonlinear partial differential equation called the BKP equation.[1]

We can now state our main result. Let $\mathcal{P}_g$ be the subvariety of $\mathcal{A}_g$ consisting of Prym varieties of stable curves: it is the closure in $\mathcal{A}_g$ of the Prym varieties of smooth curves. It has dimension $3g$ for $g \geq 5$ and contains the Jacobian locus $\mathcal{J}_g$.

Theorem IV.B.1.3. *For a principally polarized abelian variety (A, Θ), consider the following conditions:*

(*i*) *There are distinct nonzero elements $a, b \in A$ such that $\Theta \cap \Theta_a \cap \Theta_b$ is not integral.*

(*ii*) *There are distinct nonzero elements $a, b, x, y \in A$ such that, scheme-theoretically, $\Theta \cap \Theta_a \cap \Theta_b \subset \Theta_x \cup \Theta_y$.*

(*iii*) *The Kummer variety $K(A)$ admits a quadrisecant plane.*

Then, for $g \geq 7$, $\mathcal{P}_g$ is an irreducible component of the locus of $(A, \Theta) \in \mathcal{A}_g$ satisfying any one of (i)–(iii).

Remarks. 1) A quadrisecant plane to $K(A)$ means a plane in $\mathbf{P}^N$ containing four distinct points of $K(A)$.

2) In contrast to what one might hope for Jacobians, the existence of a quadrisecant plane does not characterize Prym varieties: varieties (A, Θ) containing an elliptic curve of degree 2 provide examples.

Proof. Suppose there is an irreducible subvariety Z of $\mathcal{A}_g$, strictly containing $\mathcal{P}_g$, whose general member (A, Θ) satisfies one of (i)–(iii). The main theorem of IV.A implies $\dim \operatorname{Sing} \Theta < g - 6$. Moreover, the Néron–Severi group $\mathrm{NS}(A)$ is generated by the class of Θ, since even a generic Jacobian has this property [Z]. Thus, for every nonzero $a \in A$, $\Theta \cap \Theta_a$ is integral: otherwise II.C.2.1 would give either $\dim \operatorname{Sing} \Theta \geq g - 4$ or $\operatorname{rk} \mathrm{NS}(A) \geq 2$. Consequently $\Theta \cap \Theta_a \cap \Theta_b$ has codimension 3 for every $b \neq 0, a$. By Proposition 1.2, there are therefore distinct nonzero $a, b \in A$ for which this intersection is not integral. The theorem now follows from the next two statements, proved in Sections 2 and 3:

Proposition IV.B.2.3. *Let (A, Θ) be a principally polarized abelian variety of dimension $g \geq 6$. Assume that $\mathrm{NS}(A)$ is generated by the class of Θ and that $\dim \operatorname{Sing} \Theta < g - 6$. If a, b are distinct*

[1] This observation is due to A. Beauville and G. Welters.

nonzero elements of A, at least one of infinite order, then $\Theta \cap \Theta_a \cap \Theta_b$ is integral.

Proposition IV.B.3.3. *Let (P, Θ) be a generic Prym variety of dimension at least 5, and let a, b be distinct nonzero torsion points of P. Then $\Theta \cap \Theta_a \cap \Theta_b$ is integral.*

□

IV.B.2 Singularities of $\Theta \cap \Theta_a$

Proposition IV.B.2.1. *Let (A, Θ) be a principally polarized abelian variety and $a \in A$. Assume that A is simple and a has infinite order. Then*

$$\dim(\Theta_a \cap \operatorname{Sing}\Theta) \geq \dim \operatorname{Sing}(\Theta \cap \Theta_a) - 1.$$

Recall that an abelian variety is simple if it contains no nontrivial proper abelian subvariety. We use the convention $\dim \emptyset = -1$.

Proof. Let Z be an irreducible component of maximum dimension in $\operatorname{Sing}(\Theta \cap \Theta_a)$. Let $\widetilde{Z}$ be its normalization and n the composite $\widetilde{Z} \to Z \hookrightarrow A$.

Let θ, respectively θ_a, be a nonzero section of $H^0(A, O_A(\Theta))$, respectively $H^0(A, O_A(\Theta_a))$. Each vector field D on A gives sections $D\theta \in H^0(\Theta, O(\Theta))$ and $D\theta_a \in H^0(\Theta_a, O(\Theta_a))$ (see II.C.0). Since Z lies in the singular locus of $\Theta \cap \Theta_a$, the Jacobian criterion shows that the quotients $D\theta_a/D\theta$, for $D \in H^0(A, T_A)$, define the same meromorphic section of $n^* O_A(\Theta_a - \Theta)$. It is holomorphic and nonvanishing on the open subset U complementary to $n^{-1}(\operatorname{Sing}\Theta \cup \operatorname{Sing}\Theta_a)$ in $\widetilde{Z}$. Thus $n^* O_A(\Theta_a - \Theta)$ is trivial on U.

Suppose the conclusion fails; in particular, $\dim Z \geq 1$. Then $\Theta_a \cap \operatorname{Sing}\Theta$ and, by symmetry, $\Theta \cap \operatorname{Sing}\Theta_a$ have codimension at least 2 in Z. Consequently $\widetilde{Z} \setminus U$ has codimension at least 2 in $\widetilde{Z}$. Since $\widetilde{Z}$ is normal, $n^* O_A(\Theta_a - \Theta)$ is trivial. Thus a lies in the kernel of the composite homomorphism

$$A \xrightarrow{\phi} \operatorname{Pic}^0(A) \xrightarrow{n^*} \operatorname{Pic}^0(\widetilde{Z}),$$

where ϕ is the isomorphism associated with the principal polarization, given by $\phi(x) = O_A(\Theta_x - \Theta)$. But n^* is nonzero: its transpose is the Albanese morphism $\operatorname{Alb}(\widetilde{Z}) \to A$, whose image contains a translate of Z. Since A is simple, the kernel of n^* is finite, contradicting the infinite order of a. □

Remarks. 1) With minor modifications, the proof extends to the case in which a is infinitely close to 0. If D is a nonzero vector field on A and Θ_D is the divisor on Θ defined by $D\theta = 0$, one obtains

$$\dim(\Theta_D \cap \operatorname{Sing}\Theta) \geq \dim \operatorname{Sing}\Theta_D - 1$$

when A is simple. 2) The simplicity assumption on A cannot be completely removed. For example, if A contains an elliptic curve E with $(\Theta \cdot E) = 2$, then Θ is generally smooth whereas $\Theta \cap \Theta_a$ is reducible, hence singular in codimension one, for $a \in E \setminus \{0\}$ (II.C). On the other hand, we do not know whether the infinite-order assumption is necessary. These restrictive assumptions in Proposition 2.1 prevent a result as precise as that of II.C.

3) Proposition 2.1 implies a slightly stronger statement than Theorem II.C.2.11: $\overline{\mathcal{J}}_g$ is an irreducible component of the locus of $(A, \Theta) \in \mathcal{A}_g$ for which there is a nonzero $a \in A$ with $\dim \operatorname{Sing}(\Theta \cap \Theta_a) \geq g - 3$.

Proposition IV.B.2.3. *Let (A, Θ) be a principally polarized abelian variety of dimension g. Assume that $\mathrm{NS}(A)$ is generated by the class of Θ and that $\dim \mathrm{Sing}\,\Theta < g-6$. Let a, b be distinct nonzero elements of A, at least one of infinite order. Then $\Theta \cap \Theta_a \cap \Theta_b$ is integral.*

Proof. We may assume that a has infinite order. The hypothesis on $\mathrm{NS}(A)$ implies that A is simple, so 2.1 gives $\dim \mathrm{Sing}(\Theta \cap \Theta_a) \leq g-6$. Thus $X = \Theta \cap \Theta_a$ is normal and irreducible, and its local rings are factorial ("Samuel's conjecture", [Gro 3], Exposé XI, 3.14). If the divisor $\Theta_b|_X$ is not integral, it is a sum of two nonzero effective Cartier divisors. Since $g \geq 5$, the Lefschetz theorem ([Gro 3], Exposé XII, 3.6) implies that the restriction homomorphism $\mathrm{Pic}(A) \to \mathrm{Pic}(X)$ is bijective. Thus there are two divisors on A, summing to Θ_b, whose restrictions to X are effective. By the hypothesis on $\mathrm{NS}(A)$, one must be algebraically equivalent to $m\Theta$ with $m \leq 0$, a contradiction. □

IV.B.3 The case of torsion points a and b

Lemma IV.B.3.1. *Let $g_1, g_2 \geq 1$ be integers, and set $g = g_1 + g_2 \geq 3$. Let P be a generic Prym variety of dimension g, and let a, b be distinct nonzero torsion points of P. Then P can be specialized to a product $P_1 \times P_2$, where P_i is a generic Prym variety of dimension g_i $(i = 1, 2)$, so that a, b specialize to (a_1, a_2), (b_1, b_2) with a_1, b_1 distinct and nonzero, and likewise a_2, b_2 distinct and nonzero.*

Proof. By [Be 1], P can be specialized to a product $P_1 \times P_2$ satisfying the stated conditions. The points a, b then specialize to (a_1, a_2) and (b_1, b_2); we must show that this specialization can be chosen so that $0, a_i, b_i$ are distinct $(i = 1, 2)$. Let $m \geq 2$ be an integer with $ma = mb = 0$. The group $P[m]$ of points killed by m carries a unimodular symplectic form with values in $\mathbb{Z}/m\mathbb{Z}$. Consider the étale cover of $\mathcal{D}_g$ whose fibre over P is $P[m]$. Its associated homomorphism $\pi_1(\mathcal{D}_g, P) \to \mathrm{Sp}(P[m])$ is surjective: this follows from the corresponding statement for Jacobians, itself a consequence of the connectedness of Teichmüller space. Thus for every symplectic automorphism γ of $(P_1 \times P_2)[m]$, the pair $(\gamma(a_1, a_2), \gamma(b_1, b_2))$ can again be obtained by specializing (a, b). Lemma 3.1 therefore follows from the next algebraic lemma. □

Lemma IV.B.3.2. *Let $p, q > 1$ be integers and m their least common multiple. Let Γ_1, Γ_2 be free $\mathbb{Z}/m\mathbb{Z}$-modules of even ranks $2g_1, 2g_2 \geq 2$, with $g_1 + g_2 \geq 3$, equipped with unimodular symplectic forms $\varphi_i \colon \bigwedge^2 \Gamma_i \to \mathbb{Z}/m\mathbb{Z}$. Set $\Gamma = \Gamma_1 \oplus \Gamma_2$ and $\varphi = \varphi_1 \oplus \varphi_2$. Let a, b be distinct elements of Γ of orders p, q, respectively. Then there is $\gamma \in \mathrm{Sp}(\Gamma)$ such that $\gamma a = (a_1, a_2)$, $\gamma b = (b_1, b_2)$, with $a_i \neq 0$, $b_i \neq 0$, and $a_i \neq b_i$ for $i = 1, 2$.*

Proof. Use the Chinese remainder decomposition $\mathbb{Z}/m\mathbb{Z} \simeq \prod_{\ell^e \| m} \mathbb{Z}/\ell^e\mathbb{Z}$. Fix a factor $R = \mathbb{Z}/\ell^e\mathbb{Z}$. For each nonzero component v of $a, b, a-b$ in $\Gamma \otimes R$, write $v = \ell^{\nu(v)} v_0$ with v_0 primitive, and let $\overline{v}_0$ be its nonzero reduction in the symplectic $\mathbb{F}_\ell$-space $V = \Gamma/\ell\Gamma$. This reduction is well defined.

These vectors span a subspace S of dimension at most 2. If a, b have different valuations, the leading term of $a - b$ agrees, up to sign, with that of the element of smaller valuation. If the valuations agree but the leading terms differ, the leading term of $a - b$ is their difference. If those leading terms agree, a, b contribute only one direction, and the leading term of $a - b$ contributes at most one more. The case of a zero component is immediate.

There is a linear isometric embedding $j \colon S \hookrightarrow V_1 \oplus V_2$, where $V_i = \Gamma_i/\ell\Gamma_i$, with both projections injective. If $\dim S = 1$, send a generator to the sum of two nonzero vectors. If

$\dim S = 2$, choose a basis u, v and set $c = \varphi(u, v)$. Interchange factors if necessary so that $\dim V_2 \geq 4$. In V_1, choose an independent pair with symplectic product 1; in V_2, choose an independent pair with product $c - 1$. This is possible even when $c - 1 = 0$, by using a totally isotropic plane in V_2. The sums of the pairs define the required embedding j. Witt's extension theorem extends j to a symplectic automorphism of V. The reduction map $\mathrm{Sp}(\Gamma \otimes R) \to \mathrm{Sp}(V)$ is surjective, as follows by lifting a symplectic basis; choose a lift. Each leading term has nonzero projections, and hence so does the original v. Combining these automorphisms over the prime-power factors of m by the Chinese remainder theorem gives γ for which the projections of γa, γb, and $\gamma(a - b)$ are nonzero in both factors. This is precisely the required conclusion. □

Proposition IV.B.3.3. *Let (P, Θ) be a generic Prym variety of dimension $g \geq 5$, and let a, b be distinct nonzero torsion points of P. Then $\Theta \cap \Theta_a \cap \Theta_b$ is integral.*

Proof. For $g = 5$, P is a generic principally polarized abelian variety. By A1.2.10, $\Theta \cap \Theta_a$ is smooth; since $\mathrm{NS}(P)$ is generated by Θ, the proof of 2.3 shows that $\Theta \cap \Theta_a \cap \Theta_b$ is integral.

Proceed by induction on g. For a generic Jacobian (JC, Θ), algebraic cohomology classes are integer multiples of the minimal classes $\Theta^p/p!$; this follows from a simple monodromy argument ([A–C–G–H 2], Chapter X). The same is therefore true for P. Moreover, $\Theta \cap \Theta_a$ is integral (II.C), so $\Theta \cap \Theta_a \cap \Theta_b$ has codimension 3. Let Z be an irreducible component, with its reduced structure; its cohomology class in P is $n\Theta^3/6$, where $1 \leq n \leq 6$.

