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On the geometry of EPW cubes
Sur la géométrie des EPW cubes

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Résumé

Dans cette thèse, nous étudions la géométrie des *EPW cubes,* une famille localement complète de variétés hyperkählériennes polarisées de dimension 6 construite par Iliev–Kapustka–Kapustka–Ranestad, ainsi que leurs liens avec les variétés de Gushel–Mukai de dimension 6.

Dans une première partie, nous présentons la construction des EPW cubes et étudions leur géométrie. Ils sont définis comme des revêtements doubles d'une sous-variété de dimension 6 de la grassmannienne $\mathsf{Gr}(3, V_6)$. Cette sous-variété apparaît dans une stratification par des lieux de dégénérescence lagrangiens. Nous analysons cette stratification et étudions la structure locale de chacune de ses strates. Nous construisons ensuite le revêtement double et montrons qu'il s'agit d'une variété hyperkählérienne projective de type de déformation $\mathrm{K3}^{[3]}$. Nous étudions également l'involution induite par le revêtement double et calculons des invariants topologiques de son lieu fixe.

La construction des EPW cubes se fait en famille et fournit une famille de dimension 19 de variétés hyperkählériennes projectives. Nous étudions l'application des périodes pour ces variétés hyperkählériennes polarisées et décrivons son image comme le complément de quatre diviseurs au bord, qui déterminent quatre dégénérescences singulières possibles des EPW cubes. Nous décrivons ces quatre dégénérescences singulières correspondant aux quatre diviseurs au bord. En particulier, en étudiant l'une de ces dégénérescences, nous complétons l'étude du lieu fixe de l'involution associée aux EPW cubes, en montrant qu'il s'agit d'une sous-variété lagrangienne irréductible de type général, qui ne se déforme pas à l'intérieur de l'EPW cube.

Dans une seconde partie, nous définissons les variétés de Gushel–Mukai et rappelons des résultats de Debarre et Kuznetsov montrant qu'à toute variété de Gushel–Mukai est associé canoniquement un EPW cube. Nous étudions ensuite une fibration en quadriques sur les variétés de Gushel–Mukai, appelée *seconde fibration en quadriques*, ainsi que ses lieux de dégénérescence, obtenus en intersectant la stratification définissant le EPW cube avec une sous-variété de Schubert de $\mathsf{Gr}(3, V_6)$.

Enfin, nous montrons que le schéma de Hilbert paramétrant les plans contenus dans les fibres de la seconde fibration en quadriques d'une variété de Gushel–Mukai de dimension 6 est isomorphe à une sous-variété du EPW cube associé. La correspondance d'incidence correspondante fournit une correspondance de Hodge rationnelle entre la cohomologie évanescente en degré 6 de la variété de Gushel–Mukai et la cohomologie primitive en degré 2 du EPW cube associé. Nous montrons de plus que cette correspondance est un multiple rationnel d'une isométrie.

Mots-clés : variétés hyperkählériennes, EPW cubes, familles localement complètes, involutions antisymplectiques, correspondance de Hodge.

Abstract

In this thesis, we study the geometry of *EPW cubes*, a locally complete family of polarized hyper-Kähler manifolds of dimension 6 constructed by Iliev–Kapustka–Kapustka–Ranestad, and their relation to Gushel–Mukai sixfolds.

In the first part, we present the construction of EPW cubes and study their geometry. They are double covers of a six-dimensional subvariety of the Grassmannian $\mathsf{Gr}(3, V_6)$. This subvariety arises in a stratification by Lagrangian degeneracy loci. We analyze the stratification and study the local structure of each locus. Then we construct the double cover and show that it is a projective hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type. We also study the involution induced by the double cover and compute topological invariants of its fixed locus.

The construction of EPW cubes can be done in a family and gives a 19-dimensional family of projective hyper-Kähler manifolds. We study the period map for these polarized hyper-Kähler manifolds and describe its image as the complement of four boundary divisors, which determine four possible singular degenerations of EPW cubes. We then describe the four singular degenerations of EPW cubes corresponding to the four boundary divisors. In particular, by studying one of these degenerations, we complete the study of the fixed locus of the involution associated with EPW cubes, showing that it is an irreducible Lagrangian submanifold of general type which does not deform inside the EPW cube.

In the second part, we introduce Gushel–Mukai manifolds and recall results from Debarre and Kuznetsov showing that any Gushel–Mukai manifold has a naturally associated EPW cube. We then study a fibration of quadrics on Gushel–Mukai manifolds, called the *second quadric fibration*, and its degeneracy loci, which are obtained by intersecting the stratification defining the EPW cube with a Schubert subvariety of $\mathsf{Gr}(3, V_6)$.

Finally, we show that the Hilbert scheme that parametrizes planes in the fibers of the second quadric fibration of a Gushel–Mukai sixfold is isomorphic to a subvariety of the associated EPW cube. The corresponding incidence correspondence defines a rational Hodge correspondence between the vanishing middle cohomology of the Gushel–Mukai sixfold and the primitive second cohomology of the associated EPW cube. We prove moreover that this correspondence is a rational multiple of an isometry.

Keywords: hyper-Kähler manifolds, EPW cubes, locally complete families, antisymplectic involutions, Hodge correspondence.

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Introduction

Each compact complex manifold carries a characteristic class, called *the first Chern class*, detecting how far its canonical bundle is from being topologically trivial. For complex curves, the first Chern class is an integer that determines their geometry as real surfaces, and in particular their genus. The projective line is the only complex curve with negative first Chern class; elliptic curves have trivial first Chern class; and curves of genus greater than 1 have positive first Chern class. The geometry of manifolds of higher dimension is substantially richer and the first Chern class no longer determines the differential structure (nor the topology) of the manifold.

A fundamental result in this context is the Beauville–Bogomolov decomposition theorem [**Bea83**, Théorème 2], which states that each compact Kähler manifold with *trivial* first Chern class has a finite étale cover which is a product of complex tori, strict Calabi–Yau manifolds, and hyper-Kähler manifolds. Studying the geometry of each of these building blocks provides information on the geometry of compact Kähler manifolds with trivial first Chern class.

We focus on hyper-Kähler manifolds, defined as simply connected compact Kähler manifolds whose space of holomorphic two-forms is generated by an everywhere non-degenerate two-form. Hyper-Kähler manifolds only exist in even dimensions and have trivial canonical bundle.

In dimension 2, hyper-Kähler manifolds are also known as $K3$ *surfaces*, named by Weil after Kummer, Kähler, and Kodaira. Kodaira proved that all K3 surfaces are diffeomorphic to a smooth quartic surface in the projective space [**Kod64**, Theorem 13], while very general K3 surfaces are not algebraic.

Deformations types of projective K3 surfaces are determined by a positive even number, called the *degree*, representing the self-intersection of a nondivisible ample line bundle. For $d \leq 7$ and some other values of d up to 20, Mukai described the geometry of projective K3 surfaces of degree $2d$ and Picard rank 1. In particular, for each of those values of d, he constructed a family of projective K3 surfaces of degree $2d$ such that any projective K3 surface carrying a nondivisible ample line bundle of degree $2d$ is a specialization of members of the family. Such a family is called a *locally complete family* of projective K3 surfaces of degree $2d$. For example, a locally complete family of K3 surfaces of degree 4 is given by smooth quartic surfaces in $\mathbf{P}^3$, and a locally complete family of K3 surfaces of degree 2 is given by double covers of $\mathbf{P}^2$ branched over a smooth sextic.

In higher dimensions, hyper-Kähler manifolds admit multiple deformation (and even topological) types. Beauville constructed in [**Bea83**] two families of examples in each even dimension, namely punctual Hilbert schemes of K3 surfaces and generalized Kummer varieties. O'Grady constructed two other families of hyper-Kähler manifolds of dimensions 6 ([**O'G03**]) and 10 ([**O'G99**]) that are not deformations

of the examples constructed by Beauville. This gives rise, in each even dimension, to two or three known deformation types of hyper-Kähler manifolds.

Explicit geometric constructions of locally complete families of *projective* hyper-Kähler manifolds of higher dimension are rare. The first example was given by Beauville and Donagi in [**BD85**]. They showed that the variety of lines on a smooth cubic fourfold is a hyper-Kähler manifold of dimension 4 which is a deformation of the Hilbert square of a K3 surface. Varying the smooth cubic fourfold gives a locally complete family of projective hyper-Kähler fourfolds. Moreover, there is an interesting Hodge correspondence: the primitive middle cohomology of a smooth cubic fourfold is isomorphic, as integral Hodge structures, to the primitive second cohomology of its variety of lines.

Only four other locally complete families are known, all deformations of Hilbert schemes of points on a K3 surface: two more families of fourfolds – the Debarre–Voisin varieties [**DV10**] and the double EPW sextics [**O'G06**] –, one family of sixfolds – the EPW cubes [**IKKR19**] –, and one family of eightfolds – the Lehn–Lehn–Sorger–van Straten eightfolds [**LLSv17**]. A Hodge correspondence analogous to the one studied in [**BD85**] was described for Debarre–Voisin varieties in [**DV10, BS21**], for double EPW sextics in [**DK19**], and for Lehn–Lehn–Sorger–van Straten eightfolds in [**AG23**].

This thesis is dedicated to the study of the geometry of **EPW cubes**, the locally complete family of projective hyper-Kähler sixfolds constructed by Iliev–Kapustka–Kapustka–Ranestad. The acronym "EPW" comes from the fact that their construction is similar to that of double EPW sextics, which are double covers of Eisenbud–Popescu–Walter sextics; and the term "cube" refers to the fact that they are deformations of the Hilbert cube of a K3 surface.

In the first part of the thesis, we explain the construction of EPW cubes, showing its explicit nature and the similarities with the family of double EPW sextics constructed by O'Grady. Additionally, we describe the geometry of some singular degenerations of EPW cubes.

In the second part, we discuss in detail the relation between EPW cubes and some Fano manifolds called **Gushel–Mukai manifolds**. Indeed, Debarre and Kuznetsov showed in [**DK18**] that any Gushel–Mukai manifold has a canonically associated EPW cube. Following [**DK19, DK20a**], we show that one component of the Hilbert scheme of planes in a Gushel–Mukai sixfold can be identified with a submanifold of the associated EPW cube and we study the induced Hodge correspondence between the middle cohomology of the Gushel–Mukai sixfold and the second cohomology of the associated EPW cube.

We now give a more detailed description of the contents of this thesis and the main results obtained.

In **Chapter 1**, we collect some results on and properties of hyper-Kähler manifolds, mostly for those of $\mathrm{K3}^{[n]}$ deformation type, that is, deformations of Hilbert schemes of points on a K3 surface. We present the Beauville–Bogomolov–Fujiki form, a canonically defined integral quadratic form on the second integral cohomology group of hyper-Kähler manifolds, as well as the Local Torelli Theorem and the

global period map. Moreover, we introduce moduli spaces and period maps for polarized hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$ deformation type and given *polarization type*, which is the orbit in the second integral cohomology group of the first Chern class of a nondivisible ample line bundle. In some cases, the polarization type of such a line bundle L is determined by the divisibility of its first Chern class and its square with respect to the Beauville–Bogomolov–Fujiki form. We then specialize to hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ deformation type and study the structure of their cohomology ring. The latter carries an action of the so-called Looijenga–Lunts–Verbitsky (LLV) algebra, which induces a decomposition into irreducible representations which we use to determine the Hodge classes and their relations on a very general polarized hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type. Additionally, we study the action of automorphisms of finite order on cohomology.

The construction of EPW cubes is carried out in detail in **Chapter 2**. It goes as follows. Let V_6 be a 6-dimensional complex vector space. The Lagrangian Grassmannian $\mathrm{LG}(\bigwedge^3 V_6)$ parametrizes 10-dimensional vector subspaces of $\bigwedge^3 V_6$ that are isotropic with respect to the symplectic form given by wedge product. To any $[A] \in \mathrm{LG}(\bigwedge^3 V_6)$, the authors of [**IKKR19**] associate a stratification

$$\mathsf{Z}_A^{\geq 5} \subset \mathsf{Z}_A^{\geq 4} \subset \mathsf{Z}_A^{\geq 3} \subset \mathsf{Z}_A^{\geq 2} \subset \mathsf{Z}_A^{\geq 1} \subset \mathsf{Gr}(3, V_6),$$

defined as intersection loci of two Lagrangian subbundles of the trivial vector bundle with fiber $\bigwedge^3 V_6$. We study this stratification when the Lagrangian subspace A of $\bigwedge^3 V_6$ contains no *decomposable vectors*, namely nonzero vectors of the form $u \wedge v \wedge w$. This is done in Theorem 2.5, which gives a full proof of a result stated in [**DK20a**]. It shows in particular that

- $\mathsf{Z}_A^{\geq 1}$ is a hypersurface;
- $\mathsf{Z}_A^{\geq 2}$ is the singular locus of $\mathsf{Z}_A^{\geq 1}$ and has dimension 6;
- $\mathsf{Z}_A^{\geq 3}$ is the singular locus of $\mathsf{Z}_A^{\geq 2}$ and has dimension 3;
- $\mathsf{Z}_A^{\geq 4}$ is the singular locus of Z_A^{3}; it is smooth and finite, and empty when A is general ;
- $\mathsf{Z}_A^{\geq 5}$ is empty.

The EPW cube is a canonically defined double cover $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A^{\geq 2}$ branched over $\mathsf{Z}_A^{\geq 3}$. Following the proof of [**IKKR19**], we show that, when $\mathsf{Z}_A^{\geq 4}$ is empty, $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler sixfold of $\mathrm{K3}^{[3]}$ deformation type with polarization of square 4 and divisibility 2. Finally, using the results of Chapter 1, we study the cohomology ring of EPW cubes. We describe the relations in the subring of Hodge classes of a very general EPW cube, showing in particular that these classes are all algebraic (Corollary 2.25). Moreover, we compute some topological invariants of the fixed locus of the involution ι_A associated with the double cover g_A.

In **Chapter 3**, we study the period map for EPW cubes. The construction of EPW cubes in Chapter 2 gives a rational map

$$\mu_3 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \dashrightarrow {}^{[3]}\mathscr{M}_4^{(2)},$$

where ${}^{[3]}\mathscr{M}_4^{(2)}$ is the irreducible quasiprojective moduli space of hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ deformation type with polarization of square 4 and divisibility 2. The

period map for these manifolds is an embedding

$$\wp_3 \colon {}^{[3]}\mathscr{M}_4^{(2)} \hookrightarrow {}^{[3]}\mathscr{P}_4^{(2)},$$

where ${}^{[3]}\mathscr{P}_4^{(2)}$ is a 19-dimensional irreducible quasiprojective variety. We study some special divisors contained in the period domain ${}^{[3]}\mathscr{P}_4^{(2)}$, called *Heegner divisors*, which parametrize periods of hyper-Kähler manifolds of K3$^{[3]}$ deformation type of Picard rank ≥ 2 carrying a nef and big line bundle of square 4 and divisibility 2. We also describe infinitely many Heegner divisors corresponding to birational models of Hilbert schemes of polarized K3 surfaces. Finally, we describe in Proposition 3.9 the image of the period map $\wp_3$. It is the complement of four boundary Heegner divisors, namely

$${}^{[3]}\mathscr{D}_{4,2}^{(2)}, \quad {}^{[3]}\mathscr{D}_{4,4}^{(2)}, \quad {}^{[3]}\mathscr{D}_{4,8}^{(2)}, \quad {}^{[3]}\mathscr{D}_{4,12}^{(2)}.$$

General elements of each of these boundary divisors correspond to projective hyper-Kähler manifolds of K3$^{[3]}$ deformation type with a nef and big (but not ample) line bundle of square 4 and divisibility 2, inducing a contraction whose image is a singular variety. In particular, each of these boundary divisors gives rise to a family of singular degenerations of EPW cubes.

These four families are studied in **Chapter 4**.

- The locus of Lagrangians $[A]$ in $\mathrm{LG}(\bigwedge^3 V_6)$ such that $\mathsf{Z}_A^{\geq 4} \neq \varnothing$ is a divisor Γ. For any such A with no decomposable vectors, the EPW cube $\widetilde{\mathsf{Z}}_A$ is singular. We prove in Section 4.1 that the singular locus $\mathsf{Z}_A^{\geq 4}$ of $\widetilde{\mathsf{Z}}_A$ is finite and that there exists a projective hyper-Kähler manifold Y_A and a small contraction

  $$f_A \colon \mathsf{Y}_A \longrightarrow \widetilde{\mathsf{Z}}_A$$

  contracting a $\mathbf{P}^3$ on each singular point of $\widetilde{\mathsf{Z}}_A$. When $[A]$ varies in Γ, the sixfolds Y_A correspond to general points of the boundary divisor ${}^{[3]}\mathscr{D}_{4,12}^{(2)}$. This result is analogous to the one proved in [**O'G13b**] for singular double EPW sextics, while the methods of proof are different. Indeed, the variety Y_A is not birationally isomorphic to any smooth moduli space of stable sheaves on a K3 surface (Proposition 4.9) and its existence is based on the surjectivity of the global period map.
- A general polarized K3 surface (S, L) of degree 2 is a double cover $S \to \mathbf{P}^2$ branched over a smooth sextic. The Hilbert scheme $S^{[3]}$ inherits an involution from the covering involution of the double cover and carries a nef and ample line bundle H of square 4 and divisibility 2. The contraction

  $$\varphi \colon S^{[3]} \longrightarrow X$$

  given by the linear system $|H|$ induces an involution ι on X. Letting the pair (S, L) vary, the sixfolds X correspond to general points of the boundary divisor ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$. In Section 4.2, we show that there exists a one-dimensional polarized family $\mathscr{X} \to D$ over a disk and a fiberwise involution $\iota_{\mathscr{X}}$ of $\mathscr{X}$, whose general fiber is an EPW cube with its canonical involution and whose special fiber is the singular variety X with the involution ι.

  Using results of [**BM14a**], we show that φ is a Brill–Noether contraction and deduce an explicit description of the variety X, the involution ι, and its fixed locus. The latter is a disjoint union of two components, one of

dimension 2 (which does not deform), and F_3, of dimension 3. We obtain a family $\mathscr{Y} \to B$ with semi-log-canonical fibers whose general fiber is the fixed locus of the involution of the corresponding EPW cube, and special fiber F_3. By studying the cohomology of F_3, we show that the fixed locus W_A of the canonical involution of an EPW cube $\widetilde{\mathsf{Z}}_A$ is a smooth irreducible Lagrangian which does not deform in $\widetilde{\mathsf{Z}}_A$ (Corollaries 4.20 and 4.25).

- Given a smooth quartic surface S in $\mathbf{P}^3$, let M_S be the moduli space of rank 1 degree 0 sheaves supported on a hyperplane section of the K3 surface S. There exists a Lagrangian fibration $M_S \to (\mathbf{P}^3)^\vee$ whose fiber over a general hyperplane $[H] \in (\mathbf{P}^3)^\vee$ is the Jacobian $\mathrm{Pic}^0(S \cap H)$. This fibration is associated to one extremal ray of the nef cone of M_S. The other extremal ray of this cone gives a flop

$$M_S \dashrightarrow M'_S,$$

  where M'_S is a projective hyper-Kähler manifold of K3$^{[3]}$ deformation type with a nef and big line bundle H' of square 4 and divisibility 2. The singular degenerations of EPW cubes corresponding to general points of the boundary divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,4}$ are the Li–Gieseker–Uhlenbeck contractions

$$M'_S \longrightarrow M''_S$$

  given by the linear system $|H'|$, whose exceptional divisor is a $\mathbf{P}^1$-bundle over the product of two K3 surfaces. This situation is studied in Section 4.3.
- The boundary divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$ corresponds to degenerate EPW cubes constructed from Lagrangian subspaces $A \subset \bigwedge^3 V_6$ containing a decomposable vector (these $[A]$ describe a divisor $\Sigma \subset \mathrm{LG}(\bigwedge^3 V_6)$). When $[A]$ is very general in Σ, the singular locus of $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler fourfold studied in [**IKKR17**] (Corollary 4.30). No projective resolution of $\widetilde{\mathsf{Z}}_A$ is known at the moment.

In **Chapter 5**, we introduce Gushel–Mukai manifolds. These are smooth dimensionally transverse intersections of the cone over the Grassmannian $\mathsf{Gr}(2, V_5)$ of 2-dimensional subspaces in a fixed 5-dimensional vector space V_5, with a linear subspace and a quadratic hypersurface. They occur in each dimension 1 through 6, and are Fano manifolds in dimension at least three.

Their rich geometry has been studied in [**DK18**, **DK19**, **DK20b**, **DK24**]. The authors show in particular that any Gushel–Mukai manifold X has a naturally associated triple (V_6, V_5, A), where V_6 is a 6-dimensional space containing V_5, and $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace. Moreover, when X has dimension at least 3, the Lagrangian A contains no decomposable vectors. As a consequence, any Gushel–Mukai manifold has an associated stratification $\mathsf{Z}^{\geq k}_A$ and an EPW cube $\widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$.

Let $\gamma_X \colon X \to \mathsf{Gr}(2, V_5)$ be the morphism induced by projection from the vertex of the cone over $\mathsf{Gr}(2, V_5)$ (which is not contained in X since X is smooth). In this chapter, we study the relation between the stratification $\mathsf{Z}^{\geq k}_A$ and the degeneracy loci of the quadric fibration

$$\varrho \colon \mathbf{P}(V_5/\mathcal{U}_X) \longrightarrow \mathsf{Gr}(3, V_5),$$

where $\mathcal{U}_X$ is the pullback via γ_X of the tautological subbundle of $\mathsf{Gr}(2, V_5)$. In particular, we show that for a Gushel–Mukai sixfold X, the schematic degeneracy loci

of ϱ are the loci

$$\mathsf{Z}^{\geq k}_{A,V_5} \coloneqq \mathsf{Z}^{\geq k}_{A} \cap \mathsf{Gr}(3, V_5)$$

obtained by intersecting the stratification $\mathsf{Z}^{\geq k}_{A}$ with the codimension 3 subvariety $\mathsf{Gr}(3, V_5) \subset \mathsf{Gr}(3, V_6)$ (Corollary 5.4). In the last part of the chapter, we study the stratification $\mathsf{Z}^{\geq k}_{A,V_5}$. When V_5 is general, the loci $\mathsf{Z}^{\geq k}_{A,V_5}$ have the expected dimension and the singular locus of $\mathsf{Z}^{\geq k}_{A,V_5}$ is equal to $\mathsf{Z}^{\geq k+1}_{A,V_5}$. Moreover, when X is a sixfold, Theorem 5.9 shows that $\mathsf{Z}^{\geq 1}_{A,V_5}$ and $\mathsf{Z}^{\geq 2}_{A,V_5}$ always have the expected dimension, and gives conditions for normality.

Finally, in **Chapter 6**, we study the variety of planes on Gushel–Mukai manifolds and prove a rational Hodge correspondence between a Gushel–Mukai sixfold and the associated EPW cube.

In [**DK19**], the authors show that the Hilbert scheme $F_2(X)$ of planes contained in a general GM sixfold X has two connected components. One component (the one that parametrizes the so-called "σ-planes") is a fourfold, described as a $\mathbf{P}^1$-bundle over a hyperplane section of the double EPW sextic associated to X. Moreover, the incidence correspondence for σ-planes induces a Hodge isometry between the vanishing part of the middle cohomology of X equipped with the cup product, and the primitive second integral cohomology of the double EPW sextic equipped with the opposite of the Beauville form.

The other connected component of $F_2(X)$ (the one that parametrizes the so-called "τ-planes") is related to the EPW cube $\widetilde{\mathsf{Z}}_A$. This component, denoted by $F_2^{\tau}(X)$, is isomorphic to the Hilbert scheme that parametrizes planes in the fibers of the quadric fibration ϱ. Using results from Chapter 5, we show that, when X is general, $F_2^{\tau}(X)$ is isomorphic to the smooth irreducible threefold

$$\widetilde{\mathsf{Z}}_{A,V_5} \coloneqq \widetilde{\mathsf{Z}}_A \times_{\mathsf{Gr}(3,V_6)} \mathsf{Gr}(3, V_5)$$

contained in the EPW cube $\widetilde{\mathsf{Z}}_A$.

The incidence correspondence

$$\begin{array}{ccccc} & & I = \{(x,\Pi) \in X \times F_2^{\tau}(X) \mid x \in \Pi \subset X\} & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ X & & & & F_2^{\tau}(X) \simeq \widetilde{\mathsf{Z}}_{A,V_5} \overset{i}{\hookrightarrow} \widetilde{\mathsf{Z}}_A. \end{array}$$

for τ-planes in a general Gushel–Mukai manifold X of dimension 6 yields morphisms

$$H^6(X, \mathbf{Q}) \xrightarrow{q_* p^*} H^2(F_2^{\tau}(X), \mathbf{Q}) \xleftarrow{i^*} H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q}),$$

of Hodge structures that are equivariant with respect to the $\mathbf{Z}/2\mathbf{Z}$-actions induced by the double covers $\gamma_X \colon X \to \mathsf{Gr}(2, V_5)$ and $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$. The main result of this chapter is that the above maps give a Hodge isomorphism

$$\alpha \colon H^6(X, \mathbf{Q})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q})_{\mathrm{prim}},$$

which is a rational multiple of an isometry, between the vanishing middle cohomology of the GM sixfold X and the primitive second cohomology of the EPW cube $\widetilde{\mathsf{Z}}_A$.

Part I

Geometry of EPW cubes

CHAPTER 1

General results on hyper-Kähler manifolds

1.1. Some results on hyper-Kähler manifolds

A *hyper-Kähler manifold* is a compact simply connected complex Kähler manifold, whose space of holomorphic 2-forms is generated by a symplectic form, that is, an everywhere nondegenerate holomorphic 2-form. Hyper-Kähler manifolds of dimension 2 are also called *K3 surfaces*. A hyper-Kähler manifold has trivial canonical bundle and even dimension. We recall here some properties of hyper-Kähler manifolds, as well as some important general results that we will need.

Examples of hyper-Kähler manifolds can be constructed starting from K3 surfaces. Let S be a K3 surface. Beauville proved in [**Bea83**, Section 6] that the Douady–Hilbert space $S^{[m]}$ that parametrizes analytic subspaces of S of length m is a hyper-Kähler manifold of dimension $2m$.

In the following, we will focus on hyper-Kähler manifolds obtained by deforming Hilbert powers of K3 surfaces.

Definition 1.1. For $m \geq 2$, a hyper-Kähler manifold is *of $K3^{[m]}$ deformation type* if it is a smooth deformation of $S^{[m]}$ for some K3 surface S.

1.1.1. The Beauville–Bogomolov–Fujiki form. Let X be a hyper-Kähler manifold of dimension $2m$. Since X is simply connected, by the Universal Coefficient Theorem, the group $H^2(X, \mathbf{Z})$ is torsion free.

Theorem 1.2. *(Beauville–Bogomolov-Fujiki) Let X be a hyper-Kähler manifold of dimension $2m$. There exists a nondivisible quadratic form q_X on $H^2(X, \mathbf{Z})$, called the* Beauville-Bogomolov-Fujiki form*, and a positive rational constant C_X, called the* Fujiki constant*, satisfying*

$$\forall \alpha \in H^2(X, \mathbf{Z}) \qquad \int_X \alpha^{2m} = C_X q_X(\alpha)^m, \tag{1}$$

The quadratic form q_X has signature $(3, b_2(X) - 3)$ and satisfies $q_X(\alpha) > 0$ for all Kähler classes $\alpha \in H^2(X, \mathbf{Z})$, thus in particular for all ample classes.

For more details about the Beauville-Bogomolov-Fujiki form, see [**GJH03**, Chapter 23].

Consider the Hodge decomposition

$$H^2(X, \mathbf{C}) = H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1) \oplus H^2(X, \mathcal{O}_X).$$

Equation (1) implies that $H^0(X, \Omega_X^2) \oplus H^2(X, \mathcal{O}_X)$ is q_X-orthogonal to $H^1(X, \Omega_X^1)$ and that every $\sigma \in H^0(X, \Omega_X^2)$ satisfies

$$q_X(\sigma) = 0, \qquad \text{and} \qquad q_X(\sigma, \bar{\sigma}) > 0. \tag{2}$$

When S is a K3 surface, the Beauville-Bogomolov-Fujiki form q_S is simply given by cup-product. Poincaré duality and Wu's formula show that the lattice $\Lambda_{K3} :=$

$(H^2(S,\mathbf{Z}),\smile)$ is an even unimodular lattice of signature $(3,19)$ and, by [**Ser73**, Chapter 5], it is isomorphic to

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic plane and E_8 is the unique even unimodular positive definite lattice of rank 8.

Example 1.3. (Hilbert powers of K3 surfaces) Let S be a K3 surface. Results of [**Bea83**, Section 6] show that the second integral cohomology group of the hyper-Kähler manifold $S^{[m]}$ is isomorphic to

$$H^2(S^{[m]},\mathbf{Z}) \simeq H^2(S,\mathbf{Z}) \oplus \mathbf{Z}\delta,$$

where 2δ is the class of the exceptional divisor of the natural projection $S^{[m]} \to S^{(m)}$.

This decomposition is orthogonal for the Beauville-Bogomolov-Fujiki quadratic form $q_{S^{[m]}}$ and the latter restricts to cup-product on $H^2(S,\mathbf{Z})$ and satisfies $q_{S^{[m]}}(\delta) = -2(m-1)$. Hence, the lattice $(H^2(S^{[m]},\mathbf{Z}), q_{S^{[m]}})$ is isometric to

$$\Lambda_{K3^{[m]}} \coloneqq \Lambda_{K3} \oplus \mathbf{Z}(-2(m-1)), \tag{3}$$

where $\mathbf{Z}(-2(m-1))$ is the lattice of rank 1 generated by a class of square $-2(m-1)$. Moreover, the Fujiki constant $c_{S^{[m]}}$ is equal to $(2m-1)!!$.

Since the second cohomology group is invariant under deformation, the lattice associated with a hyper-Kähler manifold of type $\mathrm{K3}^{[m]}$ is always isomorphic to $\Lambda_{K3^{[m]}}$.

We recall here some definitions for lattices. Let Λ be a lattice and denote by $\cdot$ its integral bilinear form. The *dual lattice* of Λ is

$$\Lambda^\vee \coloneqq \{x \in \Lambda_{\mathbf{Q}} \mid \forall y \in \Lambda \quad x \cdot y \in \mathbf{Z}\} = \mathrm{Hom}_{\mathbf{Z}}(\Lambda,\mathbf{Z}).$$

Clearly, there is an inclusion $\Lambda \hookrightarrow \Lambda^\vee$. The *discriminant group* of Λ is the finite abelian group

$$A_\Lambda = \Lambda^\vee/\Lambda,$$

which is trivial if and only if the lattice is unimodular.

For each $x \in \Lambda$ nonzero, the *divisibility* of x, denoted by $\mathrm{div}(x)$, is the positive generator of the ideal $x \cdot \Lambda \subset \mathbf{Z}$. Thus, the class $x_* = \left[\frac{x}{\mathrm{div}(x)}\right]$ is an element of A_Λ, whose order is equal to $\mathrm{div}(x)$ when x is primitive. For example, one has $A_{\Lambda_{K3^{[m]}}} \simeq \mathbf{Z}/(2(m-1)\mathbf{Z}$, generated by the class δ_*.

Finally, we denote by $O(\Lambda)$ the group of isometries of Λ.

1.1.1.1. *Generalized Fujiki constants.* The following is a generalization of Theorem 1.2.

Theorem 1.4 ([**GJH03**, Corollary 23.17])**.** *Let X be a hyper-Kähler manifold of dimension $2m$ and $\beta \in H^{4k}(X,\mathbf{R})$ be a class which remains of type $(2k,2k)$ on small deformations of X. There exists a real constant $C_X(\beta)$, called the* generalized Fujiki constant*, satisfying*

$$\forall \alpha \in H^2(X,\mathbf{R}) \qquad \int_X \beta\alpha^{2(m-k)} = C_X(\beta) q_X(\alpha)^{m-k}, \tag{4}$$

The constant C_X of Theorem 1.2 is equal to $C_X(1)$.

This theorem applies in particular to the even Chern classes $c_{2k}(X)$ of X (the odd Chern classes vanish because of the isomorphism $T_X \simeq T_X^\vee$ defined by the symplectic form) and to any class contained in the subring generated by the Chern classes of X. Computations of (some) generalized Fujiki constants may be found in [**Son22**, Section 2.1].

Polarizing both terms of Equation (4), we obtain the following relation valid for all $\alpha_1, \dots, \alpha_{2(m-k)} \in H^2(X, \mathbf{Q})$

$$\int_X \beta \cdot \alpha_1 \cdots \alpha_{2(m-k)} = \frac{C_X(\beta)}{(2m-2k)!} \sum_{\sigma \in \mathfrak{S}_{2m-2k}} q_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots q_X(\alpha_{\sigma(2m-2k-1)}, \alpha_{\sigma(2m-2k)}),$$

where $\mathfrak{S}_{2m-2k}$ is the group of permutations of $2m-2k$ elements.

1.1.2. Local Torelli and global period map. By results of Tian and Todorov, any hyper-Kähler manifold X admits a universal deformation $\pi\colon \mathscr{X} \to \mathrm{Def}(X)$, where $\mathrm{Def}(X)$ is smooth of dimension $h^{1,1}(X)$. Upon shrinking $\mathrm{Def}(X)$, we may assume that the local system $R^2\pi_*\mathbf{Z}$ is trivial and identify via parallel transport $H^2(\mathscr{X}_t, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$ for all $t \in \mathrm{Def}(X)$.

We define the analytic subset

$$\mathscr{D}_X := \{[x] \in \mathbf{P}(H^2(X, \mathbf{C})) \mid q_X(x) = 0, \quad q_X(x, \bar{x}) > 0\}$$

of $\mathbf{P}(H^2(X, \mathbf{C}))$. Using the relations (2), we obtain a map

$$\wp_{\mathrm{Def}(X)}\colon \mathrm{Def}(X) \longrightarrow \mathscr{D}_X \tag{5}$$

sending a point t to $[H^{2,0}(\mathscr{X}_t)]$. It follows from work of Griffiths that this map is holomorphic.

Theorem 1.5 (Local Torelli)**.** *The morphism $\wp_{\mathrm{Def}(X)}$ is a local isomorphism.*

Given a line bundle L on X, the deformation space of the pair (X, L) is the subspace

$$\mathrm{Def}(X, L) := \wp^{-1}_{\mathrm{Def}(X)}(\mathbf{P}(c_1(L)^\perp \otimes \mathbf{C}) \cap \mathscr{D}_X)$$

of $\mathrm{Def}(X)$ where the class $c_1(L)$ stays of type $(1,1)$. The restriction of the universal family π gives a family

$$\mathscr{X}_L \longrightarrow \mathrm{Def}(X, L) \tag{6}$$

which carries (after possibly shrinking $\mathrm{Def}(X, L)$) a line bundle $\mathcal{L}$ that restricts to L on the central fiber and is such that $c_1(\mathcal{L}|_{\mathscr{X}_{L,t}}) = c_1(L)$ under parallel transport for each $t \in \mathrm{Def}(X, L)$.

There exists also a global version of the local period map (5) (here presented for the $\mathrm{K3}^{[m]}$ deformation type). Let X be a hyper-Kähler manifold of $\mathrm{K3}^{[m]}$ deformation type and let $\varphi\colon H^2(X, \mathbf{Z}) \simeq \Lambda_{K3^{[m]}}$ be an isometry (called a *marking* of X). There exists a moduli space $\mathscr{M}_{\mathrm{marked}}$ of marked pairs (X, φ) of $\mathrm{K3}^{[m]}$ deformation type and a holomorphic map

$$\wp\colon \mathscr{M}_{\mathrm{marked}} \longrightarrow \mathscr{D}_{\Lambda_{K3^{[m]}}} := \{x \in \mathbf{P}(\Lambda_{K3^{[m]}} \otimes \mathbf{C}) \mid x^2 = 0, \quad x \cdot \bar{x} > 0\} \tag{7}$$

called the *global period map*.

Theorem 1.6 ([Huy99, Theorem 8.1]). *The restriction of $\wp$ to each connected component of $\mathscr{M}_{\text{marked}}$ is surjective.*

1.1.3. Moduli spaces of polarized hyper-Kähler manifolds of $K3^{[m]}$ deformation type. Let X be a hyper-Kähler manifold of $\mathrm{K3}^{[m]}$ deformation type. A *polarization* on X is an ample line bundle H on X which is primitive in the Picard group $\mathrm{Pic}(X)$. Given an isometry $\varphi\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{K3^{[m]}}$, we denote by τ the orbit of $\varphi(c_1(H))$ under the group $O(\Lambda_{K3^{[m]}})$. The orbit τ is independent of φ and is called the *polarization type* of (X,H).

Smooth polarized hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$ deformation type and polarization type τ admit a possibly reducible quasi-projective coarse moduli space which we denote by ${}^{[m]}\mathscr{M}_\tau$.

Every vector h in the orbit τ has the same square $2d$ and divisibility γ. Conversely, by [GHS10, Corollary 3.7], when $\gamma = 1$ or $\gamma = 2$ and $d + m \equiv 1 \pmod 4$, the orbit τ is determined by $2d$ and γ. It is therefore equal to the orbit of the vector

$$h = \gamma(e + bf) + \delta,$$

where $b = \frac{d+m-1}{\gamma^2}$, (e,f) is a standard basis of one of the copies of U contained in $\Lambda_{K3^{[m]}}$, and δ is the vector of square $-2(m-1)$.

In these case,s we denote by ${}^{[m]}\mathscr{M}_{2d}^{(\gamma)}$ the coarse moduli space of hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$ deformation type and polarization of square $2d$ and divisibility γ.

Theorem 1.7 (Gritsenko–Hulek–Sankaran, Apostolov). *Let d, m, γ be positive integers such that either $\gamma = 1$, or $\gamma = 2$ and $d+m \equiv 1 \pmod 4$. The quasi-projective moduli space ${}^{[m]}\mathscr{M}_{2d}^{(\gamma)}$ of polarized hyper-Kähler manifolds of deformation type $K3^{[m]}$ with a polarization of square $2d$ and divisibility γ is irreducible of dimension 20.*

1.1.4. Period map for polarized hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$ deformation type. Let (X,H) be a polarized hyper-Kähler manifold of $\mathrm{K3}^{[m]}$ deformation type and set $h := \varphi(c_1(H))$, for some isometry $\varphi\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{K3^{[m]}}$. Since the Beauville-Bogomolov-Fujiki form q_X is compatible with the Hodge decomposition, the period point $\wp(X,\varphi)$ defined in (7) lies in the analytic subset

$$\mathscr{D}_{h^\perp} := \{[x] \in \mathbf{P}(h^\perp \otimes \mathbf{C}) \mid x^2 = 0, \quad x \cdot \bar{x} > 0\}.$$

The subgroup

$$\widehat{O}(\Lambda_{K3^{[m]}}, h) := \{f \in O(\Lambda_{K3^{[m]}}) \mid f(h) = h, \quad f \text{ acts as } \pm\mathrm{id} \text{ on } A_{\Lambda_{K3^{[m]}}}\}$$

of $O(\Lambda_{K3^{[m]}})$ (where $A_{\Lambda_{K3^{[m]}}} \simeq \mathbf{Z}/(2(m-1)\mathbf{Z}$ is the discriminant group of $\Lambda_{K3^{[m]}}$) acts on $\mathscr{D}_{h^\perp}$. The quotient

$$ {}^{[m]}\mathscr{P}_\tau := \mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}}, h) \tag{8}$$

is an irreducible quasiprojective normal variety of dimension 20 which only depends on the orbit $\tau = O(\Lambda_{K3^{[m]}})h$. It is called the *period domain* of polarized hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$ deformation type with polarization type τ.

There is a well-defined algebraic period map

$$ {}^{[m]}\wp_\tau\colon {}^{[m]}\mathscr{M}_\tau \longrightarrow {}^{[m]}\mathscr{P}_\tau \tag{9}$$

sending a point $[(X,H)]$ to the class $[\varphi(H^{2,0}(X))]$, where φ is a marking on X sending $c_1(H)$ to h.

When the polarization type is determined by its degree $2d$ and its divisibility γ, we will write ${}^{[m]}\mathscr{P}_{2d}^{(\gamma)}$ and ${}^{[m]}\wp_{2d}^{(\gamma)}$ instead of ${}^{[m]}\mathscr{P}_{\tau}$ and ${}^{[m]}\wp_{\tau}$.

The following is a version of the Torelli theorem for polarized hyper-Kähler manifolds of $\mathrm{K3}^{[m]}$ deformation type, proven by Verbitsky and Markman.

Theorem 1.8 ([Mar11, Theorem 8.4])**.** *Let $m \geq 2$ be an integer and let τ be a polarization type in $\Lambda_{K3^{[m]}}$. The restriction of the period map ${}^{[m]}\wp_{\tau}$ to any irreducible component of ${}^{[m]}\mathscr{M}_{\tau}$ is an open embedding.*

1.1.5. Lagrangian subvarieties. Let X be a hyper-Kähler manifold of dimension $2n$ with holomorphic symplectic form σ_X. Since σ_X is nondegenerate, any complex submanifold $Y \subset X$ satisfying $\sigma_X|_Y = 0$ has dimension at most n.

Definition 1.9. A subvariety $L \subset X$ is *Lagrangian* if $\dim(L) = n$ and the restriction of σ_X to the smooth part L_{sm} of L vanishes.

Let $L \subset X$ be a Lagrangian submanifold, that is a smooth Lagrangian subvariety. The symplectic form induces an isomorphism

$$N_{L/X} \simeq \Omega^1_L = T^\vee_L \tag{10}$$

between its the normal bundle and its the cotangent bundle.

Proposition 1.10. *Let $L \subset X$ be a Lagrangian submanifold of a hyper-Kähler manifold X of dimension $2n$. The following equalities involving the Chern classes of L hold.*

(a) $[L]^2 = (-1)^n c_n(T_L)$;
(b) $c(T_L)\, c(T^\vee_L) = c(T_X)|_L$.

Proof. The self intersection $[L]^2$ is equal to the top Chern class of the normal bundle $N_{L/X}$ and the isomorphism of Equation (10) implies Item (a). Item (b) follows from taking Chern classes in the short exact sequence

$$0 \longrightarrow T_L \longrightarrow T_X|_L \longrightarrow N_{L/X} \longrightarrow 0$$

and the isomorphism of Equation (10). □

The following result is a cohomological characterization of Lagrangian subvarieties on projective hyper-Kähler manifolds.

Proposition 1.11. *Let X be a projective hyper-Kähler manifold of dimension $2n$ with symplectic form σ, and let $L \subset X$ be a subvariety of dimension n such that $\sigma \cdot i_*[L] = 0$. Then L is a Lagrangian subvariety.*

Proof. Let h be the first Chern class of a very ample divisor H on X. By projection formula, we have

$$\int_L i^*\sigma \cdot i^*\bar{\sigma} \cdot i^*h^{n-2} = \int_X i_*[L] \cdot \sigma \cdot \bar{\sigma} \cdot h^{n-2} = 0.$$

Therefore, the form $i^*\sigma$ is trivial on the intersection of L_{sm} with $n-2$ general hyperplanes $H_1, \ldots, H_{n-2}$ in $|H|$. Since H is very ample, the intersection $L \cap H_1 \cap \cdots \cap H_{n-2}$ covers the whole L_{sm} and $i^*\sigma$ vanishes on L_{sm}. □

1.1.6. Second primitive cohomology. Let (X, H) be a polarized hyper-Kähler manifold and set $h \coloneqq c_1(H) \in H^2(X, \mathbf{Z})$. The *primitive cohomology* of X is

$$H^2(X, \mathbf{Z})_{\mathrm{prim}} \coloneqq \{\alpha \in H^2(X, \mathbf{Z}) \mid q_X(\alpha, h) = 0\}. \tag{11}$$

The following is a consequence of Local Torelli and Theorem 1.2.

Proposition 1.12 ([Son22, Lemma 1.2.2])**.** *Let (X, H) be a polarized hyper-Kähler manifold and let q be the restriction of the Beauville–Bogomolov–Fujiki form q_X to $H^2(X, \mathbf{Z})_{\mathrm{prim}}$. By parallel transport, for any small deformation (X', H') of (X, H), we may identify $H^2(X', \mathbf{Z})_{\mathrm{prim}}$ with $H^2(X, \mathbf{Z})_{\mathrm{prim}}$.*

Up to multiplication by a scalar, the form q_X is the unique quadratic form q on $H^2(X, \mathbf{Z})_{\mathrm{prim}}$ such that

$$H^{2,0}(X') \text{ is } q\text{-orthogonal to } H^{2,0}(X') \oplus H^{1,1}(X')_{\mathrm{prim}}$$

for any deformation (X', H') of (X, H).

Let X be a projective hyper-Kähler manifold. We denote by $\mathrm{NS}(X)$ its Néron–Severi lattice and we define the *transcendental lattice* $T(X) \subset H^2(X, \mathbf{Z})$ as the q_X-orthogonal of $\mathrm{NS}(X)$. It inherits a Hodge structure from the Hodge structure on $H^2(X, \mathbf{Z})$. The following proposition shows that this Hodge structure is simple. In particular, if X has Picard rank 1, the Hodge structure on the primitive cohomology $H^2(X, \mathbf{Z})_{\mathrm{prim}}$ is simple.

Proposition 1.13. *Let X be a projective hyper-Kähler manifold. The Hodge structure on the transcendental lattice $T(X)$ is a simple Hodge structure.*

Proof. The same proof as [Huy16, Lemma 3.1] shows that $T(X)$ is the minimal primitive sub-Hodge structure $M \subset H^2(X, \mathbf{Z})$ such that $M \otimes \mathbf{C}$ contains $H^{2,0}(X)$. Since X is projective, the restriction of the quadratic form q_X to $T(X)$ is definite. Therefore, for any primitive sub-Hodge structure $T \subset T(X)$, either $T \otimes \mathbf{C}$ contains $H^{2,0}(X)$ and $T = T(X)$, or T is of pure type $(1, 1)$ hence, $T^\perp \otimes \mathbf{C}$ contains $H^{2,0}(X)$ and $T = 0$. □

1.2. Cohomology of hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ deformation type

In this section, X will be a hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type. We study its cohomology ring $H^\bullet(X, \mathbf{Q})$.

The contents of this section appeared in [Riz25a, Section 2].

The nonzero Hodge numbers of X (given by [GS93, Theorem 2]) are

$b_0 = 1$				1			
$b_2 = 23$			1	21	1		
$b_4 = 299$		1	22	253	22	1	
$b_6 = 2554$	1	21	253	2004	253	21	1

We denote by $\mathrm{SH}^{2\bullet}(X, \mathbf{Q})$ the image of the canonical map

$$\mathrm{Sym}^\bullet H^2(X, \mathbf{Q}) \longrightarrow H^{2\bullet}(X, \mathbf{Q}) \tag{12}$$

(the so-called Verbitsky component). Let c_2, c_4, c_6 be the nonzero Chern classes of X. One has ([**Mar20**, Lemma 1.5])

$$c_2 \in \mathrm{Sym}^2 H^2(X, \mathbf{Q}) = \mathrm{SH}^4(X, \mathbf{Q}) \subset H^4(X, \mathbf{Q}).$$

The nonzero Chern numbers are ([**EGL01**, Theorem 0.1])

$$c_6 = 3200\ ,\ c_2c_4 = 14720\ ,\ c_2^3 = 36800. \tag{13}$$

Let $\alpha \in H^{4k}(X, \mathbf{Q})$ be a class that remains of type $(2k, 2k)$ on all small deformations of X. By Theorem 1.4, there exists a rational number $C(\beta)$, called the Fujiki constant of β, that satisfies

$$\forall \alpha \in H^2(X, \mathbf{Q}) \qquad \int_X \beta \cdot \alpha^{6-2k} = C(\beta) q_X(\alpha)^{3-k}, \tag{14}$$

where q_X is the Beauville-Bogomolov-Fujiki form of X. From [**Son22**, Lemma 2.1.7], we get[1]

$$C(1) = 15, \quad C(c_2) = 108, \quad C(c_2^2) = 1200, \quad C(c_4) = 480. \tag{15}$$

The polarized version of Equation (14) gives, for all $\alpha_1, \dots, \alpha_{6-2k} \in H^2(X, \mathbf{Q})$,

$$\int_X \beta\alpha_1 \cdots \alpha_{6-2k} = \frac{C(\beta)}{(6-2k)!} \sum_{\sigma \in \mathfrak{S}_{6-2k}} q_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots q_X(\alpha_{\sigma(6-2k-1)}, \alpha_{\sigma(6-2k)}), \tag{16}$$

where $\mathfrak{S}_{6-2k}$ is the group of permutations of $6-2k$ elements.