Use the specialization $P \rightsquigarrow P_1 \times P_2$ of Lemma 3.1 with $\dim P_1 = 1$ and $\dim P_2 = g - 1$. The divisor Θ specializes to $\Theta' = (P_1 \times \Theta_2) + (\Theta_1 \times P_2)$, where Θ_i is a theta divisor on P_i. There is an equality of cycles

$$\begin{aligned}\Theta' \cap \Theta'_{(a_1,a_2)} \cap \Theta'_{(b_1,b_2)} = {} & P_1 \times (\Theta_2 \cap \Theta_{2,a_2} \cap \Theta_{2,b_2}) \\ & + \Theta_1 \times (\Theta_{2,a_2} \cap \Theta_{2,b_2}) \\ & + \Theta_{1,a_1} \times (\Theta_2 \cap \Theta_{2,b_2}) \\ & + \Theta_{1,b_1} \times (\Theta_2 \cap \Theta_{2,a_2}).\end{aligned}$$

The first subvariety on the right is integral by induction and has class θ_2^3. The other three are integral by II.C and have class $\theta_1\theta_2^2$. Here θ_1, θ_2 denote the classes of $\Theta_1 \times P_2$ and $P_1 \times \Theta_2$ in $H^2(P_1 \times P_2, \mathbb{Z})$. The component Z specializes to a union of some of these components. Comparing cohomology classes gives

$$n\frac{(\theta_1 + \theta_2)^3}{6} = p\theta_2^3 + q\theta_1\theta_2^2, \qquad p, q \in \mathbb{N}.$$

This forces $n = 6$. Thus $\Theta \cap \Theta_a \cap \Theta_b = Z$ and is integral. □

Remark. Let P be a generic Prym variety of dimension g, and let a, b be distinct nonzero elements of P. It seems likely that, at least for sufficiently large g, the intersection $\Theta \cap \Theta_a \cap \Theta_b$ is reducible only in the situation of 1.1:

$$a = p + q - \sigma p - \sigma q, \qquad b = p + r - \sigma p - \sigma r.$$

For $g \geq 12$, one can show that this would follow from the next conjecture, a weaker version of which is proved in [We 4]: the only $a \in P$ such that Θ_a contains $\mathrm{Sing}\,\Theta$ are those of the form $p + q - \sigma p - \sigma q$, with $p, q \in \widetilde{C}$.

Appendices

APPENDIX 1

Constructing irreducible components of N_k^g

A1.1 Definition of the subset $\mathcal{A}^{\delta}_{g',g''}$ of $\mathcal{A}_{g'+g''}$

This appendix systematically studies a situation encountered several times: in II.D.1.5.1 and II.D.2.1, we considered principally polarized abelian varieties isogenous to a product of two abelian varieties with polarizations of degree two.

Here we consider principally polarized abelian varieties isogenous to a product of two abelian varieties with polarizations of the same type. As the next proposition shows, every nonsimple principally polarized abelian variety fits this framework.

We use the notation and results of [Mu 5], recalled in I.3. The proof of the proposition is deferred to the end of this section.

Proposition A1.1.1. *Let (A, L) be a principally polarized abelian variety and $j' : X' \hookrightarrow A$ an abelian subvariety. There is a unique abelian subvariety $j'' : X'' \hookrightarrow A$ satisfying:*

i) The morphism $\pi = j' + j'' : X' \times X'' \to A$ is an isogeny.

ii) The polarization $\pi^* L$ is the product polarization $M' \boxtimes M''$, where $M' = j'^* L$ and $M'' = j''^* L$.

The following properties then hold:

iii) There is an antisymplectic isomorphism $\psi : H(M') \to H(M'')$ such that

$$\operatorname{Ker} \pi = \{(x, \psi x) \mid x \in H(M')\}, \qquad e^{M'}(x, y) e^{M''}(\psi x, \psi y) = 1$$

for all $x, y \in H(M')$.

iv) There are bases $s'_1, \dots, s'_d$ of $H^0(X', M')$ and $s''_1, \dots, s''_d$ of $H^0(X'', M'')$ such that $\sum_{i=1}^{d} s'_i \otimes s''_i$ spans $\pi^* H^0(A, L) \subset H^0(X' \times X'', \pi^* L)$.

1.2. Converse construction. Conversely, let (X', M') and (X'', M'') be polarized abelian varieties of the same type, and let $\psi : H(M') \to H(M'')$ be an antisymplectic isomorphism. Set $K = \{(x, \psi x) \mid x \in H(M')\}$. The abelian variety $A = (X' \times X'')/K$ is then principally polarized, and the canonical morphism $\pi : X' \times X'' \to A$ satisfies i) and ii). The moduli space of polarized abelian varieties (X, M) of fixed dimension and fixed type δ, equipped with a symplectic isomorphism $H(M) \simeq H(\delta)$, is irreducible. Thus the varieties (A, L) constructed from X' and X'' of fixed dimensions and fixed common type δ form an irreducible subvariety of $\mathcal{A}_g$.

Definition A1.1.3. *Write $\mathcal{A}^{\delta}_{g',g-g'}$ for the irreducible closed subset of codimension $g'(g-g')$ in $\mathcal{A}_g$ consisting of principally polarized abelian varieties of dimension g containing an abelian subvariety of dimension g' on which the induced polarization has type δ.*

By 1.1 and 1.2, $\mathcal{A}^{\delta}_{g',g''} = \mathcal{A}^{\delta}_{g'',g'}$. We shall therefore always assume $1 \le g' \le g/2$.

1.4. Singular locus. If F' and F'' are the base loci of $|M'|$ and $|M''|$, then 1.1 iv) gives

$$\pi(F' \times F'') \subset \operatorname{Sing} \Theta, \tag{A1.1.5}$$

where Θ is the theta divisor of A. In particular, if $g - g' \geq g' \geq d = \deg \delta$, then

$$\mathcal{A}^{\delta}_{g',g-g'} \subset N^{g}_{g-2d}.$$

The remainder of this appendix studies these loci $\mathcal{A}^{\delta}_{g',g-g'}$. For certain types δ, we shall show that (A1.1.5) is an equality for general X' and X'', and that $\mathcal{A}^{\delta}_{g',g-g'}$ is an irreducible component of $N^{g}_{g-2\deg\delta}$ when $g - g' \geq g' \geq \deg \delta$.

Proof of Proposition A1.1.1 and 1.2. Since j' is injective, the kernel of the dual map $\hat{j}' : \widehat{A} \to \widehat{X}'$ is connected. Let X'' be its inverse image under $\phi_L : A \xrightarrow{\sim} \widehat{A}$, and let j'' denote the inclusion $X'' \hookrightarrow A$. The matrix of the induced polarization π^*L on $X' \times X''$ is

$$\begin{pmatrix} \hat{j}'\phi_L j' & \hat{j}'\phi_L j'' \\ \hat{j}''\phi_L j' & \hat{j}''\phi_L j'' \end{pmatrix} = \begin{pmatrix} \phi_{M'} & 0 \\ 0 & \phi_{M''} \end{pmatrix},$$

because $\hat{j}'\phi_L j''$ and its dual $\hat{j}''\phi_L j'$ vanish by construction. This proves i) and ii), and uniqueness is clear. There are also a subgroup K of $H(M')$ and an injection $\psi : K \hookrightarrow H(M'')$ such that $\operatorname{Ker}\pi = \{(x, \psi x) \mid x \in K\}$. Since L is a principal polarization, we have ([Mu 5], p. 295)

$$|K|^2 = |\operatorname{Ker}\pi|^2 = |H(\pi^*L)| = |H(M')|\,|H(M'')|.$$

It follows that ψ is an isomorphism from $H(M')$ to $H(M'')$. Moreover, $\operatorname{Ker}\pi$ is isotropic for e^{π^*L} ([Mu 5], p. 291), proving iii).

By the definition of the commutator e^{π^*L}, this is equivalent to lifting ψ to an isomorphism $\widetilde{\psi} : G(M') \to G(M'')$ whose restriction to $\mathbb{C}^*$ is inversion. Let $\widetilde{G}$ and $\widetilde{K}'$ be maximal level subgroups of $G(M')$ with $\widetilde{G} \cap \widetilde{K}' = \{1\}$. Let s' (respectively s'') span the subspace of $H^0(X', M')$ (respectively $H^0(X'', M'')$) invariant under $\widetilde{K}'$ (respectively $\widetilde{\psi}(\widetilde{K}')$). In the theta representation, $\widetilde{G}$ acts transitively on the one-dimensional eigenspaces of $\widetilde{K}'$. Hence $\{\tilde{g} \cdot s' \mid \tilde{g} \in \widetilde{G}\}$ is a basis of $H^0(X', M')$; similarly, $\{\widetilde{\psi}(\tilde{g}) \cdot s'' \mid \tilde{g} \in \widetilde{G}\}$ is a basis of $H^0(X'', M'')$.

Every invariant element of $H^0(X' \times X'', \pi^*L)$ under the maximal level subgroup

$$\{(\tilde{x}, \widetilde{\psi}\tilde{x}) \mid \tilde{x} \in G(M')\} \subset G(\pi^*L) \simeq \frac{G(M') \times G(M'')}{\{(\lambda, \lambda^{-1}) \mid \lambda \in \mathbb{C}^*\}}$$

spans $\pi^*H^0(A, L)$. One checks that

$$s = \sum_{\tilde{g} \in \widetilde{G}} (\tilde{g} \cdot s') \otimes (\widetilde{\psi}(\tilde{g}) \cdot s'')$$

has the required property. Conversely, to prove 1.2, it suffices to observe that the assumptions make $K \subset H(M' \boxtimes M'')$ a maximal totally isotropic subgroup. □

A1.2 The sets $T_{g,\delta}$ and $U_{g,\delta}$

Introduce the following closed subvarieties of $\mathcal{A}_{g,\delta}$:

$$T_{g,\delta} = \{(X, M) \in \mathcal{A}_{g,\delta} \mid \text{some member of } |M| \text{ has a point of multiplicity } > 2\},$$
$$U_{g,\delta} = \{(X, M) \in \mathcal{A}_{g,\delta} \mid \text{property (P) below fails}\}.$$

(P) The base locus F of $|M|$ is smooth of dimension $g-\deg\delta$ when this integer is nonnegative, and empty otherwise. Equivalently, for a basis $s_1,\dots,s_d$ of $H^0(X,M)$, with $d=\deg\delta$, we have

$$\forall z\in F,\qquad \operatorname{rank}\left(\frac{\partial s_i}{\partial z_j}(z)\right)_{\substack{1\le i\le d\\ 1\le j\le g}}=d.$$

For example, for the principal polarization $\delta=(1)$,

$$T_{g,(1)}\subset U_{g,(1)}=N_0^g\subsetneq \mathcal{A}_g.$$

Proposition A1.2.1. *If $U_{g,\delta}=\mathcal{A}_{g,\delta}$, then $T_{g,\delta}=\mathcal{A}_{g,\delta}$.*

Proof. Set $\theta_r(z,\tau)=\theta[\begin{smallmatrix}r\\0\end{smallmatrix}](z,\tau)$ (I.3.14), for $z\in\mathbb{C}^g$, $\tau\in\mathfrak{H}_g$, and $r\in\Delta_\delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g=\Gamma_\delta$. The functions θ_r form a basis of $H^0(X,M)$, where $(X,M)\in\mathcal{A}_{g,\delta}$ corresponds to τ.

If $U_{g,\delta}=\mathcal{A}_{g,\delta}$, the method of [An–M] (especially Lemma 8, p. 203) gives an open set $U\subset\mathfrak{H}_g$ and two morphisms $\lambda\colon U\to\mathbb{C}^d\setminus\{0\}$ and $s\colon U\to\mathbb{C}^g$ such that, for $\tau\in U$,

$$\begin{cases}\theta_r(s(\tau),\tau)=0 & \text{for every } r\in\Gamma_\delta,\\ \displaystyle\sum_r\lambda_r(\tau)\frac{\partial\theta_r}{\partial z_i}(s(\tau),\tau)=0 & \text{for } 1\le i\le g.\end{cases}\tag{A1.2.2}$$

Differentiating $\sum_r\lambda_r(\tau)\theta_r(s(\tau),\tau)=0$ with respect to τ_{ij}, then using (A1.2.2) and the heat equations (I.3.14), gives

$$\forall\tau\in U,\quad \forall i,j\in\{1,\dots,g\},\qquad \sum_r\lambda_r(\tau)\frac{\partial^2\theta_r}{\partial z_i\partial z_j}(s(\tau),\tau)=0.$$

Together with (A1.2.2), this implies $T_{g,\delta}\supset U$, hence, by closedness, $T_{g,\delta}=\mathcal{A}_{g,\delta}$. □

Remark A1.2.3. We used only the heat equations for θ_r. In particular, fix constants α_r^i, for $r\in\Gamma_\delta$ and $1\le i\le n$, with $\operatorname{rank}(\alpha_r^i)=n$. If $T_{g,\delta}\ne\mathcal{A}_{g,\delta}$, then for general X in $\mathcal{A}_{g,\delta}$ the intersection

$$\bigcap_{i=1}^n\operatorname{div}\left(\sum_r\alpha_r^i\theta_r(\,\cdot\,,\tau)\right)$$

is smooth of dimension $g-n$ if $n\le g$, and empty otherwise.

Theorem A1.2.4. *We have $T_{g,\delta}\ne\mathcal{A}_{g,\delta}$ in the following cases:*

$$\delta=(2),\ g\ge1;\qquad \delta=(3),\ g\ge2;\qquad \delta=(2,2),\ g=2 \text{ or } g\ge4.$$

Proof. We argue by induction using a construction similar to 1.2. Return to the notation of the proof of 1.1 and set $M=M'\boxtimes M''$ on $X=X'\times X''$. The subgroup $\{(\tilde{x},\widetilde{\psi}\tilde{x})\mid\tilde{x}\in\widetilde{K}'\}$ of $G(M)$ is a level subgroup of order $\deg\delta$. Let H be its image in $H(M)$. There are a polarized abelian variety (Y,N) of type δ and an isogeny $p\colon(X,M)\to(Y,N)$ with kernel H. A basis of $p^*H^0(Y,N)\subset H^0(X,M)$ is

$$\{s_h=s'_h\otimes s''_h\mid h\in\widetilde{G}\},\qquad s'_h\otimes s''_h=(h\cdot s')\otimes(\widetilde{\psi}(h)\cdot s'').$$

Lemma A1.2.5. *If $(X'',M'')\notin T_{g'',\delta}$ and $\operatorname{div}s''_1$ is smooth, then $(Y,N)\notin T_{g'+g'',\delta}$ in each of the following cases:*

- $\delta = (d)$ *and* $g' = 1$;
- $\delta = (2,2)$ *and* (X', M') *is general in* $\mathcal{A}_{2,(2,2)}$.

Proof. *1)* $g' = 1$. Suppose the divisor of $s = \sum_{h \in \widetilde{G}} \lambda_h s_h$ has a point (x', x'') of multiplicity > 2. In particular, for all i, j,

$$\sum_h \lambda_h s'_h(x') s''_h(x'') = 0,$$

$$\sum_h \lambda_h s'_h(x') \frac{\partial s''_h}{\partial x''_i}(x'') = 0,$$

$$\sum_h \lambda_h s'_h(x') \frac{\partial^2 s''_h}{\partial x''_i \partial x''_j}(x'') = 0.$$

Since $(X'', M'') \notin T_{g'',\delta}$, we obtain $\lambda_h s'_h(x') = 0$ for every $h \in \widetilde{G}$. As $g' = 1$, the divisors of the s'_h are pairwise disjoint. Thus at most one λ_h is nonzero. But then

$$\operatorname{div} s = (X' \times \operatorname{div} s''_h) \cup (\operatorname{div} s'_h \times X'')$$

has no point of multiplicity > 2, since $\operatorname{div} s''_h$, a translate of $\operatorname{div} s''_1$, is smooth.

2) $g' = 2$, $\delta = (2,2)$. Again $\lambda_h s'_h(x') = 0$ for every h. From the equation of the Kummer surface, the image of X' under $|M'|$ ([Mu 5], p. 354), if three sections s'_h vanish at one point its equation has the form

$$Bxyzw + C(x^2y^2 + z^2w^2) + D(x^2w^2 + y^2z^2) + E(x^2z^2 + y^2w^2) = 0,$$

which is not a general Kummer surface (see the end of the proof of Theorem 2.4). If three coefficients λ_h vanish, case 1 applies. We may therefore assume

$$\widetilde{G} = \{1,2,3,4\}, \qquad s'_1(x') = s'_2(x') = 0, \quad \lambda_3 = \lambda_4 = 0, \quad \lambda_1\lambda_2 \neq 0.$$

For general X', the intersection $\operatorname{div} s'_1 \cap \operatorname{div} s'_2$ is smooth: it suffices to check that, for general $(A, \Theta) \in \mathcal{A}_2$ and $a \in A[2] \setminus \{0\}$, $\Theta \cap \Theta_a$ is smooth. For $1 \le j \le 2$ and $1 \le k \le g''$, we have

$$\lambda_1 \frac{\partial s'_1}{\partial x'_j}(x') s''_1(x'') + \lambda_2 \frac{\partial s'_2}{\partial x'_j}(x') s''_2(x'') = 0,$$

$$\lambda_1 \frac{\partial s'_1}{\partial x'_j}(x') \frac{\partial s''_1}{\partial x''_k}(x'') + \lambda_2 \frac{\partial s'_2}{\partial x'_j}(x') \frac{\partial s''_2}{\partial x''_k}(x'') = 0.$$

Since $\operatorname{rank}(\partial s'_i / \partial x'_j)(x') = 2$, this implies $x'' \in \operatorname{Sing}(\operatorname{div} s''_1)$, a contradiction. □

To finish the proof of the theorem, it remains to handle the following cases.

1) $\delta = (2)$, $g = 1$. This is immediate.

2) $\delta = (3)$, $g = 2$. Use the notation of the lemma with $g' = g'' = 1$. By [Mu 5], p. 350, the image of X'' under $x'' \mapsto [s''_0(x'') : s''_1(x'') : s''_2(x'')]$ has equation

$$X^3 + Y^3 + Z^3 - 3\mu'' XYZ = 0, \qquad \mu'' \notin \{1, \omega, \omega^2, \infty\}, \quad \omega^3 = 1, \ \omega \neq 1.$$

If (x', x'') has multiplicity > 2 on $\operatorname{div} s$, then either all coefficients $\lambda_j s'_j(x')$ vanish or x'' is an inflection point of the cubic, since the divisor of $\sum_{j=0}^{2} \lambda_j s'_j(x') s''_j$ has multiplicity > 2 there. Now:

$$\{\text{inflection points}\} = \{[0:1:-1], [0:1:-\omega], [0:1:-\omega^2] \text{ and permutations}\},$$

$$\{\text{tangent lines}\} = \{\mu'' X + Y + Z = 0, \ \mu'' \omega X + Y + \omega^2 Z = 0,$$
$$\mu'' \omega^2 X + Y + \omega Z = 0 \text{ and permutations}\}.$$

Thus, interpreting nonzero coordinate vectors up to a common scalar,

$$\begin{aligned}&(\lambda_0 s_0'(x'), \lambda_1 s_1'(x'), \lambda_2 s_2'(x'))\\ &\in \{0\} \cup \mathbb{C}^*\{(\mu'', 1, 1), (\mu''\omega, 1, \omega^2), (\mu''\omega^2, 1, \omega) \text{ and permutations}\}.\end{aligned} \tag{A1.2.6}$$

If the coefficient vector is zero, at most one λ_j is nonzero, since the divisors of the s_j' are disjoint. The resulting product divisor has multiplicity at most two. In the remaining case, the inflection argument and its symmetric counterpart give

$$\begin{aligned}&[s_0''(x'') : s_1''(x'') : s_2''(x'')], \quad [s_0'(x') : s_1'(x') : s_2'(x')]\\ &\in \{[0:1:-1], [0:1:-\omega], [0:1:-\omega^2] \text{ and permutations}\}.\end{aligned} \tag{A1.2.7}$$

Hence exactly one $s_j'(x')$ vanishes. Choose $\mu'' \neq 0$; then (A1.2.6) and (A1.2.7) are incompatible for a nonzero coefficient vector. This completes this case.

3) $\delta = (2,2)$, $g = 2$. It suffices to find a Kummer surface in $\mathbb{P}^3$ all of whose hyperplane sections have multiplicity at most two at every point. The quartic S_ϵ with equation

$$F_\epsilon = x_0^4 + x_1^4 + x_2^4 + x_3^4 - 2(2+\epsilon)x_0x_1x_2x_3 + \epsilon(x_0^2x_1^2 + x_2^2x_3^2) = 0$$

is a Kummer surface for $\epsilon \notin \{0, 2, -2\}$. If a hyperplane section has a point of multiplicity > 2, there is a point of S_ϵ where the quadratic form

$$\sum_{i,j} \frac{\partial^2 F_\epsilon}{\partial x_i \partial x_j} X_i X_j$$

has rank at most two. This is impossible for $\epsilon = 0$, hence also for nearby ϵ.

4) $\delta = (2,2)$, $g = 5$. Use the notation of the lemma with $g' = 2$ and $g'' = 3$. Assume:

i) $(X', M') \notin T_{2,(2,2)} \cup U_{2,(2,2)}$, possible by case 3 and Proposition 2.1;

ii) $\operatorname{div} s_1''$ and $\operatorname{div} s_1'' \cap \operatorname{div} s_2''$ are smooth, possible since $T_{3,(2)} \neq \mathcal{A}_{3,(2)}$ implies $U_{3,(2)} \neq \mathcal{A}_{3,(2)}$;

iii) $\operatorname{div} s_1'' \cap \operatorname{div} s_2'' \cap \operatorname{div} s_3''$ and $F'' = \bigcap_h \operatorname{div} s_h''$ are smooth of dimension zero, with lengths 24 and 16 respectively.

For the last assertion, as in the proof of 2.1, it suffices to show that for general $(A, \Theta) \in \mathcal{A}_3$ and $\mathbb{F}_2$-linearly independent $a, b \in A[2]$ with $e_2(a,b) = 1$, we have:

- $\Theta \cap \Theta_a \cap \Theta_b$ is smooth of dimension zero; it suffices to consider a product of three elliptic curves.
- $\Theta \cap \Theta_a \cap \Theta_b \cap \Theta_{a+b}$ is reduced and equals $\{\epsilon, \epsilon+a, \epsilon+b, \epsilon+a+b\}$ for a point ϵ of order two. The original text leaves this proof to the reader.

The latter fact implies $U_{3,(2,2)} = \mathcal{A}_{3,(2,2)}$, hence, by 2.1, also $T_{3,(2,2)} = \mathcal{A}_{3,(2,2)}$.

Now perform the construction of Lemma 2.5. If $(Y, N) \in T_{5,(2,2)}$, assumption i) gives $\lambda_h s_h''(x'') = 0$ for every h.

- If $\lambda_2 = \lambda_3 = \lambda_4 = 0$, the divisor of s is the union of $X' \times \operatorname{div} s_1''$ and $\operatorname{div} s_1' \times X''$; by ii), it has no point of multiplicity > 2.
- If $\lambda_3 = \lambda_4 = 0$ and $\lambda_1\lambda_2 \neq 0$, then $s_1''(x'') = s_2''(x'') = 0$. As at the end of 2.5, assumption ii) implies

  $$s_1'(x') = s_2'(x') = \frac{\partial s_1'}{\partial x_j'}(x') = \frac{\partial s_2'}{\partial x_j'}(x') = 0,$$

  contradicting i).

- If $\lambda_4 = 0$ and $\lambda_1\lambda_2\lambda_3 \neq 0$, then $s_1''(x'') = s_2''(x'') = s_3''(x'') = 0$. Assumption iii) gives a contradiction in the same way.
- If $\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$, then $s_h''(x'') = 0$ for every $h \in \{1, \dots, 4\}$. By iii), $\operatorname{rank}(\partial s_h''/\partial x_j'')(x'') = 3$, so
$$\operatorname{rank}\left(s_h'(x'), \frac{\partial s_h'}{\partial x_1'}(x'), \frac{\partial s_h'}{\partial x_2'}(x')\right)_{1\le h\le 4} \le 1.$$
Since $\phi_{|M'|}\colon X' \to \mathbb{P}^3$ is unramified outside $X'[2]$, we have $x' \in X'[2]$. The original text now varies X' in $\mathcal{A}_{2,(2,2)}$ and observes that $[s_1'(x') : \cdots : s_4'(x')]$ describes a subvariety spanning $\mathbb{P}^3$ (consider a product of two elliptic curves). It asserts that this vector therefore cannot remain in the one-dimensional kernel of $(\partial s_h''/\partial x_j'')(x'')$.

The original argument concludes that all λ_h vanish, and hence $(Y, N) \notin T_{5,(2,2)}$. □

Editorial note on the case $g = 5$. The last inference omits the coefficients λ_h: the actual equations require $(\lambda_h s_h'(x'))_h \in \operatorname{Ker} D''$, where D'' is the first-derivative matrix on X''. Varying $(s_h'(x'))_h$ does not exclude this condition, since the λ_h are free. More precisely, in the symmetric presentation used here, take $x' \in X'[2]$ where all $s_h'(x')$ are nonzero and locally even, and a base point x'' where the s_h'' are locally odd. For $0 \neq w \in \operatorname{Ker} D''$, choosing $\lambda_h = w_h/s_h'(x')$ makes the order-zero, first-, and second-order jets of the product vanish. Thus this particular construction does not prove the assertion for $g = 5$. This edition retains the historical statement and argument but does not claim to repair the gap; consequences using this case or its induction require an independent justification.

The results of 2.4 and 2.1 can be expressed in terms of intersections of translates of the theta divisor of a general principally polarized abelian variety. If (X, M) has type $\delta = (d)$, there are a principally polarized variety (A, Θ), an isogeny $\pi\colon X \to A$ with $\pi^*\Theta = M$, a basis $s_1, \dots, s_d$ of $H^0(X, M)$, and a point $a \in A$ of order d generating $\pi(H(M))$, such that
$$\operatorname{div} s_j = \pi^*\Theta_{ja} \qquad (1 \le j \le d).$$
The base locus of $|M|$ is an étale cover of $\Theta \cap \Theta_a \cap \cdots \cap \Theta_{(d-1)a}$.

For $\delta = (2,2)$, the image $\pi(H(M)) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is generated by $a = \pi(b)$ and $a' = \pi(b')$, with $b, b' \in X[2]$. Writing $L = O_A(\Theta)$, we have ([Mu 6], p. 228)
$$e_2(a, a') = e^{L^{\otimes 2}}(a, a') = e^{M^{\otimes 2}}(b, b') = e^M(b, 2b') = 1.$$
Thus the base locus of $|M|$ is an étale cover of $\Theta \cap \Theta_a \cap \Theta_{a'} \cap \Theta_{a+a'}$.

Before stating the corollary of 2.4, we outline a generalization. For $a \in \mathbb{C}^g \setminus \mathbb{Z}^g$, define the following closed analytic subsets of $\mathfrak{H}_g$:
$$T_{g,a} = \{\tau \in \mathfrak{H}_g \mid \exists x \in \mathbb{C}^g,\ \exists [\lambda : \mu] \in \mathbb{P}^1,\ z \mapsto \lambda\theta(z, \tau) + \mu\theta(z - a, \tau) \text{ has multiplicity at least 3 at } x\},$$
and let $U_{g,a}$ be the locus where the intersection $\theta(z, \tau) = \theta(z - a, \tau) = 0$ in $\mathbb{C}^g$ fails to be smooth of dimension $g - 2$ for $g \ge 2$, or fails to be empty for $g = 1$.

If $2a \in \mathbb{Z}^g$, the image of $T_{g,a}$ (respectively $U_{g,a}$) in $\mathcal{A}_{g,(2)}$ is $T_{g,(2)}$ (respectively $U_{g,(2)}$). As in 2.1, $U_{g,a} = \mathfrak{H}_g$ implies $T_{g,a} = \mathfrak{H}_g$. The proof of case 1 of Lemma 2.5 also shows that if $a' \in \mathbb{C} \setminus \mathbb{Z}$, $\tau' \in \mathfrak{H}_1$, $a'' \in \mathbb{C}^{g''} \setminus \mathbb{Z}^{g''}$, $\tau'' \notin T_{g'',a''}$, and $\operatorname{div} \theta(\cdot, \tau'')$ is smooth, then
$$\begin{pmatrix} \tau' & 0 \\ 0 & \tau'' \end{pmatrix} \notin T_{g''+1,(a',a'')}.$$

Consequently:

Proposition A1.2.8. *Let $g \geq 1$ and let $a \in \mathbb{C}^g$ have no integral coordinate. Then $T_{g,a} \neq \mathfrak{H}_g$.*

Proof. It remains to treat $g = 1$. Otherwise there would be an open set $U \subset \mathfrak{H}_1$ and morphisms $\lambda\colon U \to \mathbb{C}^*$ and $s\colon U \to \mathbb{C}$ satisfying, for $\tau \in U$,

$$\begin{cases} \lambda(\tau)\theta(s(\tau),\tau) + \theta(s(\tau)-a,\tau) = 0, \\ \lambda(\tau)\dfrac{\partial\theta}{\partial z}(s(\tau),\tau) + \dfrac{\partial\theta}{\partial z}(s(\tau)-a,\tau) = 0, \\ \lambda(\tau)\dfrac{\partial^2\theta}{\partial z^2}(s(\tau),\tau) + \dfrac{\partial^2\theta}{\partial z^2}(s(\tau)-a,\tau) = 0. \end{cases} \tag{A1.2.9}$$

Differentiating the first equation with respect to τ, then using the other two and the heat equation, gives

$$\frac{\partial\lambda}{\partial\tau}(\tau)\theta(s(\tau),\tau) = 0.$$

If $\theta(s(\tau),\tau) = 0$, the first equation also gives $\theta(s(\tau)-a,\tau) = 0$. This is impossible in genus one if $a \notin \mathbb{Z} + \tau\mathbb{Z}$, hence for general τ. Thus $\lambda(\tau) = \lambda$ is constant. Lemma 9 of [An–M] then implies that $\lambda\theta(z,\tau) + \theta(z-a,\tau)$ vanishes identically, which is impossible. □

Taking $a = \frac{1}{n}(1,\dots,1) \in \mathbb{Q}^g$, $n \geq 2$, yields:

Theorem A1.2.10. *Let (A,Θ) be a very general principally polarized abelian variety of dimension $g \geq 2$. For every nonzero torsion point $a \in A$, the intersection $\Theta \cap \Theta_a$ is smooth of codimension two. In particular, if Θ is symmetric, every torsion point on it belongs to $A[2]$.*

Here "very general" accounts for the countable union of exceptional loci over all torsion orders. The final assertion follows because, for symmetric Θ and $x \in \Theta$, the intersection $\Theta \cap \Theta_{2x}$ is singular at x.