1.2.1. Hodge classes on very general hyper-Kähler manifolds of K3$^{[3]}$ deformation type. Following [**GKLR22**, Section 2], we denote by $\bar{V}$ the vector space $H^2(X, \mathbf{Q})$ endowed with the (rational extension of the) Beauville–Bogomolov–Fujiki quadratic form q_X. We let $U := \mathbf{Q}^{\oplus 2}$, endowed with the quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and set

$$V := H^2(X, \mathbf{Q}) \oplus U \tag{17}$$

(orthogonal direct sum).

We follow [**Mar02**, Example 14] and [**GKLR22**, Remark 3.3]. The so-called LLV-algebra $\mathfrak{g} = \mathfrak{so}(V)$ acts on $H^\bullet(X, \mathbf{Q})$ and its decomposition into irreducible representations (which are also sub-Hodge structures) is[2]

$$H^\bullet(X, \mathbf{Q}) = V_{(3)} \oplus V_{(1,1)}, \tag{18}$$

[1] All Chern numbers and Fujiki constants for hyper-Kähler manifolds of K3$^{[n]}$ deformation type can be computed using https://github.com/8d1h/bott.

[2] See also https://webusers.imj-prg.fr/~jieao.Son22a/static/llv.html

where $V_{(3)}$ is the Verbitsky component $\mathrm{SH}^{2\bullet}(X,\mathbf{Q})$ and $V_{(1,1)}$ is $\bigwedge^2 V$. Furthermore, under the action of $\mathfrak{so}(\bar{V})$, the cohomology groups decompose further into the following representations

$$\begin{aligned} H^0(X,\mathbf{Q}) &= \mathbf{Q},\\ H^2(X,\mathbf{Q}) &= \bar{V},\\ H^4(X,\mathbf{Q}) &= \mathrm{Sym}^2\bar{V}\oplus\bar{V},\\ H^6(X,\mathbf{Q}) &= \mathrm{Sym}^3\bar{V}\oplus\textstyle\bigwedge^2\bar{V}\oplus\mathbf{Q}\eta,\\ H^8(X,\mathbf{Q}) &= \mathrm{Sym}^2\bar{V}\oplus\bar{V},\\ H^{10}(X,\mathbf{Q}) &= \bar{V},\\ H^{12}(X,\mathbf{Q}) &= \mathbf{Q}, \end{aligned}$$

where the summands in the fourth line are pairwise orthogonal with respect to the intersection pairing.[3] There are further decompositions

$$\begin{aligned} \mathrm{Sym}^2\bar{V} &= \bar{V}_{(2)}\oplus\mathbf{Q},\\ \mathrm{Sym}^3\bar{V} &= \bar{V}_{(3)}\oplus V, \end{aligned}$$

into irreducible representations, whereas $\bar{V}$ and $\bigwedge^2\bar{V}=\bar{V}_{(1,1)}$ are already irreducible.

For X very general (so nonprojective), the (special) Mumford–Tate algebra of X is equal to the algebra $\mathfrak{so}(\bar{V})$ ([**GKLR22**, Proposition 2.38]). Therefore, the space of Hodge classes in $H^\bullet(X,\mathbf{Q})$ coincides with its $\mathfrak{so}(\bar{V})$-invariant part ([**GKLR22**, Section 2.3]): it is given by

$$\begin{aligned} \mathrm{Hdg}^0(X,\mathbf{Q}) &= \mathbf{Q},\\ \mathrm{Hdg}^2(X,\mathbf{Q}) &= 0,\\ \mathrm{Hdg}^4(X,\mathbf{Q}) &= \mathbf{Q}c_2,\\ \mathrm{Hdg}^6(X,\mathbf{Q}) &= \mathbf{Q}\eta,\\ \mathrm{Hdg}^8(X,\mathbf{Q}) &= \mathbf{Q}c_4,\\ \mathrm{Hdg}^{10}(X,\mathbf{Q}) &= 0,\\ \mathrm{Hdg}^{12}(X,\mathbf{Q}) &= \mathbf{Q}. \end{aligned} \tag{19}$$

In particular, the classes c_4 and c_2^2 are proportional and the proportionality coefficient can be determined using the constants given in (14): one finds

$$c_2^2 = \frac{C(c_2^2)}{C(c_4)}\,c_4 = \frac{1200}{480}\,c_4 = \frac{5}{2}c_4. \tag{20}$$

This remains true on any hyper-Kähler manifold of K3$^{[3]}$ deformation type.

1.2.2. Action of automorphisms on the LLV algebra. Given an automorphism φ of X of finite order, we want to study the automorphism φ^* induced on $H^\bullet(X,\mathbf{Q})$.

The automorphism φ_2^* of $\bar{V}=H^2(X,\mathbf{Q})$ extends to an automorphism γ of V by acting as the identity on the hyperbolic plane U (see Equation (17)). This induces automorphisms γ_λ of any $\mathfrak{g}$-representation V_λ. We compare in Lemma 1.15 the automorphism φ^* with the automorphism $\gamma_{(3)}\oplus\gamma_{(1,1)}$ of $H^\bullet(X,\mathbf{Q})$ obtained from the isomorphism (18).

Lemma 1.14. *Let X be a hyper-Kähler manifold and let φ be an automorphism of X. For each $\mathfrak{g}$-invariant subspace W of $H^\bullet(X,\mathbf{Q})$, the space $\varphi^*(W)$ is again $\mathfrak{g}$-invariant and isomorphic to W.*

[3] The class η was studied in [**HHT12**, Proposition 6.1].

Proof. By [Flo22, Lemma 2.3], the application $h \mapsto (\varphi^*)h(\varphi^*)^{-1}$ induces an automorphism of $\mathfrak{g}$. In particular, $\mathfrak{g}\varphi^*(W) = \varphi^*\mathfrak{g}(W) = \varphi^*(W)$ for any $\mathfrak{g}$-invariant subspace W. □

In our case, the lemma implies that the automorphism φ^* of $H^\bullet(X, \mathbf{Q})$ preserves the subspaces $V_{(3)}$ and $V_{(1,1)}$, hence it decomposes as $\varphi^* = \varphi^*_{(3)} \oplus \varphi^*_{(1,1)}$.

Since the map from the Verbitsky component to $H^*(X, \mathbf{Q})$ is given by cup-product, one has $\varphi^*_{(3)} = \gamma_{(3)}$. For $\varphi^*_{(1,1)}$, the same argument as in [BGGG25, Proposition 3.3] shows the following.

Lemma 1.15. *Let $\varphi \in \mathrm{Aut}(X)$ be an automorphism of order n and let $\varphi^*_{(1,1)}$ be the induced automorphism of $V_{(1,1)}$. If n is odd, $\varphi^*_{(1,1)} = \gamma_{(1,1)}$, while if n is even, $\varphi^*_{(1,1)} = \pm\gamma_{(1,1)}$.*

Proof. We follow the proof of [BGGG25, Proposition 3.3], borrowing its notation. Since $\varphi^*_{(1,1)}$ with $\gamma_{(1,1)}$ are both automorphisms of $V_{(1,1)}$, they satisfy

$$\varphi^*_{(1,1)}(L_\alpha \cdot x) = L_{\varphi^*(\alpha)} \cdot \varphi^*_{(1,1)}(x), \quad \text{and} \quad \gamma_{(1,1)}(L_\alpha \cdot x) = L_{\varphi^*(\alpha)} \cdot \gamma_{(1,1)}(x)$$

for all $\alpha \in H^2(X, \mathbf{Z})$ that define a $\mathfrak{sl}_2$-triple. In particular,

$$\varphi^*_{(1,1)} \circ \gamma^{n-1}_{(1,1)}(L_\alpha \cdot x) = \varphi^*_{(1,1)}(L_{(\varphi^*)^{n-1}(\alpha)} \cdot \gamma^{n-1}_{(1,1)}(x)) = L_\alpha \cdot \varphi^*_{(1,1)} \circ \gamma^{n-1}_{(1,1)}(x).$$

Therefore $\varphi^*_{(1,1)} \circ \gamma^{n-1}_{(1,1)}$ is a $\mathfrak{g}$-isomorphism of $V_{(1,1)}$, hence it is equal to $\lambda \cdot \mathrm{id}$ by Schur's lemma, for some $\lambda \in \mathbf{Q}$. Comparing orders, we obtain $\lambda^n = \mathrm{id}$, hence λ is 1 for n odd, and equal to ± 1 for n even. □

1.2.3. Hodge classes on very general polarized hyper-Kähler manifolds of K3$^{[3]}$ deformation type. We now consider a very general *polarized* hyper-Kähler manifold (X, h) of K3$^{[3]}$ deformation type. In this case, the vector space $\bar{V}$ decomposes as $\bar{T} \oplus \mathbf{Q}h$, where h is the polarization and $\bar{T}$ is the transcendental subspace, that is, the orthogonal $h^\perp$. The cohomology groups decompose further as

$$\begin{aligned}
H^0(X, \mathbf{Q}) &= \mathbf{Q},\\
H^2(X, \mathbf{Q}) &= \bar{T} \oplus \mathbf{Q},\\
H^4(X, \mathbf{Q}) &= (\mathrm{Sym}^2\bar{T} \oplus \bar{T} \oplus \mathbf{Q}) \oplus (\bar{T} \oplus \mathbf{Q}),\\
H^6(X, \mathbf{Q}) &= (\mathrm{Sym}^3\bar{T} \oplus \mathrm{Sym}^2\bar{T} \oplus \bar{T} \oplus \mathbf{Q}) \oplus (\textstyle\bigwedge^2\bar{T} \oplus \bar{T}) \oplus \mathbf{Q}.
\end{aligned}$$

The (special) Mumford–Tate algebra of X is equal to $\mathfrak{so}(\bar{T})$ and the $\mathfrak{so}(\bar{T})$-invariant part of $H^\bullet(X, \mathbf{Q})$ is the space of Hodge classes.

One has $\mathrm{Hdg}^2(X, \mathbf{Q}) = \mathbf{Q}h$.

The space $\mathrm{Hdg}^4(X, \mathbf{Q})$ has dimension 3 (recall that the invariant part of $\mathrm{Sym}^2\bar{T}$ has dimension 1). We denote by λ a generator of the trivial factor outside the Verbitsky component.

The space $\mathrm{Hdg}^6(X, \mathbf{Q})$ also has dimension 3 and intersects the Verbitsky component in dimension 2. Up to multiplication by a nonzero rational number, we obtain that $h\lambda$ is the absolute Hodge class η from (19). By [HHT12, Proposition 6.1], its square is 4 (modulo squares of nonzero rational numbers).

The following lemma gives us a basis of $\mathrm{Hdg}^\bullet(X, \mathbf{Q})$.

Lemma 1.16. *The families (h^2, c_2) in $\mathrm{Hdg}^4(X, \mathbf{Q})$, and (h^3, hc_2) in $\mathrm{Hdg}^6(X, \mathbf{Q})$, are linearly independent.*

Proof. The statement in degree 4 follows from the one in degree 6, since any nontrivial linear relation between h^2 and c_2 would give a nontrivial linear relation between h^3 and hc_2.

We show that h^3 and hc_2 are linearly independent. From Equation (16), we get that there exist nonzero constants x, y, $z \in \mathbf{Q}$ such that

$$\int_X h^3\beta^3 = xq_X(h)q_X(\beta)q_X(h,\beta) + yq_X(h,\beta)^3,$$
$$\int_X hc_2\beta^3 = zq_X(h,\beta)q_X(\beta)$$

for all $\beta \in H^2(X,\mathbf{Z})$. Assume that the classes h^3 and hc_2 are proportional. Then for any β such that $q_X(\beta) = 0$, we have $q_X(h,\beta) = 0$, which is absurd since q_X is nondegenerate, hence the quadric $q_X(\beta) = 0$ is not contained in any hyperplane. □

Thus, the spaces of Hodge classes on a very general polarized hyper-Kähler sixfold (X,h) of $\mathrm{K3}^{[3]}$ deformation type are

$$\begin{aligned}
\mathrm{Hdg}^0(X,\mathbf{Q}) &= \mathbf{Q},\\
\mathrm{Hdg}^2(X,\mathbf{Q}) &= \mathbf{Q}h,\\
\mathrm{Hdg}^4(X,\mathbf{Q}) &= \mathbf{Q}h^2 \oplus \mathbf{Q}c_2 \oplus \mathbf{Q}\lambda,\\
\mathrm{Hdg}^6(X,\mathbf{Q}) &= \mathbf{Q}h^3 \oplus \mathbf{Q}hc_2 \oplus \mathbf{Q}h\lambda,\\
\mathrm{Hdg}^8(X,\mathbf{Q}) &= \mathbf{Q}h^4 \oplus \mathbf{Q}h^2c_2 \oplus \mathbf{Q}h^2\lambda,\\
\mathrm{Hdg}^{10}(X,\mathbf{Q}) &= \mathbf{Q}h^5,\\
\mathrm{Hdg}^{12}(X,\mathbf{Q}) &= \mathbf{Q}h^6,
\end{aligned} \tag{21}$$

where the class λ depends on the polarization h, while the class $h\lambda$ is the absolute Hodge class η, and is orthogonal to the Verbitsky component $\mathrm{SH}^3(X,\mathbf{Q})$.

The intersection $\mathrm{Hdg}^\bullet(X,\mathbf{Q}) \cap \mathrm{SH}^{2\bullet}(X,\mathbf{Q})$ is the $\mathbf{Q}$-subalgebra of $H^\bullet(X,\mathbf{Q})$ generated by $\{h, c_2, c_4, c_6\}$. We determine the relations in each degree.

Proposition 1.17. (a) *There are no relations in degree* ≤ 6.
(b) *In degree* 8, *the relations are*

$$c_4 = -\frac{160}{q(h)^2}h^4 + \frac{80}{3q(h)}h^2c_2, \quad c_2^2 = \frac{5}{2}c_4.$$

(c) *In degree* 10, *the relations are*

$$h^3c_2 = \frac{36}{5q(h)}\,h^5, \quad hc_2^2 = \frac{80}{q(h)^2}\,h^5, \quad hc_4 = \frac{32}{q(h)^2}\,h^5.$$

(d) *In degree* 12, *we have*

$$h^6 = 15\,q(h)^3, \quad h^4c_2 = 108\,q(h)^2, \quad h^2c_2^2 = 1200\,q(h), \quad h^2c_4 = 480\,q(h).$$

Proof. The relations in degree 12 follow from the values of the generalized Fujiki constants given in (15) and their definition (14). The relations in degree 10 then follow from the fact that the vector space $\mathrm{Hdg}^{10}(X,\mathbf{Q})$ has dimension one. For example, we obtain

$$h^3c_2 = \frac{h^4c_2}{h^6}\,h^5 = \frac{108\,q(h)^2}{15\,q(h)^3}\,h^5 = \frac{36}{5q(h)}\,h^5$$

and similarly for the other relations.

In degree 8, the second relation is (20). For the first relation, write $c_4 = xh^4 + yh^2c_2$ for some $x, y \in \mathbf{Q}$. Taking the product with h^2, we get

$$480\, q(h) = 15\, xq(h)^3 + 108\, yq(h)^2.$$

Taking the product with c_2 and using the Chern numbers given in (13), we get

$$14720 = 108\, xq(h)^2 + 1200\, yq(h).$$

Solving this system of equations for x and y, we obtain the desired relation. □

Remark 1.18. One also determine relations involving λ. The class $\eta = h\lambda$ is orthogonal to the Verbitsky component $\mathrm{SH}^3(X, \mathbf{Q})$ and, as mentioned above and up to multiplying η by a rational number, we can suppose $\eta^2 = 4$. We have the following relations in degree 12:

$$h^2\lambda^2 = 4, \qquad h^4\lambda = 0, \qquad h^2c_2\lambda = 0, \qquad c_2^2\lambda = 0,$$

where the last equality is obtain from the fact that c_2^2 is a linear combination of h^4 and h^2c_2.

Since $\mathrm{Hdg}^5(X) = \mathbf{Q}h^5$ and multiplication by h is injective on $\mathrm{Hdg}^5(X)$, the above equalities together with Proposition 1.17 imply

$$h\lambda^2 = \frac{4}{15q(h)^3}h^5, \qquad h^3\lambda = 0, \qquad hc_2\lambda = 0.$$

We conclude this section by studying Lagrangian submanifolds of very general polarized hyper-Kähler manifolds of K3$^{[3]}$ deformation type.

Proposition 1.19. *Let W be a Lagrangian submanifold of a very general polarized hyper-Kählermanifold (X, h) of* K3$^{[3]}$ *deformation type. The orthogonal projection to the Verbitsky component* $\mathrm{SH}^6(X, \mathbf{Q})$ *of the class $[W] \in H^6(X, \mathbf{Q})$ is equal to*

$$\overline{[W]} := \frac{[W] \cdot h^3}{72 \cdot q(h)^2}\left(\frac{12}{q(h)}h^3 - hc_2\right).$$

Proof. The class $[W]$ is a 3-dimensional Hodge class hence, by Equation (21), it can be written as

$$[W] = ah^3 + bhc_2 + ch\lambda,$$

where $a, b, c \in \mathbf{Q}$, and $h\lambda$ is orthogonal to the Verbitsky component. In particular, $\overline{[W]} = ah^3 + bhc_2$ and, for any class $\gamma \in \mathrm{SH}^6(X, \mathbf{Q})$, we have $\gamma \cdot [W] = \gamma \cdot \overline{[W]}$.

We apply this to the class $\gamma = h\sigma\bar{\sigma}$ in $\mathrm{SH}^6(X, \mathbf{Q})$, where σ is a symplectic form on X. Since W is Lagrangian, it satisfies $\gamma \cdot [W] = 0$ and thus $\gamma \cdot \overline{[W]} = 0$. Using Equation (16), the Fujiki constants computed in Equation (15), and the fact that $q_X(\sigma, h) = 0$, we compute

$$\begin{aligned}\gamma \cdot \overline{[W]} &= a\int_X h^4\sigma\bar{\sigma} + b\int_X c_2h^2\sigma\bar{\sigma} \\ &= a\,\frac{C(1)}{6!}6 \cdot 4!q_X(h)^2q_X(\sigma, \bar{\sigma}) + b\,\frac{C(c_2)}{4!}8q_X(h)q_X(\sigma, \bar{\sigma}) \\ &= q_X(h)q_X(\sigma, \bar{\sigma})(3q_X(h)a + 36b).\end{aligned}$$

Since $\gamma \cdot \overline{[W]} = 0$ and $q_X(h)q_X(\sigma, \bar{\sigma}) > 0$, we obtain $a = -\frac{12}{q_X(h)}b$.

Finally, we determine b in terms of the degree of W. Indeed, we have

$$[W] \cdot h^3 = \overline{[W]} \cdot h^3 = b\Big(hc_2 - \frac{12}{q_X(h)}h^3\Big) \cdot h^3 = -72\, q_X(h)^2 b$$

where, in the last equality, we use the numbers computed in Proposition 1.17. This gives the value of b and proves the proposition. □

Remark 1.20. Jieao Song pointed out to me that the above formula generalizes to any polarized hyper-Kähler manifold X of dimension $2n$ with polarization h, and any Lagrangian submanifold F deforming with the polarization. In particular, following the strategy of [**Son22**, Theorem 2.2.4], he shows that the orthogonal projection to the Verbitsky component $\mathrm{SH}^{2n}(X, \mathbf{Q})$ of the class $[F] \in H^{2n}(X, \mathbf{Q})$ is equal to

$$\overline{[F]} = \frac{[F] \cdot h^n}{q(h)^n} \cdot \frac{\mu^n}{c_X} \overline{[\exp(h/\mu)\, \mathrm{td}_X^{1/2}]_n}, \quad \text{where } \mu := \sqrt{\frac{-q(h)}{2r_X}}. \tag{22}$$

Here, $c_X = \frac{C(1)}{(2n-1)!!}$, and $r_X := \frac{C(c_2)2^n n!(2n-1)}{(2n)!24C(1)}$ is the constant introduced in [**Bec23**, Equation (3.1)].

The key point is to generalize [**Son22**, Lemma 2.2.1] by showing that, for any class $\alpha \in H^2(X, \mathbf{Q})$, there is an equality

$$(\iota^*\alpha)^n \cdot [F] = \frac{[F] \cdot h^n}{q_X(h)^n}\, q_X(\alpha, L)^n.$$

By writing $\mathrm{td}_X^{1/2} = \left(\mathrm{td}_{2k}^{1/2}\right)_k$ where $\mathrm{td}_{2k}^{1/2} \in H^{4k}(X, \mathbf{Q})$ and developing Equation (22), one obtains

$$\overline{[F]} = \frac{[F] \cdot h^n}{q_X(h)^n \cdot c_X} \sum_{2k \le n} \left(\frac{-q_X(h)}{2r_X}\right)^k \frac{h^{n-2k}}{(n-2k)!}\, \overline{\mathrm{td}_{2k}^{1/2}}. \tag{23}$$

When X is of $\mathrm{K3}^{[n]}$ deformation type, we have $c_X = 1$ and $r_X = \frac{n+3}{4}$. In particular, for $n = 3$ the above formula reduces to

$$\overline{[F]} = \frac{[F] \cdot h^3}{q_X(h)^3} \left(\frac{h^3}{3!} - \frac{q_X(h)}{3 \cdot 24} hc_2\right),$$

which is exactly Proposition 1.19.

CHAPTER 2

EPW cubes

2.1. Lagrangian intersection loci

Let V_6 be a 6-dimensional complex vector space. We endow the 20-dimensional vector space $\bigwedge^3 V_6$ with the conformal symplectic structure given by wedge product. A Lagrangian subspace of $\bigwedge^3 V_6$ is a maximal totally isotropic vector subspace. Lagrangian subspaces are parametrized by the Lagrangian Grassmannian $\mathrm{LG}(\bigwedge^3 V_6)$, which is a 55-dimensional submanifold of the Grassmannian $\mathsf{Gr}(10, \bigwedge^3 V_6)$.

For any Lagrangian subspace $A \subset \bigwedge^3 V_6$ and nonnegative integer k, we consider the degeneracy locus

$$\mathsf{L}_A^{\geq k} \coloneqq \{[L] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \dim(A \cap L) \geq k\}.$$

The scheme $\mathsf{L}_A^{\geq k}$ is the intersection of $\mathrm{LG}(\bigwedge^3 V_6)$ with the Schubert subvariety

$$\Omega_{2k-1,\dots,k}(W_\bullet) \subset \mathsf{Gr}(10, \textstyle\bigwedge^3 V_6)$$

associated with a flag

$$W_1 \subset \cdots \subset W_{10} = A$$

of totally isotropic subspaces. These schemes are studied in [**PR97**], where the authors show in particular that $\mathsf{L}_A^{\geq k}$ is Cohen-Macaulay of expected codimension $\frac{k(k+1)}{2}$ in $\mathrm{LG}(\bigwedge^3 V_6)$.

The loci $\mathsf{L}_A^{\geq k}$ can also be described as Lagrangian intersection loci over $\mathrm{LG}(\bigwedge^3 V_6)$: in the notation of [**DK20a**, Section 4], the locus $\mathsf{L}_A^{\geq k}$ is the k-th intersection locus of the subbundles

$$\mathscr{A}_1 = A \otimes \mathcal{O}_{\mathrm{LG}(\bigwedge^3 V_6)} \quad \text{and} \quad \mathscr{A}_2 = \mathcal{U},$$

where $\mathcal{U} \to \mathrm{LG}(\bigwedge^3 V_6)$ is the rank 10 tautological subbundle.

Remark 2.1. [**PR97**, Theorem 2.1] gives formulas for the classes of $\mathsf{L}_A^{\geq k}$ in the Chow group of $\mathrm{LG}(\bigwedge^3 V_6)$. In particular, one can compute

$$\begin{aligned}
[\mathsf{L}_A^{\geq 1}] &= c_1(\mathcal{U}^\vee),\\
[\mathsf{L}_A^{\geq 2}] &= (c_2c_1 - 2c_3)(\mathcal{U}^\vee),\\
[\mathsf{L}_A^{\geq 3}] &= (c_1c_2c_3 - 2c_1^2c_4 + 2c_1c_5 + 2c_2c_4 - 2c_3^2)(\mathcal{U}^\vee).
\end{aligned}$$

We consider two families of Lagrangian subspaces of $\mathrm{LG}(\bigwedge^3 V_6)$, parametrized by $\mathbf{P}(V_6)$ and $\mathsf{Gr}(3, V_6)$. For any $[V_1] \in \mathbf{P}(V_6)$ and $[U_3] \in \mathsf{Gr}(3, V_6)$, we set

$$F_{V_1} \coloneqq V_1 \wedge \textstyle\bigwedge^2 V_6 \quad \text{and} \quad T_{U_3} \coloneqq \bigwedge^2 U_3 \wedge V_6. \tag{24}$$

It is easy to verify that F_{V_1} and T_{U_3} are Lagrangian subspaces of $\bigwedge^3 V_6$. These constructions yield morphisms

$$\begin{aligned}
i_F \colon \mathbf{P}(V_6) &\to \mathrm{LG}(\textstyle\bigwedge^3 V_6) \\
[V_1] &\mapsto [F_{V_1}]
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
i_T \colon \mathsf{Gr}(3, V_6) &\to \mathrm{LG}(\textstyle\bigwedge^3 V_6) \\
[U_3] &\mapsto [T_{U_3}].
\end{aligned}$$

For every $[A] \in \mathrm{LG}(\bigwedge^3 V_6)$ and nonnegative integer k, we define degeneracy loci

$$\mathsf{Y}_A^{\geq k} \coloneqq \{[V_1] \in \mathbf{P}(V_6) \mid \dim(A \cap F_{V_1}) \geq k\}, \qquad \mathsf{Y}_A^k \coloneqq \mathsf{Y}_A^{\geq k} \smallsetminus \mathsf{Y}_A^{\geq k+1}; \tag{25}$$

and

$$\mathsf{Z}_A^{\geq k} \coloneqq \{[U_3] \in \mathsf{Gr}(3, V_6) \mid \dim(A \cap T_{U_3}) \geq k\}, \qquad \mathsf{Z}_A^k \coloneqq \mathsf{Z}_A^{\geq k} \smallsetminus \mathsf{Z}_A^{\geq k+1}, \tag{26}$$

which are the preimage of $\mathsf{L}_A^{\geq k}$ via i_F and i_T respectively.

We will study these stratifications for Lagrangian subspaces A with no decomposable vectors.

Definition 2.2. A *decomposable vector* is a nonzero element of $\bigwedge^3 V_6$ of the form $v_1 \wedge v_2 \wedge v_3$, for some vectors $v_1, v_2, v_3 \in V_6$.

The space of decomposable vectors in $\mathbf{P}(\bigwedge^3 V_6)$ is equal to $\mathsf{Gr}(3, V_6) \subset \mathbf{P}(\bigwedge^3 V_6)$, where the inclusion is given by the Plücker embedding.

O'Grady showed in [**O'G12**, Proposition 3.1] that

$$\Sigma \coloneqq \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid A \text{ contains a decomposable vector}\} \tag{27}$$

is an irreducible divisor of $\mathrm{LG}(\bigwedge^3 V_6)$.

2.2. Double EPW sextics

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace. The loci

$$\mathbf{P}(V_6) = \mathsf{Y}_A^{\geq 0} \supset \mathsf{Y}_A^{\geq 1} \supset \mathsf{Y}_A^{\geq 2} \supset \mathsf{Y}_A^{\geq 3} \supset \mathsf{Y}_A^{\geq 4}$$

defined in (25) were studied by O'Grady in [**O'G06**, **O'G08**, **O'G12**, **O'G13a**, **O'G15**].

Theorem 2.3 (O'Grady)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Then,*

- $\mathsf{Y}_A^{\geq 1}$ *is an integral normal sextic hypersurface;*
- $\mathsf{Y}_A^{\geq 2} = \mathrm{Sing}(\mathsf{Y}_A^{\geq 1})$ *is an integral normal surface of degree* 40*;*
- $\mathsf{Y}_A^{\geq 3} = \mathrm{Sing}(\mathsf{Y}_A^{\geq 2})$ *is finite and smooth, and is empty for general A;*
- $\mathsf{Y}_A^{\geq 4}$ *is empty.*

By [**O'G13a**, Proposition 2.2], the subspace

$$\Delta \coloneqq \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \mathsf{Y}_A^{\geq 3} \neq \varnothing\} \tag{28}$$

is an irreducible divisor of $\mathrm{LG}(\bigwedge^3 V_6)$, distinct from Σ.

Moreover, for any $A \in \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma$, there exists a double cover

$$f_A \colon \widetilde{\mathsf{Y}}_A \to \mathsf{Y}_A^{\geq 1}$$

branched over $\mathsf{Y}_A^{\geq 2}$, called a *double EPW sextic*. The following theorem is proven in [**O'G06**, Theorem 1.1] and [**O'G13a**, Theorem 4.15].

Theorem 2.4. *For $A \in \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Delta)$, the scheme $\widetilde{\mathsf{Y}}_A$ is a hyper-Kähler manifold of $K3^{[2]}$ deformation type. The hyperplane class $h \coloneqq f_A^{-1}(\mathcal{O}_{\mathsf{Y}_A^{\geq 1}}(1))$ is ample of square* 2 *and divisibility* 1*.*

For $A \in \Delta \smallsetminus \Sigma$ general, there exists a degree 10 K3 surface S_A and a small contraction

$$S_A^{[2]} \longrightarrow \widetilde{\mathsf{Y}}_A$$

contracting a $\mathbf{P}^2$ onto each point of $f_A^{-1}(\mathsf{Y}_A^{\geq 3})$.

The construction of the double EPW sextic can be done in families and gives a family

$$\widetilde{\mathscr{Y}} \longrightarrow \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Delta)$$

of polarized hyper-Kähler manifolds of $\mathrm{K3}^{[2]}$ deformation type with polarization of square 2 and divisibility 1, whose fiber over $[A]$ is the double EPW sextic $\widetilde{\mathsf{Y}}_A$.

This family induces a moduli morphism

$$\mu_2 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Delta) \longrightarrow {}^{[2]}\mathscr{M}_2^{(1)} \tag{29}$$

whose image is a dense open subset of the irreducible moduli space ${}^{[2]}\mathscr{M}_2^{(1)}$, defined in Theorem 1.7.

2.2.1. Dual EPW sextic. Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace. Its orthogonal

$$A^\perp := \{\phi \in \textstyle\bigwedge^3 V_6^\vee \mid \forall \omega \in A \quad \phi(\omega) = 0\}$$

is a Lagrangian subspace of $\bigwedge^3 V_6^\vee$, called the *dual* of A. This construction induces an isomorphism $\mathrm{LG}(\bigwedge^3 V_6) \simeq \mathrm{LG}(\bigwedge^3 V_6^\vee)$ which preserves the divisors of Lagrangians containing decomposable vectors.

The loci $\mathsf{Y}_{A^\perp}^{\geq k} \subset \mathbf{P}(V_6^\vee)$ obtained from the EPW sextic stratification for $A^\perp \in \mathrm{LG}(\bigwedge^3 V_6^\vee)$ can be described in terms of A as

$$\mathsf{Y}_{A^\perp}^{\geq k} = \{[V_5] \in \mathbf{P}(V_6^\vee) \mid \dim(A \cap \textstyle\bigwedge^3 V_5) \geq k\},$$

and the sextic $\mathsf{Y}_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6)$ is the projective dual of $\mathsf{Y}_A^{\geq 1} \subset \mathbf{P}(V_6^\vee)$. We will call *dual EPW sextic* the double cover $f_{A^\perp} \colon \widetilde{\mathsf{Y}}_{A^\perp} \to \mathsf{Y}_{A^\perp}^{\geq 1}$. This duality induces a rational involution on the moduli space ${}^{[2]}\mathscr{M}_2^{(1)}$.

2.3. The EPW quartic stratification

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace. The aim of this section is to prove some properties of the loci

$$\mathsf{Gr}(3, V_6) = \mathsf{Z}_A^{\geq 0} \supset \mathsf{Z}_A^{\geq 1} \supset \mathsf{Z}_A^{\geq 2} \supset \mathsf{Z}_A^{\geq 3} \supset \mathsf{Z}_A^{\geq 4} \supset \mathsf{Z}_A^{\geq 5}$$

defined in (26), following Iliev–Kapustka–Kapustka–Ranestad in [**IKKR19**].

Theorem 2.5. *If a Lagrangian subspace $A \subset \bigwedge^3 V_6$ has no decomposable vectors, the following properties hold.*

(a) $\mathsf{Z}_A^{\geq 1}$ *is an integral normal hypersurface in* $\mathsf{Gr}(3, V_6)$ *of degree* 168.
(b) $\mathsf{Z}_A^{\geq 2}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 1}$*; it is an integral normal Cohen–Macaulay sixfold of degree* 480.
(c) $\mathsf{Z}_A^{\geq 3}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 2}$*; it is a normal Cohen–Macaulay threefold of degree* 720.
(d) $\mathsf{Z}_A^{\geq 4}$ *is the singular locus of* $\mathsf{Z}_A^{\geq 3}$*; it is finite and smooth, and is empty for A general.*
(e) $\mathsf{Z}_A^{\geq 5}$ *is empty.*

The degrees are computed with respect to the Plücker embedding of $\mathsf{Gr}(3, V_6)$ in $\mathbf{P}(\bigwedge^3 V_6)$ (the Grassmannian itself has degree 42). This theorem is stated in [**DK20a**, Theorem 5.6] (but the degrees are wrong). The proof of Items (a), (b), and (c) is in [**IKKR19**, Proposition 2.6 and Corollary 2.10] while the proof of Items (d) and (e) is in [**Riz25b**, Section 2].

In Corollary 4.20, we will complete this result by showing that, under the hypotheses of the theorem, the locus $\mathsf{Z}_A^{\geq 3}$ is integral.

We fix some notation for what follows. For any point $[U_0]$ of $\mathsf{Gr}(3, V_6)$, the projective tangent space to $\mathsf{Gr}(3, V_6)$ at $[U_0]$ is $\mathbf{P}(T_{U_0}) \subset \mathbf{P}(\bigwedge^3 V_6)$ (see (24) for the definition of T_{U_0}). If we choose a subspace $U_\infty \subset V_6$ complementary to U_0, then T_{U_0} decomposes as $\bigwedge^3 U_0 \oplus \mathrm{Hom}(U_0, U_\infty)$ and the Zariski tangent space to $\mathsf{Gr}(3, V_6)$ at $[U_0]$ is isomorphic to $\mathrm{Hom}(U_0, U_\infty)$.

We also set

$$\mathcal{C}_{U_0} := \mathbf{P}(T_{U_0}) \cap \mathsf{Gr}(3, V_6), \tag{30}$$

the intersection of $\mathbf{P}(T_{U_0})$ with the space of decomposable vectors of $\bigwedge^3 V_6$. This is a 5-dimensional cone with vertex $[\bigwedge^3 U_0]$ over the smooth Segre fourfold

$$\mathscr{M}_1 \simeq \mathbf{P}(U_0^\vee) \times \mathbf{P}(U_\infty) \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$$

of morphisms of rank 1.

We start with some preliminary results. The proof of Theorem 2.5 will be done in Section 2.3.4.

2.3.1. Tangent space to $\mathsf{Gr}(3, V_6)$. In this section, we study the morphism

$$i_T \colon \mathsf{Gr}(3, V_6) \longrightarrow \mathrm{LG}(\textstyle\bigwedge^3 V_6)$$

that takes a point $[U]$ to the Lagrangian subspace $[T_U] \subset \bigwedge^3 V_6$.

Proposition 2.6. *The tangent map to i_T at a point $[U_0]$ gives an identification*

$$\Theta \colon T_{\mathsf{Gr}(3,V_6),[U_0]} \xrightarrow{\sim} H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2))$$

of 9-dimensional vector spaces, where $I_{\mathcal{C}_{U_0}}$ is the ideal sheaf of the subscheme $\mathcal{C}_{U_0}$ of $\mathbf{P}(T_{U_0})$.

Proof. Consider the affine open subsets

$$\mathcal{U} := \{[U] \in \mathsf{Gr}(3, V_6) \mid U \cap U_\infty = 0\} \subset \mathsf{Gr}(3, V_6),$$

neighborhood of $[U_0]$, and

$$\mathcal{V} := \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid A \cap T_{U_\infty} = 0\} \subset \mathrm{LG}(\textstyle\bigwedge^3 V_6).$$

We get identifications $\mathrm{Hom}(U_0, U_\infty) \simeq \mathcal{U}$ and $\mathrm{Sym}^2 T_{U_0}^\vee \simeq \mathcal{V}$ by sending a morphism to its graph and a quadratic form on T_{U_0}, viewed as a symmetric morphism $T_{U_0} \to T_{U_0}^\vee$, to the graph of the composition of this morphism with the inverse of the isomorphism $T_{U_\infty} \xrightarrow{\sim} T_{U_0}^\vee$ given by the symplectic form.

The morphism $i_T|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{V}$ then corresponds to a morphism

$$\begin{aligned} \mathrm{Hom}(U_0, U_\infty) &\longrightarrow \mathrm{Sym}^2(T_{U_0}^\vee) \\ [U] &\longmapsto [q_U], \end{aligned}$$

where q_U is described in [**IKKR19**, Lemma 2.7] as a polynomial in the entries of the matrix $(b_{ij})_{1 \leq i,j \leq 3}$ representing U in fixed bases of U_0 and U_∞. If we write an

element of $T_{U_0} = \bigwedge^3 U_0 \oplus \mathrm{Hom}(U_0, U_\infty)$ as (m, M), this description says that the part of $q_U(m, M)$ which is linear in the b_{ij} is given by $\sum_{ij} b_{i,j} M^{i,j}$, where $M^{i,j}$ is $(-1)^{i+j}$ times the determinant of the 2×2 submatrix of M in which we remove the column i and the row j.

Therefore, the image of the tangent space $T_{\mathcal{U},[U_0]}$ by Θ is generated by the cofactors $M^{i,j}$, namely, this image is, in $\mathrm{Sym}^2(T_{U_0}^\vee) = H^0(\mathbf{P}(T_{U_0}), \mathcal{O}(2))$, the linear subsystem of quadrics in $\mathbf{P}(T_{U_0})$ that vanish on the set of matrices of rank 1. □

2.3.2. The restriction map r_K. Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace and let $[U_0]$ be a point of Z_A^k, so that $K \coloneqq A \cap T_{U_0}$ has dimension k. The restriction map

$$r_K \colon H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2)) \longrightarrow H^0(\mathbf{P}(K), \mathcal{O}(2)) \simeq \mathrm{Sym}^2 K^\vee \tag{31}$$

will play a very important role in the local description of $\mathsf{Z}_A^{\geq k}$.

Proposition 2.7 ([Riz25b, Proposition 2.3])**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $[U_0]$ be a point of Z_A^k, with $k \in \{1, 2, 3, 4\}$. With the notation above,*

(a) *when $k \in \{1, 2, 3\}$, the map r_K is surjective with kernel of dimension $9 - k(k+1)/2$,*
(b) *when $k = 4$, the map r_K is injective with image a general hyperplane in $H^0(\mathbf{P}(K), \mathcal{O}(2))$.*

The meaning of "general" in Item (b) is the following: a hyperplane in the vector space $H^0(\mathbf{P}(K), \mathcal{O}(2)) \simeq \mathrm{Sym}^2 K^\vee$ is defined by a nonzero element of $\mathrm{Sym}^2 K$ and we want the corresponding tensor to be of maximal rank $k = 4$.

The proof will show that the proposition is true more generally when the linear subspace $K \subset T_{U_0}$ of dimension $k \leq 4$ has no decomposable vectors. Item (a) was already proven in [IKKR19, Lemma 2.8], but we give here a different proof.

Proof. Recall that $\mathcal{C}_{U_0}$ is the cone with vertex $\mathbf{v} \coloneqq [\bigwedge^3 U_0]$ over the smooth fourfold $\mathscr{M}_1 \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ of morphisms of rank 1. Since the latter is not contained in any hyperplane, so is $\mathcal{C}_{U_0}$, hence its projective Zariski tangent space at $\mathbf{v}$ is the whole space $\mathbf{P}^9$. Any quadric containing $\mathcal{C}_{U_0}$ is therefore a cone with vertex $\mathbf{v}$.

Since A contains no decomposable vectors, $\mathbf{P}(K)$ is disjoint from $\mathcal{C}_{U_0}$. In particular, $\mathbf{v} \notin \mathbf{P}(K)$, so we may, upon projecting from $\mathbf{v}$, consider $\mathbf{P}(K)$ as a linear subspace of $\mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ (as above, $U_\infty \subset V_6$ is a subspace complementary to U_0). All in all, this proves that r_K can be identified with the restriction morphism

$$r'_K \colon H^0(\mathbf{P}(\mathrm{Hom}(U_0, U_\infty)), I_{\mathscr{M}_1}(2)) \longrightarrow H^0(\mathbf{P}(K), \mathcal{O}(2)),$$

where $\mathbf{P}(K) \cap \mathscr{M}_1 = \varnothing$.

We start by showing that r'_K is injective when $k = 4$. Suppose by contradiction that there exists a nonzero element in the kernel of r'_K. Its zero-locus $Q \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ is a quadric that contains the disjoint $\mathbf{P}(K) = \mathbf{P}^3$ and $\mathscr{M}_1$. This contradicts the following lemma and proves the injectivity of r'_K.

Lemma 2.8. *Let $\mathscr{M}_1 \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty)) = \mathbf{P}^8$ be the smooth Segre fourfold of morphisms of rank 1 and let $Q_0 \subset \mathbf{P}^8$ be a quadric containing $\mathscr{M}_1$. Any $\mathbf{P}^3 \subset Q_0$ meets $\mathscr{M}_1$.*

Proof. The space of quadrics in $\mathbf{P}^8$ containing $\mathscr{M}_1$ is the $\mathbf{P}^8 = \mathbf{P}(\bigwedge^2 U_0 \otimes \bigwedge^2 U_\infty^\vee)$ spanned by the 9 maximal minors of elements of $\mathrm{Hom}(U_0, U_\infty)$. The group $G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ acts on it, and under the isomorphisms $\bigwedge^2 U_0 \simeq U_0^\vee$ and $\bigwedge^2 U_\infty \simeq U_\infty^\vee$, the action of G on $\mathbf{P}^8$ corresponds to the action of $\mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ on $\mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$. Therefore, there are only three orbits, given by:

(a) quadrics of (maximal) rank 9, represented for example by $M^{1,1}+M^{2,2}+M^{3,3}$;
(b) quadrics of rank 6, represented for example by $M^{2,2} + M^{3,3}$;
(c) quadrics of rank 4, represented for example by $M^{3,3}$,

where $M^{i,j}$ is the (i,j)-minor of an element $M = (x_{ij})_{1\leq i,j\leq 3}$ of $\mathrm{Hom}(U_0, U_\infty)$ (in each case, it is enough to find one quadric containing $\mathscr{M}_1$ of the correct rank).

Since the action of $\mathrm{GL}(U_0)\times\mathrm{GL}(U_\infty)$ fixes $\mathscr{M}_1$, we can suppose that the quadric Q_0 is of one in the previous list.

In case (a), the Lefschetz hyperplane theorem implies that for each $i \leq 3$, one has

$$H^{2i}(Q_0, \mathbf{Z}) \simeq H^{2i}(\mathbf{P}^8, \mathbf{Z}) = \mathbf{Z}h^i,$$

where $h \subset \mathbf{P}^8$ is a hyperplane section. Therefore, one can write $[\mathscr{M}_1] = mh^3|_{Q_0} \in H^6(Q_0, \mathbf{Z})$ for some $m \in \mathbf{Z}_{>0}$. Thus, for each $\mathbf{P}^3 \subset Q_0$, we have

$$[\mathbf{P}^3] \cdot [\mathscr{M}_1] = m(h|_{\mathbf{P}^3})^3 = m \neq 0.$$

Hence, each $\mathbf{P}^3 \subset Q_0$ meets $\mathscr{M}_1$.

In case (b), the quadric

$$Q_0 = M^{2,2} + M^{3,3} = x_{11}(x_{22} + x_{33}) - x_{12}x_{21} - x_{13}x_{31}$$

has rank 6, hence is a cone over a smooth quadric

$$Q_1 \subset \mathbf{P}^5 = \mathbf{P}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & 0 \end{pmatrix}$$

with vertex

$$\mathbf{P}^2 = (x_{11} = x_{22} + x_{33} = x_{12} = x_{21} = x_{13} = x_{31} = 0).$$

One sees that $\mathscr{M}_1 \cap \mathbf{P}^2$ is the smooth conic $x_{22}^2 + x_{23}x_{32} = 0$ and that $\mathscr{M}_1 \cap Q_1$ contains the planes

$$P_1 = \mathbf{P}(x_{11}, x_{12}, x_{13}) \quad \text{and} \quad P_2 = \mathbf{P}(x_{11}, x_{21}, x_{31})$$

that intersect at one point.

The Hilbert scheme $F_2(Q_1)$ of planes contained in Q_1 has 2 connected components, and two planes are in different components if and only if they intersect at one point (see [**GH78**, p. 35]). Therefore, P_1 and P_2 are in two different components of $F_2(Q_1)$.

Consider now our $\mathbf{P}^3 \subset Q_0$ disjoint from $\mathscr{M}_1$. Its intersection with the vertex $\mathbf{P}^2$ does not meet the conic $\mathscr{M}_1 \cap \mathbf{P}^2$ hence is finite. Since the smooth quadric Q_1 contains no 3 dimensional linear spaces, the projection of $\mathbf{P}^3$ from the vertex $\mathbf{P}^2$ to $\mathbf{P}^5$, which is contained in Q_1, is a plane contained in Q_1. Therefore, it intersects nontrivially the one P_i which is not in the same component of $F_2(Q_1)$. Since P_i is contained in $\mathscr{M}_1$, we obtain that $\mathbf{P}^3$ meets $\mathscr{M}_1$.

In case (c), the quadric

$$Q_0 = M^{3,3} = x_{11}x_{22} - x_{12}x_{21}$$

is a cone over a smooth quadric $Q_1 \subset \mathbf{P}^3(x_{11}, x_{12}, x_{21}, x_{22})$ with vertex $\mathbf{P}^4$. Hence, the projection from the vertex $\mathbf{P}^4$ to $\mathbf{P}^3$ of any linear subspace $\mathbf{P}^3 \subset Q_0$ is contained in a line on Q_1, say defined by $x_{12} = x_{22} = 0$. The preimage in $\mathbf{P}^8$ of this line is

$$\mathbf{P}^6 = \mathbf{P}\begin{pmatrix} x_{11} & 0 & x_{13} \\ x_{21} & 0 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

whose intersection with $\mathscr{M}_1$ has dimension 3. Therefore, $\mathbf{P}^3 \subset \mathbf{P}^6 \subset Q_0$ must meet $\mathscr{M}_1$. □

To finish the proof of the proposition, we need to prove that r'_K is surjective when $k \leq 3$, and that the image of r'_K is a general hyperplane when $k = 4$. This follows from the next lemma. □

Lemma 2.9. *In the above setting, the following holds.*

(i) *When $k \leq 3$, the morphism r'_K is surjective;*
(ii) *when $k = 4$, the point $\mathbf{p} \in \mathbf{P}(\mathrm{Sym}^2 K)$ defined by the hyperplane $\mathrm{Im}(r'_K) \subset H^0(\mathbf{P}(K), \mathcal{O}(2))$ has maximal rank.*

Proof. We follow the proof of [**O'G13a**, Proposition 2.5] and consider the rational morphism

$$\begin{array}{ccc} \Phi\colon \mathbf{P}^8 \simeq \mathbf{P}(\mathrm{Hom}(U_0, U_\infty)) & \dashrightarrow & |I_{\mathscr{M}_1}(2)|^\vee \simeq \mathbf{P}^8 \\ M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} & \longmapsto & \begin{pmatrix} M^{1,1} & M^{1,2} & M^{1,3} \\ M^{2,1} & M^{2,2} & M^{2,3} \\ M^{3,1} & M^{3,2} & M^{3,3} \end{pmatrix} \end{array} \tag{32}$$

induced by the linear system $|I_{\mathscr{M}_1}(2)| \subset H^0(\mathbf{P}^8, \mathcal{O}(2))$.

It is classical that

$$\Phi(M)^{i,j} = \det(M) \cdot x_{i,j}.$$

It follows that Φ is a birational involution which is regular on $\mathbf{P}^8 \smallsetminus \mathscr{M}_1$ and an isomorphism on $\mathbf{P}^8 \smallsetminus \mathscr{M}_2$, while $\mathscr{M}_2 \smallsetminus \mathscr{M}_1$ is contracted onto $\mathscr{M}_1$ (where $\mathscr{M}_2 \subset \mathbf{P}^8$ is the cubic hypersurface of noninvertible morphisms).

We show that Φ restricted to $\mathbf{P}(K) \subset \mathbf{P}^8 \smallsetminus \mathscr{M}_1$ is an embedding.

First, we note that the fibers of $\Phi\colon \mathscr{M}_2 \smallsetminus \mathscr{M}_1 \to \mathscr{M}_1$ are complements of quadric surfaces in their span $\mathbf{P}^3$. Indeed, the group $\mathrm{GL}(U_0) \times \mathrm{GL}(U_\infty)$ acts on $I_{\mathscr{M}_1}(2)$ (see the proof of Theorem 2.5) and transitively on $\mathscr{M}_1$. Therefore, all fibers of Φ are conjugate under this action. Now, the preimage of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ via $\Phi\colon \mathscr{M}_2 \smallsetminus \mathscr{M}_1 \to \mathscr{M}_1$ is, in the set of matrices M all of whose minors but $M^{1,1}$ are 0, which is equal to

$$\mathbf{P}^3 = \mathbf{P}\begin{pmatrix} 0 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix},$$

the set of those of rank exactly 2. It is defined in this $\mathbf{P}^3$ by $x_{22}x_{33} - x_{23}x_{32} \neq 0$.

The morphism $\Phi|_{\mathbf{P}(K)}$ is an embedding outside $\mathscr{M}_2$ and, at points of $\mathscr{M}_2 \smallsetminus \mathscr{M}_1$, its fiber is the linear space $\mathbf{P}(K) \cap \mathbf{P}^3$, which must be contained in the complement of a quadric surface, so it is also one reduced point. This proves our claim that the morphism $\Phi|_{\mathbf{P}(K)}$, which is defined by the linear system $\mathrm{Im}(r'_K) \subset \mathrm{Sym}^2 K^\vee$, is an embedding.

But this restriction can be identified with the composition

$$\mathbf{P}(K) \xhookrightarrow{\nu_2} \mathbf{P}(\mathrm{Sym}^2 K) \overset{h}{\dashrightarrow} \mathbf{P}(\mathrm{Im}(r'_K)^\vee),$$

where ν_2 is the second Veronese embedding and the rational map h is the projection corresponding to the inclusion $\mathrm{Im}(r'_K) \subset \mathrm{Sym}^2 K^\vee$.

The fact that it is an embedding means that the kernel of the projection does not meet the second secant variety of $\nu_2(\mathbf{P}(K))$. This shows immediately that, if $k \leq 2$, the map r'_K is surjective. When $k = 3$, this leaves two possibilities

(a) h is the projection from a point $\mathbf{p} \in \mathrm{Sym}^2 K$ of (maximal) rank 3;
(b) r'_K is surjective.

When $k = 4$, the map h is the projection from the point $\mathbf{p}$ of $\mathbf{P}(\mathrm{Sym}^2 K)$ defined by the hyperplane $\mathrm{Im}(r'_K) \subset \mathrm{Sym}^2 K^\vee$, so we only have two possibilities:

(a) $\mathrm{rank}(\mathbf{p}) = 3$;
(b) $\mathrm{rank}(\mathbf{p}) = 4$ (the hyperplane $\mathrm{Im}(r'_K)$ is general).

We show that, when k is 3 or 4, case (a) cannot happen, thereby finishing the proof of the lemma.

Since any rank 3 symmetric matrix can be written as the sum of three rank 1 symmetric matrices, this case happens if and only if there exist three distinct points $x, y, z \in \mathbf{P}(K)$ such that $\mathbf{p} = \nu_2(x) + \nu_2(y) + \nu_2(z)$, or equivalently, such that the points $\Phi(x)$, $\Phi(y)$, and $\Phi(z)$ are distinct and lie on a line ℓ.

We show that in this case, we have $\ell = \Phi(C)$ for some curve C contained in $\mathbf{P}(K)$. This leads to a contradiction, since Φ is a regular embedding on $\mathbf{P}(K)$ which is defined by quadrics, hence the image of any curve in $\mathbf{P}(K)$ has even degree.

We first consider the case $\ell \subset \mathscr{M}_1$. Note that every line in $\mathscr{M}_1$ is contained in a $\mathbf{P}^2 \subset \mathscr{M}_1$ that is given by matrices with the same 2-dimensional kernel or the same 1-dimensional image. Up to change of basis and transposition, we can suppose $\mathbf{P}^2 = \mathbf{P}\begin{pmatrix} y_1 & 0 & 0 \\ y_2 & 0 & 0 \\ y_3 & 0 & 0 \end{pmatrix}$ and $\ell \subset \mathbf{P}^2$ is defined by the equation $y_3 = 0$. Hence, the preimage of this $\mathbf{P}^2$ via $\Phi|_{\mathbf{P}^8 \smallsetminus \mathscr{M}_1}$ is the complement in

$$\mathbf{P}^5 = \mathbf{P}\begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} \subset \mathscr{M}_2$$

of the subvariety $\mathscr{M}_1 \cap \mathbf{P}^5$, and the preimage of ℓ is defined by the quadratic equation $x_{12}x_{23} - x_{22}x_{13} = 0$. Note that $\mathbf{P}(K) \cap \mathbf{P}^5$ is a linear space that contains the three distinct points x, y, z but does not meet the codimension 2 subvariety $\mathscr{M}_1 \cap \mathbf{P}^5$. Therefore, $\mathbf{P}(K) \cap \mathbf{P}^5$ is a line and, since it meets the quadric $\Phi^{-1}(\ell)$ in three distinct points, it is contained in it. In particular, ℓ is the image of this line contained in $\mathbf{P}(K)$, as wanted.

We now consider the case $\ell \not\subset \mathscr{M}_1$. In this case, Φ^{-1} defines a rational morphism on ℓ whose image C is an irreducible curve. Recall that, outside $\mathscr{M}_2$, the composition $\Phi^{-1} \circ \Phi$ is the identity. Moreover, since Φ is an automorphism of $\mathbf{P}^8 \smallsetminus \mathscr{M}_2$, we have inclusions

$$\Phi(\mathbf{P}(K)) \cap \mathscr{M}_2 = \Phi(\mathbf{P}(K) \cap \mathscr{M}_2) \subset \Phi(\mathscr{M}_2 \smallsetminus \mathscr{M}_1) = \mathscr{M}_1.$$

In particular, $\Phi(\mathbf{P}(K))\cap\mathscr{M}_2 = \Phi(\mathbf{P}(K))\cap\mathscr{M}_1$ and $\Phi^{-1}\Phi(w) = w$ for some $w \in \mathbf{P}(K)$ if and only if $\Phi(w) \notin \mathscr{M}_1$. We consider the points $\Phi(x)$, $\Phi(y)$ and $\Phi(z)$ of ℓ and distinguish four cases.

- The three points are all in $\mathscr{M}_1$. This case cannot actually happen: since $\mathscr{M}_1$ is an intersection of quadrics, each line is either contained in $\mathscr{M}_1$ or meets it in at most 2 points.
- None of the three points are in $\mathscr{M}_1$, hence the curve C contains x, y and z. Since the morphism Φ^{-1} is defined by quadrics, C is a line or a conic that meets the linear space $\mathbf{P}(K)$ in x, y, z. Since all conics are planar, it follows that C is contained in $\mathbf{P}(K)$ and, of course, $\ell = \Phi(C)$.
- Exactly one of these points, say $\Phi(x)$, is contained in $\mathscr{M}_1$. Hence, all quadric polynomials that define Φ^{-1} vanish on $\Phi(x)$, therefore the curve C is a line that meets the linear space $\mathbf{P}(K)$ in the two distinct point y and z, thus it is contained in it.
- Exactly two of these points are contained in $\mathscr{M}_1$. In this case, all quadric polynomials that define Φ^{-1} vanish at two points of ℓ, hence Φ is constant on ℓ, which is absurd.

Therefore, $\ell = \Phi(C)$ for some curve C as wanted, and we obtain a contradiction, as explained above. □

2.3.3. Local description of Z^k_A. We are now ready to give a local description of the analytic germ of Z^k_A at a point $[U_0]$ of Z^ℓ_A (with $\ell \geq k$). Actually, we will prove, following [O'G13a, Claim 3.8], a more general result that will be needed later on.

Keeping the notation (24), we define the families

$$\mathscr{Z}^{\geq k} := \{([A],[U]) \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \times \mathsf{Gr}(3,V_6) \mid \dim(A\cap T_U) \geq k\} \tag{33}$$

of degeneracy loci $\mathsf{Z}^{\geq k}_A$ and we define $\mathscr{Z}^{k}$ analogously.

As in [O'G13a, Claim 3.8], we study the local structure of $\mathscr{Z}^{\geq k}$ at a point $([A_0],[U_0])$ of $\mathscr{Z}^{\ell}$ for $\ell \geq k$.

Proposition 2.10. *Let $A_0 \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $([A_0],[U_0])$ be a point of $\mathscr{Z}^\ell$, with $\ell \in \{0,\dots,4\}$. Set $K := A_0 \cap T_{U_0}$ and let $\mathcal{S}_{\leq i} \subset \mathrm{Sym}^2 K^\vee$ be the subscheme of quadratic forms of rank $\leq i$.*

For any $k \in \{0,\dots,\ell\}$, the analytic germ of the scheme $\mathscr{Z}^{\geq k}$ at the point $([A_0],[U_0])$ is isomorphic to the product of a smooth germ $(M,0)$ and the germ $(\mathcal{S}_{\leq \ell-k},0)$.

Under this isomorphism, the germ of $\mathsf{Z}^{\geq k}_{A_0} \subset \mathscr{Z}^{\geq k}$ at $[U_0]$ corresponds to the germ of $M' \times (\mathrm{Im}(r_K)\cap \mathcal{S}_{\leq \ell-k})$ at $(0,0)$, where the morphism r_K is defined in (31) and $(M',0)\subset(M,0)$ is smooth of dimension $\max(0, 9-\ell(\ell+1)/2)$.

Proof. Choose a 3-dimensional subspace U_∞ of V_6 complementary to U_0 and such that T_{U_∞} is complementary to A_0 (and also to T_{U_0}). As in the proof of Proposition 2.6, we consider affine open neighborhoods $\mathcal{U} \subset \mathsf{Gr}(3,V_6)$ of $[U_0]$ and $\mathcal{V} \subset \mathrm{LG}(\bigwedge^3 V_6)$ of $[A_0]$, with identifications $\mathrm{Hom}(U_0,U_\infty) \simeq \mathcal{U}$ and $\mathrm{Sym}^2 T^\vee_{U_0} \simeq \mathcal{V}$.

If $([A],[U]) \in \mathcal{V}\times\mathcal{U}$, we denote by q_U and q_A the quadratic forms on T_{U_0} associated with the Lagrangian subspaces T_U and A (which are elements of $\mathcal{V}$) respectively.

Consider the smooth morphism

$$\Psi : \mathcal{V}\times\mathcal{U} \longrightarrow \mathrm{Sym}^2 T^\vee_{U_0}$$

sending $([A],[U])$ to the quadratic form $q_U - q_A$. It is then easy to check (see for example the beginning of the proof of [IKKR19, Lemma 2.9]) that the intersection $\mathscr{Z}^{\geq k} \cap (\mathcal{V} \times \mathcal{U})$ is equal to $\Psi^{-1}(\Sigma_k)$, where $\Sigma_k \subset \mathrm{Sym}^2 T^\vee_{U_0}$ is the corank k degeneracy locus.

Fix a subspace $J \subset T_{U_0}$ such that $K \oplus J = T_{U_0}$. The quadratic form $\Psi([A_0],[U_0])$ is nondegenerate on J. Consider the vector bundle $\widetilde{K} \to \mathcal{V} \times \mathcal{U}$ whose fiber

$$\widetilde{K}_{([A],[U])} := \{\omega \in T_{U_0} \mid \forall \alpha \in J \quad \Psi([A],[U])(\omega,\alpha) = 0\} \subset T_{U_0}$$

over $([A],[U])$ is the orthogonal complement of J with respect to the quadratic form $\Psi([A],[U])$.

Upon shrinking $\mathcal{V}$ and $\mathcal{U}$, we can suppose that $\widetilde{K}$ is trivial over $\mathcal{V} \times \mathcal{U}$ (with fiber K) and that $\Psi([A],[U])$ is nondegenerate on J for all $[A] \in \mathcal{V}$, $[U] \in \mathcal{U}$. Hence, the corank of $\Psi([A],[U])$ is equal to the corank of the restriction of $\Psi([A],[U])$ to $\widetilde{K}_{([A],[U])} \simeq K$. Therefore, the degeneracy loci of Ψ are equal to the degeneracy loci of the smooth morphism

$$\Psi_K \colon \mathcal{V} \times \mathcal{U} \xrightarrow{\Psi} \mathrm{Sym}^2 T^\vee_{U_0} \xrightarrow{\pi_K} \mathrm{Sym}^2 K^\vee,$$

where π_K is the restriction morphism.

In particular, $\mathscr{Z}^{\geq k} \cap (\mathcal{V} \times \mathcal{U}) = \Psi_K^{-1}(\mathcal{S}_{\leq \ell - k})$. This implies the first claim. Analogously, the scheme $\mathsf{Z}^{\geq k}_{A_0}$ is the preimage of $\mathcal{S}_{\leq \ell - k}$ via the morphism

$$\Psi_K([A_0],\cdot) \colon \mathcal{U} \longrightarrow \mathrm{Sym}^2 K^\vee.$$

Note that the morphism $\Psi([A_0],\cdot) \colon \mathcal{U} \to \mathrm{Sym}^2 T^\vee_{U_0}$ is induced by the morphism i_T from Section 2.3.1, whose tangent map induces the isomorphism Θ from Proposition 2.6. Therefore, the tangent map to $\Psi_K([A_0],\cdot)$ at $[U_0]$ is the morphism r_K of (31). Hence, analytically locally around $[U_0] \in \mathsf{Z}^\ell_{A_0}$, the image of the subscheme $\mathsf{Z}^{\geq k}_{A_0} \subset \mathscr{Z}^{\geq k}$ can be identified with $\mathrm{Im}(r_K) \cap \mathcal{S}_{\leq \ell - k} \subset \mathcal{S}_{\leq \ell - k}$.

This proves that the germ of $\mathsf{Z}^{\geq k}_{A_0}$ at $[U_0]$ is the product of the germ $\mathrm{Im}(r_K) \cap \mathcal{S}_{\leq \ell - k}$ at 0 with a smooth germ whose dimension can be computed using the fact, proved in Proposition 2.7, that, since A contains no decomposable vectors, r_K has maximal rank. □

The case $k = \ell$ gives the following corollary.

Corollary 2.11. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $[U_0] \in \mathsf{Z}^k_A$. The scheme Z^k_A is smooth at $[U_0]$, of dimension $9 - \frac{k(k+1)}{2}$ when $k \in \{0,1,2,3\}$, and of dimension 0 when $k = 4$.*

Finally, we show that for $A \in \mathrm{LG}(\bigwedge^3 V_6)$ general the scheme $\mathsf{Z}^{\geq 4}_A$ is empty.

Proposition 2.12 ([Riz25b, Proposition 3.1])**.** *The locus*

$$\Gamma := \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \mathsf{Z}^{\geq 4}_A \neq \varnothing\} \tag{34}$$

is an irreducible divisor of $\mathrm{LG}(\bigwedge^3 V_6)$.

Proof. Consider the closed subvariety

$$\widetilde{\Gamma} := \{([A],[U],[B]) \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \times \mathsf{Gr}(3,V_6) \times \mathsf{Gr}(4,\bigwedge^3 V_6) \mid B \subset A \cap T_U\},$$

where T_U was defined in (24), and the chain of morphisms

$$\begin{array}{rccccc} & & \widetilde{\Gamma} & \xrightarrow{q} & \mathsf{Gr}(4, T_{\mathcal{U}}) & \longrightarrow & \mathsf{Gr}(3, V_6) \\ ([A],[U],[B]) & \longmapsto & ([U],[B]) & \longmapsto & [U]. & & \end{array}$$

where $\mathsf{Gr}(4, T_{\mathcal{U}})$ is the $\mathsf{Gr}(4,10)$-bundle over $\mathsf{Gr}(3, V_6)$ whose fiber at $[U]$ is equal to $\mathsf{Gr}(4, T_U)$.

The fiber of q over $B \subset T_U$ is isomorphic to the set of Lagrangian subspaces A such that $B \subset A$. Since $B \subset T_U$ is totally isotropic, this fiber is isomorphic to the Lagrangian Grassmannian $\mathrm{LG}(B^\perp/B)$, smooth irreducible of dimension 21.

Since $\mathsf{Gr}(4, T_{\mathcal{U}})$ is smooth irreducible of dimension 33, it follows that $\widetilde{\Gamma}$ is also smooth irreducible, of dimension 54. Projecting $\widetilde{\Gamma}$ to $\mathrm{LG}(\bigwedge^3 V_6)$, we obtain that its image Γ is also irreducible. □

2.3.4. Proof of Theorem 2.5. We now have all the tools for proving Theorem 2.5.

We start by showing that, if A contains no decomposable vectors, $\mathsf{Z}_A^{\geq 5}$ is empty. Assume $[U_0]$ is in $\mathsf{Z}_A^{\geq 5}$, so that $A \cap T_{U_0}$ has dimension at least 5. Then, $\mathbf{P}(A \cap T_{U_0})$ has dimension at least 4, hence must meet the fivefold $\mathcal{C}_{U_0}$ in $\mathbf{P}(T_{U_0}) = \mathbf{P}^9$, which contradicts the fact that A has no decomposable vectors. Therefore, $\mathsf{Z}_A^{\geq 5}$ is empty.

The rest of the theorem follows from the tools we developed above. Recall from Section 2.1 that the schemes $\mathsf{Z}_A^{\geq k}$ are the intersection of $\mathsf{L}_A^{\geq k}$ with the image of the morphisms $i_T\colon \mathsf{Gr}(3, V_6) \to \mathrm{LG}(\bigwedge^3 V_6)$ studied in Section 2.3.1. Since i_T is a local isomorphism onto its image by Proposition 2.6, the loci Z_A^k are nonempty for $k \in \{1,2,3\}$. Moreover, Proposition 2.10 and Corollary 2.11 show that $\mathsf{Z}_A^{\geq k}$ has the expected dimension and that its singular locus is equal to $\mathsf{Z}_A^{\geq k+1}$. Since $\mathsf{Z}_A^{\geq k}$ has the expected codimension, it is then a locally complete intersection inside the Cohen-Macaulay scheme $\mathsf{L}_A^{\geq k}$, therefore $\mathsf{Z}_A^{\geq k}$ is also Cohen-Macaulay. Moreover, being Cohen-Macaulay and nonsingular in codimension 1, the scheme $\mathsf{Z}_A^{\geq k}$ is also normal and reduced.

To conclude the proof of Items (a)–(c), we need to show that $\mathsf{Z}_A^{\geq k}$ is integral (for $k \in \{1,2\}$) and compute its degree. Pulling back to $\mathsf{Gr}(3, V_6)$ the formulas of Remark 2.1, one obtains

$$\begin{aligned} [\mathsf{Z}_A^{\geq 1}] &= c_1(T^\vee), \\ [\mathsf{Z}_A^{\geq 1}] &= (c_2c_1 - 2c_3)(T^\vee), \\ [\mathsf{Z}_A^{\geq 1}] &= (c_1c_2c_3 - 2c_1^2c_4 + 2c_1c_5 + 2c_2c_4 - 2c_3^2)(T^\vee), \end{aligned}$$

where $T \to \mathsf{Gr}(3, V_6)$ is the vector bundle with fiber T_U over $[U] \in \mathsf{Gr}(3, V_6)$. Since $\mathbf{P}(T)$ is the projective tangent bundle of $\mathsf{Gr}(3, V_6)$, its dual is an extension of $\mathcal{O}_{\mathsf{Gr}(3,V_6)}(1)$ by $\Omega_{\mathsf{Gr}(3,V_6)}(1)$.

Remark 2.13. By expanding the formulas above with the aid of Macaulay2, we obtain explicit expressions for the classes $[\mathsf{Z}_A^{\geq k}]$ in the Chow ring of $\mathsf{Gr}(3, V_6)$ in terms of the Schubert basis $(\sigma_{a,b,c})_{3\geq a\geq b\geq c\geq 0}$. In particular, we have the following equalities

$$\begin{aligned} [\mathsf{Z}_A^{\geq 1}] &= 4\sigma_1, \\ [\mathsf{Z}_A^{\geq 2}] &= 16\sigma_1^3 - 12\sigma_1\sigma_2 + 12\sigma_3, \\ [\mathsf{Z}_A^{\geq 3}] &= 40\sigma_2^3 + 240\sigma_1\sigma_2\sigma_3 - 240\sigma_3^2. \end{aligned}$$

In particular, one can further compute

$$\deg(\mathsf{Z}_A^{\geq 1}) = 168 \quad , \quad \deg(\mathsf{Z}_A^{\geq 2}) = 480 \quad , \quad \deg(\mathsf{Z}_A^{\geq 3}) = 720. \tag{35}$$

This ends the proof of Item (c). In order to conclude the proof of Items (a) and (b), we need to show the integrity of the scheme $\mathsf{Z}_A^{\geq k}$ for $k = 1$ and 2. Since $\mathsf{Z}_A^{\geq k}$ is normal and reduced, it is integral if and only if it is connected.

For Item (a), it is therefore enough to note that any two hypersurfaces in $\mathsf{Gr}(3, V_6)$ meet, therefore $\mathsf{Z}_A^{\geq 1}$ is connected.

For Item (b), we follow the proof of [**IKKR19**, Lemma 3.2]: it is enough to show that the codimension-3 class $[\mathsf{Z}_A^{\geq 2}]$ in the Chow ring of $\mathsf{Gr}(3, V_6)$ does not split as sum of two nonzero classes with trivial intersection. Since $A^3(\mathsf{Gr}(3, V_6))$ is generated by the cycles $\sigma_1^3, \sigma_1\sigma_2$, and σ_3, the above condition is equivalent to asking that the equation

$$(a\sigma_1^3 - b\sigma_1\sigma_2 + c\sigma_3) \cdot ((16-a)\sigma_1^3 - (12-b)\sigma_1\sigma_2 + (12-c)\sigma_3) = 0$$

in $A^6(\mathsf{Gr}(3, V_6))$ admits only trivial integral solutions.

A computation with Macaulay2 shows that this relation is equivalent to the system

$$\begin{cases} -5a^2 + 4ab - b^2 + 56a - 20b = 0, \\ -6a^2 + 8ab - 2b^2 - 4ac + 2bc + 72a - 52b + 20c = 0, \\ 6a^2 - 6ab + b^2 + 2ac - c^2 - 72a + 36b - 4c = 0. \end{cases}$$

The first equation defines an ellipse. The set of its possible integral solutions (a, b) is therefore bounded, hence finite. Substituting all these possible solutions into the remaining two equations, one checks that the only solutions are the trivial ones $(0, 0, 0)$ and $(16, 12, 12)$, which concludes the proof of Item (b).

Finally, Item (d) follows from Proposition 2.12 and Corollary 2.11.

2.4. EPW cubes

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace containing no decomposable vectors. Since $\mathsf{Z}_A^{\geq k}$ is the pullback via i_T of $\mathsf{L}_A^{\geq k} \subset \mathrm{LG}(\bigwedge^3 V_6)$, we obtain that the schemes $\mathsf{Z}_A^{\geq k}$ are the Lagrangian intersection loci of the Lagrangian bundles

$$\mathscr{A}_1 = A \otimes \mathcal{O}_{\mathsf{Gr}(3,V_6)}, \quad \text{and} \quad \mathscr{A}_2 = T$$

over $\mathsf{Gr}(3, V_6)$, (recall that $T \to \mathsf{Gr}(3, V_6)$ is the vector bundle with fiber T_U over $[U] \in \mathsf{Gr}(3, V_6)$). Equivalently, by [**DK20a**, Section 4], they are the degeneracy loci of the morphism

$$\omega \colon \mathscr{A}_1 \hookrightarrow V \otimes \mathcal{O}_{\mathsf{Gr}(3,V_6)} \simeq V^\vee \otimes \mathcal{O}_{\mathsf{Gr}(3,V_6)} \twoheadrightarrow \mathscr{A}_2^\vee.$$

By Section 2.3, the schemes $\mathsf{Z}_A^{\geq k}$ satisfy the hypothesis of [**DK20a**, Section 4.1] for $k \leq 3$. Let $\mathscr{R}_k$ be the natural reflexive sheaf over $\widetilde{\mathsf{Z}}_A^{\geq k}$ defined as

$$\mathscr{R}_k \coloneqq (\textstyle\bigwedge^k \mathcal{C}_k)^{\vee\vee}, \tag{36}$$

where $\mathcal{C}_k$ is the cokernel of the restriction of the map ω to $\mathsf{Z}_A^{\geq k}$.

Theorem 2.14 ([**DK20a**, Theorem 5.7])**.** *Assume that A contains no decomposable vectors. For each $k \leq 3$, there exists a unique double cover $g_A^{(k)} \colon \widetilde{\mathsf{Z}}_A^{\geq k} \to \mathsf{Z}_A^{\geq k}$, branched over $\mathsf{Z}_A^{\geq k+1}$, such that*

$$(g_A^{(k)})_* \mathcal{O}_{\widetilde{\mathsf{Z}}_A^{\geq k}} = \mathcal{O}_{\mathsf{Z}_A^{\geq k}} \oplus \mathscr{R}_k(-2).$$

The scheme $\widetilde{\mathsf{Z}}_A^{\geq k}$ *is integral, normal, and smooth away from* $(g_A^{(k)})^{-1}(\widetilde{\mathsf{Z}}_A^{k+2})$.

When $[A]$ is not in Γ, the double cover $g_A^{(2)}\colon \widetilde{\mathsf{Z}}_A^{\geq 2} \to \mathsf{Z}_A^{\geq 2}$ has a more explicit construction, carried out in [**IKKR19**, Section 3]. It goes as follows.

Consider the variety $\mathsf{Z}'_A := \{([U_3], K) \in \mathsf{Gr}(3, V_6) \times \mathsf{Gr}(2, A) \mid K \subset T_U\}$. The projection to $\mathsf{Gr}(3, V_6)$ gives a morphism

$$\alpha\colon \mathsf{Z}'_A \to \mathsf{Z}_A^{\geq 2}$$

and [**IKKR19**, Lemma 3.3] shows that Z'_A is smooth and α is a resolution of singularities of $\mathsf{Z}_A^{\geq 2}$, whose exceptional locus E is a smooth divisor.

Computations with Chern classes (carried out in [**IKKR19**, Lemma 3.4] and the discussion below) show that the divisor E is even and satisfies $E = 2K_{\mathsf{Z}'_A}$. Therefore, there exists a smooth double cover $f\colon \widetilde{\mathsf{Z}}'_A \to \mathsf{Z}'_A$ branched along E. By the adjunction formula, the canonical divisor of the variety $\widetilde{\mathsf{Z}}'_A$ is equal to $\widetilde{E} := f^{-1}(E)$.

Finally, the composition $\alpha \circ f\colon \widetilde{\mathsf{Z}}'_A \to \mathsf{Z}_A$ contracts the divisor $\widetilde{E}$ to the threefold $\mathsf{Z}_A^{\geq 3}$, and is $2\colon 1$ on $\widetilde{\mathsf{Z}}'_A \smallsetminus \widetilde{E}$. The Stein factorization of $\alpha \circ f$ gives a diagram

$$\begin{array}{ccc} \widetilde{\mathsf{Z}}'_A & \xrightarrow{f} & \mathsf{Z}'_A \\ {\scriptstyle\widetilde{\alpha}}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \widetilde{\mathsf{Z}}_A & \xrightarrow[g]{} & \mathsf{Z}_A^{\geq 2}, \end{array} \tag{37}$$

where $\widetilde{\alpha}$ contracts $\widetilde{E}$ to the threefold $g^{-1}(\mathsf{Z}_A^{\geq 3})$ and g is a double cover. Since $\widetilde{\alpha}$ contracts the canonical divisor $\widetilde{E}$ of $\widetilde{\mathsf{Z}}'_A$, we obtain that $\widetilde{\mathsf{Z}}_A$ has trivial canonical bundle. Moreover, it is a smooth variety by [**IKKR19**, Proposition 3.1], irreducible by Theorem 2.5(b), and the class

$$h = g^*(\sigma_1|_{\mathsf{Z}_A^{\geq 2}}) \tag{38}$$

obtained by pulling back the hyperplane section of $\mathsf{Z}_A^{\geq 2} \subset \mathsf{Gr}(3, V_6)$ is ample, because g is finite.

Proposition 2.15 ([**IKKR19**, Lemma 4.1 and Proposition 4.5])**.** *Let* $[A]$ *be a general point in* $\Delta \smallsetminus (\Gamma \cup \Sigma)$[1] *and let* S_A *be the degree* 10 *K3 surface described in Theorem 2.4. There exist a* $2\colon 1$ *rational map*

$$\psi\colon S_A^{[3]} \dashrightarrow \mathsf{Z}_A^{\geq 2}$$

and a subset $\mathscr{B} \subset S_A^{[3]}$ *of codimension at least* 2 *such that* ψ *induces an (everywhere defined) étale double cover from* $S_A^{[3]} \smallsetminus \mathscr{B}$ *onto its image.*

The birational involution of $S_A^{[3]}$ associated with the $2\colon 1$ map ψ is described geometrically in [**Deb22**, Example 4.12].

Using Proposition 2.15, we can prove that $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type for $[A] \in \Delta \smallsetminus (\Gamma \cup \Sigma)$ general.

[1]This locally closed subset is nonempty of codimension 1 in $\mathrm{LG}(\bigwedge^3 V_6)$ by [**IKKR19**, Lemma 3.7] and [**O'G13a**, Proposition 2.2]. See Proposition 3.5 for another proof.

Lemma 2.16. *Let $[A]$ be a general point in $\Delta \setminus (\Gamma \cup \Sigma)$ and let S_A be the degree 10 K3 surface described in Theorem 2.4. The smooth variety $\widetilde{\mathsf{Z}}_A$ constructed in (37) is birational to the Hilbert scheme $S_A^{[3]}$. In particular, $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler manifold of $K3^{[3]}$ deformation type.*

Proof. Consider the rational map $\psi\colon S_A^{[3]} \dashrightarrow \mathsf{Z}_A^{\geq 2}$ and the subset $\mathscr{B} \subset S_A^{[3]}$ of Proposition 2.15. Let $\mathscr{D} \coloneqq \psi(S_A^{[3]} \setminus \mathscr{B}) \subset \mathsf{Z}_A^{\geq 2}$. Note that $S_A^{[3]} \setminus \mathscr{B}$ is simply connected, because $\mathscr{B}$ has codimension at least 2. Therefore, the étale double cover

$$\psi\colon S_A^{[3]} \setminus \mathscr{B} \to \mathscr{D}$$

is the universal cover. In particular, $\mathscr{D}$ has fundamental group $\mathbf{Z}/2\mathbf{Z}$. Moreover, $\mathscr{D}$ must be smooth like $S_A^{[3]}$, hence it does not meet the singular locus $\mathsf{Z}_A^{\geq 3}$ of $\mathsf{Z}_A^{\geq 2}$. Therefore, the restriction of $g\colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A^{\geq 2}$ to $g^{-1}(\mathscr{D})$ is an étale double cover and $g^{-1}(\mathscr{D})$ is connected, thus it is also the universal covering of $\mathscr{D}$. This implies $S_A^{[3]} \setminus \mathscr{B} \simeq g^{-1}(\mathscr{D})$, which shows that $\widetilde{\mathsf{Z}}_A$ is birationally isomorphic to $S_A^{[3]}$.

By [**Ito02**, Theorem 1.1], birationally isomorphic manifolds with trivial canonical bundles have the same Hodge numbers. We know that the manifold $\widetilde{\mathsf{Z}}_A$ is simply connected, smooth, projective, and has trivial canonical bundle, so its Hodge numbers are those of $S_A^{[3]}$. In particular, $\chi(\widetilde{\mathsf{Z}}_A, \mathcal{O}_{\widetilde{\mathsf{Z}}_A}) = 4$. We use the Decomposition Theorem [**Bea83**, Théorème 2]: since $\widetilde{\mathsf{Z}}_A$ cannot be the product of three K3 surfaces (for which $\chi(\mathcal{O}) = 8$), nor the product of two strict Calabi–Yau threefolds (for which $\chi(\mathcal{O}) = 0$), nor a strict Calabi–Yau sixfold (for which $\chi(\mathcal{O}) = 2$), nor the product of a K3 surface and a hyper-Kähler fourfold (for which $\chi(\mathcal{O}) = 6$), it must be either the product of a K3 surface and a strict Calabi–Yau fourfold, or a hyper-Kähler sixfold. If we pullback to $\widetilde{\mathsf{Z}}_A$ a symplectic form on $S_A^{[3]}$, we get a holomorphic 2-form which is holomorphic outside a closed subset of codimension at least 2 of $\widetilde{\mathsf{Z}}_A$. By Hartogs' theorem, it extends to a holomorphic 2-form on the whole of $\widetilde{\mathsf{Z}}_A$, which is symplectic on a dense open subset. Such a form does not exist on the product of a K3 surface and a strict Calabi–Yau fourfold, because nonzero 2-forms come from the K3 surface, hence have everywhere rank 2.

Therefore, $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler sixfold. Finally, by [**GJH03**, Proposition 27.8], birational hyper-Kähler manifolds are deformation equivalent hence $\widetilde{\mathsf{Z}}_A$ is of $K3^{[3]}$ deformation type. □

Consider the families $\mathscr{Z}^{\geq k} \subset \mathrm{LG}(\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_6)$ constructed in (33), whose intersection with $[A] \times \mathsf{Gr}(3, V_6)$ is $\mathsf{Z}_A^{\geq 2}$, and set

$$\mathscr{Z}_{\mathrm{sm}}^{\geq k} \coloneqq \mathscr{Z}^{\geq k} \cap (\mathrm{LG}(\textstyle\bigwedge^3 V_6) \setminus (\Gamma \cup \Sigma)) \times \mathsf{Gr}(3, V_6).$$

The construction (37) globalizes to $\mathscr{Z}_{\mathrm{sm}}^{\geq 2}$ and gives a double cover

$$\widetilde{g}\colon \widetilde{\mathscr{Z}_A} \longrightarrow \mathscr{Z}_{\mathrm{sm}}^{\geq 2} \tag{39}$$

branched over $\mathscr{Z}_{\mathrm{sm}}^{\geq 3}$. Composing with the projection to $\mathrm{LG}(\bigwedge^3 V_6)$, we obtain a smooth family

$$\widetilde{\mathscr{Z}_A} \longrightarrow \mathrm{LG}(\textstyle\bigwedge^3 V_6) \setminus (\Gamma \cup \Sigma)$$

whose fiber over $[A]$ is the manifold $\widetilde{\mathsf{Z}}_A$. Lemma 2.16 implies the following.

Corollary 2.17 ([IKKR19, Section 5]). *The family* $\widetilde{\mathscr{Z}_A} \to \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma)$ *is a smooth family of polarized hyper-Kähler manifolds of* $K3^{[3]}$ *deformation type, with polarization of Beauville–Bogomolov–Fujiki square* 4.

Proof. Since $\mathscr{Z}_{\mathrm{sm}}^{\geq 4}$ is empty, the construction (37) yields a smooth family $\widetilde{\mathscr{Z}_A}$. The relative polarization is given by the pullback of the divisor $\mathrm{LG}(\bigwedge^3 V_6) \times \Sigma_1$, where $\Sigma_1 \subset \mathsf{Gr}(3, V_6)$ is a hyperplane section. shows that Since small smooth deformations of hyper-Kähler manifolds are hyper-Kähler manifolds, an $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type for $[A]$ general in $\Delta \smallsetminus (\Gamma \cup \Sigma)$ by Lemma 2.16, the same is true for all $[A] \in \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma)$.

Finally, let h be the line bundle obtained by pullback via g of the restriction of the hyperplane section σ_1 of $\mathsf{Gr}(3, V_6)$ to Z_A. Since g is a double cover, h is ample. Using projection formula and the degrees computed in (35), we can compute $h^6 = 2 \cdot \deg(\mathsf{Z}_A) = 960$. Since $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type, it follows from Equation (15) that

$$960 = h^6 = 15 \cdot q_{\widetilde{\mathsf{Z}}_A}(h)^3,$$

hence $q_{\widetilde{\mathsf{Z}}_A}(h) = 4$. □

When $[A]$ is not in the divisor Γ, [DK20a, Lemma 5.8] shows that the double cover $g\colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A^{\geq 2}$ from (37) is equal to the double cover $g_A^{(2)}\colon \widetilde{\mathsf{Z}}_A^{\geq 2} \to \mathsf{Z}_A^{\geq 2}$ constructed in (37). To ease the notation, we will write Z_A for $\mathsf{Z}_A^{\geq 2}$, $\widetilde{\mathsf{Z}}_A$ for $\widetilde{\mathsf{Z}}_A^{\geq 2}$, and

$$g_A\colon \widetilde{\mathsf{Z}}_A \longrightarrow \mathsf{Z}_A \tag{40}$$

for the associated double cover. The hyper-Kähler sixfold $\widetilde{\mathsf{Z}}_A$ (or the double cover g_A) is called an *EPW cube*.

Remark 2.18. We keep the assumption $[A] \notin \Gamma$, so that $\widetilde{\mathsf{Z}}_A$ is smooth and $\mathsf{Z}_A^{\geq 4}$ is empty. Denote by ι_A the involution associated to the double cover g_A, whose fixed locus is the smooth threefold $\mathsf{W}_A \coloneqq g_A^{-1}(\mathsf{Z}_A^{\geq 3})$. Since the fixed locus of ι_A is odd-dimensional, the involution ι_A is antisymplectic (the fixed locus of a symplectic involution is always symplectic, hence even dimensional). In particular, W_A is a Lagrangian submanifold of $\widetilde{\mathsf{Z}}_A$. We will study it in more detail in Sections 2.5.1 and 4.2.

2.4.1. Moduli space of EPW cubes. Consider the coarse moduli space ${}^{[3]}\mathscr{M}_4$ of polarized hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ deformation type, with polarization of Beauville–Bogomolov–Fujiki square 4. By the discussion in Section 1.1.1, a primitive vector in $\Lambda_{\mathrm{K3}^{[3]}}$ of square 4 has divisibility either 1 or 2. Hence the moduli space ${}^{[3]}\mathscr{M}_4$ has two components ${}^{[3]}\mathscr{M}_4^{(1)}$ and ${}^{[3]}\mathscr{M}_4^{(2)}$, where the polarization has divisibility 1 or 2 respectively.

By Corollary 2.17, the family $\widetilde{\mathscr{Z}_A} \to \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma)$ induces a *moduli morphism*

$$\mu_3\colon \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma) \longrightarrow {}^{[3]}\mathscr{M}_4, \tag{41}$$

which sends a Lagrangian $[A] \notin \Gamma \cup \Sigma$ to the point of the moduli space corresponding to the smooth EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ with its canonical polarization.

Proposition 2.19 ([IKKR19, Proposition 5.1]). *The image of* μ_3 *is a dense open subset of a component of* ${}^{[3]}\mathscr{M}_4$ *of dimension* 20.

Together with Corollary 2.17, we can now prove the following result.

Theorem 2.20. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors such that $\widetilde{\mathsf{Z}}_A^{\geq 4} = \varnothing$. The EPW cube $\widetilde{\mathsf{Z}}_A$ constructed in (40) is a hyper-Kähler manifold of $K3^{[3]}$ deformation type and the pullback h of the hyperplane section of $\mathsf{Z}_A^{\geq 2} \subset \mathsf{Gr}(3, V_6)$ defines an ample polarization of Beauville–Bogomolov–Fujiki square 4 and divisibility 2.*

Proof. Corollary 2.17 shows that $\widetilde{\mathsf{Z}}_A$ is a hyper-Kähler manifold of $\mathrm{K3}^{[3]}$ deformation type and the line bundle h is a polarization of Beauville–Bogomolov–Fujiki square 4. Hence, we just need to show that h has divisibility 2. By the above discussion h has divisibility either 1 or 2. Remark 2.18 shows that the involution ι_A associated with the double cover $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$, is antisymplectic. Therefore, ι_A acts as $-\mathrm{id}$ on the transcendental lattice T_A of $\widetilde{\mathsf{Z}}_A$. Since by Proposition 2.19, the family of EPW cubes is a polarized family of hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ deformation type of maximal dimension, it follows that, for the very general EPW cube, the Picard group is generated by the polarization h. In particular, the transcendental lattice of the very general EPW cube is the lattice $h^\perp$. Thus, the involution ι_A preserves the polarization h and acts as $-\mathrm{id}$ on $h^\perp$, namely, it is the reflection with respect to the vector h, given by

$$\alpha \mapsto \alpha - \frac{2q(h,\alpha)}{q(h)}h,$$

where q is the Beauville–Bogomolov–Fujiki form on $H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Z})$. This implies that $q(h) = 4$ divides $2\,\mathrm{div}(h)$, hence $\mathrm{div}(h)$ is even, and by the above discussion it must be equal to 2. □

2.5. Cohomology ring of EPW cubes

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that the degeneracy locus $\mathsf{Z}_A^{\geq 4}$ is empty. Let $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$ be the associated EPW cube, defined in Equation (40), where $\widetilde{\mathsf{Z}}_A$ is a smooth hyper-Kähler sixfold.

By Remark 2.18, the involution ι_A associated to g_A is antisymplectic and its fixed locus $\mathsf{W}_A \coloneqq g_A^{-1}(\mathsf{Z}_A^{\geq 3}) \subset \widetilde{\mathsf{Z}}_A$ is a smooth Lagrangian threefold. In this section, using results and numerical relations proved in Section 1.2, we compute the Euler characteristic of W_A as well as the cohomology class $[\mathsf{W}_A] \in H^6(\mathsf{Z}_A, \mathbf{Q})$. Additionally, we determine the Chern classes of W_A.

The contents of this section appeared in [**Riz25a**, Section 3].

2.5.1. Topology of W_A. The double cover $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$ ramified along W_A gives the equality

$$\chi_{\mathrm{top}}(\widetilde{\mathsf{Z}}_A) - \chi_{\mathrm{top}}(\mathsf{W}_A) = 2\big(\chi_{\mathrm{top}}(\mathsf{Z}_A) - \chi_{\mathrm{top}}(\mathsf{Z}_A^{\geq 3})\big)$$

between topological Euler characteristics. Since $\mathsf{W}_A \simeq \mathsf{Z}_A^{\geq 3}$, we obtain

$$\chi_{\mathrm{top}}(\mathsf{W}_A) = 2\chi_{\mathrm{top}}(\mathsf{Z}_A) - \chi_{\mathrm{top}}(\widetilde{\mathsf{Z}}_A). \tag{42}$$

For a compact complex manifold, the topological Euler characteristic χ_{top} is the top Chern class so, in particular, $\chi_{\mathrm{top}}(\widetilde{\mathsf{Z}}_A) = 3200$ by Equation (13).

The covering involution ι_A of $\widetilde{\mathsf{Z}}_A$ acts on $H^\bullet(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$ and

$$H^\bullet(\mathsf{Z}_A, \mathbf{Q}) = H^\bullet(\widetilde{\mathsf{Z}}_A, \mathbf{Q})^+,$$

where $+$ means the ι_A-invariant part. As noted in Theorem 2.20, we have

$$H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q})^+ = \mathbf{Q}h \quad , \quad H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q})^- = h^\perp.$$

This is because h is ι_A-invariant and the symplectic form of $\widetilde{\mathsf{Z}}_A$ is ι_A-antiinvariant.

The discussion in Section 1.2.2 implies that either the action of ι_A on $H^\bullet(X, \mathbf{Q}) = V_{(3)} \oplus V_{(1,1)}$ is the action induced by the automorphism $\iota_A^* \oplus \mathrm{id}$ of $V = H^2(X, \mathbf{Q}) \oplus U$ (as above, U is the hyperbolic plane); or these actions coincide on the Verbitsky component $V_{(3)}$ and are opposite on $V_{(1,1)}$.

These two possibilities for the action of ι_A in cohomology give two possibilities for the cohomology groups of Z_A:

- In the first case (natural action on $V_{(1,1)}$), we can compute

$$b_0(\mathsf{Z}_A) = b_2(\mathsf{Z}_A) = 1, \quad b_4(\mathsf{Z}_A) = 255, \quad b_6(\mathsf{Z}_A) = 486,$$

and

$$\chi_{\mathrm{top}}(\mathsf{Z}_A) = 2(1 + 1 + 255) + 486 = 1000.$$

- In the second case (*opposite* action on $V_{(1,1)}$), we can compute

$$b_0(\mathsf{Z}_A) = b_2(\mathsf{Z}_A) = 1, \quad b_4(\mathsf{Z}_A) = 276, \quad b_6(\mathsf{Z}_A) = 276,$$

and

$$\chi_{\mathrm{top}}(\mathsf{Z}_A) = 2(1 + 1 + 276) + 276 = 832.$$

For example, we show how to determine $h^6(\mathsf{Z}_A, \mathbf{Q})$. We rewrite the decompositions in Section 1.2.3 as

$$H^6(X, \mathbf{Q}) \cap V_{(3)} = \mathrm{Sym}^3 \bar{T} \oplus (\mathrm{Sym}^2 \bar{T} \cdot h) \oplus (\bar{T} \cdot h^2) \oplus \mathbf{Q}h^3$$

and

$$H^6(X, \mathbf{Q}) \cap V_{(1,1)} = \bigwedge^2 \bar{T} \oplus (h \wedge \bar{T}) \oplus \bigwedge^2 U,$$

where $\bar{T} := h^\perp$.

The fixed subspace for the action of ι_A^* on $H^6(X, \mathbf{Q}) \cap V_{(3)}$ is equal to $(\mathrm{Sym}^2 \bar{T} \cdot h) \oplus \mathbf{Q}h^3$, since ι_A^* acts on $\bar{T}$ as $-\mathrm{id}$ and fixes h. Since $\bar{T}$ has dimension 22, this fixed subspace has dimension 254.

We consider now the action of ι_A^* on $H^6(X, \mathbf{Q}) \cap V_{(1,1)}$. By Lemma 1.15, this action is equal to $\pm\gamma_{(1,1)}$, where $\gamma_{(1,1)}$ is the action induced by the involution $\iota_A^* \oplus \mathrm{id}$ on V. The fixed subspace for $\gamma_{(1,1)}$ is $\bigwedge^2 \bar{T} \oplus \bigwedge^2 U$, of dimension 232, and the fixed space for $-\gamma_{(1,1)}$ is equal to $h \wedge \bar{T}$, of dimension 22.

Hence, $b_6(\mathsf{Z}_A)$ is equal to $254+232 = 486$ in the *natural* case and to $254+22 = 276$ in the *opposite* case. Therefore, the relation (42) gives

$$\chi_{\mathrm{top}}(\mathsf{W}_A) = \begin{cases} -1200 & \text{(natural case)}, \\ -1536 & \text{(opposite case)}. \end{cases} \tag{43}$$

2.5.1.1. *The class* $[\mathsf{W}_A]$ *in* $H^6(\widetilde{\mathsf{Z}}_A, \mathbf{Z})$. The class $[\mathsf{W}_A]$ is a 3-dimensional Hodge class of $\widetilde{\mathsf{Z}}_A$ hence, by Equation (21), it can be written as

$$[\mathsf{W}_A] = ah^3 + bhc_2 + c\eta,$$

where $a, b, c \in \mathbf{Q}$, and η is a class of square 4 (see Section 1.2.3).

Lemma 2.21. *The submanifold* $\mathsf{W}_A \subset \widetilde{\mathsf{Z}}_A$ *has class*

$$[\mathsf{W}_A] = \frac{5}{8}\left(3h^3 - hc_2\right) \in H^6(\widetilde{\mathsf{Z}}_A, \mathbf{Z}).$$

Moreover, the Euler characteristic of W_A *is* $\chi_{\mathrm{top}}(\mathsf{W}_A) = -1200$ *and the action of* ι_A^* *in cohomology is natural.*

Proof. Since $\mathsf{Z}_A^{\geq 2} \cdot h^3 = 720$ ([**IKKR19**, Section 2.3]), it follows from Proposition 1.19 that the class of W_A in $H^6(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$ is $[\mathsf{W}_A] = \frac{15}{8}h^3 - \frac{5}{8}hc_2 + c\eta$. By Proposition 1.10(a), the self intersection number $[\mathsf{W}_A]^2$ is equal to $-c_3(T_{\mathsf{W}_A})$. Since the Chern class $c_3(T_{\mathsf{W}_A})$ computes the topological characteristic of W_A, we obtain

$$-\chi_{\mathrm{top}}(\mathsf{W}_A) = [\mathsf{W}_A]^2 = 1200 + 4c^2.$$

We compare this with the two possible values of $\chi_{\mathrm{top}}(\mathsf{W}_A)$ computed in Equation (43) to obtain that $4c^2$ is either 0 or 336, hence $c = 0$ and $\chi_{\mathrm{top}}(\mathsf{W}_A) = -1200$. □

Remark 2.22. The restriction map

$$H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) \longrightarrow H^2(\mathsf{W}_A, \mathbf{Q})$$

has rank 1 and kernel $h^\perp$ (the symplectic form vanishes on W_A and, since the class of the latter describes a dense open subset of the intersection of $h^\perp$ and a smooth quadric, it vanishes on $h^\perp$; it is nonzero since the restriction of an ample class (such as h) is ample). This implies that $[\mathsf{W}_A]$ is an atomic Lagrangian [**Bec25**, Theorem 1.8]. Notice that its class belongs to the Verbitsky component defined in (12), according to what is expected in [**Bec25**, Proposition 3.5].