It remains to state the corollary of 2.4.

Corollary A1.2.11. *Let (A,Θ) be a general principally polarized abelian variety of dimension $g \geq 2$.*

- *If a has order three, $\Theta \cap \Theta_a \cap \Theta_{2a}$ is empty for $g = 2$ and smooth of dimension $g-3$ for $g \geq 3$.*
- *If $a, b \in A[2]$ are $\mathbb{F}_2$-linearly independent and $e_2(a,b) = 1$, then $\Theta \cap \Theta_a \cap \Theta_b$ is empty for $g = 2$ and smooth of dimension $g-3$ for $g \geq 3$; moreover, $\Theta \cap \Theta_a \cap \Theta_b \cap \Theta_{a+b}$ is smooth of dimension $g-4$ for $g \geq 4$.*

These schemes are irreducible whenever they have positive dimension.

Proof. As explained above, these assertions follow from 2.4, 2.1, and Remark 2.3. Irreducibility, also valid in 2.10, follows from the following result ([Fu–L]): if L is an ample line bundle on an irreducible projective variety X and $s_1,\dots,s_r$ are sections of L, then $\bigcap_{i=1}^{r} \operatorname{div} s_i$ is connected for $r < \dim X$. Apply this after a finite isogeny identifying the line bundles of the torsion translates. Connectedness together with smoothness gives irreducibility. □

Dependence of the argument. Conclusions using the $(2,2)$, $g = 5$, case of Theorem 2.4, or its induction in odd dimension, remain subject to the gap noted after that proof.

A1.3 The singular locus of the theta divisor of a general element of $\mathcal{A}^{\delta}_{g',g''}$

Proposition A1.3.1. *Suppose $T_{g',\delta} \neq \mathcal{A}_{g',\delta}$ and $U_{g'',\delta} \neq \mathcal{A}_{g'',\delta}$ (or the reverse conditions), with $g'' \geq g'$. The singular locus of the theta divisor of a general element of $\mathcal{A}^{\delta}_{g',g''}$ is empty if $g' < d = \deg \delta$, and otherwise is smooth of codimension $2d$, with equality in (A1.1.5). Every singularity of Θ is quadratic of rank $2d$.*

Proof. If equality in (A1.1.5) failed for every element of $\mathcal{A}^{\delta}_{g',g''}$, then, as in the proof of 2.1, there would be open sets $U' \subset \mathfrak{H}_{g'}$, $U'' \subset \mathfrak{H}_{g''}$ and holomorphic maps

$$s' : U' \times U'' \to \mathbb{C}^{g'}, \qquad s'' : U' \times U'' \to \mathbb{C}^{g''}$$

satisfying the following conditions. To simplify the formulas, write θ'_r for $\theta'_r(s'(\tau',\tau''),\tau')$ and $\theta''_{\psi r}$ for $\theta''_{\psi r}(s''(\tau',\tau''),\tau'')$, and use the same convention for their derivatives. For every $(\tau',\tau'') \in U' \times U''$,

$$\text{the two families } (\theta'_r)_r \text{ and } (\theta''_{\psi r})_r \text{ are not both identically zero;} \tag{A1.3.2}$$

$$\sum_r \theta'_r \theta''_{\psi r} = 0; \tag{A1.3.3}$$

$$\sum_r \frac{\partial \theta'_r}{\partial z'_i} \theta''_{\psi r} = 0 \quad (1 \leq i \leq g'); \tag{A1.3.4}$$

$$\sum_r \theta'_r \frac{\partial \theta''_{\psi r}}{\partial z''_j} = 0 \quad (1 \leq j \leq g''). \tag{A1.3.5}$$

Differentiating (A1.3.3) with respect to τ'_{ij} and using (A1.3.4), (A1.3.5), and the heat equations (I.3.14), gives

$$\sum_r \frac{\partial^2 \theta'_r}{\partial z'_i \partial z'_j} \theta''_{\psi r} = 0 \qquad \text{for all } i,j.$$

Since $T_{g',\delta} \neq \mathcal{A}_{g',\delta}$, we obtain $\theta''_{\psi r} = 0$ for every $r \in \Gamma_\delta$ and every $(\tau',\tau'') \in U' \times U''$. From $U_{g'',\delta} \neq \mathcal{A}_{g'',\delta}$ and (A1.3.5), we then deduce $\theta'_r = 0$ for all r, τ', τ'', contradicting (A1.3.2). Hence for general $(A,\Theta) \in \mathcal{A}^{\delta}_{g',g''}$, we have $\pi(F' \times F'') = \operatorname{Sing} \Theta$. Since $U_{g',\delta} \neq \mathcal{A}_{g',\delta}$ by 2.1 and $U_{g'',\delta} \neq \mathcal{A}_{g'',\delta}$, the singular locus is smooth of codimension $2d$, or empty if $g' < d$.

If $g' \geq d$, its Hessian matrix at a singular point is

$$\begin{pmatrix} 0 & T \\ {}^tT & 0 \end{pmatrix}, \qquad T = \left(\sum_r \frac{\partial \theta'_r}{\partial z'_i}(z') \frac{\partial \theta''_{\psi r}}{\partial z''_j}(z'') \right)_{\substack{1 \leq i \leq g' \\ 1 \leq j \leq g''}}.$$

Both first-derivative matrices have rank d, with $d \leq g' \leq g''$, so this Hessian has rank $2d$. Thus the singularities of Θ are quadratic of rank $2d$. □

Corollary A1.3.6. *For $\delta = (2),(3)$, or $(2,2)$, the singular locus of the theta divisor of a general element of $\mathcal{A}^{\delta}_{g',g''}$, with $g'' \geq g'$, is:*

- *empty if $g' < d = \deg \delta$, except possibly when $\delta = (2,2)$ and $3 \in \{g', g''\}$;*
- *a set of $(d!)^2$ points if $g' = g'' = d$;*
- *smooth, with $(d-1)!$ connected components of codimension $2d$, if $g'' > g' = d$;*
- *smooth and irreducible of codimension $2d$ if $g' > d$.*

Proof. It remains to count connected components, using the final assertion of 2.11. Suppose $g' = d$. We have $(M')^{g'} = d(g')!$ (I.3.6). The base locus F' of M' has cardinality $d\,d!$ and is stable under $H(M')$, whose order is d^2 (I.3.7). Thus the number of components is

$$\begin{cases} \dfrac{(d\,d!)^2}{d^2} = (d!)^2 & \text{if } g'' = d, \\ \dfrac{d\,d!}{d^2} = (d-1)! & \text{if } g'' > d. \end{cases}$$

□

Remark A1.3.7. The singularities of Θ are then quadratic of rank $2d$, and $\operatorname{Sing}\Theta$ is invariant under the translation group of order d^2, isomorphic to $H(\delta)$,

$$\pi(H(M') \times \{0\}) = \pi(\{0\} \times H(M'')).$$

A1.4 Applying the Andreotti–Mayer theorem

Theorem A1.4.1. *Suppose $g'' = g - g' \geq g' \geq 1$, $T_{g',\delta} \neq \mathcal{A}_{g',\delta}$ and $U_{g'',\delta} \neq \mathcal{A}_{g'',\delta}$ (see Section 2), or the reverse conditions. If $g' \geq \deg\delta$ and $g'' > \deg\delta$, then $\mathcal{A}^{\delta}_{g',g''}$ is an irreducible component of $N^g_{g-2\deg\delta}$, of codimension $g'g''$ in $\mathcal{A}_g$.*

Proof. Apply the Andreotti–Mayer theorem (II.D.1.4.3) to a general element (A,Θ) of $\mathcal{A}^{\delta}_{g',g''}$. By 3.1, $\operatorname{Sing}\Theta = \pi(F' \times F'')$ is equidimensional of codimension $2d$, where F' and F'' are the base loci of M' and M''. Moreover, F'' is irreducible. Let F'_0 be an irreducible component of F': it is a point if $g' = d$, whereas $F'_0 = F'$ if $g' > d$. By the end of 3.1, the map II.D.1.4.4 is

$$\begin{aligned} F'_0 \times F'' &\longrightarrow \mathbb{P}(T_0^\vee X' \otimes T_0^\vee X'') \subset \mathbb{P}S^2 T_0^\vee A, \\ (z',z'') &\longmapsto \left[\left(\sum_r \frac{\partial s'_r}{\partial z'_i}(z')\frac{\partial s''_r}{\partial z''_j}(z'')\right)_{i,j}\right]. \end{aligned} \tag{A1.4.2}$$

The derivatives $\partial s'_r/\partial z'_i$ and $\partial s''_r/\partial z''_j$ are sections in $H^0(F',M')$ and $H^0(F'',M'')$ ([Gr 1], p. 92, and the next lemma). It suffices to show that (A1.4.2) is nondegenerate, then apply II.D.1.4.3.

Lemma A1.4.3. *Let (X,M) be a polarized abelian variety of dimension g, and let $s_1,\dots,s_d$ be a basis of $H^0(X,M)$. If the base scheme F of $|M|$ has dimension $g-d>0$, the dg sections*

$$\left\{ \left.\frac{\partial s_r}{\partial z_j}\right|_F \;\middle|\; 1 \leq r \leq d,\ 1 \leq j \leq g \right\}$$

form a basis of $H^0(F,M)$.

Proof. For $1 \leq r \leq d$, let D_r be the scheme defined by $s_1 = \cdots = s_r = 0$, and set $D_0 = X$. We have $H^i(D_r, O_{D_r}(kM)) = 0$ for $k > 0$ and $0 < i < g - r$. Thus restriction $H^1(X,O_X) \to H^1(D_r,O_{D_r})$ is an isomorphism for $g-r>1$. Let $E_r = \operatorname{div} s_r$ on X. For $r \geq 1$ and $g-(r-1) > 1$, we have the following diagram with exact rows. For legibility, set $V_Z = H^0(Z,O_Z(M))$ and $Q_Z = H^1(Z,O_Z)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_X/\mathbb{C}s_r & \longrightarrow & V_{E_r} & \longrightarrow & Q_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\sim & & \\ 0 & \longrightarrow & V_{D_{r-1}}/\mathbb{C}s_r & \longrightarrow & V_{D_r} & \longrightarrow & Q_{D_{r-1}} & \longrightarrow & 0. \end{array}$$

The first row can be written ([Gr 1], p. 95)

$$0 \longrightarrow \langle s_1, \dots, \hat{s}_r, \dots, s_d \rangle \longrightarrow \left\langle s_1, \dots, \hat{s}_r, \dots, s_d, \frac{\partial s_r}{\partial z_1}, \dots, \frac{\partial s_r}{\partial z_g} \right\rangle \longrightarrow H^1(X, O_X) \longrightarrow 0,$$

with $\partial s_r / \partial z_j \mapsto d\bar{z}_j$, for a suitable choice of coordinates $z_1, \dots, z_g$. Induction on r shows that, for $1 \le r \le \min(d, g-1)$, a basis of $H^0(D_r, O_{D_r}(M))$ is

$$\left\{ s_{r+1}, \dots, s_d, \ \frac{\partial s_\ell}{\partial z_j} \,\middle|\, 1 \le \ell \le r, \ 1 \le j \le g \right\}.$$

Taking $r = d$ proves the lemma. □

Since $g'' > d$, the lemma applies to (X'', M''). If (A1.4.2) were degenerate, there would be $[\lambda_{ij}] \in \mathbb{P}^{g'g''-1}$ such that for every $z' \in F'_0$ and $z'' \in F''$,

$$\sum_{r,i,j} \lambda_{ij} \frac{\partial s'_r}{\partial z'_i}(z') \frac{\partial s''_r}{\partial z''_j}(z'') = 0.$$

The lemma then gives, for all $z' \in F'_0, j, r$,

$$\sum_i \lambda_{ij} \frac{\partial s'_r}{\partial z'_i}(z') = 0.$$

If $g' > d$, applying the lemma to (X', M') shows that all λ_{ij} vanish. If $g' = d$, the matrix $(\partial s'_r / \partial z'_i)(z')$ is square of rank d, giving the same conclusion. Thus (A1.4.2) is nondegenerate, and II.D.1.4.3 completes the proof. □

Remark A1.4.4. Under the assumptions of 4.1, for general $(A, \Theta) \in \mathcal{A}^{\delta}_{g',g''}$, the intersection in $\mathbb{P}T_0A$ of the tangent cones at the singular points of Θ is the disjoint union

$$\mathbb{P}T_0X' \simeq \mathbb{P}^{g'-1} \qquad \text{and} \qquad \mathbb{P}T_0X'' \simeq \mathbb{P}^{g''-1}.$$

Corollary A1.4.5.

- $\mathcal{A}^{(2)}_{g',g-g'}$ *is a component of* N^g_{g-4} *for* $2 \le g' \le g/2$ *and* $g \ge 5$.
- $\mathcal{A}^{(3)}_{g',g-g'}$ *is a component of* N^g_{g-6} *for* $3 \le g' \le g/2$ *and* $g \ge 7$.
- $\mathcal{A}^{(2,2)}_{g',g-g'}$ *is a component of* N^g_{g-8} *for* $4 \le g' \le g/2$ *and* $g \ge 9$.

Remark A1.4.6. If $\theta_{\mathrm{null},g}$ denotes the locus of principally polarized abelian varieties on which a theta constant vanishes (see III), then ([D 2])

$$\begin{aligned} &\mathcal{A}^{(2)}_{g',g-g'} \subset \theta_{\mathrm{null},g} \quad \text{for } 2 \le g' \le g/2, \\ &\mathcal{A}^{(3)}_{g',g-g'} \not\subset \theta_{\mathrm{null},g}, \\ &\mathcal{A}^{(2,2)}_{g',g-g'} \subset \theta_{\mathrm{null},g} \quad \text{for } 4 \le g' \le g/2. \end{aligned}$$

APPENDIX 2

Abelian varieties polarized of degree two and Prym varieties

A2.1 General results

Recall that an abelian variety polarized of degree 2 is an abelian variety X equipped with an algebraic equivalence class of ample line bundles M satisfying $h^0(X,M) = 2$. The group $H(M)$ is isomorphic to $(\mathbb{Z}/2)^2$, and for every nonzero $\varepsilon \in H(M)$ there is an isogeny from (X,M) onto a principally polarized abelian variety $(X^\varepsilon, M^\varepsilon)$ with kernel $\{0,\varepsilon\}$. The moduli space of abelian varieties polarized of degree 2 and dimension g is denoted by $\mathcal{A}_{g,(2)}$.

Recall that a polarized abelian variety is called irreducible if it is not isomorphic to a product of two nonzero polarized abelian varieties.