2.5.2. Chern classes. By the discussion after [**DK20a**, Theorem 5.6], and [**DK20a**, Lemma 4.3] (applied with $S = \mathsf{Gr}(3, V_6)$, $k = 3$, $c_1(\mathcal{L}) = c_1(\mathscr{A}_1) = 0$, and $c_1(\mathscr{A}_2) = \mathcal{O}_{\mathsf{Gr}(3,V_6)}(-4)$, and giving an equality of Cartier divisor classes), we have $\omega_{\mathsf{Z}_A^{\geq 3}} = \mathcal{O}_{\mathsf{Z}_A^{\geq 3}}(2)$, so that $\mathsf{Z}_A^{\geq 3}$ (hence also W_A) is a threefold of general type, and

$$c_1(T_{\mathsf{W}_A}) = -2h|_{\mathsf{W}_A}.$$

Proposition 1.10(b) gives the relation

$$c(T_{\mathsf{W}_A})c(T_{\mathsf{W}_A}^\vee) = c(T_{\widetilde{\mathsf{Z}}_A})|_{\mathsf{W}_A}$$

between Chern classes. The value of $c_1(T_{\mathsf{W}_A})$ computed above, together with the topological Euler characteristic of W_A computed Lemma 2.21, allows one to compute the Chern classes of T_{W_A}

$$\begin{aligned} c_1(T_{\mathsf{W}_A}) &= -2h|_{\mathsf{W}_A}, \\ c_2(T_{\mathsf{W}_A}) &= \frac{1}{2}c_2|_{\mathsf{W}_A} + 2h^2|_{\mathsf{W}_A}, \\ c_3(T_{\mathsf{W}_A}) &= \chi_{\mathrm{top}}(\mathsf{W}_A) = -1200, \end{aligned}$$

where $c_2 = c_2(T_{\widetilde{\mathsf{Z}}_A})$.

We can then obtain the Euler characteristics of $\mathcal{O}_{\mathsf{W}_A}$ and $\Omega^1_{\mathsf{W}_A}$.

Proposition 2.23. *One has* $\chi(\mathsf{W}_A, \mathcal{O}_{\mathsf{W}_A}) = -130$ *and* $\chi(\mathsf{W}_A, \Omega^1_{\mathsf{W}_A}) = 470$.

Proof. The Hirzebruch–Riemann–Roch formula gives

$$\chi(\mathsf{W}_A, \mathcal{O}_{\mathsf{W}_A}) = \frac{1}{24}\, c_1(T_{\mathsf{W}_A})c_2(T_{\mathsf{W}_A}) = -\frac{1}{24}\,(4h^3 + hc_2)\cdot[\mathsf{W}_A] = -130.$$

Since W_A is a smooth projective threefold, by Poincaré duality and Hodge theory, we can write

$$\begin{aligned}\frac{1}{2}\chi_{\mathrm{top}}(\mathsf{W}_A) &= 1 - 2h^{0,1}(\mathsf{W}_A) + (2h^{0,2}(\mathsf{W}_A) + h^{1,1}(\mathsf{W}_A)) - (h^{0,3}(\mathsf{W}_A) + h^{1,2}(\mathsf{W}_A))\\ &= \chi(\mathsf{W}_A, \mathcal{O}_{\mathsf{W}_A}) - h^{1,0}(\mathsf{W}_A) + h^{1,3}(\mathsf{W}_A) + h^{1,1}(\mathsf{W}_A) - h^{1,2}(\mathsf{W}_A)\\ &= \chi(\mathsf{W}_A, \mathcal{O}_{\mathsf{W}_A}) - \chi(\mathsf{W}_A, \Omega^1_{\mathsf{W}_A}),\end{aligned}$$

from which we obtain, using Lemma 2.21, the value

$$\chi(\mathsf{W}_A, \Omega^1_{\mathsf{W}_A}) = -130 - \frac{1}{2}\,(-1200) = 470.$$

□

2.5.3. Relations in $H^\bullet(\widetilde{\mathsf{Z}}_A^{\geq 2}, \mathbf{Q})$. We conclude this section by studying more relations in the cohomology ring $H^\bullet(\widetilde{\mathsf{Z}}_A^{\geq 2}, \mathbf{Q})$. In particular, we show that the Hodge classes η and λ defined in Section 1.2.1 are algebraic, by computing the pullbacks by $g_A\colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A \subset \mathsf{Gr}(3, V_6)$ of $H^\bullet(\mathsf{Gr}(3, V_6), \mathbf{Q})$.

Lemma 2.24. *In the cohomology ring* $H^*(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$*, we have*

$$\begin{aligned} &g_A^*\sigma_1 = h,\\ &g_A^*\sigma_{1,1} = \frac{1}{2}h^2 + 2\lambda, && g_A^*\sigma_2 = \frac{1}{2}h^2 - 2\lambda,\\ &g_A^*\sigma_{1,1,1} = \frac{1}{24}(h^3 + hc_2) + 2\eta, && g^*\sigma_3 = \frac{1}{24}(h^3 + hc_2) - 2\eta.\end{aligned}$$

Proof. The first equality comes from the definition (38) of h. We now study the pullback of $\sigma_{1,1,1}$ and σ_3 and $\sigma_{2,1}$.

Since $g_A^*\sigma_{1,1,1}$ is a Hodge class, (21) implies that it can be written as a linear combination of h^3, hc_2, and the absolute Hodge class $\eta = h\lambda$. The coefficients can be determined by computing some intersection numbers of $g_A^*\sigma_{1,1,1}$ with other classes. Since g_A is a double cover, the degree of $g_A^*\sigma_{1,1,1}$ is equal to twice the degree of $i_A^*\sigma_{1,1,1}$, where i_A is the inclusion $\mathsf{Z}_A \subset \mathsf{Gr}(3, V_6)$. Hence, by projection formula it follows that

$$g_A^*\sigma_{1,1,1}\cdot h^3 = 2\cdot\sigma_{1,1,1}\cdot\sigma_1^3\cdot[\mathsf{Z}_A],$$

where $[\mathsf{Z}_A]$ is the class of Z_A in $\mathrm{CH}^3(\mathsf{Gr}(3, V_6))$, computed in Remark 2.13. By expanding the product, we obtain $g_A^*\sigma_{1,1,1}\cdot h^3 = 112$.

Analogously, we can compute $g_A^*\sigma_{1,1,1}\cdot[\mathsf{W}_A]$.

The equality $g^*([\mathsf{Z}_A^{\geq 3}]) = 2[\mathsf{W}_A]$, together with the projection formula, imply

$$g_A^*\sigma_{1,1,1}\cdot[\mathsf{W}_A] = \sigma_{1,1,1}\cdot[\mathsf{Z}_A^{\geq 3}] \in H^9(\mathsf{Gr}(3, V_6)).$$

Using again the formula for $[\mathsf{Z}_A^{\geq 3}]$ computed in Remark 2.13, we obtain the equality $g_A^*\sigma_{1,1,1}\cdot[\mathsf{W}_A] = 40$. If we write $g_A^*\sigma_{1,1,1} = ah^3 + bhc_2 + d\eta$, using the expression for

$[\mathsf{W}_A]$ computed in Lemma 2.21 and the relations computed in Proposition 1.17, we obtain $a = b = \frac{1}{24}$. Finally, we use

$$(g^*\sigma_{1,1,1})^2 = 2 \cdot [\mathsf{Z}_A] \cdot \sigma_{1,1,1}^2 = 2 \cdot 16$$

and the fact that $\eta^2 = 4$ to obtain that d is equal to ± 2. Up to the choice of a sign, we obtain the desired formula for $g_A^*\sigma_{1,1,1}$. The same argument can be done for σ_3: similar computations show that the same relations $g^*\sigma_3 \cdot h^3 = 2 \cdot 56$, $g^*\sigma_3 \cdot [\mathsf{W}_A] = 40$, and $(g^*\sigma_3)^2 = 2 \cdot 16$ hold, hence again we have $g^*\sigma_3 = \frac{1}{24}(h^3 + hc_2) \pm 2\eta$. The relation $\sigma_{1,1,1} \cdot \sigma_3 = 0$ implies

$$g^*\sigma_3 = \frac{1}{24}(h^3 + hc_2) - 2\eta.$$

Let us now consider the codimension 2 classes $g^*\sigma_{1,1}$ and $g^*\sigma_2$. Using the relations $\sigma_{1,1} \cdot \sigma_1 = \sigma_{2,1} + \sigma_{1,1,1}$, $\sigma_2 \cdot \sigma_1 = \sigma_3 + \sigma_{2,1}$, and $\sigma_1^3 = \sigma_{1,1,1} + 2\sigma_{2,1} + \sigma_3$, we obtain

$$g^*\sigma_{1,1} \cdot h = \frac{1}{2}h^3 + 2\eta \ , \qquad g^*\sigma_2 \cdot h = \frac{1}{2}h^3 - 2\eta.$$

Since multiplication by h is injective on $\mathrm{Hdg}^2(\widetilde{\mathsf{Z}}_A)$ by the Hard Lefschetz Theorem, and $\eta = h\lambda$ by Remark 1.18, we obtain

$$g^*\sigma_{1,1} = \frac{1}{2}h^2 + 2\lambda \ , \qquad g^*\sigma_2 = \frac{1}{2}h^3 - 2\lambda.$$

□

Corollary 2.25. *A very general EPW cube $\widetilde{\mathsf{Z}}_A$ (with Picard rank 1) satisfies the (rational) Hodge conjecture.*

Proof. One need to show that the Hodge classes on an EPW cube $\widetilde{\mathsf{Z}}_A$ of Picard rank 1, computed in (21), are all algebraic. The Hodge classes are generated by the Chern classes of $\widetilde{\mathsf{Z}}_A$, the polarization h, and the codimension 2 class λ. Lemma 2.24 shows that the latter are algebraic. □

We conclude this section by studying the ring structure of $\mathrm{Hdg}^\bullet(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$. We have

$$\begin{aligned}
\mathrm{Hdg}^0(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}, \\
\mathrm{Hdg}^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h, \\
\mathrm{Hdg}^4(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h^2 \oplus \mathbf{Q}c_2 \oplus \mathbf{Q}\lambda, \\
\mathrm{Hdg}^6(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h^3 \oplus \mathbf{Q}hc_2 \oplus \mathbf{Q}h\lambda, \\
\mathrm{Hdg}^8(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h^4 \oplus \mathbf{Q}h^2c_2 \oplus \mathbf{Q}h^2\lambda, \\
\mathrm{Hdg}^{10}(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h^5, \\
\mathrm{Hdg}^{12}(\widetilde{\mathsf{Z}}_A, \mathbf{Q}) &= \mathbf{Q}h^6.
\end{aligned} \tag{44}$$

The classes computed in Lemma 2.24, together with the relations studied in Section 1.2.3, allow us to determine all the relations in $\mathrm{Hdg}^\bullet(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$.

Proposition 2.26. (a) *There are no relations in degree ≤ 6.*
(b) *In degree 8, the relations are*

$$c_4 = -10h^4 + \frac{20}{3}h^2c_2, \quad c_2^2 = \frac{5}{2}c_4, \quad \lambda^2 = \frac{1}{48}(2h^4 - h^2c_2), \quad c_2\lambda = 7h^2\lambda.$$

(c) *In degree* 10*, the relations are*

$$h^3c_2 = \frac{9}{5}\,h^5, \quad hc_2^2 = 5\,h^5, \quad hc_4 = 2\,h^5, \quad h\lambda^2 = \frac{1}{240}h^5, \quad h^3\lambda = 0, \quad hc_2\lambda = 0.$$

(d) *In degree* 12*, we have*

$$h^6 = 960, \quad h^4c_2 = 1728, \quad h^2c_2^2 = 4800, \quad h^2c_4 = 1920, \quad h^2\lambda^2 = 4,$$
$$h^4\lambda = 0, \quad h^2c_2\lambda = 0, \quad c_2^2\lambda = 0, \quad \lambda^3 = 0, \quad c_2\lambda^2 = -28.$$

Proof. All the relations not concerning the class λ are deduced from Proposition 1.17, using $q(h) = 4$. Analogously, Remark 1.18 computes the remaining relations in degree 8, and the degree 12 classes $h^2\lambda^2, h^4\lambda, h^2c_2\lambda$, and $c_2\lambda^2$.

We start by writing λ^2 in terms of a basis of $\mathrm{Hdg}^8(\widetilde{\mathsf{Z}}_A, \mathbf{Q})$. Recall that in the Chow ring of the Grassmannian $\mathsf{Gr}(3, V_6)$, we have the equality $\sigma_{1,1} \cdot \sigma_2 = \sigma_1(\sigma_3 + \sigma_{1,1,1})$. Pulling back by g^*, and using the expressions computed in Lemma 2.24, we obtain

$$\frac{1}{4}\,h^4 - 4\lambda^2 = \frac{1}{12}\,(h^4 + h^2c_2),$$

which in turns implies $\lambda^2 = \frac{1}{24}h^4 - \frac{1}{48}hc_2$. Multiplying everything by λ yields $\lambda^3 = 0$, while multiplying by c_2 yields $\lambda^2c_2 = 0$.

Finally, the expression for λc_2 is obtained by writing it as $ah^4 + bh_2^c + ch^2\lambda$, and using the relations $h^2c_2\lambda = 0$, $c_2^2\lambda = 0$ $c_2\lambda^2 = -28$. □

CHAPTER 3

Period map of EPW cubes

In this chapter, we study the period map and period domain of EPW cubes. We describe the Heegner divisors in the period domain of EPW cubes, which correspond to hyper-Kähler varieties of Picard rank at least two. In Section 3.3, we describe the image of the period map of EPW cubes.

The contents of Sections 3.2.1 and 3.2.2 appeared in [**Riz25b**, Appendix A], *while the contents of Section 3.3 are original.*

3.1. Period domain and period map of double EPW sextics

Let ${}^{[2]}\mathscr{M}_2^{(1)}$ be the irreducible quasiprojective 19-dimensional coarse moduli space of polarized hyper-Kähler varieties of K3$^{[2]}$-type, with polarization of Beauville–Bogomolov square 2 and divisibility 1, defined in Theorem 1.7. Consider the moduli morphism (29)

$$\mu_2 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Delta \cup \Sigma) \longrightarrow {}^{[2]}\mathscr{M}_2^{(1)}$$

sending a Lagrangian subspace $A \subset \bigwedge^3 V_6$ to the *double EPW sextic* $\widetilde{\mathsf{Y}}_A$ with its canonical polarization, defined in Section 2.2.

The polarized period map (9) for these varieties induces a morphism

$$\wp_2 \colon {}^{[2]}\mathscr{M}_2^{(1)} \hookrightarrow {}^{[2]}\mathscr{P}_2^{(1)}$$

which is an open embedding by Theorem 1.8. By [**Riz24**, Section 6.1], the period space ${}^{[2]}\mathscr{P}_2^{(1)}$ is equal to

$$ {}^{[2]}\mathscr{P}_2^{(1)} = \mathscr{D}_{h^\perp}/\widetilde{O}(h^\perp). \tag{45}$$

Here, $\Lambda_{K3^{[2]}}$ is the lattice defined in (3), $h \in \Lambda_{K3^{[2]}}$ has square 2 and divisibility 1, and $h^\perp \subset \Lambda_{K3^{[2]}}$ is the orthogonal sublattice. The index 2 subgroup $\widetilde{O}(h^\perp) \triangleleft O(h^\perp)$ consists of isometries of $h^\perp$ acting trivially on the discriminant lattice $A_{h^\perp}$.

O'Grady proved in [**O'G15**, Proposition 4.8] that the composition $\wp_2 \circ \mu_2$ extends to a regular morphism

$$\overline{\wp}_2 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \longrightarrow {}^{[2]}\mathscr{P}_2^{(1)}, \tag{46}$$

where $\Sigma' \subsetneq \Sigma$ is a proper closed subset.

3.2. Divisors in the period domain of EPW cubes

Let ${}^{[3]}\mathscr{M}_4^{(2)}$ be the irreducible quasiprojective 19-dimensional coarse moduli space of polarized hyper-Kähler varieties of K3$^{[3]}$-type, with polarization of Beauville–Bogomolov square 4 and divisibility 2, defined in Theorem 1.7. Consider the moduli morphism (41)

$$\mu_3 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Gamma \cup \Sigma) \longrightarrow {}^{[3]}\mathscr{M}_4^{(2)}$$

sending a Lagrangian subspace $A \subset \bigwedge^3 V_6$ to the *EPW cube* $\widetilde{\mathsf{Z}}_A$ with its canonical polarization, defined in Section 2.4.

The polarized period map (9) for these varieties induces a morphism

$$\wp_3 \colon {}^{[3]}\mathscr{M}_4^{(2)} \hookrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

which, again, is an open embedding by Theorem 1.8. By [**Riz24**, Section 6.1], the period space ${}^{[3]}\mathscr{P}_4^{(2)}$ is equal to

$$ {}^{[3}\mathscr{P}_4^{(2)} = \mathscr{D}_{h^\perp}/O(h^\perp), \tag{47}$$

where $h^\perp \subset \Lambda_{K3^{[3]}}$ is the orthogonal sublattice of a vector $h \in \Lambda_{K3^{[3]}}$ of square 4 and divisibility 2 (as in Section 3.2, it is independent of the choice of h). Moreover, by [**Riz24**, Section 6.1], the lattice $h^\perp$ decomposes as

$$h^\perp = M \oplus \mathbf{Z}k \oplus \mathbf{Z}\ell, \tag{48}$$

where k and ℓ are vectors of square -2 and M is a unimodular lattice. In particular, it is isomorphic to the orthogonal sublattice of a vector of square 2 and divisibility 1 in the lattice $\Lambda_{K3^{[2]}}$ and there is a double cover

$$\rho \colon {}^{[2]}\mathscr{P}_2^{(1)} \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}. \tag{49}$$

By [**LO19**, Section 2.3], the composition

$$\wp_2 \circ \mu_2 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Gamma) \longrightarrow {}^{[2]}\mathscr{P}_2^{(1)}$$

is equivariant for the duality involution $A \mapsto A^\perp$ acting on $\mathrm{LG}(\bigwedge^3 V_6)$ and the covering involution of ${}^{[2]}\mathscr{P}_2^{(1)}$ associated to ρ.

Theorem 3.1 ([**KKM24**, Corollary 6.1])**.** *The composition*

$$\mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \xrightarrow{\overline{\wp}_2} {}^{[2]}\mathscr{P}_2^{(1)} \xrightarrow{\rho} {}^{[3]}\mathscr{P}_4^{(2)}$$

is equal to $\wp_3 \circ \mu_3$ *over* $\mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma$.

This implies that the composition $\wp_3 \circ \mu_3$ extends to a regular morphism

$$\overline{\wp}_3 \coloneqq \rho \circ \overline{\wp}_2 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \smallsetminus \Sigma' \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}. \tag{50}$$

In this section, we define Heegner divisors in period domains and prove some of their properties in cases that concern us. We then analyze the images in the period domains of the divisors Δ, Γ, and Σ defined in Equations (28), (34), and (27), and prove that they are Heegner divisors. Finally, in Section 3.2.3, we show how one can construct families of EPW cubes (of Picard rank at least 2) from K3 surfaces.

3.2.1. Heegner divisors in the period domains ${}^{[2]}\mathscr{P}_2^{(1)}$ and ${}^{[3]}\mathscr{P}_4^{(2)}$. We start by introducing some notation, following [**Deb22**, Sections 3.9 and 3.10]. Let h be a primitive vector of positive square in $\Lambda_{K3^{[m]}}$, and consider the period domain ${}^{[m]}\mathscr{P}_\tau$, defined in (8), for $\tau = O(\Lambda_{\mathrm{K3^{[m]}}})$.

Definition 3.2 (Heegner divisors)**.** Let $K \subset \Lambda_{K3^{[m]}}$ be a rank 2 sublattice of signature $(1,1)$ that contains h. The *Heegner divisor* ${}^{[m]}\mathscr{D}_{\tau,K} \subset {}^{[m]}\mathscr{P}_\tau$ is the image of the hyperplane section $\mathbf{P}(K^\perp \otimes \mathbf{C}) \cap \mathscr{D}_{h^\perp}$ in the quotient $\mathscr{D}_{h^\perp}/\widehat{O}(\Lambda_{K3^{[m]}}, h)$. Moreover, for $d > 0$, we set

$${}^{[m]}\mathscr{D}_{\tau,d} \coloneqq \bigcup_{K,\ \mathrm{disc}(K^\perp)=-d} {}^{[m]}\mathscr{D}_{\tau,K},$$

where K runs over the set of rank 2 sublattices of $\Lambda_{K3^{[m]}}$ with signature $(1,1)$ that contain h. Note that the Heegner divisor ${}^{[m]}\mathscr{D}_{\tau,K}$ only depends on $\beta^\perp$, where β is any nonzero vector in $K \cap h^\perp$.

Following the notation of [DM19], we denote by ${}^{[2]}\mathscr{D}^{(1)}_{2,d}$ the Heegner divisors in the period space ${}^{[2]}\mathscr{P}^{(1)}_2$, and by ${}^{[3]}\mathscr{D}^{(2)}_{4,d}$ the Heegner divisors in the period space ${}^{[3]}\mathscr{P}^{(2)}_4$. We compare these Heegner divisors using the double cover $\rho\colon {}^{[2]}\mathscr{P}^{(1)}_2 \to {}^{[3]}\mathscr{P}^{(2)}_4$ defined in (49).

Heegner divisors in ${}^{[2]}\mathscr{P}^{(1)}_2$ are described in [Deb22, Proposition 3.29]: the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,d}$ is nonempty if and only $d = 2e$ and $e \not\equiv 3 \pmod 4$. Moreover, if e is even, the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,2e}$ is irreducible, while it has 2 components for $e \equiv 1 \pmod 4$. We give a similar description of Heegner divisors in ${}^{[3]}\mathscr{P}^{(2)}_4$.

Proposition 3.3 ([Riz25b, Proposition A.2])**.** *Let d be a positive integer. The Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,d} \subset {}^{[3]}\mathscr{P}^{(2)}_4$ is nonempty if and only if d is an even number $2e$ such that $e \not\equiv 3 \pmod 4$. When this is the case, ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$ is irreducible and is the image by ρ of the Heegner divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,2e} \subset {}^{[2]}\mathscr{P}^{(1)}_2$.*

Proof. Let $h \in \Lambda_{K3^{[3]}}$ be a vector of square 4 and divisibility 2. The period domain ${}^{[3]}\mathscr{P}^{(2)}_4$ is equal to $\mathscr{D}_{h^\perp}/\widehat{O}(h^\perp)$ by Equation (47). Recall also, from Equation (48), that the lattice $h^\perp$ is isomorphic to By the discussion above the group $O(h^\perp)$ is generated the group $M \oplus \mathbf{Z}k \oplus \mathbf{Z}\ell$, where k and ℓ are vectors of square -2 and M is a unimodular lattice. The discriminant group $A_{h^\perp}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Finally, the group $O(h^\perp)$ is generated by its subgroup $\widetilde{O}(h^\perp)$ of isometries that acts trivially on the discriminant group $A_{h^\perp}$, and the reflection r that exchanges k and ℓ and is the identity on M.

Let K be a rank-2 sublattice of $\Lambda_{\mathrm{K3}^{[3]}}$ of signature (1,1) that contains h, and denote by β a generator of $K \cap h^\perp$. We determine when the Heegner divisor defined by K is contained in ${}^{[3]}\mathscr{D}^{(2)}_{4,d}$. Since β is in $h^\perp$, we can write, with respect to the decomposition (48),

$$\beta = am + bk + c\ell,$$

where $m \in M$ is a primitive vector and $(a,b,c) = 1$. Let $2t := m^2$. We compute $\beta^2 = 2ta^2 - 2b^2 - 2c^2$ and $\mathrm{div}(\beta) = (a, 2b, 2c) = (a, 2)$. In particular, β has divisibility 2 if and only if $2 \mid a$, and in this case, either b or c must be odd.

Using [GHS13, Lemma 7.2], we compute the discriminant

$$\mathrm{disc}(\beta^\perp) = \frac{-\beta^2 \,\mathrm{disc}(h^\perp)}{\mathrm{div}(\beta)^2} = \frac{4}{\mathrm{div}(\beta)^2} \cdot 2(-ta^2 + b^2 + c^2) \tag{51}$$

of the lattice $\beta^\perp \subset h^\perp$, where we used $\mathrm{disc}(h^\perp) = 4$. Since $\mathrm{div}(\beta) \mid 2$, it follows that $\mathrm{disc}(\beta^\perp)$ is equal to $-\beta^2$ when $2 \mid a$, and to $-4\beta^2$ when $2 \nmid a$. In particular, since β^2 is even, $\mathrm{disc}(\beta^\perp)$ is an even number $2e$.

Moreover, e cannot be equal to 3 modulo 4. Indeed, if $\mathrm{disc}(\beta^\perp) \equiv 6 \pmod 8$, then a must be even and $\mathrm{disc}(\beta^\perp) = -\beta^2 = -2ta^2 + b^2 + c^2$, which is never equal to 6 modulo 8.

Conversely, we now show that for any $e \not\equiv 3 \pmod 4$, there exists a unique $O(h^\perp)$-orbit in $h^\perp$ of vectors β such that $\beta^\perp$ has discriminant $2e$. Using [Riz24, Lemma 3.3], this implies that ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$ is irreducible.

We start by studying the orbit of a primitive vector $\beta \in h^\perp$ of square β^2 and class $\beta_* = [\beta/\operatorname{div}(\beta)] = (x, y)$ in $A_{h^\perp} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. It is equal to

$$O(h^\perp)\beta = \widetilde{O}(h^\perp)\beta \cup \widetilde{O}(h^\perp)r(\beta).$$

By Eichler's Lemma, the orbit $\widetilde{O}(h^\perp)\beta$ is the set of primitive vectors v in $h^\perp$ of square β^2 and class $v_* \in A_{h^\perp}$ equal to β_*. Hence, since r acts on $A_{h^\perp}$ by exchanging the components, the orbit $O(h^\perp)\beta$ is the set of primitive vectors $v \in h^\perp$ with square β^2 and class $v_* \in A_{h^\perp}$ equal to (x, y) or (y, x). Therefore, there are, in the discriminant group, at most three $O(h^\perp)$-orbits of primitive vectors of given square, respectively of class $(0, 0)$, $(0, 1)$ or $(1, 0)$, and $(1, 1)$.

We compute the discriminant of $\beta^\perp$ in each case, using Equation (51):

(a) $\operatorname{div}(\beta) = 1$ hence $\beta_* = (0, 0)$; in this case, $\operatorname{disc}(\beta^\perp) = -4\beta^2 = 2e$ for some $e \equiv 0 \pmod 4$.
(b) $\operatorname{div}(\beta) = 2$ and $\beta_* \in \{(0, 1), (1, 0)\}$; in this case, a is even and exactly one among b and c is odd. Therefore, $\operatorname{disc}(\beta^\perp) = -\beta^2 = 2e$ for $e = -ta^2 + b^2 + c^2 \equiv 1 \pmod 4$.
(c) $\operatorname{div}(\beta) = 2$ and $\beta_* = (1, 1)$; in this case, a is even, and b and c are odd, hence $\operatorname{disc}(\beta^\perp) = -\beta^2 = 2e$ for $e = -ta^2 + b^2 + c^2 \equiv 2 \pmod 4$.

In particular, $\operatorname{disc}(\beta^\perp)$ determines β^2 and $\beta_* \in \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ up to coordinates swap, hence the $O(h^\perp)$-orbit of β.

Finally, to compare Heegner divisors of ${}^{[2]}\mathscr{P}_2^{(1)}$ and ${}^{[3]}\mathscr{P}_4^{(2)}$, recall from [**Riz24**, Proposition 6.3] that the morphism ρ is induced by the action of the reflection r on ${}^{[2]}\mathscr{P}_2^{(1)}$. Hence, ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$ is the image of ${}^{[2]}\mathscr{D}_{2,2e}^{(1)}$ by ρ. □

Remark 3.4. The previous proof also shows that the primitive vectors $\beta \in h^\perp$ associated to a Heegner divisor ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$ (where $e \not\equiv 3 \pmod 4$) satisfy the following:

- if $e \equiv 0 \pmod 4$, the vectors β have divisibility 1 and square $-e/2$;
- if $e \equiv 1, 2 \pmod 4$, the vectors β have divisibility 2 and square $-2e$.

3.2.2. Image by $\overline{\wp}_3$ of the divisors Δ, Γ, and Σ. In this section, we prove that the images of the divisors Δ, Γ, and Σ (defined in Equations (28), (34), and (27)) by the period map

$$\overline{\wp}_3 \colon \mathrm{LG}(\textstyle\bigwedge^3 V_6) \setminus \Sigma' \longrightarrow {}^{[3]}\mathscr{P}_4^{(2)}.$$

defined in (50) are distinct Heegner divisors. For this, we will use Gushel–Mukai varieties and their periods (for definitions see [**DK18**, **DK19**]).

Proposition 3.5 ([**Riz25b**, Proposition A.4])**.** *One has*

$$\overline{\overline{\wp}_3(\Delta \setminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,10}^{(2)} \quad , \quad \overline{\overline{\wp}_3(\Gamma \setminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,12}^{(2)} \quad , \quad \overline{\overline{\wp}_3(\Sigma \setminus \Sigma')} = {}^{[3]}\mathscr{D}_{4,8}^{(2)}.$$

In particular, the divisors Δ, Γ, and Σ are mutually distinct.

Proof. The images of these divisors in ${}^{[2]}\mathscr{P}_2^{(1)}$ by the period map $\overline{\wp}_2$ were determined in [**DK19**, Remark 5.29] and they are the following Heegner divisors:

$$\begin{aligned}
\overline{\overline{\wp}_2(\Delta \setminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,10}^{\prime\prime(1)} &&= \overline{\{\text{periods of GM 4folds containing a } \sigma\text{-plane}\}} \\
&&&= \overline{\{\text{periods of GM sixfolds containing a } \mathbf{P}^3\}}, \\
\overline{\overline{\wp}_2(\Gamma \setminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,12}^{(1)} &&= \overline{\{\text{periods of GM 4folds containing a } \tau\text{-plane}\}}, \\
\overline{\overline{\wp}_2(\Sigma \setminus \Sigma')} &= {}^{[2]}\mathscr{D}_{2,8}^{(1)} &&= \overline{\{\text{periods of nodal GM 4folds}\}}
\end{aligned}$$

(where ${}^{[2]}\mathscr{D}''^{(1)}_{2,10}$ is one of the components of the reducible Heegner divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,10}$). To conclude, it is then enough to apply Proposition 3.3, the relation of Theorem 3.1, and the fact that the Heegner divisors ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$, ${}^{[3]}\mathscr{D}^{(2)}_{4,12}$, and ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$ are mutually distinct. □

As a consequence, following [**O'G13a**, Section 2.1], we show that for $[A] \in \Gamma$ general, the scheme $\mathsf{Z}_A^{\geq 4}$ consists of a single point.

Lemma 3.6 ([**Riz25b**, Lemma 3.2])**.** *The subvariety*

$$\Gamma_+ = \{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid |\mathsf{Z}_A^{\geq 4}| > 1\}$$

of $\mathrm{LG}(\bigwedge^3 V_6)$ *has codimension at least* 2*. Therefore, for* $[A] \in \Gamma$ *general, the scheme* $\mathsf{Z}_A^{\geq 4}$ *consists of a single smooth point.*

Proof. We continue the parameter count started in Proposition 2.12. Consider the locally closed subvariety

$$\widetilde{\Gamma}_+ = \{([A], [U_1], [U_2]) \mid [U_1] \neq [U_2],\ \dim(A \cap T_{U_1}) = \dim(A \cap T_{U_2}) = 4\}$$

of $\mathrm{LG}(\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_6)^2$. It follows from Theorem 2.5(e) that Γ_+ is contained in the union of $\Gamma \cap \Sigma$ and the image of the first projection $\pi_+ \colon \widetilde{\Gamma}_+ \to \mathrm{LG}(\bigwedge^3 V_6)$. Since Γ and Σ are distinct irreducible divisors by Proposition 3.5, it is therefore enough to show that $\widetilde{\Gamma}_+$ has dimension at most 53.

We consider the morphism

$$\begin{aligned} \eta \colon \widetilde{\Gamma}_+ &\longrightarrow \mathsf{Gr}(4, \textstyle\bigwedge^3 V_6)^2 \times \mathsf{Gr}(3, V_6)^2 \\ ([A], [U_1], [U_2]) &\longmapsto ([A \cap T_{U_1}], [A \cap T_{U_2}], [U_1], [U_2]). \end{aligned}$$

Since A is a Lagrangian subspace, the image of η is contained in

$$N = \left\{ ([K_1], [K_2], [U_1], [U_2]) \;\middle|\; \begin{array}{lll} [U_1] \neq [U_2], & [K_1] \in \mathsf{Gr}(4, T_{U_1}), & [K_2] \in \mathsf{Gr}(4, T_{U_2}), \\ K_1 \perp K_2, & K_1 \cap T_{U_2} = K_1 \cap K_2 = K_2 \cap T_{U_1} & \end{array} \right\}.$$

Observe that the fiber of an element $([K_1], [K_2], [U_1], [U_2])$ of N is isomorphic to

$$\{[A] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid K_1 = A \cap T_{U_1},\ K_2 = A \cap T_{U_2}\},$$

which, since $K_1 + K_2$ is totally isotropic, is contained in $\mathrm{LG}((K_1+K_2)^\perp/(K_1+K_2))$. Therefore, the fiber $\eta^{-1}([K_1], [K_2], [U_1], [U_2])$ has dimension at most $\frac{(2+k)(3+k)}{2}$, where $k = \dim(K_1 \cap K_2)$.

We stratify N according to $k = \dim(K_1 \cap K_2)$ and $\ell = \dim(U_1 \cap U_2)$, and denote by

$$N_{k,\ell} = \{([K_1], [K_2], [U_1], [U_2]) \in N \mid \ell = \dim(U_1 \cap U_2),\ k = \dim(K_1 \cap K_2)\}$$

the stratum corresponding to k and ℓ. Clearly, we have the inclusion $K_1 \cap K_2 \subset T_{U_1} \cap T_{U_2}$, where the dimension $d(\ell)$ of $T_{U_1} \cap T_{U_2}$ is 0 for $\ell = 0$, and $2(\ell+1)$ for $\ell \in \{1, 2\}$. Moreover, the morphism η has equidimensional fibers over $N_{k,\ell}$. Thus, the dimension of $\widetilde{\Gamma}_+$ is the maximum of $\dim(\eta^{-1}(N_{k,\ell}))$ for $0 \leq k \leq d(\ell)$ and $0 \leq \ell \leq 2$, namely

$$\dim(\widetilde{\Gamma}_+) = \max_{0 \leq k \leq d(\ell)} \frac{(2+k)(3+k)}{2} + \dim(N_{k,\ell}). \tag{52}$$

We count parameters to determine the dimension of $N_{k,\ell}$. The space of pairs $([U_1],[U_2]) \in \mathsf{Gr}(3,V_6)^2$ whose intersection has dimension ℓ is a Schubert cell of dimension $18-\ell^2$. Having fixed such a pair $([U_1],[U_2])$, the quadruple $([K_1],[K_2],[U_1],[U_2])$ is in $N_{k,\ell}$ if and only if $K_1 \cap T_{U_2}$ has dimension k and

$$K_1 \cap T_{U_2} \subset K_2 \subset K_1^\perp \cap T_{U_2}\ , \quad K_2 \cap T_{U_1} = K_1 \cap T_{U_2}.$$

Hence, given $[K] = [K_1 \cap T_{U_2}]$ in $\mathsf{Gr}(k, T_{U_1} \cap T_{U_2})$, and $[K_1]$ in $\mathsf{Gr}(4-k, T_{U_1}/K)$, the point $[K_2]$ lies in $\mathsf{Gr}(4-k, (K_1^\perp \cap T_{U_2})/K)$. Note that, since $K_1 \cap T_{U_2}$ has dimension k, the space $K_1^\perp \cap T_{U_2}$ has dimension $6+k$. Putting all together, we obtain

$$\begin{aligned}\dim(N_{k,\ell}) &\leq 18 - \ell^2 + k(d(\ell)-k) + 6(4-k) + (4-k)(2+k)\\ &= 50 - 2(k+2)k + d(\ell)k - \ell^2.\end{aligned}$$

Hence, the dimension of $\eta^{-1}(N_{k,\ell})$ is at most $53-\ell^2-\frac{3}{2}k(k-\frac{2d(\ell)-3}{3})$, whose maximum for $0 \leq k \leq \max\{d(\ell),4\}$ and $0 \leq \ell \leq 2$ is 53. Therefore, $\widetilde{\Gamma}_+$ has dimension at most 53, as wanted. □

3.2.3. Divisors constructed from polarized K3 surfaces. We construct divisors in the moduli space ${}^{[3]}\mathscr{M}_4^{(2)}$ out of polarized K3 surfaces. Let (S,L) be a very general polarized K3 surface of degree $2e$, with $\mathrm{Pic}(S) = \mathbf{Z}[L]$. A hyper-Kähler birational model X of $S^{[3]}$, together with a class $a'L - b\delta$, define an element of ${}^{[3]}\mathscr{M}_4^{(2)}$ if and only if $a' = 2a$ with a odd, $2a^2e - b^2 = 1$, and the class $2aL - b\delta$ is ample on X.

The fact that the equation $2x^2e - y^2 = 1$ has integral solutions implies that $2e$ is not a perfect square and $e \notin \{3,9\}$. Let (a_0,b_0) be the minimal (positive) solution of that equation. By [**Deb22**, Lemma A.2], the minimal solution (z,w) of the Pell equation $z^2 - 2w^2e = 1$ is $z = 2ea_0^2 + b_0^2$ and $w = 2a_0b_0$; it satisfies $w \equiv 0 \pmod 2$, $z \equiv -1 \pmod 4$, and $z \equiv -1 \pmod{2e}$. When $e > 1$, we have the following:

- the group $\mathrm{Bir}(S^{[3]})$ of birational automorphisms of $S^{[3]}$ is generated by an antisymplectic involution ι that fixes the class $2a_0L - b_0\delta$ (of square 4 and divisibility 2) (see [**BC22**, Theorem 1.1] and its proof);
- the closure $\overline{\mathrm{Mov}(S^{[3]})}$ of the movable cone of $S^{[3]}$ is spanned by the classes L and $zL - ew\delta$ (see the first paragraph of [**BC22**, Section 5]) and the only class of square 4 and divisibility 2 in this cone is $2a_0L - b_0\delta$ (see [**Deb22**, Appendix A], which explains that all positive solutions of the equation $2x^2e - y^2 = 1$ are "generated" by (a_0,b_0));
- the cone $\overline{\mathrm{Mov}(S^{[3]})}$ is the union of $d \in \{1,3,5\}$ chambers, corresponding to d different hyper-Kähler birational models X of $S^{[3]}$; the invariant class $2a_0L - b_0\delta$ is ample on a unique such model X, and $X = S^{[3]}$ if and only if $d = 1$ (see [**BC22**, Proposition 5.6] and [**BC22**, Remark 4.2]).

Proposition 3.7. *Let $e > 1$ be an integer such that the equation*

$$2a^2e - b^2 = 1 \tag{53}$$

has integral solutions and let (a_0,b_0) be the minimal positive integral solution. There exists a rational map

$$\mathscr{F}_{2e} \dashrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

that sends a general polarized K3 surface (S, L) of degree $2e$ to the period of the pair $(X, 2a_0L - b_0\delta)$, where X is a suitable hyper-Kähler birational model of $S^{[3]}$. Its image is dense in the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$.

Proof. Let $\Lambda_{\mathrm{K3}} = M \oplus U$ be the unimodular lattice associated to K3 surfaces and fix a standard basis (u, v) of the hyperbolic plane U. The class ℓ of the polarization L on S can be represented by the vector $u + ev$. Therefore, the divisor class $2aL - b\delta$ on $S^{[3]}$ corresponds to the class $w = 2a(u + ev) - b\delta$ in the lattice $\Lambda_{\mathrm{K3}^{[3]}} = \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}\delta$, where δ has square -4.

The lattice $w^\perp$ is therefore given by

$$w^\perp = M \oplus \begin{pmatrix} -2d & 2b \\ 2b & -4a^2 \end{pmatrix},$$

where the nonunimodular part is spanned by the vectors $k_1 = u - ev$ and $k_2 = 2bv - a\delta$.

The Picard lattice of $S^{[2]}$ contains L and δ, hence also the sublattice K generated by w and $b\ell - ae\delta = bk_1 + ek_2$. Hence, the transcendental lattice of $S^{[3]}$ is contained in $K^\perp$. Since $K \cap w^\perp = bk_1 + ek_2$, we obtain that the period of $(S^{[3]}, 2aL + b\delta)$ belongs to the Heegner divisor defined by the vector $\beta := bk_1 + dk_2$. Finally, we compute $\beta^2 = -2e$ and $\mathrm{div}_{h^\perp}(\beta) = -2$. From Remark 3.4, we obtain that the wanted divisor is ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$. □

Remark 3.8. An infinite family of integers e for which the equation (53) has integral solutions is obtained by taking $a_0 = 1$, $b_0 = 2c + 1$, and $e = 2c(c + 1) + 1$.

(a) When $a_0 = 1$, $b_0 = 1$, and $e = 1$, the morphism $\phi_L \colon S \to \mathbf{P}^2 = \mathbf{P}(V_3)$ is a double cover branched along a sextic. When (S, L) is very general, the group $\mathrm{Bir}(S^{[3]})$ is generated by the biregular antisymplectic involution ι of $S^{[3]}$ induced by the involution of S associated with this double cover. By [**BC22**, Table 2] and [**Deb22**, Example 3.20], the closed movable and nef cones of $S^{[3]}$ coincide ($d = 1$ in the notation above) and are spanned by L and $2L - \delta$. The class $2L - \delta$ is therefore nef and big, but nonample on $S^{[3]}$. and the family of (quasipolarized) pairs $(S^{[3]}, 2L - \delta)$ maps onto the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,2}$ in ${}^{[3]}\mathscr{P}^{(2)}_4$ (which is *not* in the image of the period map $\wp_3$, as we will prove in Proposition 3.9).

We will study this example in details in Section 4.2.

(b) When $a_0 = 1$, $b_0 = 3$, and $e = 5$, the cone $\overline{\mathrm{Mov}(S^{[3]})}$ is spanned by L and $\iota^*(L) = 19L - 30\delta$, there are three fundamental chambers in $\mathrm{Mov}(S^{[3]})$, with walls spanned by $7L - 10\delta$ and $\iota^*(7L - 10\delta) = 13L - 20\delta$; the nef cone $\mathrm{Nef}(S^{[3]})$ is spanned by L and $7L - 10\delta$ (see [**Deb22**, Example 4.12] and [**BC22**, Table 2]).

The involution ι (which is geometrically described in [**Deb22**, Example 4.12]) is not regular on $S^{[3]}$, but $S^{[3]}$ has a unique nontrivial birational model X on which ι is regular. The proof of Corollary 2.17 shows that for $[A] \in \Delta$ general, the sixfold $\widetilde{\mathsf{Z}}^{\geq 2}_A$ is smooth and birationally isomorphic to the Hilbert scheme $S^{[3]}_A$, where S_A is a general degree 10 polarized K3 surface ([**IKKR19**, Section 5], [**Deb22**, Example 4.12]). Therefore, the pair $(X, 2L - 3\delta)$ is isomorphic to the EPW cube $\widetilde{Z}^{\geq 2}_A$ with its canonical polarization. Proposition 3.7 reproves that $\wp_3(\Delta)$ is the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$.

(c) When $c \not\equiv 1 \pmod 3$ (and $c > 0$), so that $e \equiv 1, 7, 13 \pmod{18}$, the movable and nef cones of $S^{[3]}$ coincide ([**BC22**, Proposition 5.4]), the involution ι is regular on $S^{[3]}$, and the class $2L - b_0\delta$ is ample on $S^{[3]}$. So the pairs $(S^{[3]}, 2L - b_0\delta)$ do define elements of ${}^{[3]}\mathscr{M}_4^{(2)}$, whose images in the period space cover a dense subset of the Heegner divisor ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$ by Proposition 3.7. Note that for infinitely many values of c, such a general pair will be a honest EPW cube by Proposition 3.9.

(d) When $c \equiv 1 \pmod 3$, so that $e \equiv 5 \pmod{18}$, there are 1 or 3 chambers in the closed movable cone of $S^{[3]}$ ([**BC22**, Proposition 5.4]).

3.3. Image of the period map of EPW cubes

Using [**Deb22**, Appendix B], we determine the image of the period map

$$\wp_3 \colon {}^{[3]}\mathscr{M}_4^{(2)} \hookrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

of polarized hyper-Kähler sixfolds of K3$^{[3]}$-type with polarization of square 4 and divisibility 2.

Proposition 3.9. *The complement of the image of the period map*

$$\wp_3 \colon {}^{[3]}\mathscr{M}_4^{(2)} \hookrightarrow {}^{[3]}\mathscr{P}_4^{(2)}$$

is the union

$${}^{[3]}\mathscr{D}_{4,2}^{(2)} \cup {}^{[3]}\mathscr{D}_{4,4}^{(2)} \cup {}^{[3]}\mathscr{D}_{4,8}^{(2)} \cup {}^{[3]}\mathscr{D}_{4,12}^{(2)}$$

of irreducible Heegner divisors.

Proof. We keep the notation of Section 3.2.1. The image of the period map is described in [**Deb22**, Remark B.6] (using [**Deb22**, Proposition B.5]) as the complement of the Heegner divisors of the form ${}^{[3]}\mathscr{D}_{4,K}$, where K is a primitive rank 2 sublattice of $\Lambda_{K3^{[3]}}$ such that $h \in K$ and there exist integers $0 \le k \le 2$ and $a \ge -1$ and $\lambda \in K$ with

$$\lambda^2 = 4(8a - k^2) < 0, \qquad \lambda \cdot h = 0, \qquad 4 \mid \mathrm{div}_{\Lambda_{K3^{[3]}}}(\lambda). \tag{54}$$

Let β be the primitive generator of $K \cap h^\perp$ such that $\lambda = x\beta$ for some $x \in \mathbf{Z}^{>0}$. Then Heegner divisor associated to K is therefore equal to ${}^{[3]}\mathscr{D}_{4,2e}^{(2)}$, where

$$2e = -\frac{4\beta^2}{\gamma^2},$$

for $\gamma := \mathrm{div}_{h^\perp}(\beta)$, and $4 \mid e$ if and only if $\gamma = 1$ (see Remark 3.4). Thus, the vector λ has square

$$\lambda^2 = x^2\beta^2 = \frac{-e\gamma^2x^2}{2}, \tag{55}$$

and the divisibility of λ in $\Lambda_{K3^{[3]}}$ divides $\mathrm{div}_{h^\perp}(\lambda) = x\gamma$.

Therefore the Heegner divisors in the complement of the image of ${}^{[3]}\mathscr{P}_4^{(2)}$ are the ${}^{[3]}\mathscr{D}_{4,2e}$ such that there exists a primitive vector $\beta \in h^\perp$ of divisibility γ such that

$$\frac{-4\beta^2}{\gamma^2} = 2e,$$

and there exists $x \in \mathbf{Z}$ solving

$$ex^2\gamma^2 = -2\lambda^2 = 8(k^2 - 8a) > 0, \qquad 4 \mid x\gamma \tag{56}$$

for some $0 \leq k \leq 2$ and $a \geq -1$.

Since $4 \mid x\gamma$, we obtain $2 \mid k^2 - 8a$, therefore k is either 0 or 2; moreover x is an even integer $2y$ (because γ is equal to 1 or 2). Hence, Equation (56) is equivalent to

$$ey^2\gamma^2 = 2(k^2 - 8a) > 0, \qquad 2 \mid y\gamma,$$

for $k \in \{0, 2\}$ and $a \geq -1$. We have:

- if $k = 0$, then $a = -1$ and Equation (56) is equal to $y^2 e = 16$ for $2 \mid y\gamma$. If $\gamma = 1$, y is even and $4 \mid e$ (from (55)), hence $2e = 8$. If $\gamma = 2$, then $4 \nmid e$, hence $2e$ is either 2 or 4.
- if $k = 2$, then a is either -1 or 0. In this case, γ is equal to 2: indeed 16 does not divide $2(4 - 8a)$ but for $\gamma = 1$, we have $2 \mid y$ and $4 \mid e$. Hence for $a = 0$, we obtain $ey^2 = 2$, namely $2e = 4$, and for $a = -1$ we have $ey^2 = 6$, namely $2e = 12$.

By Remark 3.4, in all cases as above, there exist a vector of divisibility 1 with associated Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$, and vectors of divisibility 2 with associated Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,2e}$ for $2e \in \{2, 4, 12\}$. □

Remark 3.10. In [**Deb22**, Appendix B], the authors explain that a Heegner divisor ${}^{[3]}\mathscr{D}_{4,K}$, with $h \in K$, is disjoint from the image of the period map if and only if $(K \cap h^\perp)^\perp$ (that is $\beta^\perp$ in the notation of the previous proof) defines a wall in the positive cone of X (see [**Deb22**, Proposition B.4 and Theorem B.1]).

Therefore, Proposition 3.9 and Remark 3.4 give vectors $\beta \in h^\perp$ that are wall divisors (in the sense of [**Mon15**, Definition 1.2]). In [**Mon15**, Section 2.2] (see also [**HT10**, Table H3]), Mongardi lists the various wall divisors that occur for hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$ type (with no polarization). These are given by primitive vectors $D \in \Lambda_{\mathrm{K3}^{[3]}}$ with square and divisibility in the following list:

$$(D^2, \mathrm{div}_{\Lambda_{\mathrm{K3}^{[3]}}}(D)) \in \{(-12, 2), \quad (-36, 4), \ (-2, 1), \ (-4, 4), \ (-4, 2)\}. \tag{57}$$

Writing down the embedding $h^\perp \subset \Lambda_{\mathrm{K3}^{[3]}}$, we can relate the Heegner divisors of Proposition 3.9 to the wall divisors listed in (57). The situation is the following.[1]

	β^2	$\mathrm{div}_{\Lambda_{\mathrm{K3}^{[3]}}}(\beta)$
${}^{[3]}\mathscr{D}^{(2)}_{4,2}$	-2	1
${}^{[3]}\mathscr{D}^{(2)}_{4,4}$	-4	2
${}^{[3]}\mathscr{D}^{(2)}_{4,8}$	-2	1
${}^{[3]}\mathscr{D}^{(2)}_{4,12}$	-12	2

[1] Note that there are no primitive vectors in $h^\perp$ with divisibility 4 in $\Lambda_{\mathrm{K3}^{[3]}}$, since $\mathrm{div}_{\Lambda_{\mathrm{K3}^{[3]}}}(v) \mid \mathrm{div}_{h^\perp}(v)$ and the divisibility of any primitive vector in $h^\perp$ divides 2. So the second and fourth possibilities in the list (57) do not occur for us.

CHAPTER 4

Degenerations of EPW cubes

The aim of this chapter is to study the geometry of some (singular) degenerations of EPW cubes, whose period is a general point of some of the Heegner divisors described in Chapter 3.

Theorem 3.1 shows that the following diagram is commutative

$$\begin{array}{ccccc} \mathrm{LG}(\bigwedge^3 V_6) & \overset{\mu_2}{\dashrightarrow} & {}^{[2]}\mathscr{M}_2^{(1)} & \overset{\wp_2}{\hookrightarrow} & {}^{[2]}\mathscr{P}_2^{(1)} \\ & \searrow^{\mu_3} & & & \downarrow \rho \\ & & {}^{[3]}\mathscr{M}_4^{(2)} & \underset{\wp_3}{\hookrightarrow} & {}^{[3]}\mathscr{P}_4^{(2)}, \end{array} \tag{58}$$

where μ_2 and μ_3 are the moduli morphisms constructed in (29) and (41), and $\wp_2$ and $\wp_3$ are the polarized period maps.

We summarize here the known information about special Heegner divisors in ${}^{[3]}\mathscr{P}_4^{(2)}$, using the notation introduced in Section 3.2.1.

- The ramification divisor of the double cover ρ is the Heegner divisor ${}^{[2]}\mathscr{D}_{2,4}^{(1)}$ ([**Riz24**, Corollary 6.1]), which does not meet the image of μ_2 by [**O'G15**, Theorem 1.3].
- [**DM19**, Example 6.3] and Proposition 3.3 show that the image of $\wp_2$ is the complement of the union of the Heegner divisors ${}^{[2]}\mathscr{D}_{2,2}^{(1)}$, ${}^{[2]}\mathscr{D}_{2,8}^{(1)}$, and one component of ${}^{[2]}\mathscr{D}_{2,10}^{(1)}$.
- Proposition 3.9 shows that the image of $\wp_3$ is the complement of the union of the Heegner divisors ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$, ${}^{[3]}\mathscr{D}_{4,4}^{(2)}$, ${}^{[3]}\mathscr{D}_{4,8}^{(2)}$, and ${}^{[3]}\mathscr{D}_{4,12}^{(2)}$, which we will call boundary divisors.
- Proposition 3.5 shows that the map $\wp_3 \circ \mu_3$ (which has been extended to $\mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma'$ in Section 3.2) sends the divisor Σ to ${}^{[3]}\mathscr{D}_{4,8}^{(2)}$, the divisor Δ to ${}^{[3]}\mathscr{D}_{4,10}^{(2)}$, and the divisor Γ to ${}^{[3]}\mathscr{D}_{4,12}^{(2)}$.

In Section 4.1, we study the singular EPW cubes obtained from Lagrangian subvarieties A in the divisor Γ, so that their (generalized) periods are in the boundary divisor ${}^{[3]}\mathscr{D}_{4,12}^{(2)}$. By Theorem 2.14, these varieties have finite singular locus. We show that they admit a projective small resolution by a hyper-Kähler manifold contracting a $\mathbf{P}^3$ over each singular point.

In Section 4.2, we study in detail the geometry of the singular hyper-Kähler varieties whose (generalized) periods are general points in the boundary divisor ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$.

We then use this degeneration to study the geometry of the fixed locus of the canonical antisymplectic involution of EPW cubes. We show in particular that it is irreducible and rigid inside the EPW cube.

Finally, in Section 4.3, we discuss the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$ and the other two boundary divisors ${}^{[3]}\mathscr{D}^{(2)}_{4,4}$ and ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$.

4.1. The divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,12}$: singular EPW cubes

In this section, we study the singular sixfold $\widetilde{\mathsf{Z}}^{\geq 2}_A$ when $[A] \in \Gamma \setminus \Sigma$. Adapting the methods of [**O'G13a**], we prove the following results:

(a) The scheme $\mathsf{Z}^{\geq 4}_A = \{z_1, \dots, z_r\}$ is finite and smooth, and equals Z^4_A; moreover, if $[A]$ is general in Γ, then Z^4_A is a singleton ($r = 1$).

(b) The singular set of the sixfold $\widetilde{\mathsf{Z}}^{\geq 2}_A$ is equal to

$$g^{-1}(\mathsf{Z}^4_A) = \{g^{-1}(z_1), \dots, g^{-1}(z_r)\}.$$

For each $i \in \{1, \dots, r\}$, the tangent cone to $\widetilde{\mathsf{Z}}^{\geq 2}_A$ at the point $g^{-1}(z_i)$ is isomorphic to the (affine) cone over the incidence variety $I \subset \mathbf{P}^3 \times (\mathbf{P}^3)^\vee$. Therefore, the exceptional divisor of the blowup $\widetilde{X}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ along $g^{-1}(\mathsf{Z}^4_A)$ is the disjoint union of smooth divisors $E_1, \dots, E_r$, all isomorphic to I and with normal bundles $\mathcal{O}_{E_i}(-1,-1)$. For any choice $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ of contractions of each E_i onto either $\mathbf{P}^3$ or $(\mathbf{P}^3)^\vee$, we obtain an (analytic) small resolution $\mathsf{X}^\varepsilon_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ with exceptional locus a disjoint union of r copies of $\mathbf{P}^3$.

(c) There exists a choice of ε such that X^ε_A is a projective smooth quasipolarized hyper-Kähler sixfold with a projective contraction $\mathsf{X}^\varepsilon_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ of r copies of $\mathbf{P}^3$.

Item (a) has already been proved in Theorem 2.5 and Theorem 3.6. For Item (b), we study the desingularization of $\widetilde{\mathsf{Z}}^{\geq 2}_A$ in families. Thanks to the local description carried out in Proposition 2.10, we reduce ourselves to studying the "universal double cover" of the scheme $\mathcal{S}_{\leq 2}$ of (4×4)-matrices of rank ≤ 2. This is done in Section 4.1.1. In Section 4.1.2, we obtain a description of the blowup in families, from which we deduce a description of the blowup of $\widetilde{\mathsf{Z}}^{\geq 2}_A$.

The construction of the small analytic resolutions $\mathsf{X}^\epsilon_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ is analogous to the one in [**O'G13a**, Sections 3.1 and 3.2]. We do it in Proposition 4.4.

The most subtle part of our results is Item (c), that is, proving that there exists a small *projective* resolution of $\widetilde{\mathsf{Z}}^{\geq 2}_A$ (as in the double EPW sextic case, if Z^4_A has more than one point, we do not expect the small resolutions $\mathsf{X}^\varepsilon_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ to be all projective – that is, Kähler). For double EPW sextics, this was done by O'Grady, by explicitly constructing an isomorphism between the Hilbert square $S^{[2]}_A$ and one of these analytic resolutions (see Theorem 2.4). We cannot hope for an analogous construction in our case, because there are no associated K3 surfaces, and Y_A cannot be realized, even birationally, as a moduli space of sheaves over a K3 surface (we prove this in Proposition 4.9).

However, using the properties of the period map for EPW cubes and the surjectivity of the period map for hyper-Kähler sixfolds, we produce in Proposition 4.5 a

smooth projective hyper-Kähler sixfold Y_A with a big and nef line bundle H that induces a small contraction $\mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$; the pair (Y_A, H) is a deformation of standard (smooth) EPW cubes. Finally, using results from [**KP25**], we show in Theorem 4.7 that Y_A is actually a divisorial contraction of the blowup $\widetilde{\mathsf{X}}_A$, hence one of the small analytic resolutions $\mathsf{X}_A^{\varepsilon}$.

The contents of this section appeared in [**Riz25b**]

4.1.1. Local model of double cover. In this section, we consider a "canonical example" of double cover. Let V be a vector space of dimension n. Consider the universal quadratic form

$$q\colon V \otimes \mathcal{O}_{\mathrm{Sym}^2 V^\vee} \longrightarrow V^\vee \otimes \mathcal{O}_{\mathrm{Sym}^2 V^\vee}.$$

Denote by $\mathcal{S}_{\leq i} \subset \mathrm{Sym}^2 V^\vee$ the rank i degeneracy locus of q, so that

- $\mathcal{S}_{\leq 2}$ is the affine cone over the secant variety to the second Veronese embedding $\mathbf{P}(V^\vee) \subset \mathbf{P}(\mathrm{Sym}^2 V^\vee)$,
- $\mathcal{S}_{\leq 1}$ is the affine cone over the same embedding $\mathbf{P}(V^\vee) \subset \mathbf{P}(\mathrm{Sym}^2 V^\vee)$,
- $\mathcal{S}_{\leq 0} = \{0\}$.

In this situation, [**DK20a**, Theorem 3.1] provides a canonical double cover

$$\widetilde{f}\colon \widetilde{\mathcal{S}}_{\leq 2} \to \mathcal{S}_{\leq 2}.$$

Lemma 4.1. *The affine variety* $(V^\vee \otimes V^\vee)_1 \coloneqq \{\mu \in V^\vee \otimes V^\vee \mid \mathrm{rk}(\mu) \leq 1\}$ *is the cone over the Segre variety* $\mathbf{P}(V^\vee) \times \mathbf{P}(V^\vee)$. *The map*

$$\begin{aligned} g_2\colon (V^\vee \otimes V^\vee)_1 &\longrightarrow \mathcal{S}_{\leq 2} \\ \mu &\longmapsto \mu + \iota(\mu), \end{aligned}$$

where ι *is the involution that exchanges the two factors, is the canonical double cover* $\widetilde{f}\colon \widetilde{\mathcal{S}}_{\leq 2} \to \mathcal{S}_{\leq 2}$. *In particular, the unique singular point of the variety* $\widetilde{\mathcal{S}}_{\leq 2}$ *is* 0.

Proof. We follow the proof of [**DK20a**, Lemma 3.4]. Note that f is a double cover because it is the quotient by the action of the involution ι on $(V^\vee \otimes V^\vee)_1$; it is moreover étale over $\mathcal{S}^0_{\leq 2}$. Thus, since $(V^\vee \otimes V^\vee)_1$ is normal, in order to show that g_2 and $\widetilde{f}$ are isomorphic, it is enough to show that their associated reflexive sheaves are the same.

Let R be the ring of functions of the affine variety $(V^\vee \otimes V^\vee)_1$. If we denote by $S^\bullet(V \otimes V)$ the symmetric algebra of $V \otimes V$, we can identify R with the quotient $S^\bullet(V \otimes V)/I$, where I is the ideal generated by

$$\{(v \otimes w)(v' \otimes w') - (v \otimes w')(v' \otimes w) \mid v, w, v', w' \in V\}.$$

Note that there is an isomorphism

$$R \simeq \bigoplus_{i=0}^{\infty} S^i(V) \otimes S^i(V) \tag{59}$$

that sends the class of a monomial $(v_1 \otimes w_1) \cdots (v_j \otimes w_j) \in S^j(V \otimes V)$ to $(v_1 \cdots v_j) \otimes (w_1 \cdots w_j)$.

Given a subset P of $S^\bullet(V \otimes V)$, we denote by $[P]$ its image in $S^\bullet(V \otimes V)/I$.

The action of $\mathbf{Z}/2\mathbf{Z}$ on $(V^\vee \otimes V^\vee)_1$ induces an action on R given by the involution ι that sends $[v \otimes w]$ to $[w \otimes v]$. We have a decomposition

$$R = R_+ \oplus R_-, \tag{60}$$

where R_+ is the invariant part and R_- is the antiinvariant part with respect to this action. Under the isomorphism (59), their graded parts are given by

$$R_+ = \bigoplus_{i=0}^{\infty} S^2(S^i V) \quad \text{and} \quad R_- = \bigoplus_{i=0}^{\infty} \bigwedge^2(S^i V).$$

Note that R_- is the vector space generated by $[\alpha - \iota(\alpha)]$, for all monomials α in $S^\bullet(V \otimes V)$. Any monomial α in $S^j(V \otimes V)$ can be written as $\alpha = (v \otimes w)\beta$, where $\beta \in S^{j-1}(V \otimes V)$. Thus, by induction on j, using the equality

$$2(\alpha - \iota(\alpha)) = (v \otimes w - w \otimes v)(\beta + \iota(\beta)) + (v \otimes w + w \otimes v)(\beta - \iota(\beta)),$$

we see that R_- is generated by $[\bigwedge^2 V]$ as an R_+-module.

We have then $\mathcal{S}_{\leq 2} = \operatorname{Spec}(R_+)$, the direct sum (60) provides the decomposition of the sheaf $g_{2,*}\mathcal{O}_{(V^\vee \otimes V^\vee)_1}$ into invariant and antiinvariant parts, and the reflexive sheaf associated to g_2 is the sheaf associated with the module R_-.

The restriction of q to $\mathcal{S}_{\leq 2}$ is induced by the morphism

$$\begin{array}{rcl} q\colon V \otimes R_+ & \longrightarrow & V^\vee \otimes R_+ = \operatorname{Hom}(V \otimes R_+, R_+) \\ v \otimes 1 & \longmapsto & (w \otimes 1 \mapsto \frac{1}{2}[v \otimes w + w \otimes v]) \end{array}$$

of free R_+-modules. Set $\mathcal{S}_2 := \mathcal{S}_{\leq 2} \smallsetminus \mathcal{S}_{\leq 1}$. We show that the sheaf $\det(\operatorname{Im}(q|_{\mathcal{S}_2}))$ (which is, by the construction of $\widetilde{f}$ in [**DK20a**, Theorem 3.1], the reflexive sheaf associated to $\widetilde{f}$) is the sheaf associated to the R_+-module R_-. Indeed, over $\mathcal{S}_2$, the image of q is locally free of rank 2, thus

$$\det(\operatorname{Im}(q|_{\mathcal{S}_2})) = \bigwedge^2 \operatorname{Im}(q|_{\mathcal{S}_2}) = \operatorname{Im}\big(\bigwedge^2 q|_{\mathcal{S}_2}\big).$$

The morphism

$$\bigwedge^2 q\colon \bigwedge^2 V \otimes R_+ \longrightarrow \operatorname{Hom}(\bigwedge^2 V \otimes R_+, R_+)$$

sends an element $(v_1 \wedge v_2) \otimes 1 \in \bigwedge^2 V \otimes R_+$ to the dual form defined by

$$(w_1 \wedge w_2) \otimes 1 \longmapsto q(v_1, w_1)q(v_2, w_2) - q(v_1, w_2)q(v_2, w_1).$$

Upon expanding the latter expression, we obtain that $\bigwedge^2 q$ factors (up to a constant) as

$$\begin{array}{ccccc} \bigwedge^2 V \otimes R_+ & \longrightarrow & R_- & \longrightarrow & \operatorname{Hom}(\bigwedge^2 V \otimes R_+, R_+) \\ (v_1 \wedge v_2) \otimes 1 & \longmapsto & v_1 \wedge v_2 & \longmapsto & \big((w_1 \wedge w_2) \otimes 1 \mapsto -(w_1 \wedge w_2)(v_1 \wedge v_2)\big), \end{array}$$

where we identify $\bigwedge^2 V$ with $(R_-)_1$.

Note that the first morphism is surjective, since R_- is generated by $\bigwedge^2 V$ as an R_+-module, while the second morphism is injective. Thus $\operatorname{Im}\big(\bigwedge^2 q|_{\mathcal{S}_2}\big)$ is the sheaf associated to R_-, as wanted. □

Let $\rho\colon \widetilde{T} \to \widetilde{\mathcal{S}}_{\leq 2}$ be the blowup of the unique singular point 0 of $\widetilde{\mathcal{S}}_{\leq 2}$ and let $H \subset \operatorname{Sym}^2 V^\vee$ be a general hyperplane.

Proposition 4.2 (proposition)**.** *The exceptional divisor E of ρ is isomorphic to $\mathbf{P}(V^\vee) \times \mathbf{P}(V^\vee)$ and the normal bundle to E in $\widetilde{T}$ is isomorphic to $\mathcal{O}_{\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)}(-1,-1)$.*

Moreover, the intersection of the strict transform of $g_2^{-1}(H)$ with E is a general hyperplane section of E.

Proof. The description in Lemma 4.1 of $\widetilde{\mathcal{S}}_{\leq 2}$ as the cone over the Segre variety $\mathbf{P}(V^\vee) \times \mathbf{P}(V^\vee)$ implies the description of the exceptional divisor E and its normal bundle.

Choose coordinates $(x_i)_{1\leq i\leq n}$ on $V^\vee$, which induce coordinates $(s_{ij})_{1\leq i\leq j\leq n}$ on $\operatorname{Sym}^2 V^\vee$. The inverse image $g_2^{-1}(H)$ can then be identified with the cone over the preimage of the hyperplane $s_{11}+\cdots+s_{nn}=0$ by the morphism

$$\begin{aligned}\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee) &\longrightarrow \mathbf{P}(\mathcal{S}_{\leq 2}) \subset \mathbf{P}(\operatorname{Sym}^2 V^\vee)\\ ([(x_i)],[(y_j)]) &\longmapsto [(x_iy_j+y_jx_i)]_{1\leq i\leq j\leq n}.\end{aligned}$$

It is therefore given by the equation $x_1y_1+\cdots+x_ny_n=0$. Thus, the strict transform of $g_2^{-1}(H)$ intersects the exceptional divisor $\mathbf{P}(V^\vee)\times\mathbf{P}(V^\vee)$ in a general hyperplane section. □

4.1.2. Singularities of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ for $[A]$ in $\Gamma\smallsetminus\Sigma$ and simultaneous resolution. We go back to EPW cubes. Consider the families

$$\mathscr{Z}^{\geq k} \longrightarrow \mathrm{LG}(\textstyle\bigwedge^3 V_6)$$

defined in Section 2.3.3. Let $[A_0]\in\Gamma\smallsetminus\Sigma$ and let $\mathcal{V}\subset \mathrm{LG}(\bigwedge^3 V_6)\smallsetminus\Sigma$ be a (sufficiently small classical) open neighborhood of $[A_0]$. We set $\mathscr{Z}_{\mathcal{V}}^{\geq k} := \mathscr{Z}^{\geq k}\cap(\mathcal{V}\times\mathsf{Gr}(3,V_6))$. As in the case of EPW sextics (see the proof of [**DK20a**, Theorem 5.2]), there exists a unique double cover

$$f_{\mathcal{V}}\colon \widetilde{\mathscr{Z}_{\mathcal{V}}} \longrightarrow \mathscr{Z}_{\mathcal{V}}^{\geq 2} \tag{61}$$

branched along $\mathscr{Z}_{\mathcal{V}}^{\geq 3}$, whose restriction over each $\mathsf{Z}_A^{\geq 2}$ is the morphism $g\colon \widetilde{\mathsf{Z}}_A^{\geq 2}\to \mathsf{Z}_A^{\geq 2}$.

The next theorem explains the geometry of $\widetilde{\mathscr{Z}_{\mathcal{V}}}$; note that the situation is completely analogous (in dimension 6 instead of 4) to the situation for double EPW sextics described in [**O'G13a**, Claim 3.8].

Theorem 4.3. *With notation as previously, we have the following.*

(a) *The singular locus of $\widetilde{\mathscr{Z}_{\mathcal{V}}}$ is $f_{\mathcal{V}}^{-1}(\mathscr{Z}_{\mathcal{V}}^{\geq 4})$ and it is finite over $\mathcal{V}$.*

(b) *The blowup*

$$\rho_{\mathcal{V}}\colon \widetilde{\mathscr{X}_{\mathcal{V}}} \longrightarrow \widetilde{\mathscr{Z}_{\mathcal{V}}}$$

of $f_{\mathcal{V}}^{-1}(\mathscr{Z}_{\mathcal{V}}^{\geq 4})$ in $\widetilde{\mathscr{Z}_{\mathcal{V}}}$ is smooth.

(c) *The morphism $f_{\mathcal{V}}\circ\rho_{\mathcal{V}}$ restricted to the exceptional divisor $E_{\mathcal{V}}$ of $\rho_{\mathcal{V}}$ induces a locally trivial fibration*

$$E_{\mathcal{V}} \longrightarrow \mathscr{Z}_{\mathcal{V}}^{\geq 4}$$

with fibers $\mathbf{P}^3\times\mathbf{P}^3$, and the restriction of the normal bundle to $E_{\mathcal{V}}$ in $\widetilde{\mathscr{X}_{\mathcal{V}}}$ to each fiber is $\mathcal{O}(-1,-1)$.

(d) *For any $[A]\in\mathcal{V}\cap\Gamma$, the strict transform $\widetilde{\mathsf{X}}_A$ of $\widetilde{\mathsf{Z}}_A^{\geq 2}\subset\widetilde{\mathscr{Z}_{\mathcal{V}}}$ is smooth and intersects $E_{\mathcal{V}}$ in a general hyperplane section of $\mathbf{P}^3\times\mathbf{P}^3$.*

Proof. For any $([A],[U])\notin\mathscr{Z}_{\mathcal{V}}^{\geq 4}$, consider an open neighborhood $\mathcal{V}'\times\mathcal{U}\subset\mathcal{V}\times\mathsf{Gr}(3,V_6)$ of $([A],[U])$ such that $\mathscr{Z}^{\geq 4}\cap(\mathcal{V}'\times\mathcal{U})=\varnothing$. By uniqueness of the double cover constructed in [**DK20a**], the variety $\widetilde{\mathscr{Z}_{\mathcal{V}}}\cap(\mathcal{V}'\times\mathcal{U})$ is the double cover of $\mathscr{Z}^{\geq 2}\cap(\mathcal{V}'\times\mathcal{U})$, hence is smooth by [**DK20a**, Corollary 4.8]. Therefore, the singular locus of $\widetilde{\mathscr{Z}_{\mathcal{V}}}$ is contained in $f_{\mathcal{V}}^{-1}(\mathscr{Z}_{\mathcal{V}}^{\geq 4})$.

Let us work now around a point $([A_0],[U_0])$ of $\mathscr{Z}_{\mathcal{V}}^{\geq 4}$. By Proposition 2.10 (with $k=2$ and $\ell=4$), the analytic germ of $\mathscr{Z}_{\mathcal{V}}^{\geq 2}$ at that point is the product of a smooth germ and the germ of $\mathcal{S}_{\leq 2}$ at 0, where $\mathcal{S}_{\leq 2}\subset \mathrm{Sym}^2(A_0\cap T_{U_0})^\vee$ is the subscheme of quadratic forms of rank ≤ 2. In particular, locally, we have a diagram

$$\begin{array}{ccc} \widetilde{\mathscr{Z}_{\mathcal{V}}} & \longrightarrow & \widetilde{\mathcal{S}}_{\leq 2} \\ {\scriptstyle f_{\mathcal{V}}}\downarrow & & \downarrow{\scriptstyle g_2} \\ \mathscr{Z}_{\mathcal{V}}^{\geq 2} & \xrightarrow{\Psi_K} & \mathcal{S}_{\leq 2}, \end{array}$$

where Ψ_K is smooth and g_2 is the canonical double cover of $\mathcal{S}_{\leq 2}$ (see Lemma 4.1). Moreover, the variety $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is the product of a smooth germ with $g_2^{-1}(\mathrm{Im}(r_K)\cap\mathcal{S}_{\leq 2})$, where $\mathrm{Im}(r_K)\subset \mathrm{Sym}^2(A_0\cap T_{U_0})^\vee$ is a general hyperplane (Lemma 2.9).

Therefore, it is enough to show (a), (b), (c), and (d) for the scheme $\widetilde{\mathcal{S}}_{\leq 2}$ described in Lemma 4.1) and the preimage of a general hyperplane section of $\mathcal{S}_{\leq 2}$. This was done in Section 4.1.1. □

Given $[A_0]\in\Gamma\smallsetminus\Sigma$ with $\mathsf{Z}_{A_0}^{\geq 4}=\{z_1,\dots,z_r\}$, we have shown that the blowup of $\widetilde{\mathsf{Z}}_{A_0}^{\geq 2}$ along it singular locus $\mathsf{Z}_{A_0}^{\geq 4}$ is

$$\tau\colon \widetilde{\mathsf{X}}_{A_0}\longrightarrow \widetilde{\mathsf{Z}}_{A_0}^{\geq 2},$$

with exceptional divisor $E=E_1\sqcup\cdots\sqcup E_r$, the disjoint union of r copies of the incidence variety $I\subset \mathbf{P}^3\times(\mathbf{P}^3)^\vee$ (this is a general hyperplane section), and $\mathcal{O}(E)|_{E_i}=\mathcal{O}_{E_i}(-1,-1)$.

The situation is therefore exactly as in [**O'G13a**, Section 3.2], which we follow without repeating the proofs. For each $i\in\{1,\dots,r\}$, choose a projection $E_i\to\mathbf{P}^3$. We call that a choice ε of $\mathbf{P}^3$-fibrations for $\mathsf{Z}_{A_0}^{\geq 2}$.

Proposition 4.4. *Let $[A_0]\in\Gamma\smallsetminus\Sigma$ and let ε be a choice of $\mathbf{P}^3$-fibrations for $\mathsf{Z}_{A_0}^{\geq 2}$. There exists a classical open neighborhood $\mathcal{V}$ of $[A_0]$ in $\mathrm{LG}(\bigwedge^3 V_6)\smallsetminus\Sigma$ such that*

(a) *There is a factorization*

$$\rho_{\mathcal{V}}\colon \widetilde{\mathscr{X}_{\mathcal{V}}}\xrightarrow{d_\varepsilon}\mathscr{X}_{\mathcal{V}}^{\varepsilon}\xrightarrow{c_\varepsilon}\widetilde{\mathscr{Z}_{\mathcal{V}}^{\geq 2}},$$

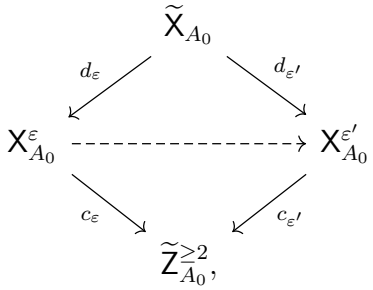
where $\mathscr{X}_{\mathcal{V}}^{\varepsilon}$ is a complex manifold and d_ε induces on each $E_i\subset\widetilde{\mathsf{X}}_{A_0}\subset\widetilde{\mathscr{X}_{\mathcal{V}}}$, the chosen projection $E_i\to\mathbf{P}^3$.

(b) *The composition*

$$\varphi_{\mathcal{V}}\colon \mathscr{X}_{\mathcal{V}}^{\varepsilon}\xrightarrow{c_\varepsilon}\widetilde{\mathscr{Z}_{\mathcal{V}}^{\geq 2}}\longrightarrow\mathcal{V}$$

is smooth and
- *the fiber of $[A]\in\mathcal{V}\smallsetminus\Gamma$ is the EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$,*
- *the fiber $\mathsf{X}_A^{\varepsilon}$ of $[A]\in\mathcal{V}\cap\Gamma$ is a small resolution of the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$, with exceptional locus a disjoint union of $\mathbf{P}^3$ (one for each singular point of $\widetilde{\mathsf{Z}}_A^{\geq 2}$).*

Given another choice ε' of $\mathbf{P}^3$-fibrations, there exists a commutative diagram



where the horizontal dashed arrow is a birational morphism which is the Mukai flop of the exceptional $\mathbf{P}^3$ for which $\varepsilon'_i \neq \varepsilon_i$. Note that we do not claim that $\mathsf{X}^{\varepsilon}_{A_0}$ is projective. In Section 4.1.3, we will show that at least one $\mathsf{X}^{\varepsilon}_{A_0}$ (for a suitable choice of ε) is indeed projective.

4.1.3. Projective resolutions of $\widetilde{\mathsf{Z}}^{\geq 2}_A$. Given a Lagrangian $[A] \in \Gamma \setminus \Sigma$, we described in Section 4.1.2 the blowup $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ of the singular locus of the EPW cube $\widetilde{\mathsf{Z}}^{\geq 2}_A$ and, *in the analytic category,* some small resolutions $h_\varepsilon\colon \mathsf{X}^{\varepsilon}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ together with factorizations

$$\tau\colon \widetilde{\mathsf{X}}_A \xrightarrow{d_\varepsilon} \mathsf{X}^{\varepsilon}_A \xrightarrow{c_\varepsilon} \widetilde{\mathsf{Z}}^{\geq 2}_A.$$

The aim of this section is to show that there exists ε such that the analytic contraction c_ε of Proposition 4.4 is projective. We do so by constructing a projective hyper-Kähler manifold Y_A with a small contraction $\mathsf{Y}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ and then showing that the blowup map τ factors though it. Our construction will use properties of the period map for EPW cubes which we now explain.

Proposition 4.5. *Let $[A] \in \Gamma \setminus \Sigma$. There exists a smooth quasipolarized projective hyper-Kähler manifold (Y_A, H) such that the big and nef line bundle H defines a small contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$.*

Proof. We use the following lemma, whose proof goes as the proof of [**DM19**, Theorem 6.1] and is based on the surjectivity of the global period map (Theorem 1.6).

Lemma 4.6. *Given a point $x \in {}^{[3]}\mathscr{P}^{(2)}_4$, there exists a smooth projective hyper-Kähler sixfold Y with a big and nef line bundle H of square 4 and divisibility 2 with period x.*

Let (Y_A, H) be the quasipolarized hyper-Kähler sixfold associated to the period point $x \coloneqq \overline{\wp}_3([A])$ given by the lemma.

Consider the local universal deformation $f\colon (\mathscr{Y}, \mathcal{H}) \to (B, 0)$ of the pair (Y_A, H), defined in (6). For b general in B, the line bundle $\mathcal{H}_b$ is ample on $\mathscr{Y}_b$ and the pair $(\mathscr{Y}_b, \mathcal{H}_b)$ defines an element of the moduli space ${}^{[3]}\mathscr{M}^{(2)}_4$. Up to shrinking B to a smooth curve containing 0, we can suppose that for each $b \in B$, the point $\wp_3([\mathscr{Y}_b])$ is contained in the image of $\overline{\wp}_3$ and is, for all $b \neq 0$, the period of a (smooth) EPW cube.

Since the line bundle $\mathcal{H}$ is f-big and f-nef, the relative Kawamata–Viehweg vanishing theorem ([**KMM87**, Theorem 1-2-5 and Remark 1-2-6]) implies $\mathbf{R}^{>0} f_* \mathcal{H} = 0$. Furthermore, by the relative Kawamata base point-free theorem ([**KMM87**, Theorem 3-1-1 and Remark 3-1-2]), the $\mathcal{O}_B$-algebra $\bigoplus_{m \geq 0} f_* \mathcal{H}^m$ is finitely generated and

there is a B-morphism $\varphi\colon \mathscr{Y} \to \mathscr{Z}'$ to its projective spectrum such that the line bundle $\mathcal{H}$ is the pullback of a relatively ample line bundle on $\mathscr{Z}'$. The morphism ϕ then induces isomorphisms $\mathscr{Y}_b \xrightarrow{\sim} \mathscr{Z}'_b$ for $b \in B \smallsetminus \{0\}$, and a projective contraction $\mathsf{Y}_A \to \mathsf{Z}'$ on central fibers.

Consider the families

$$\begin{array}{ccccc} & \widetilde{\mathscr{Z}_B^{\geq 2}} & & \mathscr{Z}' & \\ & \searrow & & \swarrow & \\ & & B, & & \end{array} \tag{62}$$

where the morphism $\widetilde{\mathscr{Z}_B^{\geq 2}} \to B$, with central fiber $\widetilde{\mathsf{Z}}_A^{\geq 2}$, is obtained by lifting the curve B to $\mathrm{LG}(\bigwedge^3 V_6)$ using the period map $\wp_3$ as in [**O'G15**], and comes with a relatively ample line bundle (which is the pullback of the hyperplane section of $\mathsf{Z}_A^{\geq 2} \subset \mathsf{Gr}(3, V_6)$ on each fiber).

Over each point of $B \smallsetminus \{0\}$, the fibers of the two morphisms in the diagram (62) are isomorphic polarized hyper-Kähler manifolds with the same period map hence, by the Torelli theorem, there exists an isomorphism between the two families that respects the polarizations. By separatedness of the moduli space of polarized varieties with trivial canonical bundle (see for example [**Bou14**, Theorem 2.1]), the central fibers $\widetilde{\mathsf{Z}}_A^{\geq 2}$ and Z' of these two morphisms are isomorphic.

Therefore, we have a projective birational contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$, defined by the sections of a sufficiently large multiple mH of H. Taking the top self-intersections of both sides of the equality $c^* H_A = mH$, we obtain $m = 1$, so in fact, H defines c.

In particular, $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is isomorphic outside a closed subset of codimension ≥ 2 to an open subset of Y_A, which has trivial canonical bundle. It follows that its smooth locus $(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}$ also has trivial canonical bundle.

It remains to show that c is small. For that, we first show that the image of the exceptional locus of c is the singular locus of $\widetilde{\mathsf{Z}}_A^{\geq 2}$. Consider the restriction

$$c'\colon c^{-1}\big((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}\big) \longrightarrow (\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}$$

of c to the preimage of the smooth locus of $\widetilde{\mathsf{Z}}_A^{\geq 2}$. It is a birational morphism between smooth varieties, hence

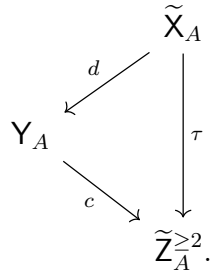
$$K_{c^{-1}((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}})} \underset{\mathrm{lin}}{\equiv} K_{(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}} + \mathrm{Exc}(c').$$

Since the canonical divisors $K_{c^{-1}((\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}})}$ and $K_{(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}}$ are trivial, we obtain that c' is an isomorphism. Therefore, c is an isomorphism over $(\widetilde{\mathsf{Z}}_A^{\geq 2})_{\mathrm{sm}}$. This implies that the image of the exceptional locus of c is the finite set of singular points of $\widetilde{\mathsf{Z}}_A^{\geq 2}$. The contraction is therefore small by [**Nam01**, Proposition (1.4)]. □

Given $[A] \in \Gamma \smallsetminus \Sigma$, we now prove that the blowup $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ along its finite singular locus Z_A^4 factors through the small contraction c constructed above.

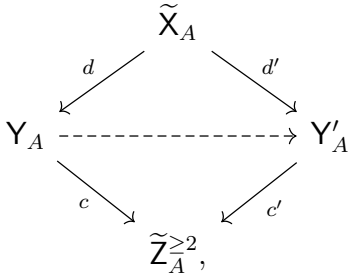
Theorem 4.7. *Let $[A] \in \Gamma \smallsetminus \Sigma$ be a Lagrangian subspace of $\bigwedge^3 V_6$. The resolution $\tau\colon \widetilde{\mathsf{X}}_A \to \widetilde{\mathsf{Z}}_A^{\geq 2}$ obtained by blowing up the finite singular locus of the EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$*

factors through the small contraction c constructed in Proposition 4.5:



The morphism $d\colon \widetilde{\mathsf{X}}_A \to \mathsf{Y}_A$ contracts each component $E_i \subset \mathbf{P}^3 \times \mathbf{P}^3$ of the exceptional locus of τ to a copy of $\mathbf{P}^3$ and the small contraction $c\colon \mathsf{Y}_A \to \widetilde{\mathsf{Z}}^{\geq 2}_A$ contracts each such $\mathbf{P}^3$ to a singular point of $\widetilde{\mathsf{Z}}^{\geq 2}_A$.

Moreover, there exists another projective variety Y'_A and a diagram



where $\mathsf{Y}_A \dashrightarrow \mathsf{Y}'_A$ is the flop of every $\mathbf{P}^3$ in the exceptional locus of c.

We will need the following lemma.

Lemma 4.8. *Let D be a divisor on $\widetilde{\mathsf{X}}_A$ such that the line bundle $\mathcal{O}_{E_i}(D|_{E_i})$ has nonzero sections for each component E_i of E. For any sufficiently ample divisor H on $\widetilde{\mathsf{Z}}^{\geq 2}_A$, the line bundle $\mathcal{O}_{\widetilde{\mathsf{X}}_A}(D + \tau^* H)$ is generated by its global sections.*

Proof. Consider the divisor $\tau_* D$ on $\widetilde{\mathsf{Z}}^{\geq 2}_A$. For H sufficiently ample, $\tau_* D + H$ is globally generated on $\widetilde{\mathsf{Z}}^{\geq 2}_A$. Write $\tau^*(\tau_* D) = D + D'$, where the support of D' is contained in E. For H sufficiently ample, we may also suppose that the linear system $|\tau^* H - D'|$ has no base points outside of E.

The first condition on H implies that, for any $x \notin E$, there exists an effective divisor F on $\widetilde{\mathsf{Z}}^{\geq 2}_A$ linearly equivalent to $\tau_* D + H$, whose support does not contain $\tau(x)$. We then have $\tau^* F = D + D' + \tau^* H$, hence $D + 2\tau^* H \underset{\text{lin}}{\equiv} \tau^* F + \tau^* H - D'$. The second condition on H implies that x is not in the base locus of $D + 2\tau^* H$.

It remains to show that $D + 2\tau^* H$ has no base points on E. The divisor $(D + 2\tau^* H)|_{E_i} = D|_{E_i}$ is effective by assumption hence is globally generated on E_i (like every effective divisor on E_i). Therefore, it is enough to show that we can lift the sections of $(D + 2\tau^* H)|_E$ to $\widetilde{\mathsf{X}}_A$. For that, the vanishing

$$H^1(\widetilde{\mathsf{X}}_A, D + 2\tau^* H - E) = 0$$

is enough. Using the Leray spectral sequence, it is enough to show the vanishings

$$H^1\big(\widetilde{\mathsf{Z}}^{\geq 2}_A, (\tau_* \mathcal{O}_{\widetilde{\mathsf{X}}_A}(D - E))(2H)\big) = H^0\big(\widetilde{\mathsf{Z}}^{\geq 2}_A, (R^1 \tau_* \mathcal{O}_{\widetilde{\mathsf{X}}_A}(D - E))(2H)\big) = 0.$$

The vanishing of the H^1 holds for H sufficiently ample ([**Har77**, Proposition 5.3]). For the vanishing of the H^0, we need to show $R^1\tau_*\mathcal{O}_{\widetilde{\mathsf{X}}_A}(D-E)=0$. This follows from the theorem on formal functions ([**Har77**, Theorem 11.1]), noticing that $H^1(E, D-mE)=0$ for all $m\geq 0$ (because $E|_{E_i}=\mathcal{O}(-1,-1)$ and, again, $D|_{E_i}$ is effective). □

Proof of the theorem. By [**KP25**, Corollary A.2], small normal partial resolutions of $\widetilde{\mathsf{Z}}_A^{\geq 2}$ correspond to nontrivial classes in $\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})/\mathrm{Pic}(\widetilde{\mathsf{Z}}_A^{\geq 2})$ modulo multiplication by positive integers. In particular, the existence of c implies that there exists a Weil divisor class $[D]$ on $\widetilde{\mathsf{Z}}_A^{\geq 2}$ which is not Cartier at any singular point (and such that Y_A is the blowup of $[D]$).

As before, we denote by $z_1,\dots,z_r$ the points of Z_A^4 and by E_i the component of the exceptional divisor E of τ over the point z_i.

Construction of a divisor L on $\widetilde{\mathsf{X}}_A$ whose restriction to each E_i is of the form $\mathcal{O}_{E_i}(a_i,b_i)$ with $a_i\neq b_i$. The blowup τ induces an isomorphism

$$\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})\simeq \mathrm{Pic}(\widetilde{\mathsf{X}}_A)/\bigoplus_i \mathbf{Z}E_i.$$

Let L be a divisor on $\widetilde{\mathsf{X}}_A$ whose class corresponds to the nontrivial Weil divisor class $[D]\in\mathrm{Cl}(\widetilde{\mathsf{Z}}_A^{\geq 2})$ and whose support does not contain any E_i.

Choose a neighborhood $\mathsf{Z}_i\subset\widetilde{\mathsf{Z}}_A^{\geq 2}$ of $g^{-1}(z_i)$ that contains no other singular points and set $\mathsf{X}_i:=\widetilde{\mathsf{X}}_A\times_{\widetilde{\mathsf{Z}}_A^{\geq 2}}\mathsf{Z}_i$ and $\mathsf{Y}_i:=\mathsf{Y}_A\times_{\widetilde{\mathsf{Z}}_A^{\geq 2}}\mathsf{Z}_i$. Then, $\mathsf{X}_i\to\mathsf{Z}_i$ is the blowup of the unique singular point $g^{-1}(z_i)$ of Z_i and $c_i\colon\mathsf{Y}_i\to\mathsf{Z}_i$ is a small projective resolution.

Again by [**KP25**, Corollary A.2], the contraction c_i corresponds to the nontrivial class of the Weil divisor $D_i:=D\cap\mathsf{Z}_i$ in $\mathrm{Cl}(\mathsf{Z}_i)/\mathrm{Pic}(\mathsf{Z}_i)$. As in [**KP25**, Equation (1.1)], there is an injection

$$j_i\colon\ \mathrm{Cl}(\mathsf{Z}_i)/\mathrm{Pic}(\mathsf{Z}_i)\hookrightarrow\mathrm{Pic}(E_i)/\mathbf{Z}(E_i|_{E_i})\simeq\mathbf{Z},$$

where the last isomorphism follows from the Lefschetz Hyperplane theorem.

Therefore, $\mathrm{Cl}(\mathsf{Z}_i)/\mathrm{Pic}(\mathsf{Z}_i)\simeq\mathbf{Z}$ has rank 1 and is generated by the class of D_i. Since j_i is injective, the divisor class $j_i([D_i])=[L|_{E_i}]$ is not a multiple of $[E_i|_{E_i}]\simeq\mathcal{O}(-1,-1)$, hence is of the form $\mathcal{O}_{E_i}(a_i,b_i)$ with $a_i\neq b_i$.

Construction of smooth projective varieties Y_+ and Y_- and divisorial contractions $d_\pm\colon\widetilde{\mathsf{X}}_A\to\mathsf{Y}_\pm$. Let L be the divisor constructed above. There exist integers p_i and q_i such that $L+p_iE_i$ and $-L+q_iE_i$ restrict on E_i to $\mathcal{O}_{E_i}(c_i,0)$ and $\mathcal{O}_{E_i}(0,c_i)$, with $c_i>0$.

We set $L_1:=L+p_1E_1+\cdots+p_rE_r$ and $L_2:=-L+q_1E_1+\cdots+q_rE_r$. These divisors restrict on each E_i to positive multiples of $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. By Lemma 4.8, after adding to L_1 and L_2 the pullback of a sufficiently ample divisor on $\widetilde{\mathsf{Z}}_A^{\geq 2}$, we can suppose that they are globally generated. Therefore, by [**Ish77**, Theorem 1 and Corollary 2], they define morphisms $d_\pm\colon\widetilde{\mathsf{X}}_A\to\mathsf{Y}_\pm$, with $\mathsf{Y}_\pm$ smooth projective, that map each $E_i\subset\mathbf{P}^3\times\mathbf{P}^3$ to either one of the two $\mathbf{P}^3$ and fit into factorizations

$$\tau\colon\widetilde{\mathsf{X}}_A\xrightarrow{d_\pm}\mathsf{Y}_\pm\xrightarrow{c_\pm}\widetilde{\mathsf{Z}}_A^{\geq 2},$$

where c_+ and c_- are small projective contractions that are not isomorphic over each Z_i.

End of the proof. Because of the isomorphism $\mathrm{Cl}(\mathsf{Z}_i)/\operatorname{Pic}(\mathsf{Z}_i) \simeq \mathbf{Z}$, the results of [**KP25**, Appendix A] imply that Z_i has exactly two projective small resolutions, namely $\mathsf{Y}_i^+ = \mathrm{Bl}_{[D_i]}(\mathsf{Z}_i)$ and $\mathsf{Y}_i^- = \mathrm{Bl}_{[-D_i]}(\mathsf{Z}_i)$, and Y_i is one of those. Over each Z_i, the contractions c_+ and c_- are two distinct normal small projective resolutions of Z_i, hence they must coincide with Y_i^+ and Y_i^-. Since Y_i is one of those, we have a morphism $\mathsf{X}_i \to \mathsf{Y}_i$. In this way, we extend the rational map $d \coloneqq c^{-1} \circ \tau \colon \widetilde{\mathsf{X}}_A \dashrightarrow \mathsf{Y}_A$ to each X_i, hence to $\widetilde{\mathsf{X}}_A$, as a regular morphism.

Finally, Y_A is the blowup of, say, $[D]$, and [**KP25**, Corollary A.4.] shows that the blowup Y'_A of $[-D]$ is the flop of Y_A, and the same argument shows that there exists a morphism $d' \colon \widetilde{\mathsf{X}}_A \to \mathsf{Y}'_A$. □

4.1.4. No associated K3 surface. In this section we prove that, when $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}^{\geq 2}_A$ is *not* birationally isomorphic to any smooth moduli space of stable sheaves on a K3 surface. For this, we use the determination of the image by the period map of the divisor Γ obtained in Proposition 3.5 and results from [**PRO25**].

Proposition 4.9. *When $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}^{\geq 2}_A$ is not birationally isomorphic to any smooth moduli space of stable sheaves on a K3 surface.*

Proof. By [**PRO25**, Theorem 3.7], a hyper-Kähler manifold X of $\mathrm{K3}^{[3]}$-type is birationally isomorphic to a smooth moduli space $M_v(S, H)$ for some K3 surface S if and only if X and S have Hodge isometric transcendental lattices. We will determine these lattices when X is a smooth model of a very general singular EPW cube $\widetilde{\mathsf{Z}}^{\geq 2}_A$ as in the proposition and when S is a (necessarily very general) polarized K3 surface, and prove that they cannot be isomorphic.

As shown in Proposition 4.5, when $[A] \in \Gamma \smallsetminus \Sigma$ is very general, the singular EPW cube $\widetilde{\mathsf{Z}}^{\geq 2}_A$ is birationally isomorphic to a quasipolarized projective hyper-Kähler manifold (Y_A, H) of $\mathrm{K3}^{[3]}$-type.

Let $\Lambda_{\mathrm{K3}^{[3]}} = \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}(-4)$ be the lattice associated with hyper-Kähler manifolds of $\mathrm{K3}^{[3]}$-type and let h be a vector of square 4 and divisibility 2 in $\Lambda_{\mathrm{K3}^{[3]}}$. By Proposition 3.5, the period point of Y_A is a very general point in the Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,12}$, hence the transcendental lattice of Y_A is isometric to $K^\perp \subset \Lambda$, where $K \subset \Lambda$ is a discriminant 12 lattice that contains h.