Proposition A2.1.1. *Let (X,M) be an irreducible abelian variety polarized of degree 2. Suppose that some $D \in |M|$ is reducible and that $\dim X \geq 2$. Then there exist two principally polarized abelian varieties (A_1,Θ_1) and (A_2,Θ_2) and an isogeny $\pi : X \to A_1 \times A_2$ of degree 2 such that*

$$D = \pi^*(\Theta_1 \times A_2 + A_1 \times \Theta_2).$$

In particular, there exist nonzero abelian varieties polarized of degree 2, (X_1,M_1) and (X_2,M_2), and an isogeny $\varphi : X_1 \times X_2 \to X$ of degree 2 such that $\varphi^*M \simeq M_1 \boxtimes M_2$.

Proof. Write $D = D_1 + D_2$ and set (see I.3.3)

$$K_i = H(D_i), \qquad p_i : X \longrightarrow X/K_i^0 = B_i.$$

There is a divisor Θ_i on B_i such that $p_i^*\Theta_i = D_i$ and $H^0(B_i,\Theta_i) \simeq H^0(X,D_i)$. Since (X,M) is irreducible, the polarizations Θ_i are principal. The kernel of $f : X \to B = B_1 \times B_2$ is

$$K_1^0 \cap K_2^0 \subset \{x \mid D_x = D\} \subsetneq H(M);$$

it therefore has at most 2 elements.

If f is injective, the restriction of the product polarization on B to its image X has degree 2. As in A1.1.1, there is an isogeny

$$\pi : X \times Y \longrightarrow B_1 \times B_2,$$

where $Y = \ker(B_1 \times B_2 \xrightarrow{\sim} \hat{B}_1 \times \hat{B}_2 \to \widehat{X})$ is an abelian variety polarized of degree 2.

Since K_i^0 is connected, $\hat{p}_i : \hat{B}_i \to \widehat{X}$ is injective and $Y \cap (\{0\} \times B_2) = Y \cap (B_1 \times \{0\}) = \{0\}$. Thus the maps $j_i : Y \to B_1 \times B_2 \to B_i$, for $i = 1,2$, are injective, and $j_i^*\Theta_i$ is ample on Y. If $\dim Y = n$, then (I.3.6)

$$\begin{aligned} 2 &= \frac{1}{n!}[(\mathrm{pr}_1^*\,\Theta_1 + \mathrm{pr}_2^*\,\Theta_2)|_Y]^n \\ &= \frac{1}{n!}(j_1^*\Theta_1 + j_2^*\Theta_2)^n \geq \deg j_1^*\Theta_1 + \deg j_2^*\Theta_2, \end{aligned}$$

with strict inequality if $n > 1$. Hence $n = \deg j_1^*\Theta_1 = \deg j_2^*\Theta_2 = 1$. The polarizations $j_i^*\Theta_i$ are therefore principal, and we may write $B_i \simeq Y \times A_i$ with A_i principally polarized. If $g : Y \to Y \times Y$ is the diagonal map, then

$$X \simeq \ker(\widehat{Y} \times \widehat{A}_1 \times \widehat{Y} \times \widehat{A}_2 \xrightarrow{\hat{g}\circ \mathrm{pr}_{1,3}} \widehat{Y}).$$

Since X is assumed irreducible, $A_1 = A_2 = 0$, and $X \simeq Y$ has dimension 1, contrary to the hypothesis.

Thus f has degree 2 onto its image A. The restriction Θ of the product polarization on B to A is principal and $D = f^*\Theta$. The divisor Θ is reducible, and so is the principally polarized abelian variety (A, Θ), proving the first assertion.

Finally, the kernel of the dual isogeny $\widehat{\pi} : \widehat{A}_1 \times \widehat{A}_2 \to \widehat{X}$ is generated by $(\varepsilon_1, \varepsilon_2)$, where ε_i is a nonzero point of order 2 on $\widehat{A}_i$. If $\widehat{Y}_i$ is the quotient of $\widehat{A}_i$ by ε_i, there is an isogeny of degree 2 from $\widehat{X}$ onto $\widehat{Y}_1 \times \widehat{Y}_2$. Its transpose is the required isogeny of degree 2 from $Y_1 \times Y_2$ onto X. □

1.2. Choose generators α, β of $H(M)$ and lifts $\tilde{\alpha}, \tilde{\beta} \in \mathcal{G}(M)$ satisfying

$$\tilde{\alpha}^2 = \tilde{\beta}^2 = 1, \qquad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta} = -1.$$

The space of sections of M invariant under $\tilde{\alpha}$ has dimension 1. Choose a generator s and set $t = \tilde{\beta} \cdot s$; then $\{s, t\}$ is a basis of $H^0(X, M)$.

The preceding proposition shows that if (X, M) is irreducible of dimension ≥ 2, the only potentially reducible members of $|M|$ are:

- $\mathrm{div}(s)$ and $\mathrm{div}(t) = \mathrm{div}(s)_\beta$, in which case $X^\alpha = X/\{0, \alpha\}$ is reducible;
- $\mathrm{div}(s + t)$ and $\mathrm{div}(s - t) = \mathrm{div}(s + t)_\alpha$, in which case X^β is reducible;
- $\mathrm{div}(s + it)$ and $\mathrm{div}(s - it) = \mathrm{div}(s + it)_\alpha$, in which case $X^{\alpha+\beta}$ is reducible.

1.3. We place ourselves in the situation of the proposition: let (X_1, M_1), (X_2, M_2), and (X, M) be abelian varieties polarized of degree 2, and let $\pi : X_1 \times X_2 \to X$ be an isogeny whose kernel is generated by (α_1, α_2), where $\alpha_i \in H(M_i)$ is nonzero. Then (X, M) is irreducible if and only if both (X_1, M_1) and (X_2, M_2) are irreducible.

It suffices to show that if (X, M) is reducible, so that there is a decomposition

$$(X, M) \simeq (A, L) \times (X', M'),$$

where (A, L) is irreducible and principally polarized and (X', M') is polarized of degree 2, then (X_1, M_1) or (X_2, M_2) is reducible.

Let (A_i, L_i) be the principally polarized quotient of (X_i, M_i) by α_i. The composite map $p : A \to X \to A_1 \times A_2$ satisfies $p^*(L_1 \boxtimes L_2) \simeq L$. Since (A, L) is irreducible, it is a direct factor of, say, (A_1, L_1). It is then also a direct factor of (X_1, M_1), which is therefore reducible.

A2.2 The loci $\mathcal{A}^2_{g_1,\dots,g_r}$

We now study in greater detail the loci $\mathcal{A}^2_{g_1,g_2}$ introduced in A1.1.

Recall their construction. For $i = 1, 2$, choose $(X_i, M_i) \in \mathcal{A}_{g_i,(2)}$ and generators α_i, β_i of $H(M_i)$. The quotient of $X_1 \times X_2$ by the subgroup generated by (α_1, α_2) and (β_1, β_2) is a principally polarized abelian variety (A, Θ) belonging to $\mathcal{A}^2_{g_1,g_2}$.

Suppose now that some $D \in |M_2|$ (or $|M_1|$) is reducible. The preceding proposition expresses (A, Θ) as the quotient of $X_1 \times X_2' \times X_3'$ by the subgroup generated, if for example $D = D_{\alpha_2}$, by

$$(0, \alpha_2', \alpha_3'), \quad (\alpha_1, \alpha_2', 0), \quad (\beta_1, \beta_2', \beta_3'),$$

where

$$(X_2', M_2') \in \mathcal{A}_{g_2', (2)}, \quad (X_3', M_3') \in \mathcal{A}_{g_3', (2)}, \quad g_2' + g_3' = g_2,$$

and α_2', β_2' generate $H(M_2')$, while α_3', β_3' generate $H(M_3')$.

This process can of course be continued if one of $|M_1|$, $|M_2'|$, or $|M_3'|$ contains a reducible divisor of dimension > 0. The resulting quotients are no longer all of the same type: there are two cases.

i) (A, Θ) is a quotient of $X_1 \times X_2 \times X_3 \times X_4$ by the subgroup generated by

$$(\alpha_1, \alpha_2, 0, 0), \quad (\alpha_1, 0, \alpha_3, 0), \quad (\alpha_1, 0, 0, \alpha_4), \quad (\beta_1, \beta_2, \beta_3, \beta_4).$$

This occurs if a member of $|M_3'|$ invariant under translation by α_3' is reducible. ii) (A, Θ) is the quotient of $X_1 \times X_2 \times X_3 \times X_4$ by the subgroup generated by

$$(\alpha_1, \alpha_2, 0, 0), \quad (\alpha_1, 0, \alpha_3, \alpha_4), \quad (0, 0, \beta_3, \beta_4), \quad (\beta_1, \beta_2, \beta_3, 0).$$

This occurs if a member of $|M_3'|$ invariant under translation by β_3' is reducible.

2.1. We consider only the first type. Define the subsets $\mathcal{A}^2_{g_1, \dots, g_r}$ of $\mathcal{A}_{g_1 + \cdots + g_r}$ as follows.

Let $(X_i, M_i) \in \mathcal{A}_{g_i, (2)}$ for $i = 1, \dots, r$; let α_i, β_i generate $H(M_i)$; and choose a basis $\{s^i, t^i\}$ of $H^0(X_i, M_i)$ as in 1.2. The quotient of $X_1 \times \cdots \times X_r$ by the subgroup generated by

$$(\alpha_1, 0, \dots, 0, \alpha_i, 0, \dots, 0), \quad 2 \le i \le r, \qquad (\beta_1, \beta_2, \dots, \beta_r)$$

is a principally polarized abelian variety (A, Θ). The subset of $\mathcal{A}_{g_1 + \cdots + g_r}$ so obtained is denoted by $\mathcal{A}^2_{g_1, \dots, g_r}$.

The space

$$H^0(A, \Theta) \subset H^0(X_1 \times \cdots \times X_r, M_1 \boxtimes \cdots \boxtimes M_r)$$

is generated by $s^1 s^2 \cdots s^r + t^1 t^2 \cdots t^r$.

Factors can be regrouped: if X_1' is the quotient of $X_1 \times X_2$ by the subgroup generated by (α_1, α_2), then (A, Θ) is again a quotient of $X_1' \times X_3 \times \cdots \times X_r$ of the above type, with $\alpha_1' = (\alpha_1, 0)$ and $\beta_1' = (\beta_1, \beta_2)$, interpreted as images in the quotient. In particular,

$$\mathcal{A}^2_{g_1, \dots, g_r} \subset \mathcal{A}^2_{g_1 + g_2, g_3, \dots, g_r},$$

and so on.

2.2. Conversely, the discussion at the start of this section shows that every member of $\mathcal{A}^2_{g_1, g_2}$ can be expressed as a member of

$$\mathcal{A}^2_{g_1^1, \dots, g_1^r, g_2^1, \dots, g_2^s}, \qquad r, s \ge 1, \quad \sum_{j=1}^{r} g_1^j = g_1, \quad \sum_{j=1}^{s} g_2^j = g_2,$$

that is, as a quotient of $X_1^1 \times \cdots \times X_1^r \times X_2^1 \times \cdots \times X_2^s$, with the additional property that $\operatorname{div} s_j^i$ is irreducible whenever $g_j^i \ge 2$.

The following proposition determines when a member of $\mathcal{A}^2_{g_1,\dots,g_r}$ is irreducible. The assumption that (X_i, M_i) is irreducible is clearly necessary.

Proposition A2.2.3. *Let $(A,\Theta) \in \mathcal{A}^2_{g_1,\dots,g_r}$ be expressed as a quotient of a product of $r \geq 2$ irreducible abelian varieties (X_i, M_i) polarized of degree 2. Then (A,Θ) is irreducible except in the following case: $r = 2$, $g_1 = g_2 = 1$, and there is an isomorphism $\psi : (X_1, M_1) \xrightarrow{\sim} (X_2, M_2)$ satisfying $\psi(\alpha_1) = \alpha_2$ and $\psi(\beta_1) = \beta_2$. In this case (A,Θ) is isomorphic to $X_1 \times X_1$ with its product polarization.*

Proof. It suffices to treat $r = 2$. With the notation of 2.1, an equation of $D = \pi^*\Theta$ is $s^1 s^2 + t^1 t^2$. Suppose D is reducible. If a component Z of D did not project onto X_1, it would be of the form $Z = D_1 \times X_2$, with D_1 a divisor on X_1. Both s^1 and t^1 would vanish on D_1. Thus $\dim X_1 \geq 2$ and 1.2 would imply that (X_1, M_1) is reducible.

Every component of D therefore projects onto both X_1 and X_2. Setting aside $g_1 = g_2 = 1$, suppose, for instance, that $g_2 > 1$. For generic $x \in X_1$, $D \cap (\{x\} \times X_2)$ is reducible, so $\operatorname{div}(ps^2 + qt^2)$ is reducible for every $(p : q) \in \mathbf{P}^1$. Again this contradicts 1.2.

Now consider $g_1 = g_2 = 1$. If (A,Θ) is reducible, write $\pi : X_1 \times X_2 \to E_1 \times E_2$. If an induced map $u^i_j : X_i \to E_j$ were zero, u^i_{3-j} would be an isomorphism because $X_i \to E_1 \times E_2$ is injective. But the induced polarization on X_i would then be principal, contrary to the hypothesis.

Thus all u^i_j are surjective, and

$$\deg((u^i_1)^*\Theta_1 + (u^i_2)^*\Theta_2) = 2.$$

The polarizations $(u^i_j)^*\Theta_j$ are therefore principal, and the u^i_j are isomorphisms. Through these, we identify X_2 with X_1, matching the chosen generators. The kernel of π is then generated by (α_1, α_1) and (β_1, β_1), and the quotient π is identified with

$$X_1 \times X_1 \longrightarrow X_1 \times X_1, \qquad (x, y) \longmapsto (x + y, x - y).$$

□

The following consequence of 2.3 supplements 1.1 and 1.2.

Proposition A2.2.4. *Let (X, M) be an irreducible abelian variety of dimension g, polarized of degree 2. At most one of the three quotients $(X^\varepsilon, M^\varepsilon)$, $\varepsilon \in H(M) \setminus \{0\}$, is reducible, unless X is the quotient of $E \times E$, for an elliptic curve E, by $\{0, (\alpha, \alpha)\}$ with $0 \neq \alpha \in E[2]$, so that $\dim X = 2$.*

In this exceptional case, all three quotients are reducible if and only if E has complex multiplication by i ($j(E) = 1728$) and α is the unique nonzero element of $E[2]$ fixed by this multiplication.

Proof. Suppose that

$$(X^\alpha, M^\alpha) = (X, M)/\{0, \alpha\} \simeq (A_1 \times A_2, L_1 \boxtimes L_2).$$

As at the end of the proof of 1.1, there is an isogeny $\pi : X_1 \times X_2 \to X$ of degree 2, where the (X_i, M_i) are irreducible and polarized of degree 2, $H(M_i)$ is generated by α_i, β_i, and $\ker \pi$ is generated by (α_1, α_2), with $\alpha = \pi(\alpha_1, 0) = \pi(0, \alpha_2)$ and $\beta = \pi(\beta_1, \beta_2)$.