As explained in the proof of Proposition 3.3, the lattice $h^\perp$ is of the form $M \oplus \mathbf{Z}k \oplus \mathbf{Z}\ell$, where $M = N \oplus U$ is unimodular and k and ℓ have square -2, and the lattice $K^\perp$ is equal to $\beta^\perp \subset h^\perp$ for some primitive vector $\beta \in h^\perp$ of square -12, divisibility 2 and class $\beta_* = (1, 1) \in A_{h^\perp}$. Up to isometries of $h^\perp$, this vector is unique by [**Eic74**, Satz 10.4] (see also the discussion in the proof of Proposition 3.3). Therefore, we can take $\beta = 2m + k + l$ with $m = u - v$, where (u, v) is a basis of the lattice $U \subset M$. The transcendental lattice of X is equal to

$$T = N \oplus \begin{pmatrix} 2 & 0 & 1 \\ 0 & -4 & -2 \\ 1 & -2 & -2 \end{pmatrix},$$

where a basis of the nonunimodular part is given by $(u + v, k - \ell, v + k)$.

The transcendental lattice of a very general polarized K3 surface of degree $2e$ is isomorphic to $N \oplus U \oplus \mathbf{Z}(-2e)$. We show that the lattice T defined above is not of this form by proving that there exist no vectors $w \in T$ of divisibility 1 and square 0 orthogonal to N. Such a vector w would be of the form $x(u+v)+y(k-\ell)+z(v+k)$. We compute its divisibility and square in terms of x, y, z:

$$\operatorname{div}(w) = (2x + z, -4y - 2z, x - 2y - 2z)$$

and

$$w^2 = 2x^2 - 4y^2 - 2z^2 + 2xz - 4yz = 2(x^2 - z^2 + xz - 2(y^2 + yz)).$$

We now impose $w^2 = 0$ and $\operatorname{div}(w) = 1$. The first condition implies

$$x^2 - z^2 + xz - 2(y^2 + yz) = 0.$$

In particular, $x^2 + z^2 + xz$ is even, hence x and z are both even, which is absurd since w has divisibility 1. □

Remark 4.10. More generally, [**BMW25**, Corollary 4.8] shows that for $[A] \in \Gamma \setminus \Sigma$ very general, the singular EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is not birationally isomorphic to any moduli space of *twisted* sheaves on a K3 surface.

4.2. The divisor ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$ and geometry of the fixed locus

The goal of this section is twofold:

- describe the family of singular degenerations of EPW cubes arising from the divisor ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$;
- use this family and the degeneration methods of Flapan–Macrì–O'Grady–Saccà, who studied in [**FMOS22**] and [**FMOS23**] the fixed loci of antisymplectic involutions on hyper-Kähler varieties with a polarization of square 2, to compute some numerical invariants of the fixed locus.

In Section 4.2.1, we construct a family $\mathscr{X} \to D$ over a smooth curve, whose general fibers are EPW cubes, and whose special fiber X is a singular degeneration of EPW cubes whose period lies in the Heegner divisor ${}^{[3]}\mathscr{D}_{4,2}^{(2)}$. This family carries an involution $\iota_{\mathscr{X}}$ which restricts to the canonical involution of EPW cubes on general fibers. In Sections 4.2.2 and 4.2.3, we describe in detail the geometry of X, and the action of the restriction ι of the involution $\iota_{\mathscr{X}}$ to the central fiber. Moreover, in Section 4.2.4, we give a local description of X and of $\operatorname{Fix}(\iota)$. Finally, in Section 4.2.5, we study the fixed locus of $\iota_{\mathscr{X}}$ and, in Section 4.2.6, we compute some "external" Hodge numbers of the fixed locus of the antisymplectic involutions of EPW cubes. As corollaries, we obtain that this fixed locus is irreducible (see Corollary 4.20) and does not deform inside the EPW cube (see Corollary 4.25).

The contents of this section appeared in [**Riz25a**, Section 4 and 5].

4.2.1. Hilbert cube of degree 2 K3 surfaces. Let (S, L) be a *very general* polarized K3 surface of degree 2. The polarization L induces a double cover

$$\varphi := \varphi_L \colon S \longrightarrow \mathbf{P}^2$$

branched over a smooth sextic $\Gamma \subset \mathbf{P}^2$ and defines an involution j of S. We have

$$\operatorname{NS}(S^{[3]}) = \mathbf{Z}L \oplus \mathbf{Z}\delta.$$

By [**BC22**, Proposition 4.2 and Theorem 1.1], the closed movable and nef cones of $S^{[3]}$ coincide and, by [**BM14a**, Proposition 13.1], they are spanned by the classes L and $2L-\delta$. The class $2L-\delta$, of square 4 and divisibility 2, is therefore nef and big, but not ample on $S^{[3]}$. Therefore, the extremal ray $\mathbf{R}_{>0}(2L-\delta)$ induces a projective contraction

$$\pi\colon S^{[3]} \longrightarrow X. \tag{63}$$

We show in Lemma 4.11 that this contraction is divisorial and describe its exceptional divisor $\Delta \subset S^{[3]}$.

Let $\bar{L}$ be the ample line bundle on the (singular) variety X such that $\pi^*\bar{L} = L$. The biregular antisymplectic involution $j^{[3]}$ of $S^{[3]}$ induced by the involution j of S acts trivially on $\mathrm{NS}(S^{[3]}) = \mathbf{Z}L \oplus \mathbf{Z}\delta$ (and by $-\mathrm{Id}$ on its orthogonal complement in $H^2(S^{[3]}, \mathbf{Z})$), so it descends to an involution ι on X.

The data $S^{[3]}$, L, Δ, and $j^{[3]}$ satisfy requirements (a)–(d) of [**FMOS22**, Section 2]. Hence, we can apply [**FMOS22**, Proposition 2.1 and Corollary 2.2] (as in [**FMOS23**, Section 3.1]) to obtain a family

$$f_{\mathscr{X}}\colon \mathscr{X} \longrightarrow D \tag{64}$$

over a smooth pointed curve $(D, 0)$ with a relatively ample line bundle $\mathcal{L}$ on $\mathscr{X}$ and a fiberwise involution $\iota_{\mathscr{X}}$ of $\mathscr{X}$ such that:

- the fiber over 0 is the variety X with the ample line bundle $\bar{L}$ and involution ι;
- for any $t \in D \smallsetminus \{0\}$, the pair $(\mathscr{X}_t, \mathcal{L}_t)$ is a smooth EPW cube $\widetilde{\mathsf{Z}}_{A_t}$ (for some Lagrangian $A_t \subset \bigwedge^3 V_6$) with its canonical polarization and ι_t is the natural EPW involution ι_{A_t}.

In this section, we study the special fiber (X, ι) and the fixed locus of ι.

4.2.2. The special fiber of $\mathscr{X} \to D$. The central fiber of the family $\mathscr{X} \to D$ is the singular variety X obtained by the contraction (63) of $S^{[3]}$ defined by the nef and big extremal ray $\mathbf{R}_{\geq 0}(2L-\delta)$.

The manifold $S^{[3]}$ is the moduli space $M_S(1, 0, -2)$ of stable sheaves on S with Mukai vector $v \coloneqq (1, 0, -2)$, where the vector v lies in the Mukai lattice

$$\widetilde{\Lambda} = H^0(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z}) \oplus H^0(S, \mathbf{Z}).$$

Consider the orthogonal complement $v^\perp \subset \widetilde{\Lambda}$ of v in $\widetilde{\Lambda}$. There is an identification

$$\Theta\colon v^\perp \xrightarrow{\sim} H^2(S^{[3]}, \mathbf{Z})$$

such that $L = \Theta(0, -L, 0)$ and $\delta = \Theta(-1, 0, -2)$ ([**BM14a**, Section 13]).

In [**Bri08**], Bridgeland showed that $S^{[3]}$ is also isomorphic to the moduli space $M_\sigma(v)$ of σ-stable objects in $D^b(S)$, where σ is a stability condition contained in the distinguished connected component $\mathrm{Stab}^\dagger(S)$ (see [**Bay18**, Section 3] for the definition).

Contractions of moduli spaces of sheaves on K3 surfaces are described in [**BM14a**]. Their main result is that the wall and chamber decomposition of the movable cone of $M_\sigma(v)$ corresponds to the wall and chamber decomposition induced by v on $\mathrm{Stab}^\dagger(S)$.

Moreover, the contraction associated to a certain wall $\mathcal{W} \subset \mathrm{Stab}^\dagger(S)$ is characterized via [**BM14a**, Theorem 5.7] in terms of the associated hyperbolic lattice $H_{\mathcal{W}} \in H^*_{\mathrm{alg}}(S, \mathbf{Z})$ (see [**BM14a**, Proposition 5.1] for the definition of $H_{\mathcal{W}}$). By

definition, the hyperbolic lattice $H_{\mathcal{W}}$ contains v, and $\mathcal{W}$ corresponds to the wall $\Theta(H_{\mathcal{W}}^{\perp} \cap v^{\perp})$ of the movable cone of $M_\sigma(v)$.

In particular, in our case, contractions of $S^{[3]}$ are given by the two extremal rays $\mathbf{R}_{\geq 0}L$ and $\mathbf{R}_{\geq 0}(2L-\delta)$ of $\mathrm{Nef}(S)$. The extremal ray $\mathbf{R}_{\geq 0}L$ is orthogonal to the isotropic class $w := (0,0,-1)$, which satisfies $w \cdot v = 1$. The induced contraction is the Hilbert–Chow morphism $S^{[3]} \to S^{(3)}$ ([**BM14a**, Theorem 5.7(a)]).

The next lemma describes the contraction π induced by the other extremal ray $\mathbf{R}_{\geq 0}(2L-\delta)$.

Lemma 4.11. *Let (S,L) be a polarized K3 surface of degree 2. The extremal ray $\mathbf{R}_{\geq 0}(2L-\delta)$ induces a divisorial contraction $\pi\colon S^{[3]} \to X$ with smooth exceptional divisor*

$$\Delta = \{Z \in S^{[3]} \mid Z \text{ is contained in some curve } C \in |L|\}.$$

Moreover, the restriction of π to Δ is a $\mathbf{P}^1$-fibration

$$\pi|_\Delta\colon \Delta \longrightarrow T,$$

where the smooth hyper-Kähler fourfold T is the moduli space of stable sheaves on S with Mukai vector $(0,L,-4)$.

Proof. We use the notation of [**BM14a**]. Under the isomorphism Θ, the class $2L-\delta$ is the image of the vector $w = (1,-2L,2)$. In particular, the hyperbolic lattice associated to the contraction induced by $2L-\delta$ is $H = \langle v, (1,-L,2)\rangle = w^\perp \subset H^2_{\mathrm{alg}}(S,\mathbf{Z})$. It is a lattice with intersection form $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$ and it contains the spherical class $s = (1,-L,2)$.

The corresponding potential wall $\mathcal{W} = \mathcal{W}_H$ inside the connected component $\mathrm{Stab}^\dagger(S) \subset \{(\alpha,\beta) \in \mathbf{R}_{>0} \times \mathbf{R}\}$ is a connected component of the (semi)circular wall cut out by the equation

$$(\beta+2)^2 + \alpha^2 = 2. \tag{65}$$

Note that, by [**Bri08**, Lemma 6.2], every $(\alpha,\beta) \in \mathbf{R}_{>0} \times \mathbf{R} \smallsetminus \{(1,-1\}$ that satisfies Equation (65) defines a stability condition, hence the wall $\mathcal{W}$ is equal to

$$\mathcal{W} = \{\sigma_{\alpha,\beta} \mid (\beta+2)^2 + \alpha^2 = 2,\ \alpha \in \mathbf{R}_{>0} \times (-\infty,-1)\}. \tag{66}$$

We analyze the wall $\mathcal{W}$ using [**BM14a**, Theorem 5.7]. We first remark that H contains no isotropic classes: indeed a vector $u = xv + ys \in H$, with $x,y \in \mathbf{Z}$, has square $4x^2 - 2y^2$ which is never 0.

Moreover, the wall $\mathcal{W}$ is not totally semistable. Since $\mathcal{W}$ is nonisotropic, to check this, we only need to show that there are no *effective* spherical classes u with $(u,v) < 0$. A spherical class $u \in H$ is a vector $u = xv + ys$ with $x,y \in \mathbf{Z}$ such that $2x^2 - y^2 = -1$, and the condition $(u,v) < 0$ is equal to $x < 0$. By [**BM14a**, Proposition 5.5], a class u is effective if and only if $\mathrm{Re}\left(\frac{Z_{\alpha,\beta}(u)}{Z_{\alpha,\beta}(v)}\right) > 0$ for one (hence for all) $(\alpha,\beta) \in \mathcal{W}$. In particular, we can choose $\beta = -2$ and $\alpha = \sqrt{2}$.

Using [**Bay18**, Equation (3)], we compute $Z_{\alpha,\beta}(v) = -2\beta(\beta+2+i\alpha)$ and $Z_{\alpha,\beta}(s) = -2(\beta+1)(\beta+2+i\alpha)$ for $(\alpha,\beta) \in \mathcal{W}$. We obtain

$$\frac{Z_{\sqrt{2},-2}(u)}{Z_{\sqrt{2},-2}(v)} = x + y\frac{Z_{\sqrt{2},-2}(s)}{Z_{\sqrt{2},-2}(v)} = x + \frac{y}{2}.$$

Note that for each pair (x, y) with $x < 0$ that satisfies $2x^2 - y^2 = -1$, we have $x + \frac{y}{2} < 0$, thus there are no effective spherical classes u with $(u, v) < 0$.

Therefore the wall $\mathcal{W}$ is nonisotropic not totally semistable, and the lattice H contains the spherical class s, which is effective on $\mathcal{W}$ since $\frac{Z_{\sqrt{2},-2}(s)}{Z_{\sqrt{2},-2}(v)} = 1/2$. By [**BM14a**, Theorem 5.7(a)], the associated contraction π is divisorial of Brill–Noether type. In particular, by [**BM14a**, Lemma 7.1], π contracts a divisor Δ of class $\Theta(s) = L - \delta$. Moreover, since s is the Mukai vector of the line bundle $\mathcal{O}_S(-L)$, the divisor Δ is given by the condition $\mathrm{Hom}(\mathcal{O}_S(-L), \bullet) \neq 0$. In particular, this corresponds to length-3 subschemes $Z \subset S$ that are contained in some curve $C \in |L|$.

More precisely, following the proof of [**BM14a**, Lemma 7.4], the image $T := \pi(\Delta)$ is isomorphic to the moduli space $M_{\sigma_0}(a)$, where $a := v - s = (0, L, -4)$ and $\sigma_0 = \sigma_{\alpha_0,\beta_0}$ is a stability condition lying on the wall $\mathcal{W}$. By [**BM14b**, Example 9.7], we have $M_{\sigma_0}(a) = M_S(a)$ (by choosing $A = 1$, $B = \beta_0$), hence T is a smooth hyper-Kähler fourfold that parametrizes sheaves of the form $i_{C,*}(\xi)$, where $i_C \colon C \hookrightarrow S$ is the inclusion of a curve $C \in |L|$ in S and ξ is a torsion-free rank 1 sheaf on C of degree -3.

Finally, the fiber of π over a point $[A] \in M_S(a)$ is isomorphic to the projective space $\mathbf{P}(\mathrm{Ext}^1(\mathcal{O}_S(-L), A))$. Since

$$\mathrm{ext}^1(\mathcal{O}_S(-L), A) = \langle s, v - s\rangle = -s^2 = 2,$$

the map $\pi|_E \colon E \to M_S(a)$ is a $\mathbf{P}^1$-fibration.

If $C \in |L|$ is smooth (of genus 2), a torsion-free rank 1 sheaf on C of degree -3 is of the form $\mathcal{O}_C(-Z)$, with $Z \in C^{(3)}$. If $A := i_{C,*}(\mathcal{O}_C(-Z))$, the preimage $\pi^{-1}([A])$ is equal to $|\mathcal{O}_C(Z)|$ (they have the same dimension by Riemann–Roch, and any degree 3 effective divisor on C equivalent to Z defines a point in $S^{[3]}$ that is sent to $[A]$ via π). □

We keep the notation of the proof. Let σ_0 be a stability condition on the wall $\mathcal{W}$ defined in Equation (66). The moduli space $M_{\sigma_0}(v)$ has been constructed as a good moduli space in the sense of Alper (see [**Taj23**, Theorem 3.3]). By [**BM14b**, Proposition 8.1], there exists a morphism

$$f \colon X \longrightarrow M_{\sigma_0}(v) \tag{67}$$

which is the identity on $X \setminus \pi(\Delta) \simeq M_\sigma(v) \setminus \Delta$ to $[F]$ and sends $[F] \in M_S(a) \simeq \pi(\Delta)$ to the polystable object $[\mathcal{O}_S(H) \oplus F]$.

In particular, the morphism f is a bijection of sets. We will show in Section 4.2.4 that $M_{\sigma_0}(v)$ is normal; this will imply that f is an isomorphism by Zariski's Main Theorem. By [**AS25**, Theorem 2.3], we obtain that X can be described as a (singular) Bridgeland moduli space, with respect to a stability condition σ_0 lying on the wall $\mathcal{W}$.

4.2.3. Fixed locus on the special fiber. We determine the set-theoretic fixed locus of the involution ι of X. We still denote by $\Gamma \subset S$ the ramification locus of the double cover $\varphi \colon S \to \mathbf{P}^2$, that is, the fixed curve of associated involution j of S.

Proposition 4.12. *The fixed locus of the involution ι of X has two connected components:*

$$\mathrm{Fix}(\iota) = F_2 \sqcup F_3,$$

where

- *F_2 is irreducible of dimension 2 and contained in the fourfold $T = \pi(\Delta)$;*
- *F_3 is irreducible of dimension 3 and its normalization is given by the restriction*
$$\pi|_{\Gamma^{(3)}} : \Gamma^{(3)} \longrightarrow F_3,$$
which is an étale double cover on the smooth divisor $\Delta \cap \Gamma^{(3)} \subset \Gamma^{(3)}$.

Proof. We start by computing the fixed locus of $j^{[3]}$ on $S^{[3]}$. Since $S^{[3]}$ is a hyper-Kähler manifold and $j^{[3]}$ is an antisymplectic regular involution, $\mathrm{Fix}(j^{[3]})$ is smooth Lagrangian. In particular, each component has dimension 3.

Clearly, $\Gamma^{[3]} \subset S^{[3]}$ is fixed by $j^{[3]}$, hence it is one connected component of $\mathrm{Fix}(j^{[3]})$. We write
$$\mathrm{Fix}(j^{[3]}) = \Gamma^{[3]} \sqcup F'.$$
Note that F' is contained in the exceptional divisor Δ. Indeed, if $Z \in S^{[3]}$ is fixed by $j^{[3]}$, its support $\mathrm{Supp}(Z)$ is either contained in Γ or is of the form $\mathrm{Supp}(Z) = \{x + j(x) + y\}$. In particular, if $Z \notin \Gamma^{[3]}$ is fixed by $j^{[3]}$, then $\varphi(Z)$ is supported on two points, hence lies on a line in $\mathbf{P}^2$, so that $Z \in \Delta$.

This characterizes the fixed locus of ι outside the smooth fourfold $W = \pi(\Delta) \subset X$. We study the action of ι on $T = M_S(0, L, -4)$.

Lemma 4.13. *The action of the involution ι on the smooth fourfold $T = M_S(0, L, -4)$ is given by*
$$\iota(i_{C,*}(\xi_3)) = i_{C,*}(\omega_C^{\otimes -3} \otimes \xi_3^\vee), \tag{68}$$
for any $C \in |L|$ and any degree -3 torsion-free rank 1 sheaf ξ_3 on C.

Proof. Suppose that C is smooth and $\xi_3 = \mathcal{O}_C(-Z)$, where $Z = p + q + r \in C^{(3)}$. Then it satisfies $\iota\left(i_{C,*}\left(\mathcal{O}_C(-Z)\right)\right) = i_{C,*}\left(\mathcal{O}_C\left(-j^{[3]}(Z)\right)\right)$.

By the adjunction formula, the canonical bundle of C is equal to $\mathcal{O}_C(L)$. Since $\varphi|_C \colon C \to \mathbf{P}^1$ is a double cover, for any point $p \in C$, we have isomorphisms
$$\omega_C \simeq \mathcal{O}_C(L) \simeq \varphi|_C^*(\mathcal{O}_{\mathbf{P}^1}(\varphi(p))) \simeq \mathcal{O}_C(p + j(p)).$$
Therefore, we can rewrite
$$\mathcal{O}_C\left(-j^{[3]}(Z)\right) = (\mathcal{O}_C(-j(p) - j(q) - j(r))) = \omega_C^{\otimes -3} \otimes \mathcal{O}_C(p + q + r).$$
In particular, formula (68) is verified for any element of T of the form $i_{C,*}(\mathcal{O}_C(-p - q - r))$ where p, q, r are points on C and $C \in |H|$ is a smooth curve.

By Bertini's theorem, these elements form a dense open subset of T, therefore the involution on T is given by formula (68) everywhere. □

Note that upon tensoring by $\mathcal{O}_S(L)$, we obtain an isomorphism
$$T \simeq M_S(0, L, 0),$$
where elements of $M_S(0, L, 0)$ are isomorphism classes of sheaves $i_{C,*}(\xi)$, where $i_C \colon C \to S$ is the inclusion of a curve $C \in |L|$ and ξ is a torsion-free sheaf of rank 1 and degree 1 on C.

Under this isomorphism, the involution ι on T corresponds to the involution ι_0 on $M_S(0, L, 0)$ given by
$$\iota_0(i_{C,*}(\xi)) = i_{C,*}(\omega_C \otimes \xi^\vee),$$
which is one of the involutions studied in [**FMOS22**].

Using the description of the fixed locus of ι_0 in [**FMOS22**, Proposition 4.2], we obtain that the fixed locus of $\iota|_T$ has two connected components W_+ and W_-, both smooth irreducible and 2-dimensional, which are the closures of the loci of $i_{C,*}(\xi\otimes\mathcal{O}(-L))$ where ξ is an even, or odd, theta characteristic on a smooth curve C. Since C has genus 2, a thetacharacteristic ξ on C is odd if and only if $h^0(C,\xi)=1$ (the 6 odd thetacharacteristics correspond to the 6 Weierstrass points $x_1,\dots,x_6$ of C; the 10 even thetacharacteristics correspond to the $x_i+x_j+x_k-g^1_2$, where $i,j,k\in\{1,\dots,6\}$ are distinct—this description counts each of them twice, using $x_1+\cdots+x_6=3g^1_2$).

To conclude, we show that $\mathrm{Fix}(\iota)$ is equal to the image $\pi(\mathrm{Fix}(j^{[3]}))$. Clearly, we have an inclusion $\pi(\mathrm{Fix}(j^{[3]}))\subset\mathrm{Fix}(\iota)$ and they are equal on $X\smallsetminus T$. We restrict ourselves to T, where $\mathrm{Fix}(\iota)\cap T=\mathrm{Fix}(\iota|_T)=W_+\sqcup W_-$ and

$$\pi\left(\mathrm{Fix}(j^{[3]})\right)\cap W=\pi\left(\mathrm{Fix}(j^{[3]})\cap\Delta\right)=\pi\left(\Gamma^{(3)}\cap\Delta\right)\cup\pi(F').$$

Now, $\pi(F')$ is contained in $W_+\sqcup W_-$, which has dimension 2. This implies that F' is contracted by π. We show that $\pi(F')=W_-$. By definition of W_-, it is enough to show that for $i_{C,*}(\xi_3)\in\pi(F')$ general, the element $\xi=\xi_3\otimes\mathcal{O}_C(L)$ is an odd theta characteristic. The general element of $\pi(F')$ is $\iota_{C,*}\mathcal{O}_C(-x-j(x)-y)\in T$, where C is the pullback of the line through $\varphi(x)$ and $\varphi(y)$. Therefore, we obtain $\xi=\mathcal{O}_C(-x-j(x)-y)\otimes\mathcal{O}_C(x+j(x))=\mathcal{O}_C(-y)$, which is an odd theta characteristic by the discussion above, since y belongs to $C\cap\Gamma$. Therefore, the morphism π restricts to a $\mathbf{P}^1$-fibration $F'\to W_-$.

The component $\Gamma^{(3)}$ intersects Δ in a divisor and is disjoint from F', hence $\pi(\Gamma^{(3)}\cap\Delta)$ is equal to W_+. The restriction of π to $\Gamma^3\cap\Delta$ is finite, since the fiber over $i_{C,*}(\xi)$ is contained in $C^{(3)}\cap\Gamma^{(3)}\subset(C\cap\Gamma)^3$ Moreover, for any element $F=i_{C,*}(-Z)\in W_+$, for a smooth curve C, the intersection $\pi^{-1}(F)\cap\Gamma^{(3)}=|\mathcal{O}_C(-Z)|\cap\Gamma^{(3)}$ is given by 2 points, namely Z and the residual intersection $Z':=\Gamma\cdot C-Z$. Therefore, the induced morphism

$$\Gamma^{(3)}\cap\Delta\xrightarrow{2:1}W_+ \tag{69}$$

is a double cover and the corresponding involution sends collinear points $x+y+z\in\Gamma^{(3)}$ on a line ℓ to the residual intersection $\Gamma\cap\ell-(x+y+z)$. A general sextic $\Gamma\subset\mathbf{P}^2$ has no tritangent lines, so this involution is base point-free. The double cover (69) is étale and, since the surface W_+ is smooth, so is $\Gamma^{(3)}\cap\Delta$.

Therefore, $\pi\left(\mathrm{Fix}(j^{[3]})\right)\cap T=\mathrm{Fix}(\iota|_T)$. In particular, the fixed locus of ι is the image of the fixed locus of $j^{[3]}$.

This ends the proof of the proposition, with $F_2:=W_-$ and $F_3:=\pi(\Gamma^{[3]})$. □

Remark 4.14. The scheme F_3 is semi-log-canonical (see [**Kol23**, Chapter 3, Definition 12]). Indeed, its normalization is the smooth scheme $\Gamma^{(3)}$ and the normalization morphism

$$\pi\colon\Gamma^{(3)}\longrightarrow F_3$$

restricts to an étale double cover over the divisor $E\subset\Gamma^{(3)}$ of collinear points of $\Gamma^{(3)}$.

4.2.4. Local description of X. Following [**FMOS23**], we give a local description of the variety X and the involution ι. Moreover, we show that the schematic fixed locus $\mathrm{Fix}(\iota)$ is a reduced local complete intersection.

The first step is to show that the morphism $f\colon X \to M_{\sigma_0}(v)$ constructed in Equation (67) is an isomorphism. We have already noted that f is a bijection hence, by Zariski's Main Theorem, it is enough to show that $M_{\sigma_0}(v)$ is normal.

By the analysis in Section 4.2.2, singular points of $M_{\sigma_0}(v)$ are given by S-equivalence classes of polystable sheaves of the form $E = \mathcal{O}(-L) \oplus F$, where F is a stable vector sheaf in the fourfold $M_S(0, L, -4)$.

We study the local structure of $M_{\sigma_0}(v)$ at $[E]$ using the Kuranishi map. Consider the map

$$\mu_2\colon \operatorname{Ext}^1(E,E) \longrightarrow \operatorname{Ext}^2(E,E)$$

given by the Yoneda product. By [**AS25**, Corollary 4.1], there is an isomorphism of germs

$$(M_{\sigma_0}(v), [E]) \simeq (\mu_2^{-1}(X)/\!\!/G, [0]), \tag{70}$$

where the group G is equal to $\operatorname{Aut}(\mathcal{O}(-L)) \times \operatorname{Aut}(F)$ and acts on $\operatorname{Ext}^1(E,E)$ by conjugation.

The situation is analogous to that of [**FMOS23**, Remark 2.7 and Lemma 5.4]. Using their methods, we show the following lemma.

Lemma 4.15. *Let $[E]$ be a singular point of $M_{\sigma_0}(v)$. There is an isomorphism*

$$(X, [E]) \simeq (\mathbf{A}^4_{\mathbf{C}} \times Q, 0)$$

of analytic germs, where $Q = \operatorname{Spec}(\mathbf{C}[u_1,u_2,u_3]/u_1^2 - u_2u_3)$.

In particular, the good moduli space $M_{\sigma_0}(v)$ is normal.

Proof. Upon changing the singular point $[E]$ in its S-equivalence class, we can suppose that E is of the form $T \oplus F$, where T is the stable spherical sheaf $\mathcal{O}(-L)$ and F is stable with Mukai vector $a = (0, L, -4)$. By Equation (70), the germ of X at the singular point E is equal to $\mu_2^{-1}(0)/\!\!/G$.

We describe the morphism μ_2 on $\operatorname{Ext}^1(E,E)$. Since T is spherical, we have decompositions

$$\operatorname{Ext}^1(E,E) = \operatorname{Ext}^1(F,F) \oplus \operatorname{Ext}^1(T,F) \oplus \operatorname{Ext}^1(F,T)$$

and

$$\operatorname{Ext}^2(E,E) = \operatorname{Ext}^2(F,F) \oplus \operatorname{Ext}^2(T,T) \simeq \mathbf{C}^2,$$

where we used that $\operatorname{Ext}^2(T,F) = \operatorname{Ext}^2(F,T) = 0$, since T and F are nonisomorphic stable sheaves with same phase. The Yoneda product μ_2 is given by

$$(\eta, a, b) \longmapsto (\eta^2 + b \circ a, a \circ b).$$

Note that, since F is a stable point of the moduli space $M_{\sigma_0}(a)$, the Kuranishi map for F is trivial, hence $\eta^2 = 0$ for any $\eta \in \operatorname{Ext}^1(F,F)$. Additionally, the trace map defines isomorphisms $\operatorname{Ext}^2(F,F) \simeq \mathbf{C}$ and $\operatorname{Ext}^2(T,T) \simeq \mathbf{C}$ such that $a \circ b = -b \circ a$ for any (a,b) in $\operatorname{Ext}^1(T,F) \oplus \operatorname{Ext}^1(F,T)$.

Therefore, the kernel of μ_2 is equal to

$$\mu_2^{-1}(0) = \operatorname{Ext}^1(F,F) \times \{(a,b) \in \operatorname{Ext}^1(T,F) \oplus \operatorname{Ext}^1(F,T) \mid a \circ b = 0\}.$$

The vector space $\operatorname{Ext}^1(F,F)$ has dimension $a^2 + 2 = 4$ and the spaces $\operatorname{Ext}^1(T,F)$ and $\operatorname{Ext}^1(F,T)$ have dimensions $a \cdot (v - a) = 2$. Given a basis (a_1, a_2) of $\operatorname{Ext}^1(T,F)$,

let (b_1, b_2) be a basis of $\mathrm{Ext}^2(F,T)$ such that $a_1 \circ b_2 = a_2 \circ b_1 = 0$. We obtain an isomorphism

$$\mu_2^{-1}(0) \simeq \mathbf{A}^4_{\mathbf{C}} \times \mathrm{Spec}\left(\mathbf{C}[a_1,a_2,b_1,b_2]/(a_1b_1+a_2b_2)\right).$$

Now the action of the group $G = \mathrm{Aut}(T) \oplus \mathrm{Aut}(F) \simeq (\mathbf{C}^*)^2$ is given by conjugation on elements of $\mathrm{Ext}^1(E,E)$. Therefore, it is trivial on $\mathrm{Ext}^1(F,F)$, and $(g,h) \in (\mathbf{C}^*)^2$ acts as the multiplication by gh^{-1} on $\mathrm{Ext}^1(T,F)$ and as the multiplication by hg^{-1} on $\mathrm{Ext}^1(F,T)$. Therefore, as in [**FMOS23**, Remark 2.7], the quotient $\mu_2^{-1}(0)/\!\!/G$ is equal to $\mathbf{A}^4_{\mathbf{C}} \times Q$, where

$$Q = \mathrm{Spec}\left(\mathbf{C}[u_1,u_2,u_3]/(u_1^2 - u_2u_3)\right)$$

and $u_1 = a_1b_1$, $u_2 = -a_1b_2$ and $u_3 = a_2b_1$.

This shows in particular that $M_{\sigma_0}(v)$ is a local complete intersection. Since its singular locus is parametrized by $M_S(v)$, it has codimension 2. Therefore the moduli space $M_{\sigma_0}(v)$ is normal. □

As already noted, the above lemma implies the desired modular description for the special fiber of $\mathscr{X} \to B$.

Corollary 4.16. *The morphism f given in Equation (67) is an isomorphism, hence the central fiber X of the family $\mathscr{X} \to B$ is isomorphic to $M_{\sigma_0}(v)$.*

This allows us to determine the local structure of the schematic fixed locus $\mathrm{Fix}(\iota)$.

Lemma 4.17. *Let $[E]$ be a singular point of X and consider the isomorphism of germs*

$$(X,[E]) \simeq (\mathbf{A}^4_{\mathbf{C}} \times Q, 0)$$

from Lemma 4.15. If the point $[E]$ lies on the component F_3 of $\mathrm{Fix}(\iota)$ (see Proposition 4.12), this isomorphism restricts to an isomorphism of germs

$$(\mathrm{Fix}(\iota),[E]) \simeq (\mathbf{A}^2_{\mathbf{C}} \times B, 0),$$

where $B = \mathrm{Spec}\left(\mathbf{C}[u_2,u_3]/(u_2u_3)\right)$. In particular, $\mathrm{Fix}(\iota)$ is reduced at points of F_3.

Proof. By Corollary 4.16, the variety X is isomorphic to $M_{\sigma_0}(v)$; this gives the first isomorphism of germs. We go back to the notation of the proof of Lemma 4.15.

We start by describing the action of the involution on the germ $\mathbf{A}^4_{\mathbf{C}} \times Q$. Since $[E]$ is fixed by ι, the factor $F \in M_S(a)$ is fixed by the restriction of ι to $M_S(a) = W \subset X$ (see Section 4.2.2). Since $M_S(a)$ is a smooth hyper-Kähler fourfold and ι restricts to an antisymplectic involution on it, its fixed locus is smooth of dimension 2. Hence, on germs, the fixed locus of $\iota|_{\mathbf{A}^4_{\mathbf{C}}}$ is equal to $\mathbf{A}^2_{\mathbf{C}}$.

We now analyze the action of ι on Q. Note that the fixed eigenspace for the action of ι on $\mathrm{Ext}^1(T,F)$ has dimension 1. Indeed, since $\mathbf{P}(\mathrm{Ext}^1(T,F))$ coincide with the fiber of π over the point E (see Lemma 4.13), this is equivalent to say the fixed locus of $j^{(3)}|_{\mathbf{P}(\mathrm{Ext}^1(T,F))}$ is finite (when F is $j^{(3)}$-invariant), which follows from the proof of Proposition 4.12.

Moreover, the action of ι on $\mathrm{Ext}^1(F,T)$ is dual to the one on $\mathrm{Ext}^1(T,F)$, therefore the action of ι on Q is the one described in [**FMOS23**, Proposition 5.4]. In particular, its fixed locus is equal to $B = \mathrm{Spec}(\mathbf{C}[u_2,u_3]/u_2u_3)$.

This implies that the schematic fixed locus $\mathrm{Fix}(\iota)$ is reduced. Indeed, it is smooth at any smooth point of X, since the action ι restricts to a regular action on the

smooth locus of X, and the reducedness at singular points follows from its local description, since B is reduced. □

Remark 4.18. A similar argument shows that the whole fixed locus of ι is reduced.

4.2.5. Degeneration of W_A. Given the family $\mathscr{X} \to D$ studied in Section 4.2.2, we consider the schematic fixed locus $\mathrm{Fix}(\iota_{\mathscr{X}}) \to \mathscr{D}$ of $\iota_{\mathscr{X}}$ on $\mathscr{X}$ (see the proof of [**FMOS23**, Proposition 5.6] for the definition of the schematic structure on the fixed locus). For $t \neq 0$, the fiber $\mathrm{Fix}(\iota_{\mathscr{X}})_t = \mathrm{Fix}(\iota_t)$ is the smooth Lagrangian threefold W_{A_t}. The reducedness of the fixed locus of ι studied in Section 4.2.4 implies $\mathrm{Fix}(\iota_{\mathscr{X}})_0 = \mathrm{Fix}(\iota) = F_3 \cup F_2$.

Let $\mathscr{X}^*$ be the pullback of $\mathscr{X} \to D$ to $D \smallsetminus \{0\}$. The family $\mathscr{X}^*$ is smooth, hence the fixed locus $\mathrm{Fix}_{\mathscr{X}^*}(\iota_{\mathscr{X}^*})$ of $\iota_{\mathscr{X}^*}$ is also smooth and its fibers are equal to $\mathrm{Fix}(\iota_t)$ for $t \neq 0$. Let $\mathscr{Y}$ be the closure of $\mathrm{Fix}_{\mathscr{X}^*}(\iota_{\mathscr{X}^*})$ inside $\mathscr{X}$, with induced morphism

$$f \colon \mathscr{Y} \longrightarrow D. \tag{71}$$

The morphism f is flat by [**Liu02**, 4.3 Proposition 3.9] since it is dominant and $\mathscr{Y}$ is integral. Therefore, it is equidimensional and $\mathscr{Y}_0$ has dimension 3. Moreover, $\mathscr{Y}$ is contained in $\mathrm{Fix}(\iota_{\mathscr{X}})$, hence $\mathscr{Y}_0$ is equal to F_3 (scheme-theoretically).

Theorem 4.19. *Let $[A] \in \mathrm{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma \cup \Gamma$ be a Lagrangian with associated smooth EPW cube $\widetilde{\mathsf{Z}}_A$. The external Hodge numbers of W_A can be computed as*

$$h^{p,0}(\mathsf{W}_A) = h^p(F_3, \mathcal{O}_{F_3}).$$

Proof. Hodge numbers are invariant under smooth deformations, hence it is enough to compute $h^{p,0}(\mathsf{W}_A)$ when W_A is a smooth fiber of the morphism $f \colon \mathscr{Y} \to D$ defined in (71).

All fibers of f are semi-log-canonical: the central fiber is isomorphic to F_3, which is semi-log-canonical by Remark 4.14, and the other fibers W_A are smooth. Therefore, since f is flat and proper, [**Kol23**, Corollary 2.64] implies that $h^p(\mathscr{Y}_t, \mathcal{O}_{\mathscr{Y}_t})$ is independent of $t \in D$. The theorem follows. □

As a first consequence, the above theorem allows us to complement Theorem 2.5(c).

Corollary 4.20. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace containing no decomposable vectors. The schemes W_A and $\mathsf{Z}_A^{\geq 3}$ are irreducible.*

Proof. Theorem 2.5 shows that the scheme $\mathsf{Z}_A^{\geq 3}$ is normal and reduced, therefore it is irreducible if and only if it connected. Since W_A and $\mathsf{Z}_A^{\geq 3}$ are isomorphic, it suffices to prove that W_A is connected. If $[A] \notin \Sigma \cup \Gamma$, the connectedness of W_A follows from the connectedness of F_3 via Theorem 4.19.

We use a "standard trick," as in [**DK18**, Theorem B.2], to show that $\mathsf{Z}_A^{\geq 3}$ is connected for any $[A] \in \Gamma \smallsetminus \Sigma$. Let $(T, 0)$ be a smooth pointed connected curve parametrizing Lagrangians $(A_t)_{t \in T}$ with no decomposable vectors, such that $[A_t]$ does not lie in Γ for $t \neq 0$, and $A_0 = A$. For any $t \neq 0$, the scheme $\mathsf{Z}_{A_t}^{\geq 3}$ is a smooth irreducible threefold. Consider the subvariety

$$\mathscr{Z}_T^{\geq 3} := \bigcup_{t \in T} \{t\} \times \mathsf{Z}_{A_t}^{\geq 3}$$

inside $T \times \mathsf{Gr}(3, V_6)$. It is a third determinantal locus, therefore it has codimension at most 6 everywhere. Hence each connected component of $\mathscr{Z}_T^{\geq 3}$ has dimension at

least 4. Since the scheme $\mathsf{Z}^{\geq 3}_{A_0}$ has dimension 3, it follows that every connected component of $\mathsf{Z}^{\geq 3}_{A_0}$ deforms in a neighborhood of 0. As we have already proved that $\mathsf{Z}^{\geq 3}_{A_t}$ is connected for $t \neq 0$, we conclude that $\mathsf{Z}^{\geq 3}_{A_0}$ is connected as well. □

Remark 4.21. The same argument shows that for any $[A] \in \mathrm{LG}(\bigwedge^3 V_6)$, the scheme $\mathsf{Z}^{\geq k}_A$, for $k \in \{1,2,3\}$, is connected as soon as it has the expected dimension.

4.2.6. Cohomology groups $H^p(F_3, \mathcal{O}_{F_3})$. We study the cohomology of the structure sheaf of F_3. We keep the notation of Section 4.2.3.

Let $\Gamma \subset \mathbf{P}^2$ be a smooth plane sextic and let $E \subset \Gamma^{(3)}$ be the (smooth) divisor of collinear 3-points on Γ, with involution σ defined by

$$\sigma(x+y+z) = (C \cdot \ell) - x - y - z,$$

for any $x+y+z \in E$ that lies on a line ℓ.

By Proposition 4.12, the scheme F_3 is obtained from $\Gamma^{(3)}$ via the morphism

$$\eta \colon \Gamma^{(3)} \longrightarrow F_3 \tag{72}$$

given by the restriction of the contraction $\pi \colon S^{[3]} \to X$ to $\Gamma^{(3)} \subset S^{[3]}$. The morphism η is an isomorphism on $\Gamma^{(3)} \smallsetminus E$, and $\eta|_E \colon E \to W := \eta(E)$ is the (étale) double cover induced by the involution σ defined above. Notice that η is in particular the normalization of F_3.

Consider the morphism of sheaf $\mathcal{O}_{F_3} \to \eta_* \mathcal{O}_{\Gamma^{(3)}}$. It is injective, as η is dominant. Moreover, since η is an isomorphism outside E, the cokernel of η is supported on W. Since the restriction $\eta|_E$ is a double cover, there is a decomposition

$$(\eta|_E)_* \mathcal{O}_E = \mathcal{O}_W \oplus \mathscr{R},$$

where $\mathcal{O}_W$ and $\mathscr{R}$ are respectively the invariant and antiinvariant parts with respect to the action of σ^*. Since $\mathcal{O}_{F_3}|_W = \mathcal{O}_W$ and $\eta_* \mathcal{O}_{\Gamma^{(3)}}|_W = (\eta|_E)_* \mathcal{O}_E$, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{F_3} \longrightarrow \eta_* \mathcal{O}_{\Gamma^{(3)}} \longrightarrow j_* \mathscr{R} \longrightarrow 0, \tag{73}$$

where j is the embedding $W \hookrightarrow F_3$ and the map $\eta_* \mathcal{O}_{\Gamma^{(3)}} \to j_* \mathscr{R}$ is induced by taking the restriction to W and projecting to $\mathscr{R}$. This induces a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^1(F_3, \mathcal{O}_{F_3}) \to H^1(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \xrightarrow{\beta_1} H^1(W, \mathscr{R}) \to H^2(F_3, \mathcal{O}_{F_3}) \to \\ \to H^2(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \xrightarrow{\beta_2} H^2(W, \mathscr{R}) \to H^3(F_3, \mathcal{O}_{F_3}) \to H^3(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \to 0, \end{aligned} \tag{74}$$

where we used $H^3(W, \mathscr{R}) = 0$ since W has dimension 2, and $H^0(W, \mathscr{R}) = 0$ because of the equalities $\mathbf{C} = H^0(E, \mathcal{O}_E) = H^0(W, (\eta|_E)_* \mathcal{O}_E) = H^0(W, \mathcal{O}_W)$.

The map $\beta_i \colon H^i(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \to H^i(W, \mathscr{R})$ factors as

$$H^i(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \xrightarrow{\mathrm{res}_i} H^i(E, \mathcal{O}_E) = H^i(W, (\eta|_E)_* \mathcal{O}_E) \longrightarrow H^i(W, \mathscr{R}),$$

where the morphism res_i is induced by the restriction to $E \subset \Gamma^{(3)}$ and the second map is induced by the projection $(\eta|_E)_* \mathcal{O}_E \to \mathscr{R}$.

Lemma 4.22. *The restrictions*

$$\mathrm{res}_{0,q} \colon H^0(\Gamma^{(3)}, \Omega^q_{\Gamma^{(3)}}) \longrightarrow H^0(E, \Omega^q_E)$$

are injective for $q \in \{0,1,2\}$.

Proof. Since Γ is a smooth plane sextic curve, it is not trigonal, hence the symmetric product $\Gamma^{(3)}$ embeds into $\mathrm{Jac}(\Gamma)$ and the restrictions

$$H^0(\mathrm{Jac}(\Gamma), \Omega^q_{\mathrm{Jac}(\Gamma)}) \longrightarrow H^0(\Gamma^{(3)}, \Omega^q_{\Gamma^{(3)}}) \tag{75}$$

are isomorphisms for $q \in \{0,1,2,3\}$ ([**Mac62**, (11.1)]). Therefore, it is enough to consider the restriction morphisms induced by the inclusion $E \subset \Gamma^{(3)} \subset \mathrm{Jac}(\Gamma)$. Moreover, since

$$H^0(\mathrm{Jac}(\Gamma), \Omega^q_{\mathrm{Jac}(\Gamma)}) = \bigwedge^q H^0(\mathrm{Jac}(\Gamma), \Omega^1_{\mathrm{Jac}(\Gamma)}), \tag{76}$$

it is enough to show the injectivity of the restriction

$$r\colon H^0(\mathrm{Jac}(\Gamma), \Omega^2_{\mathrm{Jac}(\Gamma)}) \longrightarrow H^0(E, \Omega^2_E).$$

We show that the surface $E \subset \mathrm{Jac}(\Gamma)$ is nondegenerate in the sense of [**Ran80**, Section II], which implies by [**Ran80**, Lemma II.1] that r is injective.

Let θ be the canonical principal polarization on $\mathrm{Jac}(\Gamma)$. By [**Mac62**, (16.2)], the cohomology class of E in $H^2(\Gamma^{(3)}, \mathbf{Z})$ is equal to

$$-6\eta + \theta|_{\Gamma^{(3)}} \in H^2(\Gamma^{(3)}, \mathbf{Z}),$$

where $\eta \in H^2(\Gamma^{(3)}, \mathbf{Z})$ is the class of $\Gamma^{(2)} \subset \Gamma^{(3)}$.[1] By the projection formula, the class of E in $\mathrm{Jac}(\Gamma)$ is therefore given by

$$[E] = -6[\Gamma^{(2)}] + [\Gamma^{(3)}] \cdot \theta.$$

Since $[\Gamma^{(i)}] \in H^*(\mathrm{Jac}(\Gamma), \mathbf{Z})$ is equal to $\frac{\theta^{10-i}}{(10-i)!}$, it follows that $[E] = 2\theta^8/8!$ in $H^*(\mathrm{Jac}(\Gamma), \mathbf{Z})$. Therefore, $[E]$ is a positive multiple of θ^8, hence E is nondegenerate by [**Ran80**, Corollary II.2]. □

Proposition 4.23. *The cohomology groups of the structure sheaf of F_3 are*

$$\begin{aligned} H^1(F_3, \mathcal{O}_{F_3}) &= 0, \\ H^2(F_3, \mathcal{O}_{F_3}) &= H^{2,0}(\Gamma^{(3)}) \oplus \mathrm{Coker}(\mathrm{res}_1), \\ H^3(F_3, \mathcal{O}_{F_3}) &= H^{3,0}(\Gamma^{(3)}) \oplus H^2(W, \mathscr{R}). \end{aligned}$$

Proof. We start by showing that in the long exact sequence (74), the morphism β_1 is injective and β_2 is the zero map. We have a factorization

$$\beta_i\colon H^i(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \hookrightarrow H^i(E, \mathcal{O}_E) \twoheadrightarrow H^i(W, \mathscr{R}),$$

where the first map is injective by Lemma 4.22 and Hodge theory, since $\Gamma^{(3)}$ and E are smooth by Proposition 4.12, and the second map is the projection to the anti-invariant part of $H^i(E, \mathcal{O}_E)$ with respect to the action of σ^*.

Consider the embedding $E \subset \Gamma^{(3)} \subset \mathrm{Jac}^3(\Gamma)$, as in the proof of Lemma 4.22. By (75) and (76), the involution σ^* acts by multiplication by $(-1)^q$ on the subspace $H^0(\Gamma^{(3)}, \Omega^q_{\Gamma^{(3)}}) \subset H^0(E, \Omega^q_E)$, hence also on $H^q(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \subset H^q(E, \mathcal{O}_E)$.

In particular, $H^1(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}})$ is anti-invariant for the action of σ^* on $H^1(E, \mathcal{O}_E)$, hence is contained in $H^1(W, \mathscr{R})$. Therefore, the morphism β_1 is injective. Analogously, $H^2(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}})$ is all invariant for the action of σ^*, therefore the morphism β_2 is the zero map.

[1] The divisor E in $\Gamma^{(3)}$ is not ample. Indeed, by [**Kou93**, Lemma (1)], we have $(\theta - 6\eta)^3 = \sum_{i=0}^3 \binom{3}{i}(-6)^{3-i}\frac{10!}{(10-i)!} = -36$. So we cannot use the Lefschetz Hyperplane Theorem to determine its cohomology.

Therefore, the long exact sequence (74) gives $H^1(F_3, \mathcal{O}_{F_3}) = 0$ and exact sequences

$$0 \to H^1(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \xrightarrow{\beta_1} H^1(W, \mathscr{R}) \to H^2(F_3, \mathcal{O}_{F_3}) \to H^2(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \to 0$$

and

$$0 \to H^2(W, \mathscr{R}) \to H^3(F_3, \mathcal{O}_{F_3}) \to H^3(\Gamma^{(3)}, \mathcal{O}_{\Gamma^{(3)}}) \to 0,$$

from which we deduce the desired statements. □

Using Theorem 4.19, Proposition 2.23, (75), and (76), we obtain information on some Hodge numbers of the smooth threefold W_A.

Corollary 4.24. *Assume that the EPW cube $\widetilde{\mathsf{Z}}_A$ is smooth. We have*

$$\begin{aligned} h^{0,1}(\mathsf{W}_A) &= 0, \\ h^{0,2}(\mathsf{W}_A) &\geq 45, \\ h^{0,3}(\mathsf{W}_A) &= 131 + h^{0,2}(\mathsf{W}_A), \\ h^{1,2}(\mathsf{W}_A) &= 470 + h^{0,2}(\mathsf{W}_A) + h^{1,1}(\mathsf{W}_A). \end{aligned}$$

Corollary 4.25. *Assume that the EPW cube $\widetilde{\mathsf{Z}}_A$ is smooth. Its submanifold W_A is rigid in $\widetilde{\mathsf{Z}}_A$.*

Proof. Since W_A is Lagrangian, the normal sheaf $N_{\mathsf{W}_A/\widetilde{\mathsf{Z}}_A}$ is isomorphic to $\Omega^1_{\mathsf{W}_A}$. Therefore,

$$h^0(\mathsf{W}_A, N_{\mathsf{W}_A/\widetilde{\mathsf{Z}}_A}) = h^0(\mathsf{W}_A, \Omega^1_{\mathsf{W}_A}) = h^{0,1}(\mathsf{W}_A) = 0$$

and the submanifold W_A does not deform in $\widetilde{\mathsf{Z}}_A$. □

4.2.7. A moving Lagrangian subvariety of $\widetilde{\mathsf{Z}}_A$. Let $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$ be a smooth EPW cube and let W_A be the fixed locus of the associated involution ι_A of $\widetilde{\mathsf{Z}}_A$. By Corollary 4.25, the Lagrangian W_A does not deform in $\widetilde{\mathsf{Z}}_A$. In this section, we show that there exists a family

$$\begin{array}{ccc} \widetilde{\mathcal{W}} & \longrightarrow & \widetilde{\mathsf{Z}}_A \\ \downarrow & & \\ \mathscr{D} & & \end{array}$$

such that the fibers of $\widetilde{\mathcal{W}} \to \mathscr{D}$ are Lagrangian submanifolds of $\widetilde{\mathsf{Z}}_A$ of class $2[\mathsf{W}_A]$ that cover $\widetilde{\mathsf{Z}}_A$.

Consider the subvariety

$$\mathscr{D} = \{[B] \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \mid \dim(A \cap B) \geq 9\}$$

of Lagrangian subspaces of $\bigwedge^3 V_6$ intersecting A in a hyperplane and let

$$\mathscr{X}^{\geq 3}_{\mathscr{D}} := \mathscr{X}^{\geq 3} \times_{\mathrm{LG}(\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_6)} (\mathscr{D} \times \mathsf{Gr}(3, V_6)),$$

be the pullback of the family (33) to $\mathscr{D} \times \mathsf{Gr}(3, V_6)$. The fiber of the projection $\mathscr{X}^{\geq 3}_{\mathscr{D}} \to \mathscr{D}$ over a point $[B] \in \mathscr{D}$ is the locus $\mathsf{Z}^{\geq 3}_B$.

By definition of $\mathscr{D}$, for any $[B] \in \mathscr{D}$, the locus $\mathsf{Z}^{\geq 3}_B$ is contained in $\mathsf{Z}^{\geq 2}_A$, hence the image of the projection $\mathscr{X}^{\geq 3}_{\mathscr{D}} \to \mathsf{Gr}(3, V_6)$ is contained in $\mathsf{Z}^{\geq 2}_A$.

Lemma 4.26. *The morphism $f \colon \mathscr{X}^{\geq 3}_{\mathscr{D}} \to \mathsf{Z}^{\geq 2}_A$ is surjective.*

Proof. We want to show that for any $[U_0] \in \mathsf{Z}_A^{\geq 2}$, there exists $[B] \in \mathscr{D}$ such that the point $[U_0]$ lies in the stratum $\mathsf{Z}_B^{\geq 3}$. This is clearly true for $[U_0] \in \mathsf{Z}_A^{\geq 3}$, with $B = A$.

Let $[U_0]$ be a point in Z_A^2 and denote by K the 2-dimensional subspace $A \cap T_{U_0}$. We show that for any hyperplane $H \subset A$ containing K, there exists a Lagrangian subspace $B \subset \bigwedge^3 V_6$ such that $A \cap B = H$ and $[U_0]$ lies in $\mathsf{Z}_B^{\geq 3}$.

The space of Lagrangian subspaces B such that $A \cap B$ contains H is parametrized by the line $\mathbf{P}(H^\perp/H)$, where $H^\perp$ has dimension 11. Since K is contained in H and in T_{U_0}, it follows that both $H^\perp$ and $T_{U_0}^\perp = T_{U_0}$ contain K and are contained in $K^\perp$. Note that the quotient T_{U_0}/K has dimension 8 and $H^\perp/K$ has codimension 7 in $K^\perp/K$. Therefore, there exists a nonzero 3 form $\omega \in T_{U_0} \cap H^\perp$ that does not lie in K. Finally, since H is contained in A and $A \cap T_{U_0} = K$, ω is not contained in H. Thus, the subspace $B = H \oplus \mathbf{C}\omega \subset \bigwedge^3 V_6$ is Lagrangian and intersects A in H, and the intersection $B \cap T_{U_0}$ is the three-dimensional space $K \oplus \mathbf{C}\omega$. This concludes the proof. □

We denote by $\widetilde{\mathcal{W}}_{\mathscr{D}}$ be the pullback

$$\begin{array}{ccc} \widetilde{\mathcal{W}}_{\mathscr{D}} & \xrightarrow{\tilde{f}} & \widetilde{\mathsf{Z}}_A \\ \downarrow & & \downarrow{\scriptstyle g_A} \\ \mathscr{Z}_{\mathscr{D}}^{\geq 3} & \longrightarrow & \mathsf{Z}_A^{\geq 2} \end{array}$$

of $\mathscr{Z}_{\mathscr{D}}^{\geq 3}$ via the double cover g_A, and by $\widetilde{f}_B$ the morphism

$$\widetilde{f}_B \colon \widetilde{\mathsf{W}}_B \hookrightarrow \widetilde{\mathcal{W}} \xrightarrow{\tilde{f}} \widetilde{\mathsf{Z}}_A,$$

where $\widetilde{\mathsf{W}}_B$ is the fiber of $\widetilde{\mathcal{W}} \to \mathscr{D}$ over $[B] \in \mathscr{D}$. Note that $\widetilde{f}_B$ is the pullback of the inclusion $f_B \colon \mathsf{Z}_B^{\geq 3} \subset \mathscr{Z}_{\mathscr{D}}^{\geq 3} \xrightarrow{f} \mathsf{Z}_A^{\geq 2}$.

Proposition 4.27. *The morphism $\widetilde{f} \colon \widetilde{\mathcal{W}}_{\mathscr{D}} \to \widetilde{\mathsf{Z}}_A$ is surjective. Moreover, for each $B \in \mathscr{D}$, the subvariety $\widetilde{f}_B(\widetilde{\mathsf{W}}_B)$ is a Lagrangian subvariety of $\widetilde{\mathsf{Z}}_A$ whose cohomology class is equal to $2[\mathsf{W}_A]$.*

Proof. The surjectivity of $\widetilde{f}$ comes from the fact that it is the pullback of the surjective morphism f (Lemma 4.26) via the surjective map g_A. Consider a point $[B] \in \mathscr{D} \smallsetminus \{[A]\}$, and set $H = B \cap A$. The line $\mathbf{P}(H^\perp/H)$ is contained in $\mathscr{D}$, and induces an equality of cycles

$$[\widetilde{f}_B(\widetilde{\mathsf{W}}_B)] = [\widetilde{f}_A(\widetilde{\mathsf{W}}_A)] = g_A^*[\mathsf{Z}_A^{\geq 3}]$$

in $\widetilde{\mathsf{Z}}_A$. Since $\mathsf{Z}_A^{\geq 3}$ is the branch locus of g_A, we obtain the equality $[\widetilde{f}_B(\widetilde{\mathsf{W}}_B)] = 2[\mathsf{W}_A]$. Since W_A is Lagrangian, it satisfies $\sigma \cdot [\mathsf{W}_A] = 0$. Therefore, Proposition 1.11 shows that $\widetilde{f}_B(\widetilde{\mathsf{W}}_B)$ is a Lagrangian subvariety of $\widetilde{\mathsf{Z}}_A$. □

4.3. The divisors ${}^{[3]}\mathscr{D}_{4,4}^{(2)}$, ${}^{[3]}\mathscr{D}_{4,8}^{(2)}$, and ${}^{[3]}\mathscr{D}_{4,10}^{(2)}$

In this section, we consider the remaining Heegner divisors from the list considered at the beginning of the chapter.

The Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,4}$. This divisor does not meet the image of the period map $\wp_3$ (although the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,4}$ does meet the image of $\wp_2$). We describe below the quasipolarized hyper-Kähler sixfolds with period a very general point of this divisor.

Let (S, L) be a polarized K3 surface of degree 4 such that $\mathrm{Pic}(S) = \mathbf{Z}L$. Consider the sixfold $M_S := M_S(v)$, where $v := (0, L, -2)$ is the Mukai vector. It is the moduli space of sheaves of the form $i_{C,*}(\xi)$, where $i_C\colon C \hookrightarrow S$ is the inclusion of a curve $C \in |L|$ in S and ξ is a torsion-free rank 1 sheaf on C of degree 0. There is an identification

$$\Theta\colon v^\perp \xrightarrow{\sim} H^2(M_S, \mathbf{Z})$$

and the class $H := \Theta(2, -L, 0)$ has square 4 and divisibility 2. The class $H_0 := \Theta(0, 0, -1)$ has square 0 and its sections define a Lagrangian fibration

$$\pi\colon M_S \longrightarrow |L| \simeq \mathbf{P}^3,$$

where the fiber of a smooth $C \in |L|$ is its Jacobian $\mathrm{Pic}^0(C)$ ([**BM14a**, Lemma 11.3]). The morphism π has a section given by the trivial bundle. Finally, the Néron–Severi group is equal to

$$\mathrm{NS}(M_S) = v^\perp \cap (\mathbf{Z} \oplus \mathbf{Z}L \oplus \mathbf{Z}) = \mathbf{Z}H \oplus \mathbf{Z}H_0,$$

with intersection matrix $\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$ and determinant -4.

The class H_0 on M_S is nef but not ample; therefore, it spans an extremal ray in both cones $\overline{\mathrm{Mov}(M_S)}$ and $\mathrm{Nef}(M_S)$. We show that the other extremal ray of $\overline{\mathrm{Mov}(M_S)}$ is the class H. To determine it, following [**BM14a**, Theorem 12.3], we need to look for nonzero classes $s, w \in \mathbf{Z} \oplus \mathbf{Z}L \oplus \mathbf{Z}$ such that $s^2 = -2$ and $s \cdot v = 0$, or $w^2 = 0$ and $w \cdot v = 2$ (the vector v has divisibility 2 in $\mathbf{Z} \oplus \mathbf{Z}L \oplus \mathbf{Z}$). The intersection matrix of $\mathrm{NS}(M_S)$ shows that every vector in $v^\perp \cap (\mathbf{Z} \oplus \mathbf{Z}L \oplus \mathbf{Z})$ has square divisible by four, hence we only need to look for vectors $w = (a, b, c)$ satisfying

$$\begin{cases} w \cdot v = 4b + 2a = 2, \\ w^2 = 4b^2 - 2ac = 0. \end{cases} \tag{77}$$

Solving the system gives two vectors $w_1 = (1, 0, 0)$ and $w_2 = v - w_1$. The other extremal ray of $\overline{\mathrm{Mov}(M_S)}$ is thus given by $\Theta(v^\perp \cap w_i^\perp) = \mathbf{R}_{\geq 0}H$.

The walls and chamber structure of the movable cone is given by walls defining flops. Following [**BM14a**, Theorem 5.7(b)], such a wall $\mathcal{W}$ is equal to

(a) $\Theta(s^\perp \cap v^\perp)$, for some spherical vector s such that $s \cdot v = 2$ (we use again $s \cdot v$ is even),

(b) or $\Theta(\mathcal{H}^\perp \cap v^\perp)$, where $\mathcal{H}$ is a hyperbolic lattice containing two *positive* vectors a and b such that $v = a + b$ (following [**BM14a**, Definition 5.4], a vector a of positive square is *positive* if $a \cdot v > 0$).

We first show that case (b) does not happen. Let $v = a + b$ be a decomposition of v as the sum of two *positive* vectors. Since v has square 4 and divisibility 2 in $\mathbf{Z} \oplus \mathbf{Z}L \oplus \mathbf{Z}$, the conditions $a \cdot v > 0$ and $(v - a) \cdot v > 0$ imply $a \cdot v = b \cdot v = 2$. Using again $v^2 = 4$, we obtain $a^2 = b^2$ and $a^2 + a \cdot b = 2$. Since a^2 is even and $a \cdot b$ cannot be 0 because (a, b) span a lattice of signature $(1, -1)$, it follows that a and b are isotropic vectors whose intersection with v is equal to 2. Those are exactly the solutions to Equation (77), hence determine the boundary wall $\mathbf{R}_{\geq 0}H$.

Therefore, we are left with case (a). We look for integral vectors $s = (a, b, c)$ which are solutions of the system

$$\begin{cases} s \cdot v = 4b + 2a = 2, \\ s^2 = 4b^2 - 2ac = -2 \end{cases} \iff \begin{cases} 2b = 1 - a, \\ a^2 - 2(c+1)a + 3 = 0. \end{cases} \tag{78}$$

Looking at the second equation, we obtain that a must divide 3 and by substitution, we find four solutions, listed in the table below.

s	$\mathcal{W}$
$(1, 0, 1)$	$\mathbf{R}_{\geq 0}(H + 2H_0)$
$(-1, 1, -3)$	$\mathbf{R}_{\geq 0}(H + 2H_0)$
$(3, -1, 1)$	$\mathbf{R}_{\geq 0}(H - \frac{2}{3}H_0)$
$(-3, 2, -3)$	$\mathbf{R}_{\geq 0}(H - \frac{2}{3}H_0)$

The second column is the potential wall defined by each vector, using that for a vector $s = (a, b, c)$, the ray $\Theta(s^\perp \cap v^\perp)$ is spanned by

$$H + \frac{2(c + 2b)}{a} H_0.$$

The last two walls are outside the movable cone of M_S, hence the only wall is given by $H + 2H_0$, and the cones are finally equal to

$$\overline{\mathrm{Mov}(M_S)} = \mathbf{R}_{\geq 0}H_0 + \mathbf{R}_{\geq 0}H \ , \qquad \mathrm{Nef}(M_S) = \mathbf{R}_{\geq 0}H_0 + \mathbf{R}_{\geq 0}(H + 2H_0).$$

The extremal ray $\mathbf{R}_{\geq 0}H_0$ defines the Lagrangian fibration π. The extremal ray $\mathbf{R}_{\geq 0}(H + 2H_0)$ defines the flop $M_S \dashrightarrow M'_S$ of the zero section $\mathbf{P}^3 \subset M_S$ of π to another smooth hyper-Kähler model M'_S of M_S, on which the class H is nef and big. Finally, the extremal ray $\mathbf{R}_{\geq 0}H$ defines a Li–Gieseker–Uhlenbeck divisorial contraction $M'_S \to M''_S$ whose exceptional divisor is a $\mathbf{P}^1$-fibration over the product of two smooth K3 surfaces.

The Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,8}$. This divisor does not meet the image of the period map $\wp_3$ (nor does the divisor ${}^{[2]}\mathscr{D}^{(1)}_{2,4}$ meet the image of $\wp_2$). It corresponds to the divisor $\Sigma \subset \bigwedge^3 V_6$ of Lagrangian subspaces containing a decomposable vector. We describe here the singularities of the corresponding degenerate EPW cubes.

Let $[A]$ be a general element in Σ and consider the associated stratification

$$\mathsf{Gr}(3, V_6) = \mathsf{Z}_A^{\geq 0} \supset \mathsf{Z}_A^{\geq 1} \supset \mathsf{Z}_A^{\geq 2} \supset \mathsf{Z}_A^{\geq 3} \supset \mathsf{Z}_A^{\geq 4}$$

defined in Equation (26). Since $[A]$ lies in $\Sigma \smallsetminus \Gamma$, the scheme $\mathsf{Z}_A^{\geq 4}$ is empty.

For any point $[U_0]$ in $\mathsf{Gr}(3, V_6)$, we defined in (30) the 5-dimensional cone

$$\mathcal{C}_{U_0} \coloneqq \mathbf{P}(T_{U_0}) \cap \mathsf{Gr}(3, V_6) \subset \mathbf{P}(T_{U_0}) = \mathbf{P}^9 \tag{79}$$

as the intersection of $\mathbf{P}(T_{U_0})$ with the space of decomposable vectors of $\bigwedge^3 V_6$. We begin by proving a result stated in [**IKKR17**, Lemma 3.2], but whose proof is incomplete.

Proposition 4.28. *Let $[A]$ be a general point in the divisor $\Sigma \subset \bigwedge^3 V_6$. Then,*

(a) *the space A contains a unique decomposable vector $\bigwedge^3 U$;*
(b) *the scheme $\mathsf{Z}_A^{\geq 1}$ contains $\mathcal{C}_U$;*

(c) *the intersection* $\mathsf{Z}_A^{\geq 2} \cap \mathcal{C}_U$ *is a hypersurface in* $\mathcal{C}_U$*;*
(d) *the intersection* $\mathsf{Z}_A^{\geq 3} \cap \mathcal{C}_U$ *is a surface.*

Proof. Item (a) is a consequence of [**IKKR17**, Lemma 3.10]. For any point $[U_0] \in \mathcal{C}_U$, the intersection $A \cap T_{U_0}$ contains $\bigwedge^3 U$, therefore $\mathcal{C}_U$ is contained in $\mathsf{Z}_A^{\geq 1}$. This shows Item (b).

To show Items (c) and (d), we count parameters and define the variety

$$V \coloneqq \{([A/\textstyle\bigwedge^3 U], [U_0]) \in \mathrm{LG}((\bigwedge^3 U)^\perp/\bigwedge^3 U) \times \mathcal{C}_U \mid (A \cap T_{U_0})/\bigwedge^3 U \neq 0\}.$$

The fiber over $[A/\bigwedge^3 U]$ of the projection

$$\mathrm{pr}_1 \colon V \longrightarrow \mathrm{LG}((\textstyle\bigwedge^3 U)^\perp/\bigwedge^3 U)$$

is the scheme $\mathsf{Z}_A^{\geq 2} \cap \mathcal{C}_U$. The cohomology class computations of [**IKKR17**, Lemma 3.2] implies that this fiber has dimension at least 4. In particular, pr_1 is surjective and V has dimension at least

$$\dim(\mathrm{LG}((\textstyle\bigwedge^3 U)^\perp/\bigwedge^3 U)) + 4 = 49.$$

Item (c) will follow if we prove that V has dimension at most 49.

We also define the variety

$$V' \coloneqq \{([A/\textstyle\bigwedge^3 U], [U_0], [\omega]) \in V \times \mathbf{P}((\bigwedge^3 U)^\perp/\bigwedge^3 U) \mid \omega \in (A \cap T_{U_0})/\bigwedge^3 U\}.$$

The projection $p \colon V' \to V$ is surjective and its fibers are projective spaces. Let $W \subset V$ be the locus where these fibers have positive dimensions. The fiber over $[A/\bigwedge^3 U]$ of the restriction

$$\mathrm{pr}_1|_W \colon W \longrightarrow \mathrm{LG}((\textstyle\bigwedge^3 U)^\perp/\bigwedge^3 U)$$

is the scheme $\mathsf{Z}_A^{\geq 3} \cap \mathcal{C}_U$. The cohomology class s of [**IKKR17**, Lemma 3.2] implies that this fiber has dimension at least 2. Again, Item (d) will follow if we prove that W has dimension at most 47.

We count parameters to determine the dimension of V'. A triple $([A/\bigwedge^3 U], [U_0], [\omega])$ lies in V' if and only if

- the point $[U_0]$ lies in the irreducible 5-dimensional variety $\mathcal{C}_U$;
- the point $[\omega]$ lies in the 8-dimensional projective space $\mathbf{P}(T_{U_0}/\bigwedge^3 U)$;
- the Lagrangian A contains the subspace $K = \bigwedge^3 U \oplus \mathbf{C}\omega$. Therefore, A/K lies in the irreducible Lagrangian Grassmannian $\mathrm{LG}(K^\perp/K)$, which has dimension 36.

Putting all together, we obtain that V' is irreducible of dimension 49. This implies first that V has dimension 49 and that p is a birational isomorphism. In particular, W has codimension at least 2 in V. As explained above, this finishes the proof of the proposition. □

We now study the schemes $\mathsf{Z}_A^{\geq k}$. As in Section 2.3.2, the restriction morphism

$$r_K \colon H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2)) \longrightarrow H^0(\mathbf{P}(K), \mathcal{O}(2)),$$

where $K \coloneqq A \cap T_{U_0}$ has dimension k if $[U_0]$ is in Z_A^k, will play an essential role.

Proposition 4.29. *Let* $[A]$ *be a general element in the divisor* $\Sigma \subset \bigwedge^3 V_6$ *and let* $\bigwedge^3 U$ *be the unique decomposable vector in* A*. We have:*

(a) *The scheme* $\mathsf{Z}_A^{\geq 1}$ *is a normal integral hypersurface in* $\mathsf{Gr}(3, V_6)$ *that contains the* 5*-dimensional cone* $\mathcal{C}_U$*. The singular locus of* $\mathsf{Z}_A^{\geq 1}$ *is* $\mathsf{Z}_A^{\geq 2} \cup \mathcal{C}_U$*.*

(b) *The scheme* $\mathsf{Z}_A^{\geq 2}$ *is a normal integral Cohen-Macaulay sixfold and* $\mathsf{Z}_A^{\geq 2} \cap \mathcal{C}_U$ *is a hypersurface in* $\mathcal{C}_U$*. The singular locus of* $\mathsf{Z}_A^{\geq 2}$ *is* $\mathsf{Z}_A^{\geq 3} \cup (\mathsf{Z}_A^{\geq 2} \cap \mathcal{C}_U)$.

(c) *The scheme* $\mathsf{Z}_A^{\geq 3}$ *is a nonnormal connected reduced Cohen-Macaulay threefold whose singular locus is the surface* $\mathsf{Z}_A^{\geq 3} \cap \mathcal{C}_U$.

Proof. We follow the proof of Proposition 2.10 and consider a point $[U_0]$ in Z_A^{ℓ}. Let $U_\infty \subset V_6$ be a 3-dimensional subspace complementary to U_0 and consider the affine open neighborhood $\mathcal{U} = \mathrm{Hom}(U_0, U_\infty)$ of $[U_0]$ in $\mathsf{Gr}(3, V_6)$. Finally, set $K := A \cap T_{U_0}$ and consider the morphism

$$\Psi_K([A], \cdot) := \mathcal{U} \longrightarrow \mathrm{Sym}^2 K^\vee \tag{80}$$

introduced in that proof, whose differential is the morphism r_K. The first part of the proof of Proposition 2.10 (which is true with no hypothesis on A) shows that, for any $k \leq \ell$, the scheme $\mathsf{Z}_A^{\geq k} \cap \mathcal{U}$ is the preimage by $\Psi_K([A], \cdot)$ of the subscheme $\mathcal{S}_{\ell-k} \subset \mathrm{Sym}^2 K^\vee$ of quadratic forms of rank $\leq \ell - k$. When r_K has maximal rank, this description determines the local structure of $\mathsf{Z}_A^{\geq k}$ at $[U_0]$. When $k \in \{1, 2, 3\}$ (which we assume in the rest of the proof), having maximal rank for r_K is the same as being surjective.

Proposition 2.7 shows that, when $K \subset T_{U_0}$ contains no decomposable vectors, the morphism r_K has maximal rank. Otherwise, the image of r_K is contained in the subspace of quadrics in $\mathrm{Sym}^2 K^\vee$ vanishing on the subset $K \cap \mathcal{C}_{U_0}$ of decomposable vectors in K, and in particular r_K is not surjective.

The only decomposable vector in A is $\bigwedge^3 U$, thus $K = A \cap T_{U_0}$ contains a decomposable vector if and only if the point $[U]$ is in the cone $\mathcal{C}_{U_0}$, and this happens if and only if $\dim(U \cap U_0) \geq 2$. Therefore, the map r_K is not surjective if and only if the point $[U_0]$ is in the cone $\mathcal{C}_U \subset \mathsf{Gr}(3, V_6)$.

For $[U_0] \in \mathsf{Z}_A^k \smallsetminus \mathcal{C}_U$, the morphism r_K is therefore surjective. As in the proof of Theorem 2.5, we obtain that the scheme $\mathsf{Z}_A^{\geq k} \smallsetminus \mathcal{C}_U$ is a scheme of expected codimension $\frac{k(k+1)}{2}$, whose singular locus is equal to $\mathsf{Z}_A^{\geq k+1} \smallsetminus \mathcal{C}_U$. Using the dimension of $\mathsf{Z}_A^{\geq k} \cap \mathcal{C}_U$ computed in Proposition 4.28, we see that $\mathsf{Z}_A^{\geq k}$ also has expected dimension, hence is Cohen-Macaulay, and connected by Remark 4.21. Moreover, $\mathsf{Z}_A^{\geq 1}$ and $\mathsf{Z}_A^{\geq 2}$ are nonsingular in codimension 1, hence normal.

To conclude the proof of the lemma, we need to prove that $\mathsf{Z}_A^{\geq k} \cap \mathcal{C}_U$ is contained in the singular locus of $\mathsf{Z}_A^{\geq k}$.

Consider a point $[U_0] \in \mathsf{Z}_A^k \cap \mathcal{C}_U$. By the above discussion, the scheme $\mathsf{Z}_A^{\geq k} \cap \mathcal{U}$ is the zero locus of the morphism $\Psi_K([A], \cdot)$ of Equation (80). By the Jacobian criterion, it is smooth at $[U_0]$ if and only if the differential of $\Psi_K([A], \cdot)$ has rank equal to the codimension of $\mathsf{Z}_A^{\geq k}$ at $[U_0]$. We already proved that Z_A^k has the expected codimension, hence the codimension at each point is equal to the dimension of $\mathrm{Sym}^2 K^\vee$. We saw that for $[U_0] \in \mathcal{C}_U$, the morphism r_K is not surjective, therefore the scheme $\mathsf{Z}_A^{\geq k}$ is not smooth at $[U_0]$.

This shows that $\mathsf{Z}_A^k \cap \mathcal{C}_U$ is contained in the singular locus of $\mathsf{Z}_A^{\geq k}$. Since the singular locus of $\mathsf{Z}_A^{\geq k}$ is closed and contained in $\mathsf{Z}_A^{\geq k} \cap \mathcal{C}_U$, this concludes the proof. □

Since $\mathsf{Z}_A^{\geq 2}$ is normal, [**DK20a**, Theorem 4.2] shows that there exists a double cover $g_A \colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A^{\geq 2}$ branched over $\mathsf{Z}_A^{\geq 3}$, as in Theorem 2.14. Results of [**IKKR17**] imply the following.

Corollary 4.30. *The singular locus of $\widetilde{\mathsf{Z}}_A$ is a polarized hyper-Kähler fourfold of $K3^{[2]}$ deformation type with a polarization of degree 4.*

Proof. Consider the fourfold $H \coloneqq \mathsf{Z}_A^{\geq 2} \cap \mathcal{C}_U$. The analysis of Proposition 4.29 implies that the singular locus of $\widetilde{\mathsf{Z}}_A$ is the preimage $\widetilde{H}$ of H via g_A. [**IKKR17**, Theorem 1.1 and Proposition 3.8] shows that $\widetilde{H}$ is a hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$ deformation type with a polarization of degree 4. □

The Heegner divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$. This divisor meets the image of the period map $\wp_3$. Remark 3.8(b) describes EPW cubes with period a very general point of this divisor: they are obtained by flopping $S_A^{[3]}$, where (S_A, L) is a very general polarized K3 surface of degree 10. An analysis similar to the one carried out for the divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,4}$ shows that the flop is induced by the wall $7L - 10\delta$, corresponding to the spherical vector $s = (2, -1, 3)$.

Part II

Relation with Gushel–Mukai manifolds

CHAPTER 5

Gushel–Mukai manifolds

5.1. Definition and cohomology

Definition 5.1 ([DK18, Definition 2.1])**.** A Gushel–Mukai (GM for short) manifolds of dimension $n \in \{3,4,5,6\}$ is a *smooth* transverse intersection

$$X = \mathsf{CGr}(2,V_5) \cap \mathbf{P}(W) \cap Q \subset \mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 V_5),$$

where V_5 is a vector space of dimension 5, $\mathsf{CGr}(2,V_5)$ is the cone in $\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ with vertex $\nu \coloneqq \mathbf{P}(\mathbf{C})$ over the Grassmannian $\mathsf{Gr}(2,V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$ in its Plücker embedding, $W \subset \mathbf{C} \oplus \bigwedge^2 V_5$ is a vector space of dimension $n+5$, and $Q \subset \mathbf{P}(W)$ is a quadratic hypersurface.

Since X is smooth, it does not contain the vertex ν of the cone $\mathsf{CGr}(2,V_5)$. Therefore, the projection from ν defines a morphism

$$\gamma_X \colon X \longrightarrow \mathsf{Gr}(2,V_5). \tag{81}$$

If the vertex ν is not contained in $\mathbf{P}(W)$, the morphism γ_X can be identified X with a smooth transverse intersection

$$\mathbf{P}(W) \cap Q \cap \mathsf{Gr}(2,V_5) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5),$$

where Q is a quadratic hypersurface of $\mathsf{Gr}(2,V_5)$ and W is a vector subspace of $\bigwedge^2 V_5$ of dimension $n+5$. In this case, X is called *ordinary*.

GM manifolds such that ν is contained in $\mathbf{P}(W)$ are called *special* (this is necessarily the case when $n=6$). In this case, γ_X induces a double cover

$$X \longrightarrow \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W') \subset \mathbf{P}(\textstyle\bigwedge^2 V_5),$$

where $W' \coloneqq W \cap \bigwedge^2 V_5$.

In [DK18, Theorem 3.10], the authors associate to any GM manifolds of dimension $n \in \{3,4,5,6\}$ a triples (V_6, V_5, A) such that

- V_6 is a vector space of dimension 6, which is the space of quadratic equations of X inside $\mathbf{P}(W)$;
- $V_5 \subset V_6$ is a hyperplane, called the *Plücker hyperplane*, which can naturally be identified with the space of quadrics cutting out $\mathsf{CGr}(2,V_5) \cap \mathbf{P}(W)$ in $\mathbf{P}(W)$;
- $A \subset \bigwedge^3 V_6$ is a subspace which is Lagrangian with respect to the symplectic form given by wedge product; it contains no decomposable vectors if $n \geq 3$ ([DK18, Theorem 3.16]).

Moreover, $\dim(A \cap \bigwedge^3 V_5)$ is equal to $5-n$ if X is ordinary, and to $6-n$ if X is special ([DK18, Proposition 3.13]).

Conversely, given a Lagrangian data set (V_6, V_5, A) where A has no decomposable vectors, the dimension ℓ of $A \cap \bigwedge^3 V_5$ is at most 3. One can then construct

- an ordinary GM manifold of dimension $5-\ell$,
- a special GM manifold of dimension $6-\ell$,

with associated Lagrangian data set (V_6, V_5, A).

In particular, to any GM manifold X of dimension ≥ 3, one may associate the stratifications

$$\mathsf{Y}_A^{\geq 3} \subset \mathsf{Y}_A^{\geq 2} \subset \mathsf{Y}_A^{\geq 1} \subset \mathbf{P}(V_6), \qquad \mathsf{Y}_{A^\perp}^{\geq 3} \subset \mathsf{Y}_{A^\perp}^{\geq 2} \subset \mathsf{Y}_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee),$$

and

$$\mathsf{Z}_A^{\geq 4} \subset \mathsf{Z}_A^{\geq 3} \subset \mathsf{Z}_A^{\geq 2} \subset \mathsf{Z}_A^{\geq 1} \subset \mathsf{Gr}(3, V_6)$$

defined in Equations (25) and (26).

We will study some relations between the geometry of X and the geometry of the EPW varieties.

5.1.1. Cohomology of Gushel–Mukai manifolds. In [DK19, Section 3], the authors compute the integral cohomology ring $H^\bullet(X, \mathbf{Z})$ of a Gushel–Mukai manifold X. They prove that it is torsion-free and that the pullback map

$$\gamma_X^* \colon H^d(\mathsf{Gr}(2, V_5), \mathbf{Z}) \longrightarrow H^d(X, \mathbf{Z})$$

is an isomorphism for $d < \dim(X)$, and injective for $d = \dim(X)$.

Let X be a Gushel–Mukai sixfold. In this case, the cohomology ring $H^\bullet(X, \mathbf{Z})$ is concentrated in even degrees and the nonzero Hodge numbers are

$$\begin{array}{lccccccccc} b_0 = 1 & & & & & 1 & & & \\ b_2 = 1 & & & & 0 & 1 & 0 & & \\ b_4 = 2 & & & 0 & 0 & 2 & 0 & 0 & \\ b_6 = 24 & 0 & 0 & 1 & 22 & 1 & 0 & 0, \end{array}$$

computed in [DK19, Proposition 3.1].

In this case, the morphism γ_X^* is an isomorphism in each degree $d \neq 6$, hence the only part of the cohomology of X which does not come from $H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Z})$ lies in $H^6(X, \mathbf{Z})$. The cup-product defines on $H^6(X, \mathbf{Z})$ the structure of an even unimodular lattice and we will be interested in the *vanishing cohomology* of X, defined as

$$H^6(X, \mathbf{Z})_{\mathrm{van}} \coloneqq \{x \in H^6(X, \mathbf{Z}) \mid x \cdot \gamma_X^*(H^6(\mathsf{Gr}(2, V_5), \mathbf{Z})) = 0\}. \tag{82}$$

Over $\mathbf{Q}$, there is an orthogonal sum decomposition

$$H^6(X, \mathbf{Q}) = \gamma_X^*(H^6(\mathsf{Gr}(2, V_5), \mathbf{Q})) \oplus H^6(X, \mathbf{Q})_{\mathrm{van}}. \tag{83}$$

Finally, by [DK19, Proposition 3.9], there exists an isometry of lattices

$$H^6(X, \mathbf{Z})_{\mathrm{van}} \simeq E_8^{\oplus 2}(-1) \oplus U^{\oplus 2} \oplus \mathbf{Z}(-2) \oplus \mathbf{Z}(-2). \tag{84}$$

5.2. The second quadric fibration

If X is a GM manifold, we denote by $\mathcal{U}_X$ the pullback via the morphism γ_X (see (81)) of the rank 2 tautological subbundle $\mathcal{U}_2$ of $\mathsf{Gr}(2, V_5)$. The *second quadric fibration* is the morphism

$$\varrho \colon \mathcal{Q} \coloneqq \mathbf{P}_X((V_5 \otimes \mathcal{O}_X)/\mathcal{U}_X) \longrightarrow \mathsf{Gr}(3, V_5) \tag{85}$$

whose fiber over $[U_3] \in \mathsf{Gr}(3, V_5)$ is the intersection $\mathsf{CGr}(2, U_3) \cap X = \gamma_X^{-1}(\mathsf{Gr}(2, U_3))$ in $\mathsf{CGr}(2, V_5)$.

When X is a GM sixfold, by [DK18, Proposition 4.10], ϱ is indeed a family of quadrics in the projective bundle

$$\varphi\colon \mathbf{P}(\mathbf{C}\oplus\textstyle\bigwedge^2\mathcal{U}_3)\longrightarrow \mathsf{Gr}(3,V_5),$$

where $\mathcal{U}_3$ is the tautological subbundle of $\mathsf{Gr}(3,V_5)$: if $X=\mathsf{CGr}(2,V_5)\cap Q$ as in Definition 5.1, the fiber $\varrho^{-1}([U_3])$ over $[U_3]\in\mathsf{Gr}(3,V_5)$ is the quadric surface $\mathbf{P}(\mathbf{C}\oplus\bigwedge^2U_3)\cap Q$.

This quadric surface in $\mathbf{P}^3$ has

- maximal rank 4 if $[U_3]\in\mathsf{Z}^0_{A,V_5}$ (general fiber),
- rank 3 if $[U_3]\in\mathsf{Z}^1_{A,V_5}$ (expected codimension 1 in $\mathsf{Gr}(3,V_5)$),
- rank 2 if $[U_3]\in\mathsf{Z}^2_{A,V_5}$ (expected codimension 3 in $\mathsf{Gr}(3,V_5)$),
- rank 1 if $[U_3]\in\mathsf{Z}^3_{A,V_5}$ (expected codimension 6 in $\mathsf{Gr}(3,V_5)$),

where $\mathsf{Z}^{\geq k}_{A,V_5}$ is the intersection of $\mathsf{Z}^{\geq k}_A$ with $\mathsf{Gr}(3,V_5)$ in $\mathsf{Gr}(3,V_6)$, and Z^k_{A,V_5} is defined analogously. The loci $\mathsf{Z}^{\geq k}_{A,V_5}\subset\mathsf{Gr}(3,V_5)$ will be studied in more detail in Section 5.3.

The aim of this section is to show that the loci $\mathsf{Z}^{\geq k}_{A,V_5}$ are the *schematic* degeneracy loci of ϱ. Although we will only use this property for (special) GM sixfolds, we prove it in a more general setting.

For the proof, we need the following definition.

Definition 5.2. Let X be a Gushel–Mukai manifold X with associated Lagrangian data set (V_6,V_5,A). Its *isotropic locus* is

$$\Sigma_2(X)=\Sigma_2(V_5)=\{[U_3]\in\mathsf{Gr}(3,V_5)\mid A\cap(\textstyle\bigwedge^2U_3\wedge V_5)\neq 0\}\subset\mathsf{Gr}(3,V_5).$$

Clearly the isotropic locus is contained in the scheme $\mathsf{Z}^{\geq 1}_{A,V_5}$.

The following proposition is an analogue of [DK20a, Lemma 5.4].

Proposition 5.3. *Let X be a Gushel-Mukai manifold of dimension n. Assume $n\geq 3$ if X is ordinary and $n\geq 4$ if X is special. Let $\varrho\colon\mathcal{Q}\to\mathsf{Gr}(3,V_5)$ be the associated second quadric fibration, as defined in (85). The schematic degeneracy loci of ϱ coincide with the schemes $\mathsf{Z}^{\geq k}_{A,V_5}$ away from $\Sigma_2(X)$.*

Proof. We follow the proof of [DK20a, Lemma 5.4]. We first consider the case of a ordinary GM manifold X of dimension n. Consider the restriction to $\mathsf{Gr}(3,V_5)$ of the symplectic bundle $\mathcal{V}=\bigwedge^3V_6\otimes\mathcal{O}_{\mathsf{Gr}(3,V_6)}$, with Lagrangian subbundles

$$\mathscr{A}_1=A\otimes\mathcal{O}_{\mathsf{Gr}(3,V_5)},\quad \mathscr{A}_2=V_6\wedge\textstyle\bigwedge^2\mathcal{U}_3,\quad\text{and}\quad \mathscr{A}_3=\bigwedge^3V_5\otimes\mathcal{O}_{\mathsf{Gr}(3,V_5)},$$

where $\mathcal{U}_3$ is the tautological subbundle of $\mathsf{Gr}(3,V_5)$. Consider also the isotropic subbundles

$$\mathscr{I}_1\coloneqq\mathscr{A}_1\cap\mathscr{A}_3=(A\cap\textstyle\bigwedge^3V_5)\otimes\mathcal{O}_{\mathsf{Gr}(3,V_5)}\subset\mathscr{A}_1\quad\text{and}\quad\mathscr{I}_2\coloneqq\mathscr{A}_2\cap\mathscr{A}_3=V_5\wedge\textstyle\bigwedge^2\mathcal{U}_3\subset\mathscr{A}_2,$$

of respective ranks $5-n$ and 7. Outside $\Sigma_2(X)$, the morphisms $\mathscr{I}_1\oplus\mathscr{A}_2\to\mathcal{V}$ and $\mathscr{A}_1\oplus\mathscr{I}_2\to\mathcal{V}$ are embeddings, and the bundle $\mathscr{I}\coloneqq\mathrm{Im}(\mathscr{I}_1\oplus\mathscr{I}_2\to\mathcal{V})$ is isotropic. Therefore, the bundles $\mathcal{V},\mathscr{A}_1,\mathscr{A}_2$, and $\mathscr{I}$ satisfy the hypothesis of [DK20a, Proposition 4.5] and we can perform isotropic reduction over $\mathsf{Gr}(3,V_5)\smallsetminus\Sigma_2(X)$ with respect to $\mathscr{I}$.

Let $\widetilde{\mathcal{V}}$ be the kernel of the morphism $\mathcal{V}\longrightarrow\mathscr{I}^\vee$ obtained from the identification $\mathcal{V}\simeq\mathcal{V}^\vee$ induced by the symplectic form. Following [DK20a, Section 4.2], the

quotient $\overline{\mathscr{V}} \coloneqq \widetilde{\mathscr{V}}/\mathscr{I}$ is a vector bundle of rank $2(n-2)$ over $\mathsf{Gr}(3,V_5) \smallsetminus \Sigma_2(X)$. The isotropic reduction $\overline{\mathscr{A}}_k$ of $\mathscr{A}_k$ is defined as the image of $\mathscr{A}_k \cap \widetilde{\mathscr{V}}$ in $\overline{\mathscr{V}}$. By [**DK20a**, Proposition 4.5], $\overline{\mathscr{A}}_1$ and $\overline{\mathscr{A}}_2$ are Lagrangian subbundles of $\overline{\mathscr{V}}$ satisfying

$$\mathsf{Z}^{\geq k}_{A,V_5} \smallsetminus \Sigma_2(X) = S^k(\mathscr{A}_1,\mathscr{A}_2) \smallsetminus \Sigma_2(X) = S^k(\overline{\mathscr{A}}_1,\overline{\mathscr{A}}_2),$$

where the first equality follows from [**DK20a**, Equation (28)].

Since $\mathscr{I}$ is contained in $\mathscr{A}_3$, the isotropic reduction $\overline{\mathscr{A}}_3$ is equal to $\mathscr{A}_3/\mathscr{I}$ and is a vector bundle of rank $n-2$ over $\mathsf{Gr}(3,V_5) \smallsetminus \Sigma_2(X)$. Moreover, the inclusions $\mathscr{A}_1 \cap \mathscr{A}_3 \subset \mathscr{I}$ and $\mathscr{A}_2 \cap \mathscr{A}_3 \subset \mathscr{I}$ induce morphisms

$$\omega_{1,3}\colon \overline{\mathscr{A}}_3 \longrightarrow \overline{\mathscr{A}}_1^\vee \quad \text{and} \quad \omega_{2,3}\colon \overline{\mathscr{A}}_3 \longrightarrow \overline{\mathscr{A}}_2^\vee,$$

which are isomorphisms outside $\Sigma_2(X)$. Let q be the quadratic fibration associated with $\overline{\mathscr{A}}_1, \overline{\mathscr{A}}_2, \overline{\mathscr{A}}_3$ via [**DK20a**, Equation (24)]. By the proof of [**DK18**, Proposition 4.10], the quadric fibration q coincides with the restriction of ϱ to $\mathsf{Gr}(3,V_5) \smallsetminus \Sigma_2(X)$. Then, [**DK20a**, Proposition 4.7] implies the equalities

$$S_k(\varrho|_{\mathsf{Gr}(3,V_5)\smallsetminus\Sigma_2(X)}) = S_k(q) = S_k(\overline{\mathscr{A}}_1,\overline{\mathscr{A}}_2) = \mathsf{Z}^{\geq k}_{A,V_5} \smallsetminus \Sigma_2(X).$$

Finally, let us turn to the case where X special and let X_0 be the ordinary GM manifold obtained from the Lagrangian data set associated to X. The isotropy locus $\Sigma_2(X)$ is equal to $\Sigma_2(X_0)$, as it only depends on (V_6,V_5,A). The argument of [**DK20a**, Lemma 5.4] proves that the quadric fibrations $\mathcal{Q}(X)$ and $\mathcal{Q}(X_0)$ have the same degeneracy loci. The lemma therefore follows from the ordinary case. □

Corollary 5.4. *If X is a GM manifold of dimension 6, the schematic degeneracy loci of ϱ are the schemes $\mathsf{Z}^{\geq k}_{A,V_5} \subset \mathsf{Gr}(3,V_5)$.*

Proof. Since X is a sixfold, the pair (A,V_5) satisfies $\dim(A \cap \bigwedge^3 V_5) = 0$. In particular, the isotropic locus $\Sigma_2(X)$ is empty and the corollary follows from the proposition. □

5.2.1. Cohomology of the second quadric fibration. We go back to the case of a Gushel–Mukai sixfold X and study the cohomology of the second quadric fibration $\varrho\colon \mathcal{Q} \to \mathsf{Gr}(3,V_5)$. Denote by $\pi\colon \mathcal{Q} \to X$ the $\mathbf{P}^2$-bundle defining $\mathcal{Q}$ and by $j\colon \mathcal{Q} \hookrightarrow \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3)$ the inclusion of $\mathcal{Q}$ as a family of quadrics. The diagram

(86)
$$\begin{array}{ccccc} & & \mathcal{Q} & \xrightarrow{\ j\ } & \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3) \\ & \swarrow^{\pi} & & \searrow^{\varrho} & \downarrow{\scriptstyle \varphi} \\ \mathsf{Gr}(2,V_5) & \xleftarrow{\ \gamma_X\ } & X & & \mathsf{Gr}(3,V_5) \end{array}$$

relates the two projective bundles π and φ.

Following [**Nag07**, Section 2.3], we consider the *variable cohomology*

$$H^k(\mathcal{Q},\mathbf{Q})_{\mathrm{var}} \coloneqq \mathrm{Coker}(j^*\colon H^k(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3),\mathbf{Q}) \longrightarrow H^k(\mathcal{Q},\mathbf{Q})). \tag{87}$$

The dual of the variable cohomology of $\mathcal{Q}$ is related to the vanishing cohomology of X (defined in Equation (82)) via the $\mathbf{P}^2$-bundle π.

Proposition 5.5. *The $\mathbf{P}^2$-bundle $\pi\colon \mathcal{Q} \to X$ induces an isomorphism*

$$\pi^*\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \xrightarrow{\ \sim\ } H^6(\mathcal{Q},\mathbf{Q})_{\mathrm{var}}.$$

Proof. For any projective bundle $\mathbf{P}(\mathcal{E})$ over a manifold S, with $\operatorname{rank}(\mathcal{E}) = r$, the cohomology of $\mathbf{P}(\mathcal{E})$ is an $H^*(S, \mathbf{Q})$-algebra generated by the degree 2 class $\xi = c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$, satisfying the polynomial relation

$$\xi^r + c_1(\mathcal{E})\xi^{r-1} + \cdots + c_r(\mathcal{E}) = 0.$$

In particular, we obtain

$$H^*(\mathcal{Q}, \mathbf{Q}) = \pi^* H^*(X, \mathbf{Q})[\xi]/(\xi^3 + c_1\xi^2 + c_2\xi + c_3), \tag{88}$$

where ξ is a π-relatively ample class, and the c_i are the pullbacks of the Chern classes of the vector bundle $(V_5 \otimes \mathcal{O}_X)/\mathcal{U}_X$ on X. In particular, the class c_i is equal to $\pi^*(\gamma^*\sigma_i)$, where $\sigma_i \in H^{2i}(\mathsf{Gr}(2, V_5), \mathbf{Z})$ is a Schubert class. Therefore, the subspace $\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}]$ of $H^\bullet(\mathcal{Q}, \mathbf{Q})$ is a subalgebra.