The principally polarized variety X^β (respectively $X^{\alpha+\beta}$) is the quotient of $X_1 \times X_2$ by the subgroup generated by (α_1, α_2) and (β_1, β_2) (respectively (α_1, α_2) and $(\alpha_1 + \beta_1, \beta_2)$). By 2.3 it is irreducible unless there is an isomorphism $\psi : X_1 \xrightarrow{\sim} X_2$ with $\psi(\alpha_1) = \alpha_2$, $\psi(\beta_1) = \beta_2$ (respectively $\psi(\alpha_1) = \alpha_2$, $\psi(\alpha_1 + \beta_1) = \beta_2$), and $\dim X_i = 1$.

These two possibilities occur simultaneously only if there is an automorphism τ of X_1 satisfying $\tau(\alpha_1) = \alpha_1$ and $\tau(\beta_1) = \alpha_1 + \beta_1$. This happens if and only if X_1 is the elliptic curve with complex multiplication by i ($j = 1728$). □

A2.3 The families $\mathcal{P}^2_{g_1,\dots,g_r}$

The aim of this section is to study $\mathcal{P}^2_{g_1,\dots,g_r}$, the subset of $\mathcal{A}^2_{g_1,\dots,g_r}$ consisting of varieties constructed as in 2.1 from elements (X_i, M_i) of a subset $\mathcal{P}_{g_i,(2)} \subset \mathcal{A}_{g_i,(2)}$ that we now define. An *admissible cover ramified at four points* is a cover identified with the quotient of a stable curve $\widetilde{C}$ of genus $2g-1$ by an involution σ satisfying:

1. σ has exactly four fixed points smooth on $\widetilde{C}$, denoted by p_1, p_2, p_3, p_4;
2. at any other fixed points, necessarily nodes, the two branches are not interchanged;
3. the number of nodes interchanged by σ equals the number of components of $\widetilde{C}$ interchanged by σ.

The Prym variety, in the sense of [Mu 2], of an admissible cover $\pi : \widetilde{C} \to C = \widetilde{C}/\sigma$ is the abelian variety $\ker(J\widetilde{C} \xrightarrow{\mathrm{Nm}} JC)$ with the polarization induced by the principal polarization of $J\widetilde{C}$. Its dual, isomorphic to $J\widetilde{C}/\pi^*JC$, is then equipped with a polarization of degree two.

Let $\mathcal{P}_{g,(2)} \subset \mathcal{A}_{g,(2)}$ consist of the duals of Prym varieties of admissible covers ramified at four points. These are abelian varieties polarized of degree 2. In dimension 1, we use the usual convention allowing smooth genus-one double covers of $\mathbf{P}^1$ ramified at four points. Thus $\mathcal{P}_{1,(2)} = \mathcal{A}_{1,(2)}$; the stable-curve argument below concerns $g \geq 2$.

3.1. The curve $\widetilde{C}' = \widetilde{C}/(p_1 \sim p_2,\ p_3 \sim p_4)$, obtained by identifying p_1 with p_2 and p_3 with p_4, with its induced involution σ', defines an admissible cover in the sense of [Be 1]. There is a diagram ([Be 1], p. 159):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{Z}/2)^2 & \to & P' \cup P'^{-} & \to & \widehat{X} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{C}^*)^2 & \longrightarrow & J\widetilde{C}' & \longrightarrow & J\widetilde{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{C}^*)^2 & \longrightarrow & JC' & \longrightarrow & JC & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P' is the principally polarized Prym variety associated with $(\widetilde{C}', \sigma')$ and $\widehat{X}$ is the Prym variety associated with $(\widetilde{C}, \sigma)$. Thus there is an isogeny of degree 2 from X onto $\widehat{P}'$. The abelian variety (X, M) belongs to $\mathcal{P}_{g,(2)}$. The three isogenies obtained from $\widetilde{C}'$, $\widetilde{C}'' = \widetilde{C}/(p_1 \sim p_3,\ p_2 \sim p_4)$, and $\widetilde{C}''' = \widetilde{C}/(p_1 \sim p_4,\ p_2 \sim p_3)$ are the quotients of X by the three nonzero elements of $H(M)$.

Proposition A2.3.2. *The subvariety $\mathcal{P}_{g,(2)}$ of $\mathcal{A}_{g,(2)}$ is closed and irreducible.*

Proof. First consider a connected stable curve $\widetilde{C}$ of genus $2g-1$, with at worst ordinary nodes, equipped with an involution σ. Write C for the quotient curve and $P = (\ker \mathrm{Nm})^0$ for the associated Prym variety. We have the following analogue of Lemma 5.1 of [Be 1].

Lemma A2.3.3. *The cover $\widetilde{C} \to C$ is admissible in the above sense if and only if:*

- *σ is not the identity on any component of $\widetilde{C}$;*

- $p_a(C) = g - 1$;
- *P is an abelian variety.*

Proof. Assume these three conditions. The proof of Lemma 5.1 of loc. cit. gives, in Beauville's notation,

$$n_e + n'_f = c_e, \qquad \frac{r}{2} + n'_f = 2.$$

To show admissibility, it therefore suffices to prove that the number n'_f of nodes of $\widetilde{C}$ fixed by σ with the branches interchanged is zero.

Let x be such a node and $\widetilde{N}$ the normalization of $\widetilde{C}$ at x. Again $\widetilde{N}/\sigma \simeq C$, and $\ker(J\widetilde{C} \to J\widetilde{N})^0 \subset P$. Since P is an abelian variety, $J\widetilde{C} \to J\widetilde{N}$ is an isomorphism, so $\widetilde{N}$ is disconnected. Write $\widetilde{N} = \widetilde{N}_1 \cup \widetilde{N}_2$, with each $\widetilde{N}_i$ connected. The set $\{y \in \widetilde{N}_1 \mid \sigma y \in \widetilde{N}_2\}$ is open and closed and contains one of the points above x. Hence $\widetilde{N} = \widetilde{N}_1 \sqcup \sigma\widetilde{N}_1$, and $g(\widetilde{C}) = 2g(\widetilde{N}_1)$, a contradiction. Thus $n'_f = 0$, proving the lemma. □

We now proceed as in Section 6 of loc. cit. For the same reasons, there is a complete irreducible variety $\overline{S}$, a family of stable curves $\widetilde{C} \to \overline{S}$, and an $\overline{S}$-involution σ, such that:

- for every $s \in \overline{S}$, the induced involution on $\widetilde{C}_s$ is not the identity on any component;
- $\widetilde{C}_s$ has genus $2g - 1$, and $\widetilde{C}_s/\sigma_s$ has genus $g - 1$;
- every connected smooth curve $\widetilde{C}$ of genus $2g - 1$ with an involution having four fixed points gives a pair $(\widetilde{C}, \sigma)$ isomorphic to $(\widetilde{C}_s, \sigma_s)$ for some $s \in \overline{S}$.

Define, in the same way,

$$\overline{\mathcal{P}} = \ker(\mathrm{Nm} : \mathrm{Pic}^0(\widetilde{C}/\overline{S}) \longrightarrow \mathrm{Pic}^0(C/\overline{S}))^0.$$

This is a smooth algebraic space over $\overline{S}$. The set S of points $s \in \overline{S}$ for which $\overline{\mathcal{P}}_s$ is an abelian variety is open, and the induced map $p : S \to \mathcal{A}_{g,\delta_g}$ is proper, where $\delta_g = (1, 2, \dots, 2)$ is the polarization type of degree 2^{g-1} (Proposition 6.3 of loc. cit.). It follows that $\mathcal{P}_{g,(2)}$ is closed in $\mathcal{A}_{g,(2)}$.

To prove irreducibility, a study of local deformations of an admissible pair (see [H–M], pp. 29 and 33) shows that it deforms to an admissible pair $(\widetilde{C}, \sigma)$ with $\widetilde{C}$ smooth and connected. The irreducibility of the family of these pairs follows from the fact already proved in the introduction to II.D: if $C \to S$ is an irreducible family of smooth curves, and $C^4_S = C \times_S C \times_S C \times_S C$ and $\mathcal{J}^d = \mathrm{Pic}^d(C/S)$ denotes the relative Picard scheme in degree d, with $m : \mathcal{J}^2 \to \mathcal{J}^4$ the squaring map, then the S-scheme $\mathcal{J}^2 \times_{\mathcal{J}^4} C^4_S$ is irreducible. □

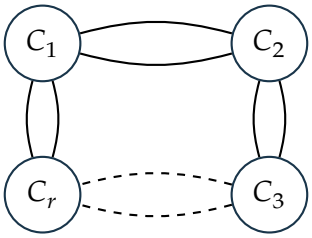
3.4. Define $\mathcal{P}^2_g \subset \mathcal{A}_g$ as the set of quotients of members $(P, M) \in \mathcal{P}_{g,(2)}$ by a nonzero element of $H(M)$. By 3.1 this is also the set of Prym varieties of admissible covers of stable curves of genus $g + 1$ with at least two nodes. It is a closed irreducible subset of $\mathcal{P}_g$.

Similarly, define $\mathcal{P}^2_{g_1,\dots,g_r}$ as the set of members of $\mathcal{A}^2_{g_1,\dots,g_r}$ constructed as in 2.1 from $(X_i, M_i) \in \mathcal{P}_{g_i,(2)}$. The following proposition and its proof give:

$$\mathcal{P}^2_{g_1,\dots,g_r} \subset \mathcal{P}_{g_1+\cdots+g_r}. \tag{A2.3.5}$$

3.6. Every member of $\mathcal{P}^2_{g_1,g_2}$ can be expressed as a quotient of $X^1_1 \times \cdots \times X^r_1 \times X^1_2 \times \cdots \times X^s_2$, with, in the notation of 2.1, $\mathrm{div}\, s^i_j$ irreducible whenever $\dim X^i_j \geq 2$, and $X^i_j \in \mathcal{P}_{g^i_j,(2)}$.

3.7. $\mathcal{P}^2_{g_1,\dots,g_r}$ is the set of Prym varieties of admissible covers of stable curves that are unions of r curves $C_1,\dots,C_r$ of respective genera $g_1-1,\dots,g_r-1$, meeting as follows. The incidence graph represents a component by a vertex and a node by an edge: consecutive components in the cycle meet at two points.



Proposition A2.3.8. *Let* $(X_1,M_1)\in\mathcal{A}_{g_1,(2)}$, $(X_2,M_2)\in\mathcal{A}_{g_2,(2)}$, $g=g_1+g_2$, *and let* $(X,M)\in\mathcal{A}_{g,(2)}$ *be the quotient of* $X_1\times X_2$ *by the subgroup generated by* (α_1,α_2), *where* $\alpha_i\in H(M_i)$ *is nonzero. Then*

$$X\in\mathcal{P}_{g,(2)} \quad\Longleftrightarrow\quad X_1\in\mathcal{P}_{g_1,(2)} \text{ and } X_2\in\mathcal{P}_{g_2,(2)}.$$

Proof. If $(X_i,M_i)\in\mathcal{P}_{g_i,(2)}$, then $\widehat{X}_i$ is the Prym variety of an admissible double cover $\pi_i:\widetilde{C}_i\to C_i$, ramified at four smooth points $p_1^i,\dots,p_4^i$ of $\widetilde{C}_i$. As explained in 3.1, the element $\alpha_i\in H(M_i)$ corresponds to a choice of two of these points, say p_1^i,p_2^i.

Let $\widetilde{C}=(\widetilde{C}_1\cup\widetilde{C}_2)/(p_1^1\sim p_1^2,\ p_2^1\sim p_2^2)$, with the induced involution σ. The associated cover is admissible, ramified at p_3^1,p_4^1,p_3^2,p_4^2. Its Prym variety P fits into an exact sequence

$$0\longrightarrow\mathbb{Z}/2\longrightarrow P\longrightarrow\widehat{X}_1\times\widehat{X}_2\longrightarrow 0,$$

and X is isomorphic to $\widehat{P}$.

Conversely, if $X\in\mathcal{P}_{g,(2)}$, then $\widehat{X}$ is a Prym variety associated with $(\widetilde{C},\sigma)$. If $p_1,\dots,p_4$ are the smooth fixed points of σ on $\widetilde{C}$, one of the three admissible covers $\widetilde{C}'\to\widetilde{C}'/\sigma'$, $\widetilde{C}''\to\widetilde{C}''/\sigma''$, $\widetilde{C}'''\to\widetilde{C}'''/\sigma'''$ (see 3.1) has Prym variety $P_1\times P_2$, with $P_i\simeq\widehat{P}_i=X_i/\{0,\alpha_i\}$.

We argue by induction, assuming the proposition in dimension $<g$. *First case: σ interchanges some components of $\widetilde{C}$.* By Theorem 5.4 of [Be 1], we may write

$$\widetilde{C}=A\cup\widetilde{B}\cup\sigma A,\quad p_1,\dots,p_4\in\widetilde{B},\qquad X\simeq JA\times Y,$$

where $Y=\widehat{\mathrm{Prym}(\widetilde{B},\sigma)}\in\mathcal{P}_{h,(2)}$, $h<g$. By moving factors into Y if necessary, we may assume JA irreducible. The proof of 1.3 then gives, for example, $X_1\simeq JA\times Y_1$. Applying the inductive hypothesis to $Y_1\times X_2\to Y$ proves the result.

Second case: σ interchanges no components of $\widetilde{C}$. Theorem 4.10 of [Be 1] gives one of the following cases.

- The curve C' is hyperelliptic. Then C is hyperelliptic, and if $\varphi:C\to\mathbf{P}^1$ is its hyperelliptic map, $\varphi p_1=\varphi p_2$ and $\varphi p_3=\varphi p_4$. It follows readily from II.D.1, or from an analysis as in [Mu 2], p. 346, that $X_i\in\mathcal{H}_{g_i,(2)}$, in the notation of Section 4. Together with 4.1, this gives the conclusion.
- The curve C' is the union of two curves C_1',C_2' meeting at two points.

1) If $p_1,p_2,p_3,p_4\in\widetilde{C}_2$, then X is reducible: $X\simeq P'\times X'$, where $P'=\mathrm{Prym}(\widetilde{C}_1'\to C_1')$ is principally polarized and $X'=\widehat{\mathrm{Prym}(\widetilde{C}_2\to C_2)}$ is polarized of degree 2.

Every irreducible factor (A, L) of P' is again a Prym variety. By the proof of 1.3 it is also a factor of, say, (X_1, M_1), so $X_1 \simeq A \times Y_1$. By induction, X_2 and Y_1 are duals of Prym varieties. The same is true of X_1, finishing this case.

2) If instead, for example, $p_1, p_2 \in \widetilde{C}_1$ and $p_3, p_4 \in \widetilde{C}_2$, then the abelian varieties $X_i' = \operatorname{Prym}(\widetilde{C}_i \to C_i)$ are polarized of degree 2 and admit an isogeny of degree 2 onto $P_i' = \operatorname{Prym}(\widetilde{C}_i' \to C_i')$. Since $P_1 \times P_2 \simeq P_1' \times P_2'$, we may write $P_i \simeq P_{i,1} \times P_{i,2}$ and $P_j' \simeq P_{1,j} \times P_{2,j}$, where the $P_{i,j}$ are Prym varieties. The isogeny $X_j' \to P_j'$ corresponds to

$$\varepsilon_j' = (\varepsilon_{1,j}, \varepsilon_{2,j}) \in P_j'[2] = P_{1,j}[2] \times P_{2,j}[2].$$

For $\varepsilon_{i,j} \neq 0$, let $X_{i,j} \to P_{i,j}$ be the corresponding isogeny of degree 2. If $\varepsilon_{i,j} = 0$, simply set $X_{i,j} = P_{i,j}$: this principally polarized Prym factor splits off and is carried along directly, as in the first case. When both components of a column are nonzero, there is an isogeny of degree 2, $X_{1,j} \times X_{2,j} \to X_j'$, so induction gives $X_{i,j} \in \mathcal{P}_{*,(2)}$. When only one component is nonzero, the decomposition is a product, and its principally polarized factor is split off in the same way. To reassemble each row, apply the forward implication of the proposition to $X_{i,1} \times X_{i,2} \to X_i$ when both components are nonzero, carrying along the principally polarized factors directly in the other cases. This completes the proof. □

A2.4 The families $\mathcal{H}^2_{g_1,\dots,g_r}$

Again, $\mathcal{H}^2_{g_1,\dots,g_r}$ is the subset of $\mathcal{A}^2_{g_1,\dots,g_r}$ constructed as in 2.1 from $(X_i, M_i) \in \mathcal{H}_{g_i,(2)}$, where $\mathcal{H}_{g_i,(2)} \subset \mathcal{A}_{g_i,(2)}$ is defined as follows.

For distinct Weierstrass points Q_0, Q_1 on a smooth hyperelliptic curve H of genus g, the point $O_H(Q_0 - Q_1)$ has order 2 in JH. It therefore determines an isogeny $X \to JH$ of degree 2. Let $\mathcal{H}_{g,(2)}$ be the family of abelian varieties polarized of degree 2 constructed in this way. It is irreducible and is not closed in $\mathcal{A}_{g,(2)}$ unless $g = 1$.

Let $\varphi : H \to \mathbf{P}^1$ be the hyperelliptic map, ramified at $P_0 = \varphi(Q_0), P_1 = \varphi(Q_1), P_2, \dots, P_{2g+1}$, and let $\psi : C \to \mathbf{P}^1$ be the double cover ramified at $P_2, \dots, P_{2g+1}$. By II.E.1, the variety $X \in \mathcal{H}_{g,(2)}$ above is dual to the Prym variety of the cover of C ramified along $\psi^*(P_0 + P_1)$ and associated with $\psi^* O_{\mathbf{P}^1}(1)$. Thus

$$\mathcal{H}_{g,(2)} \subset \mathcal{P}_{g,(2)}. \tag{A2.4.1}$$

For $(X, M) \in \mathcal{H}_{g,(2)}$, the isogeny $X \to JH$ singles out a nonzero element $\alpha_X \in H(M)$. The family of quotients of members $(X, M) \in \mathcal{H}_{g,(2)}$ by nonzero elements of $H(M)$ has, for $g \geq 3$, two irreducible components: the hyperelliptic Jacobian family $\mathcal{H}_g$ and a family denoted by $\mathcal{H}_g^2$, arising from the other two quotients. In small dimensions these images may coincide; $\mathcal{H}_g^2$ still denotes the image of the two quotients other than the quotient by α_X.

In the interpretation of 3.1, the first case corresponds to $C' = C/(x_0 \sim \tau x_0,\ x_1 \sim \tau x_1)$, where $\psi^* P_i = x_i + \tau x_i$ for $i = 0, 1$. The second corresponds to $C'' = C/(x_0 \sim x_1,\ \tau x_0 \sim \tau x_1)$ and $C''' = C/(x_0 \sim \tau x_1,\ x_1 \sim \tau x_0)$.

4.2. For $g \geq 3$, the curves C'' and C''' are irreducible and nonhyperelliptic. The list in Theorem 4.10 of [Be 1] therefore shows that the members of $\mathcal{H}_g^2$ are irreducible principally polarized abelian varieties. In dimension 1, irreducibility is automatic. In dimension 2, the exceptions in 2.4 must be retained: the assertion without a dimension restriction does not apply.

4.3. Define $\mathcal{H}^2_{g_1,\dots,g_r} \subset \mathcal{A}^2_{g_1,\dots,g_r}$ as the set of quotients of $\prod_{i=1}^r (X_i, M_i)$, with $(X_i, M_i) \in \mathcal{H}_{g_i,(2)}$, by the subgroup generated by

$$(\alpha_1, 0, \dots, 0, \alpha_i, 0, \dots, 0), \quad 2 \le i \le r, \qquad (\alpha_{X_1}, \alpha_{X_2}, \dots, \alpha_{X_r}),$$

where $\alpha_i \in H(M_i) \setminus \{0, \alpha_{X_i}\}$.

By 3.5 and 4.1,

$$\mathcal{H}^2_{g_1,\dots,g_r} \subset \mathcal{P}^2_{g_1,\dots,g_r}. \tag{A2.4.4}$$

4.5. Finally, the members of $\mathcal{H}^2_{g_1,\dots,g_r}$, $r \ge 2$, are irreducible principally polarized abelian varieties except in the precise exceptional case of 2.3: $r = 2$, $g_1 = g_2 = 1$, and the two gluing generators are matched by an isomorphism of the polarized elliptic curves. In that exceptional case the quotient is a product of two elliptic curves. Apply 1.3, 2.3, and 4.3.

APPENDIX 3

Special subvarieties and intersections of theta divisors in Prym varieties

A3.1 Special subvarieties of Prym varieties

In [Be 3], Beauville introduced the special subvariety of the Prym variety P of an étale double cover of smooth curves $\pi\colon \widetilde{C} \to C$, associated with a complete g^r_d on C. If L is the corresponding line bundle of degree d on C and $\tilde{L}$ is a line bundle on $\widetilde{C}$ such that $\operatorname{Nm}\tilde{L} = L$, it is defined by

$$V_0 = P \cap \{O_{\widetilde{C}}(E) \otimes \tilde{L}^{-1} \mid E \geq 0,\ \pi_* E \in |L|\}.$$

The definition extends to an admissible cover of stable curves by replacing "effective divisor" with "effective Cartier divisor" ([Be 3], Remark 4). The g^r_d need not be complete.

The following irreducibility criterion generalizes the corollary on p. 365 of [Be 3].

Theorem A3.1.1. *Let $\widetilde{C}$ be an irreducible stable curve, let $\pi\colon \widetilde{C} \to C$ be an admissible cover, and let $\phi\colon C \dashrightarrow \mathbb{P}^r$ be a rational map birational onto its image. Then the special subvariety associated with the linear system defined by ϕ is irreducible.*

Proof. It suffices to find an open subset Ω of smooth points of C such that

$$\left\{(\tilde{x}_1, \dots, \tilde{x}_d) \in \widetilde{C}^d \;\middle|\; \pi\tilde{x}_i \in \Omega,\quad \sum_{i=1}^{d} \pi\tilde{x}_i \in |L|\right\}$$

has at most two components. By [Be 3], p. 359, and with its notation, these are dense in S_0 and S_1, respectively, so that S_0 and $V_0 = \widetilde{\phi}(S_0)$ are irreducible. This follows from the next theorem. □

Theorem A3.1.2. *Let $C \subset \mathbb{P}^r$ be an irreducible curve of degree d, let $\pi\colon \widetilde{C} \to C$ be a degree-2 morphism with $\widetilde{C}$ irreducible, and let $U \subset (\mathbb{P}^r)^*$ be the open set of hyperplanes meeting C transversely in d smooth points over which π is unramified. For $1 \leq m \leq d$, set*

$$I(m) = \{(\tilde{x}_1, \dots, \tilde{x}_m, H) \in \widetilde{C}^m \times U \mid \pi\tilde{x}_i \in H,\ \pi\tilde{x}_i \neq \pi\tilde{x}_j\ (i \neq j)\}.$$

Then $I(m)$ is irreducible for $m < d$, and $I(d)$ has at most two irreducible components.

Proof. As in II.B.6, consider the monodromy group G of the étale morphism $p\colon I(1) \to U$. It acts on the fibre $\{a_1^+, \dots, a_d^+, a_1^-, \dots, a_d^-\}$ of a base point of U, where $a_i^+ = \sigma a_i^-$ and σ is the involution of $\widetilde{C}$ associated with π.

The case $d = 1$ is immediate. For $d \geq 2$, projection from $I(2)$ onto the open subset of $\widetilde{C}^2$ given by $\pi\tilde{x}_1 \neq \pi\tilde{x}_2$ has fibres that are nonempty open subsets of projective spaces of dimension $r - 2$. Thus $I(2)$ is irreducible. Consequently G acts transitively on the ordered quadruples $(a_i^\alpha, a_j^\beta, a_i^{-\alpha}, a_j^{-\beta})$, with $i \neq j$ and $\alpha, \beta \in \{-, +\}$. As in II.B.6, simple tangency gives a double transposition in G, which we may take to be $(a_1^+, a_2^+)(a_1^-, a_2^-)$.

Conjugating shows that G contains every permutation

$$(a_i^{\alpha}, a_j^{\beta})(a_i^{-\alpha}, a_j^{-\beta}) \qquad (i \neq j,\ \alpha, \beta = \pm).$$

Their products give simultaneous permutations of the indices and sign changes at any two indices. We must therefore consider ordered m-tuples $(a_{j(1)}^{\alpha_1}, \dots, a_{j(m)}^{\alpha_m})$, with j injective. A simultaneous permutation sends $j(k)$ to k. Negative signs can then be changed in pairs. For $m < d$, an unused index also allows a single sign among the chosen indices to be changed, so every tuple lies in the orbit of $(a_1^+, \dots, a_m^+)$. For $m = d$, there are at most the two orbits represented by $(a_1^+, \dots, a_d^+)$ and $(a_1^+, \dots, a_{d-1}^+, a_d^-)$. □

A3.2 The intersections $\Xi \cdot \Xi_{r+s-\sigma r-\sigma s}$ in Prym varieties

As already observed in IV.A.5.1 and IV.B.1.1, if $\pi\colon \widetilde{C} \to \widetilde{C}/\sigma = C$ is a cover of smooth irreducible curves with Prym variety (P, Ξ), and $r, s \in \widetilde{C}$ satisfy $r \neq \sigma s$, then $\Xi \cdot \Xi_{r+s-\sigma r-\sigma s}$ is isomorphic to the special subvariety associated with $L = \omega_C(-\pi r - \pi s)$.

This remains true when π is admissible, $\widetilde{C}, C$ are stable, and r, s are smooth points of $\widetilde{C}$. As a smooth point x tends to a singular point r of $\widetilde{C}$, the element $O_{\widetilde{C}}(x - \sigma x) \in P^-$ tends to an element $\varepsilon(r) \in P^-$ of order two. We are therefore led to study $\Xi \cdot \Xi_{\varepsilon(r)+s-\sigma s}$ with r singular and s smooth.

Lemma A3.2.1. *Let $n\colon N \to C$ be the normalization of C at $\pi(r)$. Then $\Xi \cdot \Xi_{\varepsilon(r)+s-\sigma s}$ is the special subvariety associated with the rational map $\phi_{|\omega_N(-\pi s)|} \circ n^{-1}$.*

Proof. It suffices to show that

$$\Xi \cdot \Xi_{\varepsilon(r)+s-\sigma s} = \{L \in P^+ \mid \exists u \in H^0(L) \setminus \{0\},\ u(s) = u(r) = 0\}.$$

Let $r = w_1, w_2, \dots, w_n$ be the singular points of $\widetilde{C}$, and let L_r be the line bundle associated with $\varepsilon(r)$. Put $a = \varepsilon(r) + s - \sigma s \in P$. We have the following exact sequences, where $\tilde{n}$ denotes the normalization of $\widetilde{C}$:

$$0 \longrightarrow H^0(L \otimes L_r^{-1}(\sigma s - s)) \longrightarrow H^0(\tilde{n}^* L(\sigma s - s)) \xrightarrow{\delta_r} \bigoplus_{j=1}^{n} \mathbb{C}_{w_j},$$

$$\delta_r(u) = (u(p_1) + u(q_1), u(p_2) - u(q_2), \dots),$$

$$0 \longrightarrow H^0(L(\sigma s - s)) \longrightarrow H^0(\tilde{n}^* L(\sigma s - s)) \xrightarrow{\delta} \bigoplus_{j=1}^{n} \mathbb{C}_{w_j},$$

$$\delta(u) = (u(p_1) - u(q_1), u(p_2) - u(q_2), \dots),$$

where p_j, q_j are the points of the normalization of $\widetilde{C}$ over w_j.

First take $L \in P^+$ with a nonzero section $u \in H^0(L)$ vanishing at $r = w_1$ and at s. It defines a section of $L \otimes L_r^{-1}(-s)$, so $h^0(L \otimes L_r^{-1}(\sigma s - s)) \geq 1$. Since this last line bundle lies in P^+, it belongs to Ξ, and $L \in \Xi_a$. Similarly, $h^0(L) \geq 1$ implies $L \in \Xi$.

Conversely, if $L \in \Xi \cdot \Xi_a$, then $h^0(L) \geq 2$ and $h^0(L \otimes L_r^{-1}(\sigma s - s)) \geq 2$. If $h^0(L(\sigma s - s)) \geq 3$, simply choose a section u vanishing at σs and at r. If $h^0(L(\sigma s - s)) = 1$, then σs is fixed in $|L(\sigma s - s)|$. Choose a section u of $L \otimes L_r^{-1}(\sigma s - s)$ vanishing at r. We have

$$\tilde{n}^* u \in H^0(\tilde{n}^* L(\sigma s - s)), \qquad \tilde{n}^* u(p_1) = \tilde{n}^* u(q_1) = 0,$$
$$\tilde{n}^* u(p_j) = \tilde{n}^* u(q_j) \quad (j \geq 2).$$

This defines a section of $L(\sigma s - s)$ vanishing at r and, like every section of this bundle, at σs. □

Theorem 1.1 now gives the following criterion for irreducibility of $\Xi \cdot \Xi_{[r,s]}$. As in IV.B, $[r,s] \in P$ denotes the element associated with $O_{\widetilde{C}}(r+s-\sigma r-\sigma s)$ when r is smooth, and with $O_{\widetilde{C}}(s-\sigma s) \otimes L_r$ otherwise; s is always assumed smooth.

Proposition A3.2.2. *Let $\pi\colon \widetilde{C} \to C$ be an admissible cover of irreducible stable curves, where C is nonhyperelliptic of genus $g+1 \geq 5$. Let $r, s \in \widetilde{C}$, with s smooth and $r \neq \sigma s$. Let $\phi\colon C \to \mathbb{P}^g$ be the canonical map and Γ its image. Suppose projection $\Gamma \dashrightarrow \mathbb{P}^{g-2}$ from the line joining $\phi(\pi r)$ to $\phi(\pi s)$ is birational onto its image (use the tangent line if $r = s$). Then $\Xi \cdot \Xi_{[r,s]}$ is integral.*

APPENDIX 4

Some results on superelliptic curves

Recall that a superelliptic structure on a smooth curve C is an involution τ whose quotient C/τ is an elliptic curve E. Write $p\colon C \to E$ for the quotient, $O_E(-\delta) = \bigwedge^2 p_* O_C$, and $\Delta \in |2\delta|$ for the branch divisor on E (I.2.6).

Terminological note. The historical term "superelliptic" used here means "bielliptic" in current terminology.

A superelliptic curve is one admitting such a structure. The structure is unique in genus ≥ 6 [Sho 2], but there can be as many as five in genus 5 [Te]. In every genus ≥ 2, there are only finitely many, since the automorphism group of the curve is finite. We collect the results needed below.

Proposition A4.1. *Let τ be a superelliptic structure on a curve C of genus $g \geq 6$, let $p\colon C \to E$ be the quotient, and fix $A \in \mathrm{Pic}^{g-1} E$. Put*

$$Z = \{p^*M \otimes O_C(D) \mid M \in \mathrm{Pic}^2 E,\ D \in C^{(g-5)},\ M^2 \otimes O_E(p_*D) \simeq A\}.$$

Then Z is irreducible of dimension $g-5$, and

$$\{a \in \mathrm{Pic}^0 C \mid Z + a \subset \Theta\} = \{O_C(p^*e - x - y) \mid x, y \in C,\ e \in E\}.$$

Proof. Irreducibility of Z follows from II.D.1.2.10, and the inclusion $\supset$ is clear.

Every $a \in \mathrm{Pic}^0 C$ can be written as $a = p^*M^{-1} \otimes O_C(D)$ with D effective of degree $2r$. If $2r \geq g+2$, then $h^0(C, D) \geq 3$, so $D \equiv p^*N + D'$ with D' effective of degree $< 2r$.

If $2r = g+1$, the ample divisor $\Theta \subset J^{g-1}C$ meets the curve $a + p^*J^{r-1}E$. This gives a representation $a = p^*M^{-1} \otimes O_C(D)$ with D effective of degree $g-1$. We may therefore assume $2r \leq g$, that D is p-simple (definition preceding II.D.1.2.7), and that $Z + a \subset \Theta$. It suffices to obtain a contradiction when $r \geq 2$.

1) $2r \leq g-2$. For generic $N \in \mathrm{Pic}^r E$, there is $G \geq 0$ on C such that $D+G$ is p-simple and $A \simeq N^2 \otimes O_E(p_*G)$. Since $r \geq 2$, $L = p^*N \otimes O_C(G)$ belongs to Z, so

$$H^0(L \otimes a) = H^0(p^*(N \otimes M^{-1}) \otimes O_C(G+D)) \neq 0.$$

By II.D.1.2.7, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(N \otimes M^{-1}) &\longrightarrow H^0(L \otimes a) \\ &\longrightarrow H^0(N \otimes M^{-1} \otimes O_E(p_*G + p_*D - \delta)) \\ &\simeq H^0(M \otimes N^{-1} \otimes \mathrm{Nm}\, a \otimes A(-\delta)). \end{aligned}$$

Thus $N \simeq M$ or $N \simeq M \otimes \mathrm{Nm}\, a \otimes A(-\delta)$, impossible since M is fixed and N varies.

2) $2r = g-1$. Choose N with $N^2 \simeq A$. As above, $L = p^*N \in Z$ implies $L \otimes a \in \Theta$ and $N \simeq M$ or $N \simeq M \otimes \alpha$, where $\alpha = \mathrm{Nm}\, a \otimes A(-\delta)$. But we may replace N with $N \otimes \varepsilon$ for any $\varepsilon \in \mathrm{Pic}^0 E[2]$, whereas

$$\bigcap_{\varepsilon \in \mathrm{Pic}^0 E[2]} \{M \otimes \varepsilon, M \otimes \varepsilon \otimes \alpha\} = \emptyset.$$

This is a contradiction.

3) $2r = g$. Take $x \in \operatorname{Supp} D$ and put $G = \tau x$. Choose N with $N^2 \otimes O_E(px) \simeq A$. Then $\deg N = r - 1 \geq 2$ and $L = p^*N \otimes O_C(\tau x) \in Z$. Now

$$L \otimes a = p^*(N \otimes M^{-1}(px)) \otimes O_C(F), \qquad F = D - x.$$

The same exact sequence and $H^0(L \otimes a) \neq 0$ give $N \simeq M(-px)$ or $N \simeq M(-px) \otimes \operatorname{Nm} a \otimes A(-\delta)$. Again conclude by replacing N with $N \otimes \varepsilon$, where $\varepsilon^2 \simeq O_E$. □

The following result is proved in the same way.

Proposition A4.2. *Let τ be a superelliptic structure on a curve C of genus $g \geq 3$ (respectively $g \geq 5$), and let $p\colon C \to E$ be the quotient. For $j = 1, 2$, put*

$$T_j = \{p^*M \otimes O_C(D) \mid M \in \operatorname{Pic}^j E,\ D \in C^{(g-1-2j)}\}.$$

Then T_1 (respectively T_2) is irreducible of dimension $g - 2$ (respectively $g - 4$), and

$$\{a \in \operatorname{Pic}^0 C \mid T_1 + a \subset \Theta\} = p^* \operatorname{Pic}^0 E,$$

respectively

$$\{a \in \operatorname{Pic}^0 C \mid T_2 + a \subset \Theta\} = \{O_C(p^*e - x - y) \mid x, y \in C,\ e \in E\}.$$

Proposition A4.3. *Let τ be a superelliptic structure on a curve C of genus $g \geq 5$, let $p\colon C \to E$ be the quotient, and fix $a \in \operatorname{Pic}^0 C \setminus \{0\}$ and $A \in \operatorname{Pic}^{g-1} E$. Put*

$$Y = \{L \in \Theta \cdot \Theta_a \mid \operatorname{Nm} L \simeq A\}.$$

i) Y is of pure dimension $g - 3$, unless there is $\alpha \in \operatorname{Pic}^0 E$ with $a = p^*\alpha$ and $A \simeq \alpha \otimes O_E(\delta)$. In the latter case, Y is irreducible of dimension $g - 2$.

ii) If $a = O_C(-p^*e + x + y)$, with $x, y \in C$ and $e \in E$, then Y is irreducible, except possibly in one of the following cases:

- *$e = px$ or $e = py$, hence $a \in C - C$;*
- *$A \simeq O_E(px - e + \delta)$ or $A \simeq O_E(py - e + \delta)$;*
- *$g = 5$, x, y are ramification points of p, and $A \simeq O_E(2px + 2py)$ or $A \simeq O_E(px + 3py)$, with $2px \equiv 2py$ in the latter case.*

Proof. By ampleness of Θ, Θ_a, and since $g \geq 3$, their intersection meets every translate of $K = \{L \mid \operatorname{Nm} L \simeq A\}$. It cannot therefore be contained in K. Thus, if $Y = \Theta \cap \Theta_a \cap K$ has dimension $g - 2$, then $\Theta \cap \Theta_a$ is reducible. By II.B.1, one of the following occurs.

- $a \in C - C$. One checks that Y has two irreducible components of dimension $g - 3$.
- $a = p^*\alpha$. Then

$$\Theta \cdot \Theta_a = \{L \in \Theta \mid \operatorname{Nm} L \simeq \alpha \otimes O_E(\delta)\} \cup \{O_C(p^*x + D) \mid x \in E,\ D \in C^{(g-3)}\}.$$

  If $A \simeq \alpha \otimes O_E(\delta)$, Y is irreducible of dimension $g - 2$; otherwise it is irreducible of dimension $g - 3$.

- $g = 5$, and $q\colon C \to F$ is another superelliptic structure with $a = q^*\beta$, $\beta \in \operatorname{Pic}^0 F$. The intersection $\Theta \cdot \Theta_a$ has the same form as above, and Y is reducible of dimension $g - 3$.

From now on, assume $a = O_C(-p^*e + x + y)$, $a \notin C - C$ (hence $e \neq px, py$), and $a \notin p^* \operatorname{Pic}^0 E$ (hence $y \neq \tau x$).

First suppose $A \simeq O_E(px - e + \delta)$. Then $a = p^*(A(-\delta)) \otimes O_C(y - \tau x)$. If $L \in \Theta$ and $\operatorname{Nm} L \simeq A$, then

$$L \otimes \tau^* L \simeq p^* A, \qquad \omega_C \otimes L^{-1} \simeq \tau^* L \otimes p^*(A^{-1}(\delta)),$$

so $H^0(L \otimes p^*(A^{-1}(\delta))) \neq 0$. For $L \in Y$, we also have

$$H^0(a^{-1} \otimes L) = H^0(p^*(A^{-1}(\delta)) \otimes L(\tau x - y)) \neq 0.$$

Thus Y is the union of the following two sets, with $\operatorname{Nm} L \simeq A$ understood throughout:

$$\{L \mid H^0(p^*(A^{-1}(\delta)) \otimes L(-y)) \neq 0\} = \{L \mid H^0(\omega_C \otimes L^{-1}(-\tau y)) \neq 0\},$$
$$\{L \mid h^0(p^*(A^{-1}(\delta)) \otimes L(\tau x)) > 1\} = \{L \mid H^0(L(-x)) \neq 0\}.$$

We therefore impose the additional hypotheses

$$A \not\simeq O_E(px - e + \delta), \qquad A \not\simeq O_E(py - e + \delta).$$

Consider $Y \cap W$, where $W = \{O_C(x + D) \mid D \in C^{(g-2)}\}$. If $O_C(x + D) \in Y$ and $\tau y + D$ is p-simple, then

$$O_C(x + D) \otimes a^{-1} = O_C(D + \tau y + p^*(e - py)).$$

The exact sequence of II.D.1.2.7 gives

$$\begin{aligned} 0 \longrightarrow H^0(e - py) = 0 \longrightarrow H^0(O_C(x + D) \otimes a^{-1}) \\ \longrightarrow H^0(e - py + p_*D + py - \delta) \\ = H^0(e + A - px - \delta) = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} Y \cap W = \{O_C(x + y + D) \mid D \in C^{(g-3)},\ O_E(p_*D) \simeq A(-px - py)\} \\ \cup \{O_C(x + p^*m + D) \mid D \in C^{(g-4)},\ m \in E,\ O_E(p_*D + 2m) \simeq A(-px)\}. \end{aligned}$$

These two sets are irreducible of dimension $g - 4$. Choose in their intersection

$$L_0 = O_C(x + y + \tau y + D),$$

with $D + x$ and $D + \tau y$ p-simple. For $g \geq 6$, this is always possible: $p_*D \in |A(-px - 2py)|$, a base-point-free linear system of degree $g - 4 \geq 2$. For $g = 5$, it is still possible except in the cases

- $A \simeq O_E(2px + 2py)$ and x is ramified;
- $A \simeq O_E(px + 3py)$ and y is ramified.

Interchanging x, y shows that, if L_0 cannot be chosen, then both x, y are ramified and $A \in \{O_E(2px + 2py), O_E(px + 3py)\}$. This is the exceptional case in the statement.

Assume, then, that L_0 has been chosen. We have

$$\begin{aligned}\omega_C \otimes L_0^{-1} &\simeq p^*(A^{-1}(\delta)) \otimes \tau^* L_0 \\ &\simeq p^*(A^{-1}(\delta + py)) \otimes O_C(\tau D + \tau x), \\ L_0 \otimes a^{-1} &\simeq O_C(p^* e + \tau y + D), \\ \omega_C \otimes L_0^{-1} \otimes a &\simeq p^*(A^{-1}(\delta - e + py + px)) \otimes O_C(y + \tau D).\end{aligned}$$

By II.D.1.2.7 and the p-simplicity of $D + x, D + \tau y$, $h^0(L_0) = h^0(L_0 \otimes a^{-1}) = 1$. Thus Θ, Θ_a are smooth at L_0.

The tangent hyperplanes to $\{L \mid \operatorname{Nm} L \simeq A\}$, Θ, and Θ_a at L_0 in $T_{L_0} JC \simeq H^0(C, \omega_C)^*$ correspond respectively to the following sections of

$$H^0(C, \omega_C) \simeq \mathbb{C}r \oplus p^* H^0(E, \delta) :$$

$$r, \qquad p^*[px + p_* D + py + u], \qquad p^*[py + p_* D + e + v].$$

Here r is a section of $O_C(p^*\delta) \simeq \omega_C$ whose divisor is the ramification divisor of p, and

$$|A^{-1}(\delta + py)| = \{u\}, \qquad |A^{-1}(\delta - e + py + px)| = \{v\}.$$

Our hypotheses imply $e \notin \{px, py, u\}$. The three hyperplanes are therefore linearly independent, and Y is smooth at L_0. To prove that Y is irreducible, it now suffices to show that every component Z of Y contains L_0. Since L_0 lies on both components of $Y \cap W$, each of dimension $g - 4$, it suffices to show that

$$\dim(Z \cap W) \geq g - 4.$$

Let $\phi \colon C^{(g-1)} \to \Theta$ be the canonical map, and let

$$C_x^{(g-2)} = \{D \in C^{(g-1)} \mid D \geq x\}, \qquad \psi \colon C_x^{(g-2)} \to W$$

be its restriction. By II.D.1.2.7, the locus of positive-dimensional fibres is

$$\begin{aligned}U &= \{z \in C_x^{(g-2)} \mid \dim \psi^{-1}(\psi z) \geq 1\} \\ &= \{x + D \mid D \in C^{(g-2)},\ h^0(D) \geq 2\} \\ &= \{x + p^* e_1 + p^* e_2 + G \mid e_1, e_2 \in E,\ G \in C^{(g-6)}\}.\end{aligned}$$

It has dimension $g - 4$ for $g \geq 6$ and is empty for $g = 5$. For $g \geq 6$, the map

$$U \longrightarrow \operatorname{Pic}^{g-1} E, \qquad D \longmapsto O_E(p_* D)$$

is surjective. If $\widetilde{Z}$ is a component of $\phi^{-1}(Y)$ dominating Z, it follows that

$$\dim(U \cap \widetilde{Z}) \leq g - 5.$$

Now $C_x^{(g-2)}$ is an ample Cartier divisor on $C^{(g-1)}$ [Fu–L]. Hence

$$\dim(C_x^{(g-2)} \cap \widetilde{Z}) \geq \dim \widetilde{Z} - 1 \geq g - 4.$$

Together with the preceding upper bound, this gives

$$\dim(W \cap Z) \geq \dim \psi(C_x^{(g-2)} \cap \widetilde{Z}) \geq g - 4,$$

completing the proof. □

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