Similarly, we have

$$H^*(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}) = \varphi^* H^*(\mathsf{Gr}(3, V_5), \mathbf{Q})[\eta]/(\eta^4 + d_1\eta^3 + d_2\eta^2 + d_3\eta), \tag{89}$$

where η is a φ-relatively ample class and the d_i are the Chern classes of the vector bundle $\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3$ on $\mathsf{Gr}(3, V_5)$ (note that $d_4 = 0$).

Equation (88), together with the description of the cohomology of X done in Section 5.1.1, imply that the cohomology of $\mathcal{Q}$ is concentrated in even degrees and, for all $i \leq 6$, we have

$$H^i(\mathcal{Q}, \mathbf{Q}) = \bigoplus_{k=0}^{2} \bar{\xi}^k \cdot \pi^* H^{i-2k}(X, \mathbf{Q}).$$

For all $i \leq 5$, the cohomology groups $H^i(\mathcal{Q}, \mathbf{Q})$ are contained in the subalgebra $\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}]$ and there is an isomorphism

$$H^6(\mathcal{Q}, \mathbf{Q})/(\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}])_6 = \pi^*(H^6(X, \mathbf{Q})_{\mathrm{van}}).$$

The variable cohomology of $\mathcal{Q}$ being defined as the cokernel of the morphism

$$j^* \colon H^6(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}) \longrightarrow H^6(\mathcal{Q}, \mathbf{Q}),$$

it is therefore enough to show that the image of j^* is $(\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}])_6$.

We first show that the image of j^* is contained in $\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}]$. Indeed, by (89), the cohomology group $H^6(\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3), \mathbf{Q})$ splits as

$$\varphi^* H^6(\mathsf{Gr}(3, V_5), \mathbf{Q}) \oplus \bar{\eta} \cdot H^4(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q})$$

and, since $H^6(\mathsf{Gr}(3, V_5), \mathbf{Q}) = \sigma_1 \cdot H^4(\mathsf{Gr}(3, V_5), \mathbf{Q})$, we obtain

$$H^6(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}) = H^2(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}) \cdot H^4(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}).$$

Therefore, the image of j^* is contained in $H^2(\mathcal{Q}, \mathbf{Q}) \cdot H^4(\mathcal{Q}, \mathbf{Q})$, and thus it is contained in $\pi^*\gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}]$ by the discussion above.

In particular, we have $\operatorname{Im}(j^*) \subset (\pi^*\gamma_X^* H^*(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}])_6$. The right side of this inclusion is isomorphic to

$$H^6(\mathsf{Gr}(2, V_5), \mathbf{Q}) \oplus \bar{\xi} \cdot H^4(\mathsf{Gr}(2, V_5), \mathbf{Q}) \oplus \bar{\xi}^2 \cdot H^2(\mathsf{Gr}(2, V_5), \mathbf{Q}),$$

hence it has dimension 5. To conclude, we show that the image of j^* has dimension at least 5. Indeed, the rank of j^* is greater than or equal to the rank of

$$j_* j^* \colon H^6(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}) \longrightarrow H^8(\mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 \mathcal{U}_3), \mathbf{Q}).$$

This morphism is simply the intersection with the class $[\mathcal{Q}] \in H^2(\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3), \mathbf{Q})$. Since $\mathcal{Q}$ is a quadric section, the class $[\mathcal{Q}]$ is equal to $2(\bar{\eta} + \varphi^* x_1)$, for some $x_1 \in H^2(\mathsf{Gr}(3, V_5), \mathbf{Q})$. Therefore, the morphism $j_* j^*$ is multiplication by $2(\eta + \varphi^* x_1)$ on

$$H^6(\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3)) = \varphi^* H^6(\mathsf{Gr}(3, V_5)) \oplus \bar{\eta} \cdot \varphi^* H^4(\mathsf{Gr}(3, V_5)) \oplus \bar{\eta}^2 \cdot \varphi^* H^2(\mathsf{Gr}(3, V_5)) \oplus \mathbf{Q}\bar{\eta}^3,$$

where all the cohomology groups above are with rational coefficients. The image of $j^* j_*$ therefore contains the sum

$$(\bar{\eta} + \varphi^* x_1)\varphi^* H^6(\mathsf{Gr}(3, V_5)) + (\bar{\eta} + \varphi^* x_1)\bar{\eta} \cdot \varphi^* H^4(\mathsf{Gr}(3, V_5)) + (\bar{\eta} + \varphi^* x_1)\bar{\eta}^2 \cdot \varphi^* H^2(\mathsf{Gr}(3, V_5))$$

and, since $\bar{\eta}$ satisfies no polynomial relations of degree smaller than 4, this sum is direct and the rank of $j_* j^*$ is at least 5.

This finishes the proof of the claim $\mathrm{Im}(j^*) = (\pi^* \gamma_X^* H^*(\mathsf{Gr}(2, V_5), \mathbf{Q})[\bar{\xi}])_6$, and therefore the proof of the proposition. □

Remark 5.6. Actually, one may also explicitly determine the morphism

$$j^* \colon H^6(\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3), \mathbf{Z}) \longrightarrow H^6(\mathcal{Q}, \mathbf{Z})$$

and show that its image is equal to $(\pi^* \gamma_X^* H^\bullet(\mathsf{Gr}(2, V_5), \mathbf{Z})[\bar{\xi}])_6$. In particular, this proves that π^* induces an inclusion

$$H^6(X, \mathbf{Z})_{\mathrm{van}} \hookrightarrow H^6(X, \mathbf{Z})/(\mathrm{Im}(\gamma_X^*)) \simeq H^6(\mathcal{Q}, \mathbf{Z})_{\mathrm{var}}.$$

5.3. The stratification $\mathsf{Z}^{\geq k}_{A,V_5}$

Let X be a Gushel–Mukai manifold of dimension at least 3, with associated Lagrangian data (A, V_5, V_6). Proposition 5.3 shows that the schematic degeneracy loci of the second quadric fibration ϱ of X are, away from the isotropy locus $\Sigma_2(X)$ (see Definition 5.2), the schemes

$$\mathsf{Z}^{\geq k}_{A,V_5} = \mathsf{Z}^{\geq k}_{A} \cap \mathsf{Gr}(3, V_5) \subset \mathsf{Gr}(3, V_6).$$

The aim of this section is to study the stratification $\mathsf{Z}^{\geq k}_{A,V_5}$ of $\mathsf{Gr}(3, V_5)$, in order to determine the dimensions and study the normality of these loci. We first show in Proposition 5.7 that the stratification behaves as expected when $V_5 \subset V_6$ is a general hyperplane namely, $\mathsf{Z}^{\geq k}_{A,V_5} \subset \mathsf{Gr}(3, V_5)$ is normal of expected codimension $\frac{k(k+1)}{2}$ and its singular locus is equal to $\mathsf{Z}^{\geq k+1}_{A,V_5}$.

In Section 5.3.2, we carry out a finer argument in the case $[V_5] \in \mathsf{Y}^0_{A^\perp}$, which corresponds to the case when X is a sixfold. In particular, we show that $\mathsf{Z}^{\geq 1}_{A,V_5}$ and $\mathsf{Z}^{\geq 2}_{A,V_5}$ are integral varieties of expected dimensions and that $\mathsf{Z}^{\geq 2}_{A,V_5}$ is normal when Y^3_{A,V_5} is empty.

The contents of Section 5.3.1 will appear in [**Riz26**, Section 1.2]*, while Section 5.3.2 is original.*

5.3.1. The general case. We first study the loci $\mathsf{Z}^{\geq k}_{A,V_5}$ when A is a Lagrangian subspace of $\bigwedge^3 V_6$ and $V_5 \subset V_6$ is a *general* hyperplane. We show that in this case the schemes $\mathsf{Z}^{\geq k}_{A,V_5}$ have expected dimensions.

We also study the associated double covers

$$\begin{array}{ccc} \widetilde{\mathsf{Z}}^{\geq k}_{A,V_5} \coloneqq \widetilde{\mathsf{Z}}^{\geq k}_{A} \times_{\mathsf{Gr}(3,V_6)} \mathsf{Gr}(3,V_5) & & \\ \Big\downarrow g^{(k)}_{A,V_5} & & (90) \\ \mathsf{Z}^{\geq k}_{A,V_5} \coloneqq \mathsf{Z}^{\geq k}_{A} \times_{\mathsf{Gr}(3,V_6)} \mathsf{Gr}(3,V_5) & & \end{array}$$

obtained as pullbacks of the double covers constructed in Theorem 2.14.

Proposition 5.7. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $V_5 \subset V_6$ be a general hyperplane.*

(a) *The scheme $\mathsf{Z}^{\geq 1}_{A,V_5}$ is an irreducible normal Cohen–Macaulay hypersurface in $\mathsf{Gr}(3,V_5)$, of degree 20, smooth outside $\mathsf{Z}^{\geq 2}_{A,V_5}$. Moreover, $\widetilde{\mathsf{Z}}^{\geq 1}_{A,V_5}$ is an irreducible normal Cohen–Macaulay scheme of (expected) dimension 5, smooth outside of the preimage by $g^{(1)}_{A,V_5}$ of $\mathsf{Z}^{\geq 3}_{A,V_5}$.*

(b) *The scheme $\mathsf{Z}^{\geq 2}_{A,V_5}$ is an irreducible normal Cohen–Macaulay threefold of degree 56, smooth outside $\mathsf{Z}^{\geq 3}_{A,V_5}$. Moreover, $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ is an irreducible normal Cohen–Macaulay scheme of (expected) dimension 3, smooth outside of the preimage by $g^{(2)}_{A,V_5}$ of $\mathsf{Z}^{\geq 4}_{A,V_5}$.*

(c) *The scheme $\mathsf{Z}^{\geq 3}_{A,V_5}$ is smooth finite of cardinality 40.*

The degrees are computed with respect to the Plücker embedding of $\mathsf{Gr}(3,V_5)$ in $\mathbf{P}(\bigwedge^3 V_5)$ (the Grassmannian itself has degree 5).

Proof. All the smoothness statements follow from Theorem 2.5 and Kleiman's version of the Bertini theorem ([**Har77**, Theorem III.10.8]) applied to Z^{k}_{A,V_5}. The statements about the dimensions and Cohen–Macaulayness also follow and normality is given by Cohen–Macaulayness and smoothness in codimension 1.

For the irreducibility in cases (a) and (b), we apply the Bertini theorem for Grassmannians ([**Deb96**, Theorem 6.2]) to the morphisms $\mathsf{Z}^{\geq k}_{A} \hookrightarrow \mathsf{Gr}(3,V_6)$ and $\widetilde{\mathsf{Z}}^{\geq k}_{A} \to \mathsf{Z}^{\geq k}_{A} \hookrightarrow \mathsf{Gr}(3,V_6)$: to prove that the preimage of a general $\mathsf{Gr}(3,V_5)$ is irreducible of the expected dimension, it is enough to check that $\mathsf{Z}^{\geq k}_{A}$ meets $\mathsf{Gr}(3,V_4)$ for a general subspace V_4 of V_6. The class of $\mathsf{Gr}(3,V_4)$ in the Chow ring $A^\bullet(\mathsf{Gr}(3,V_6))$ is represented by the Schubert cycle $\sigma_{2,2,2}$. Using Schubert calculus and Remark 2.13, we compute

$$[\mathsf{Z}^{\geq 1}_{A}] \cdot [\mathsf{Gr}(3,V_4)] = 4\sigma_1 \cdot \sigma_{2,2,2} = 4\sigma_{3,2,2}$$

and

$$[\mathsf{Z}^{\geq 2}_{A}] \cdot [\mathsf{Gr}(3,V_4)] = (16\sigma_1^3 - 12\sigma_1\sigma_2 + 12\sigma_3) \cdot \sigma_{2,2,2} = 16\sigma_{3,3,3},$$

which are nonzero. Hence the desired intersections are not empty.

The pullback of σ_1 under the inclusion $\mathsf{Gr}(3,V_5) \subset \mathsf{Gr}(3,V_6)$ is still the hyperplane class σ_1. Therefore, Remark 2.13 implies that the class of $\mathsf{Z}^{\geq 1}_{A,V_5}$ is equal to $4\sigma_1 \in A^1(\mathsf{Gr}(3,V_5),\mathbf{Q})$, hence we can compute

$$\deg_{\mathsf{Gr}(3,V_5)}(\mathsf{Z}^{\geq 1}_{A,V_5}) = 4\sigma_1^6 = 20$$

and

$$\deg_{\mathsf{Gr}(3,V_5)}(\mathsf{Z}^{\geq 2}_{A,V_5}) = [\mathsf{Z}^{\geq 2}_{A,V_5}] \cdot \sigma_1^3 \cdot [\mathsf{Gr}(3,V_5)] = 56.$$

Finally, in (c), the cardinality of $\mathsf{Z}^{\geq 3}_{A,V_5}$ is given by the intersection product

$$[\mathsf{Z}^{\geq 3}_{A}] \cdot [\mathsf{Gr}(3,V_5)] = (40\sigma_2^3 + 240\sigma_1\sigma_2\sigma_3 - 240\sigma_3^2) \cdot \sigma_{1,1,1} = 40.$$

This finishes the proof of the proposition. □

Remark 5.8. By a properness argument, the proposition implies that for any Lagrangian subspace $A \subset \bigwedge^3 V_6$ and any hyperplane $V_5 \subset V_6$, the schemes $\mathsf{Z}^{\geq k}_{A,V_5}$ are nonempty for $k \in \{1,2,3\}$, of dimension at least their expected dimension $6 - k(k+1)/2$.

5.3.2. Local study. We now consider the (more general) case where the hyperplane $[V_5]$ belongs to the open subset $\mathsf{Y}^0_{A^\perp} \subset \mathbf{P}(V_6^\vee)$. This corresponds to the case where the GM manifold X associated to the Lagrangian data (V_6, V_5, A) is either a GM sixfold or an ordinary GM fivefold.

We carry out a local study of the schemes $\mathsf{Z}^{\geq k}_{A,V_5}$, showing in particular the following improvement of Proposition 5.7.

Theorem 5.9. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $[V_5] \in \mathsf{Y}^0_{A^\perp}$.*

(a) *The scheme $\mathsf{Z}^{\geq 1}_{A,V_5}$ is an integral normal hypersurface in $\mathsf{Gr}(3,V_5)$ of degree 20.*
(b) *The scheme $\mathsf{Z}^{\geq 2}_{A,V_5}$ is the singular locus of $\mathsf{Z}^{\geq 1}_{A,V_5}$; it is an integral Cohen–Macaulay threefold of degree 56, which is normal when $\mathsf{Y}^3_{A,V_5} = \varnothing$.*

Remark 5.10. In the hypothesis of the theorem, $\mathsf{Z}^{\geq 2}_{A,V_5}$ has the expected dimension, therefore its cohomology class can be computed as the pullback of $[\mathsf{Z}^{\geq 2}_{A}]$ via the inclusion map $i\colon \mathsf{Gr}(3,V_5) \hookrightarrow \mathsf{Gr}(3,V_6)$. Using Remark 2.13, we obtain

$$[\mathsf{Z}^{\geq 2}_{A,V_5}] = i^*(16\sigma_1 - 12\sigma_1\sigma_2 + 12\sigma_3).$$

Recall that the classes σ_i are the Chern classes of the quotient bundle $\mathcal{Q}_6$ of $\mathsf{Gr}(3,V_5)$, and that there is an isomorphism $i^*\mathcal{Q}_6 \simeq \mathcal{Q}_5 \oplus \mathcal{O}_{\mathsf{Gr}(3,V_5)}$, where $\mathcal{Q}_5$ is the quotient bundle of $\mathsf{Gr}(3,V_5)$. Since Chern classes commutes with pullback, we obtain

$$[\mathsf{Z}^{\geq 2}_{A,V_5}] = 16\sigma_1 - 12\sigma_1\sigma_2 = 20\sigma_{2,1} \in H^6(\mathsf{Gr}(3,V_5), \mathbf{Q}).$$

5.3.2.1. *Tangent space to $\mathsf{Gr}(3,V_5)$.* We follow the strategy of the proof of Theorem 2.5. We start by studying the morphism

$$i_{T,5}\colon \mathsf{Gr}(3,V_5) \xhookrightarrow{i_5} \mathsf{Gr}(3,V_6) \xrightarrow{i_T} \mathrm{LG}(\textstyle\bigwedge^3 V_6)$$

sending a point $[U] \in \mathsf{Gr}(3,V_5)$ to the Lagrangian subspace $[T_U]$.

Let $U_0 \subset V_6$ be a three-dimensional subspace. We denote by U_∞ a complementary vector subspace to U_0 in V_6, so that $T_{U_0} = \bigwedge^2 U_0 \wedge V_6$ decomposes as $\bigwedge^3 U_0 \oplus \mathrm{Hom}(U_0, U_\infty)$.

Section 2.3.1 shows that i_T is an embedding and that the tangent map to i_T at a point $[U_0]$ gives an identification

$$\Theta\colon T_{\mathsf{Gr}(3,V_6),[U_0]} \xrightarrow{\sim} H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2)),$$

where $I_{\mathcal{C}_{U_0}}$ is the ideal sheaf of the cone $\mathcal{C}_{U_0} \subset \mathbf{P}(T_{U_0})$ with vertex $[\bigwedge^3 U_0]$ over the subscheme $\mathscr{M}_1 \subset \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$ of morphisms of rank 1.

Let

$$P_5 := \mathbf{P}(\mathrm{Hom}(U_0, U_\infty \cap V_5)) \tag{91}$$

be the 5-dimensional linear space of linear maps $U_0 \to U_\infty$ whose image is contained in V_5.

Proposition 5.11. *The tangent map to $i_{T,5}$ at a point $[U_0] \in \mathsf{Gr}(3, V_5)$ gives an identification*

$$\Theta_5 \colon T_{\mathsf{Gr}(3,V_5),[U_0]} \xrightarrow{\sim} H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0,V_5}}(2)) \subset H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2)),$$

where $\mathcal{C}_{U_0,V_5}$ is the cone in $\mathbf{P}(T_{U_0})$ with vertex $[\bigwedge^3 U_0]$ over $\mathscr{M}_1 \cup P_5$.

Proof. The tangent map to $i_{T,5}$ at a point $[U_0]$ factors as $\Theta \circ T_{i_5,[U_0]}$, where $T_{i_5,[U_0]}$ is the tangent map to i_5 at $[U_0]$. Section 2.3.1 shows that the tangent space $T_{\mathsf{Gr}(3,V_6),[U_0]}$ can be identified with $\mathcal{U} := \mathrm{Hom}(U_0, U_\infty)$ and that Θ sends a matrix $(b_{i,j}) \in \mathcal{U}$ to the quadratic form given by

$$[\textstyle\bigwedge^3 V_6] \oplus \mathrm{Hom}(U_0, U_\infty) \ni (m, M) \mapsto \sum_{ij} b_{i,j} M^{i,j},$$

where $M^{i,j}$ isV times the determinant of the (2×2)-submatrix of M in which we remove the ith column and the jth row.

The image via $T_{i_5,[U_0]}$ of the tangent space $T_{\mathsf{Gr}(3,V_5),[U_0]}$ can be identified with $\mathcal{U}_5 = \mathrm{Hom}(U_0, U_\infty \cap V_5) \subset \mathcal{U}$. If we choose a basis (u'_1, u'_2, u'_3) of U_∞ such that (u'_1, u'_2) is a basis of $U_\infty \cap V_5$, the space $\mathcal{U}_5$ is the space of matrices (b_{ij}) such that $b_{31} = b_{32} = b_{33} = 0$.

Therefore, the image of $T_{\mathsf{Gr}(3,V_5),[U_0]}$ by Θ_5 is spanned by the cofactors $M^{i,j}$ such that $j \neq 3$. This is, in $\mathrm{Sym}^2(T^\vee_{U_0}) = H^0(\mathbf{P}(T_{U_0}), \mathcal{O}(2))$, the linear system of quadrics in $\mathbf{P}(T_{U_0})$ that vanish on the cone over the set of matrices of rank 1 or with image contained in V_5. □

5.3.2.2. *The restriction map $r_{K,5}$.* We will show in Section 5.3.2.4 that the local structure of Z^k_{A,V_5} at a point $[U_0]$ is controlled by the restriction morphism

$$r_{K,5} \colon H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0,V_5}}(2)) \hookrightarrow H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2)) \xrightarrow{r_K} H^0(\mathbf{P}(K), \mathcal{O}(2)), \tag{92}$$

where K is the k-dimensional space $T_{U_0} \cap A$ and $I_{\mathcal{C}_{U_0,V_5}}$ is the ideal of the cone $\mathcal{C}_{U_0,V_5}$ defined in Proposition 5.11.

Recall from that proposition that the space $H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0,V_5}}(2))$ has dimension 6, while the space $\mathrm{Sym}^2 K$ has dimension $\frac{k(k+1)}{2}$.

Proposition 5.12. *Let $[U_0] \in \mathsf{Z}^k_{A,V_5}$ and let $\mathbf{P}(K) \subset \mathbf{P}(T_{U_0})$ be a linear space of dimension $k-1$ that does not meet the cone $\mathcal{C}_{U_0,V_5}$. Then,*

- (a) *when $k = 1$, the map $r_{K,5}$ is surjective;*
- (b) *when $k = 2$, one of the following holds:*
  - (b1) *the map $r_{K,5}$ is surjective;*
  - (b2) *there exists $v \in U_0$ such that $\mathbf{P}(K) \subset \mathbf{P}(v \wedge U_0 \wedge V_6)$, and the map $r_{K,5}$ has corank 1;*
- (c) *when $k = 3$, one of the following holds:*
  - (c1) *the map $r_{K,5}$ has corank ≤ 1 (its kernel has dimension ≤ 1);*
  - (c2) *there exists $v \in U_0$ such that $\mathbf{P}(K) \cap \mathbf{P}(v \wedge U_0 \wedge V_6)$ has dimension 1, and the map $r_{K,5}$ has corank 2 (its kernel has dimension 2).*

If A contains no decomposable vectors and the point $[V_5]$ lies in $\mathsf{Y}^0_{A^\perp}$ (so that $A \cap \bigwedge^3 V_5 = \{0\}$), the projective space $\mathbf{P}(A \cap T_{U_0})$ does not meet the cone $\mathcal{C}_{U_0,V_5}$ and we can apply the above proposition to study the morphism $r_{K,5}$.

Proof. As in Proposition 2.7, upon projecting from the vertex $[\bigwedge^3 U_0] \in \mathbf{P}(T_{U_0})$ of the cone $\mathcal{C}_{U_0}$, we can consider $\mathbf{P}(K)$ as a linear subspace of $\mathbf{P}^8 := \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$. Under this identification, for every nonzero vector $v \in U_0$, the intersection $\mathbf{P}(K) \cap \mathbf{P}(v \wedge U_0 \wedge V_6)$ is the subspace of $\mathbf{P}(K)$ of morphisms whose kernel contains v. Furthermore, the morphism $r_{K,5}$ has the same image as the composition

$$r'_{K,5} \colon H^0(\mathbf{P}^8, I_{\mathscr{M}_1 \cup P_5}(2)) \hookrightarrow H^0(\mathbf{P}^8, I_{\mathscr{M}_1}(2)) \xrightarrow{r'_K} H^0(\mathbf{P}(K), \mathcal{O}(2)),$$

when $\mathbf{P}(K) \cap (\mathscr{M}_1 \cup P_5) = \varnothing$. Recall that P_5, defined in (91), is the space of morphisms whose image is contained in V_5.

The statements (a)-(c) are then equivalent to

- (a′) when $k = 1$, the map $r'_{K,5}$ is surjective;
- (b′) when $k = 2$, one of the following holds:
  - (b1′) the map $r'_{K,5}$ is surjective;
  - (b2′) there exists a nonzero vector $v \in U_0$ in the kernel of all morphisms in $\mathbf{P}(K)$, and the map $r'_{K,5}$ has corank 1.
- (c′) when $k = 3$, one of the following holds:
  - (c1′) the map $r'_{K,5}$ has corank ≤ 1;
  - (c2′) there exists a nonzero vector $v \in U_0$ such that the space of morphisms in $\mathbf{P}(K)$ whose kernel contains v has dimension 1, and the map $r'_{K,5}$ has corank 2.

We follow the strategy of Lemma 2.9 and consider the rational map Φ_5 induced by the linear system $|I_{\mathscr{M}_1 \cup P_5}(2)|$ on $\mathbf{P}^8 = \mathbf{P}(\mathrm{Hom}(U_0, U_\infty))$. The linear system $|I_{\mathscr{M}_1 \cup P_5}(2)|$ of quadrics containing $\mathscr{M}_1 \cup P_5$ is contained in $|I_{\mathscr{M}_1}(2)|$. Therefore, the morphism Φ_5 factors as

$$\Phi_5 \colon \mathbf{P}^8 \overset{\Phi}{\dashrightarrow} |I_{\mathscr{M}_1}(2)|^\vee \overset{\pi}{\dashrightarrow} |I_{\mathscr{M}_1 \cup P_5}(2)|^\vee \simeq \mathbf{P}^5,$$

where the morphism Φ (defined in Equation (32)) is such that $\Phi(M)_{i,j} = M^{i,j}$ and π is the projection from the linear subspace

$$\mathbf{P}^2 = \mathbf{P} \begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \subset \mathbf{P} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \simeq |I_{\mathscr{M}_1}(2)|^\vee. \tag{93}$$

Remark 5.13. Note that $\Phi^{-1}(\mathbf{P}^2) \smallsetminus \mathscr{M}_1$ is equal to $P_5 \smallsetminus \mathscr{M}_1$. Indeed, the conditions

$$M^{31} = M^{32} = 0 \quad \text{and} \quad M^{21} = M^{22} = 0$$

mean

$$x_{31} x_{\bullet 2} - x_{32} x_{\bullet 1} = 0 \quad \text{and} \quad x_{31} x_{\bullet 3} - x_{33} x_{\bullet 1} = 0,$$

where $x_{\bullet j}$ is the jth column of the matrix $M = (x_{ij})$. If moreover M has rank at least 2, one has $x_{31} = 0$ and, either $x_{32} = x_{33} = 0$ or the column $x_{\bullet 1}$ is 0. Using $M^{11} = M^{12} = 0$, one obtains $x_{31} = x_{32} = x_{33} = 0$, namely $[M]$ lies in P_5. The reverse inclusion $\Phi(P_5 \smallsetminus \mathscr{M}_1) \subset \mathbf{P}^2$ is obvious.

As in the proof of Lemma 2.9, the restriction of Φ_5 to $\mathbf{P}(K)$ can be identified with

$$\Phi_{5,K} \colon \mathbf{P}(K) \overset{\nu_2}{\hookrightarrow} \mathbf{P}(\mathrm{Sym}^2 K) \overset{h_5}{\dashrightarrow} \mathbf{P}(\mathrm{Im}(r'_{K,V_5})^\vee),$$

where ν_2 is the second Veronese embedding and the rational map h_5 is the projection corresponding to the inclusion $\mathrm{Im}(r'_{K,5}) \subset \mathrm{Sym}^2 K^\vee$.

Note that $\Phi_{5,K}$ also factors as

$$\Phi_{5,K}\colon \mathbf{P}(K) \overset{\Phi}{\hookrightarrow} \mathbf{P}(\mathrm{Sym}^2 K) \overset{\pi}{\dashrightarrow} \mathbf{P}^5.$$

Since $k \leq 3$, by Lemma 2.9, the map r'_K is surjective and the morphism Φ coincides with ν_2 on $\mathbf{P}(K)$. It follows that the morphism $r'_{K,5}$ has corank c if and only if the intersection of the span

$$\Pi \coloneqq \langle \Phi(\mathbf{P}(K)) \rangle = \mathbf{P}(\mathrm{Sym}^2 K)$$

with the center $\mathbf{P}^2$ of the projection π has dimension $c-1$ (if $c=0$, it should be empty).

Note that, since $\mathbf{P}(K)$ does not meet P_5, the Veronese $\Phi(\mathbf{P}(K))$ does not meet $\mathbf{P}^2$.

Case (a) ($k=1$) then follows, since $\mathbf{P}(K)$ is a point.

We now consider case (b) ($k=2$), where $\mathbf{P}(K)$ is a line ℓ and $\Phi(\ell)$ is a conic, whose linear span Π is therefore a plane. The center $\mathbf{P}^2$ does not meet the conic $\Phi(\ell)$, hence the corank c of $r'_{K,5}$ is either 0 (if $\mathbf{P}^2 \cap \Pi$ is empty; this is case (b1′)) or 1 (if $\mathbf{P}^2 \cap \Pi$ is a point; this is case (b2′)).

The line ℓ meets the subscheme $\mathscr{M}_2$ of noninvertible morphisms in at least one point p. Since p is a rank 2 matrix that does not lie in P_5, up to a change of basis preserving $U_\infty \cap V_5$, we can suppose

$$p = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right], \quad \text{hence} \quad \Phi(p) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right].$$

Every point in Π lies either on a line through $\Phi(p)$ and another point $\Phi(q)$ of $\Phi(\ell)$, or on the tangent line to $\Phi(\ell)$ at $\Phi(p)$.

In the first case, a linear combination of $\Phi(p)$ and $\Phi(q)$ lies in $\mathbf{P}^2$. Because of the definition of P given in (93), this is equivalent to saying that there exists $q \in \ell$ of the form

$$q = \left[\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \right], \quad \text{satisfying} \quad \Phi(q) = \left[\begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \right],$$

where the $*$ represent possibly nonzero entries.

Since q does not lie in P_5, the entries q_{31}, q_{32}, and q_{33} cannot be all 0. The equalities $q^{3,1} = q^{3,2} = 0$ and $q^{2,1} = q^{2,2} = 0$ then imply that the column $q_{\bullet,1}$ is 0 (by the computations of Remark 5.13). Namely, q and p have same nontrivial kernel.

Consider now the case where the tangent to $\Phi(\ell)$ at $\Phi(p)$ meets $\mathbf{P}^2$. Let q be a second point in ℓ. Then the plane Π is spanned by $(\Phi(p), \Phi(q), B(p,q))$, where

$$B(p,q) \coloneqq \Phi(p+q) - \Phi(q) - \Phi(p),$$

and $\Phi(\ell)$ is the zero locus in Π of the equation $s^2\Phi(p) + stB(p,q) + t^2\Phi(q)$. The tangent line to $\Phi(\ell)$ at $\Phi(p)$ is then spanned by $\Phi(p)$ and $B(p,q)$. The special form of the matrix p allows one to show that the matrix $B(p,q)$ has the form

$$B(p,q) = \left[\begin{pmatrix} q_{22}+q_{33} & -q_{21} & -q_{31} \\ -q_{21} & q_{11} & 0 \\ -q_{31} & 0 & q_{11} \end{pmatrix} \right].$$

Therefore, if the tangent to $\Phi(\ell)$ at $\Phi(p)$ lies in $\mathbf{P}^2$, then q_{11}, q_{21}, and q_{31} vanish, and that again implies that q and p have the same nontrivial kernel.

Putting all together, this proves that if $r'_{K,5}$ is not surjective, all the morphisms in ℓ have the same one dimensional kernel and, by the discussion above, the map $r'_{K,5}$ has corank 1.

Finally, we consider case (c) ($k = 3$), where $\mathbf{P}(K)$ is a plane P and $\Phi(P)$ is a Veronese surface, whose linear span Π is a 5-dimensional space. Case (c1′) happens when $\Pi \cap \mathbf{P}^2$ is either empty or a point. Suppose now that we are in case (c2′), namely that the map $r'_{K,5}$ has corank at least 2, and $\Pi \cap \mathbf{P}^2$ has dimension at least 1. We show that in this case, $\Pi \cap \mathbf{P}^2$ has dimension exactly 1 and P contains a unique line ℓ of morphisms with the same kernel.

Consider the secant variety to the Veronese surface $\Phi(P)$, which is a cubic hypersurface $H \subset \Pi$. Since $\Pi \cap \mathbf{P}^2$ has dimension at least 1, the intersection $H \cap \mathbf{P}^2$ is nonempty. Therefore, there exists a point of $\mathbf{P}^2$ lying on a secant or a tangent line to $\Phi(P)$, and this is equivalent to saying that there exists a line $\ell \subset P$ such that $\mathbf{P}^2$ meets the linear span of $\Phi(\ell)$. By the proof in the case $k = 2$, since $\Phi(P)$ does not meet the center $\mathbf{P}^2$, this implies that all morphisms in ℓ have the same kernel and $\langle \Phi(\ell) \rangle \cap \mathbf{P}^2$ is a point.

To finish the proof of the proposition, we need to show that $\mathbf{P}^2$ cannot be contained in Π. Indeed, if this is the case, the intersection $H \cap \mathbf{P}^2$ contains at least two distinct points, which determine two distinct lines ℓ and ℓ' in P of morphisms with same kernel. Since P is linear and contains no rank 1 matrices, all the morphisms in P have the same kernel. Therefore, there exists a 2-dimensional subspace $U_2 \subset U_0$ such that P is contained in $\mathbf{P}(\mathrm{Hom}(U_2, U_\infty)) = \mathbf{P}^5$. Thus, P must meet the 3-dimensional space $\mathbf{P}(\mathrm{Hom}(U_2, U_\infty \cap V_5))$, which is absurd since $\mathbf{P}(K) \cap P_5$ is empty. □

5.3.2.3. *The conditions* (b2′) *and* (c2′). In this section, we study the locus of points $[U_0] \in \mathsf{Z}^{\geq 2}_{A,V_5}$ for which there exists a nonzero vector $v \in U_0$ such that the subspace

$$A \cap (v \wedge U_0 \wedge V_6) \subset A \cap T_{U_0}$$

has dimension 2. When A contains no decomposable vectors and $[V_5]$ lies in $\mathsf{Y}^0_{A^\perp}$, this is exactly the set of points $[U_0] \in \mathsf{Z}^{\geq 2}_A$ for which conditions (b2′) or (c2′) of Proposition 5.12 hold.

As in [**KKM24**, Section 3], we consider the variety

$$\Omega \coloneqq \bigcup_{v \in V_6} \mathbf{P}(v \wedge \textstyle\bigwedge^2 V_6) \subset \mathbf{P}(\bigwedge^3 V_6) \tag{94}$$

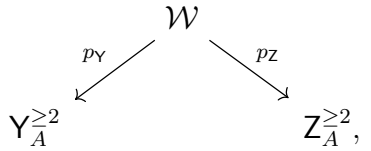
of points in $\mathbf{P}(\bigwedge^3 V_6)$ of the form $[v \wedge \eta]$ for some $v \in V_6$ and $\eta \in \bigwedge^2 V_6$.

Consider also the variety

$$\mathcal{W} \coloneqq \{([U_0], [K]) \in \mathsf{Gr}(3, V_6) \times \mathsf{Gr}(2, A) \mid \mathbf{P}(K) \subset \mathbf{P}(T_{U_0}) \cap \Omega\} \tag{95}$$

(recall that T_{U_0} is the Lagrangian subspace $\bigwedge^2 U_0 \wedge V_6$ in $\bigwedge^3 V_6$).

Lemma 5.14. *Suppose that A contains no decomposable vectors. There are morphisms*



where p_{Z} is induced by the projection to $\mathsf{Gr}(3, V_6)$ and p_{Y} is an isomorphism over Y^2_A and contracts a $\mathbf{P}^2$ over each point of $\mathsf{Y}^{\geq 3}_A$. In particular, $\mathcal{W}$ has dimension 2.

Proof. Let $([U_0], [K]) \in \mathcal{W}$. Since A contains no decomposable vectors, the condition $\mathbf{P}(K) \subset \Omega$ implies that there exists a unique nonzero vector $v \in U_0$ such that $\mathbf{P}(K)$ is contained in $\mathbf{P}(v \wedge \bigwedge^2 V_6)$. We define the morphism p_{Y} by sending the point $([U_0], [K])$ to $[v] \in \mathsf{Y}^{\geq 2}_A$.

To study the morphism p_{Y}, we show that for each point $[v] \in \mathsf{Y}^{\geq 2}_A$ and each 2-dimensional subspace $K \subset A \cap (v \wedge \bigwedge^2 V_6)$, there exists a unique $[U_0] \in \mathsf{Gr}(3, V_6)$ that contains v and satisfies

$$K \subset T_{U_0} \cap (v \wedge \textstyle\bigwedge^2 V_6) = v \wedge U_0 \wedge V_6.$$

This shows that p_{Y} is an isomorphism over $\mathsf{Y}^{\geq 2}_A$ and that the fiber over each point $[v] \in \mathsf{Y}^3_A$ is isomorphic to $\mathbf{P}((A \cap (v \wedge \bigwedge^2 V_6))^\vee)$, proving the lemma.

We start by proving that, if such a space U_0 exists, it is unique. Let $U_0, U_0' \subset V_6$ be two distinct 3-dimensional subspaces containing v. We show that every two-dimensional subspace of

$$L \coloneqq (v \wedge U_0 \wedge V_6) \cap (v \wedge U_0' \wedge V_6)$$

contains a decomposable vector.

Assume first that $U_0 \cap U_0'$ has dimension 1 (spanned by v). In this case, the vector space L is contained in $\bigwedge^3(U_0 + U_0')$, where $U_0 + U_0'$ has dimension 5. By [**DK18**, Remark B.4], any two-dimensional subspace of $\bigwedge^3(U_0 + U_0')$ contains a decomposable vector. Assume now that $U_2 \coloneqq U_0 \cap U_0'$ has dimension 2. In this case,

$$L = (\textstyle\bigwedge^2 U_2 \wedge V_6) \oplus \big(v \wedge U_0/U_2 \wedge U_0'/U_2\big)$$

has dimension 5, therefore any 2-dimensional subspace of L meets the hyperplane $\bigwedge^2 U_2 \wedge V_6$ of decomposable vectors.

To conclude the proof of the lemma, we show that for every 2-dimensional subspace $K \subset A \cap (v \wedge \bigwedge^2 V_6)$, there exists a point $[U_0] \in \mathsf{Gr}(3, V_6)$ satisfying $K \subset T_{U_0}$.

By [**KKM24**, Equation 3.1], for any vector of the form $v \wedge \alpha$ in A, the locus of points $[U_0] \in \mathsf{Gr}(3, V_6)$ such that $v \wedge \alpha$ belongs to T_{U_0} is a hyperplane section of the Schubert variety $\Sigma(v, W_5(v, \alpha))$ of subspaces containing v and contained in the hyperplane $W_5(v, \alpha) \subset V_6$ corresponding to the point $v \wedge \alpha \wedge \alpha \in \bigwedge^5 V_6 \simeq V_6^\vee$.

Let $(v \wedge \alpha, v \wedge \beta)$ be a basis of K. The vector space K is contained in T_{U_0} if the point $[U_0]$ belongs to the intersection of two hyperplane sections of the Schubert variety $\Sigma(v, W_5(v, \alpha) \cap W_5(v, \beta))$. By [**DK18**, Remark B.4], $W_5(v, \alpha)$ and $W_5(v, \beta)$ are distinct hyperplanes of V_6, hence that Schubert variety is isomorphic to $\mathbf{P}^3$. In particular, any two hyperplane sections intersect nontrivially and there exists $[U_0]$ such that K is contained in T_{U_0}. □

We apply Lemma 5.14 to give a bound on the dimension of the locus where conditions (b2′) or (c2′) of Proposition 5.12 hold. Consider the loci

$$B_2 := \{[U_0] \in \mathsf{Z}^2_{A,V_5} \mid r_{K,5} \text{ is not surjective}\},$$
$$B_3 := \{[U_0] \in \mathsf{Z}^3_{A,V_5} \mid r_{K,5} \text{ has corank} > 1\}, \tag{96}$$

where $K := A \cap T_{U_0}$ and the map $r_{K,5}$ was defined in (92).

Corollary 5.15. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian with no decomposable vectors and let $[V_5]$ be a point in $\mathsf{Y}^0_{A^\perp}$.*

(i) *the loci B_2 and B_3 both have dimensions at most 2;*
(ii) *if $\mathsf{Y}^{\geq 3}_{A,V_5}$ is empty, the loci B_2 and B_3 both have dimensions at most 1.*

Proof. Since A contains no decomposable vectors and $[V_5]$ belongs to $\mathsf{Y}^0_{A^\perp}$, we can apply Proposition 5.12 to study the corank of $r_{K,5}$. Since Z^4_{A,V_5} is finite (Theorem 2.5(d)), that proposition says that it is enough to show that the locus

$$B = \{[U_0] \in \mathsf{Z}^{\geq 2}_{A,V_5} \smallsetminus \mathsf{Z}^4_{A,V_5} \mid \exists v \in U_0 \quad \dim(A \cap (v \wedge U_0 \wedge V_6)) \geq 2\}$$

has dimension at most 2, and dimension at most 1 when $\mathsf{Y}^{\geq 3}_{A,V_5}$ is empty.

The condition defining B implies that for every point $[U_0] \in B$, there exists a 2-dimensional space K such that

$$K \subset A \cap (v \wedge U_0 \wedge V_6) = (A \cap T_{U_0}) \cap (v \wedge \textstyle\bigwedge^2 V_6).$$

In other words, the pair $([U_0], [K])$ is in the variety $\mathcal{W}$ defined in (95). Therefore, B is contained in $p_{\mathsf{Z}}(\mathcal{W}) \subset \mathsf{Z}^{\geq 2}_A$. By Lemma 5.14, $\mathcal{W}$ has dimension 2, hence B has dimension at most 2.

Assume now $\mathsf{Y}^3_{A,V_5} = \varnothing$. By definition, the set B is contained in

$$p_{\mathsf{Z}}(\mathcal{W}) \cap \mathsf{Z}^{\geq 2}_{A,V_5} = p_{\mathsf{Z}}\big(\mathcal{W} \cap (\mathsf{Gr}(3, V_5) \times \mathsf{Gr}(2, A))\big).$$

The image by p_{Y} a point $([U_0], [K])$ of $\mathcal{W} \cap (\mathsf{Gr}(3, V_5) \times \mathsf{Gr}(2, A))$ is a nonzero vector $v \in U_0$. Since $U_0 \subset V_5$, we see that the image by p_{Y} of $\mathcal{W} \cap (\mathsf{Gr}(3, V_5) \times \mathsf{Gr}(2, A))$ is contained in $\mathsf{Y}^{\geq 2}_{A,V_5}$. Since the latter is equal to Y^2_{A,V_5}, we obtain by Lemma 5.14 a morphism

$$p = p_{\mathsf{Z}} \circ p_{\mathsf{Y}}^{-1} \colon \mathsf{Y}^2_{A,V_5} \longrightarrow \mathsf{Z}^{\geq 2}_A$$

whose image contains B. By [**DK19**, Lemma 2.6], Y^2_{A,V_5} is a curve, hence B has dimension at most 1 in this case. □

5.3.2.4. *Local description of Z^k_{A,V_5}.* Let $V_5 \subset V_6$ be a hyperplane. We set

$$\mathscr{Z}^{\geq k}_5 := \{([A], [U]) \in \mathrm{LG}(\textstyle\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_5) \mid \dim(T_U \cap A) \geq k\}.$$

This is the intersection of the variety $\mathscr{Z}^{\geq k} \subset \mathrm{LG}(\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_6)$ studied in (33) with $\mathrm{LG}(\bigwedge^3 V_6) \times \mathsf{Gr}(3, V_5)$. The fiber of $[A] \in \mathrm{LG}(\bigwedge^3 V_6)$ under the first projection $\mathscr{Z}^{\geq k} \to \mathrm{LG}(\bigwedge^3 V_6)$ is the scheme Z^k_{A,V_5}. For $k \in \{1, 2, 3\}$, it is nonempty of dimension $\geq 6 - k(k+1)/2$ by Remark 5.8.

The same proof as that of Proposition 2.10 shows the following.

Proposition 5.16. *Let $A_0 \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $([A_0], [U_0])$ be a point of $\mathscr{Z}^\ell_5$, with $\ell \in \{0, \dots, 3\}$. Set $K := A_0 \cap T_{U_0}$ and let $\mathcal{S}_{\leq i} \subset \mathrm{Sym}^2 K^\vee$ be the subscheme of quadratic forms of rank $\leq i$.*

For any $k \in \{0,\dots,\ell\}$, the analytic germ of the scheme $\mathscr{Z}_5^{\geq k}$ at the point $([A_0],[U_0])$ is isomorphic to the product of a smooth germ $(M_5,0)$ and the germ $(\mathcal{S}_{\leq \ell-k},0)$.

When the map $r_{K,5}$ is surjective, under this isomorphism, the germ of $\mathsf{Z}^{\geq k}_{A_0,V_5} \subset \mathscr{Z}_5^{\geq k}$ at $[U_0]$ corresponds to the germ of $M_5' \times \mathcal{S}_{\leq \ell-k}$ at $(0,0)$, where $(M_5',0) \subset (M_5,0)$ is smooth of dimension $6 - \frac{\ell(\ell+1)}{2}$.

The following lemma allow us to treat the case in which the map $r_{K,5}$ is not surjective.

Lemma 5.17. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and let $([A],[U_0])$ be a point of $\mathscr{Z}_5^k$, with $k \in \{0,\dots,3\}$. Set $K \coloneqq A \cap T_{U_0}$. The tangent space to Z^k_{A,V_5} at $[U_0]$ has dimension lower than or equal to the dimension of the kernel of the map $r_{K,5}$.*

Proof. Since Z^k_{A,V_5} is the intersection of Z^k_A and $\mathsf{Gr}(3,V_5)$, we have an inclusion

$$T_{\mathsf{Z}^k_{A,V_5},[U_0]} \subset T_{\mathsf{Z}^k_A,[U_0]} \cap T_{\mathsf{Gr}(3,V_5),[U_0]},$$

for any $[U_0] \in \mathsf{Z}^k_{A,V_5}$. We showed in Section 5.3.2.1 that the tangent space $T_{\mathsf{Gr}(3,V_5),[U_0]}$ can be identified with the subspace $H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0,V_5}}(2)) \subset H^0(\mathbf{P}(T_{U_0}), I_{\mathcal{C}_{U_0}}(2))$, and Proposition 2.10 shows that the tangent space $T_{\mathsf{Z}^k_A,[U_0]}$ can be identified with the kernel of the morphism r_K for $K \coloneqq A \cap T_{U_0}$. The lemma follows from description of $r_{K,5}$ of Equation (92). □

We now have all the tools for the proof of Theorem 5.9.

Proof of Theorem 5.9. Proposition 5.16 and the study of $r_{K,5}$ carried out in Proposition 5.12 imply that the scheme Z^1_{A,V_5} is smooth of dimension 5, the scheme $\mathsf{Z}^2_{A,V_5} \smallsetminus B_2$ is smooth of dimension 3, where $B_2 \subset \mathsf{Z}^2_{A,V_5}$ is the locus defined in (96), where $r_{K,5}$ is not surjective. Lemma 5.17 and Proposition 5.16((c1′)) imply that the scheme $\mathsf{Z}^3_{A,V_5} \smallsetminus B_3$ has dimension at most 1, where $B_3 \subset \mathsf{Z}^3_{A,V_5}$ is the locus defined in (96), where the kernel of $r_{K,5}$ has dimension strictly greater than 1. By Corollary 5.15, the loci B_2 and B_3 have dimension at most 2, hence $\mathsf{Z}^{\geq 1}_{A,V_5}$ has (expected) dimension 5, $\mathsf{Z}^{\geq 2}_{A,V_5}$ has (expected) dimension 3, and $\mathsf{Z}^{\geq 3}_{A,V_5}$ has dimension at most 2, and at most 1 if Y^3_{A,V_5} is empty. Moreover, Proposition 5.16 shows that the singular locus of $\mathsf{Z}^{\geq 1}_{A,V_5}$ is equal to $\mathsf{Z}^{\geq 2}_{A,V_5}$.

Consider now the 3-dimensional scheme $\mathsf{Z}^{\geq 2}_{A,V_5}$. It is a local complete intersection inside $\mathsf{Z}^{\geq 2}_A$, since it is locally defined by three equations and has codimension 3. Since $\mathsf{Z}^{\geq 2}_A$ is Cohen–Macaulay by Theorem 2.5(b), so is $\mathsf{Z}^{\geq 2}_{A,V_5}$. Moreover, Proposition 5.16 shows that the singular locus of $\mathsf{Z}^{\geq 2}_{A,V_5}$ is contained in $\mathsf{Z}^{\geq 3}_{A,V_5} \cup B_2$. When Y^3_{A,V_5} is empty, the latter has dimension at most 1 by Corollary 5.15, hence we obtain normality in this case.

Finally, since $\mathsf{Z}^{\geq 1}_{A,V_5}$ and $\mathsf{Z}^{\geq 2}_{A,V_5}$ have expected dimensions, their degrees can be computed as in Proposition 5.7.

The irreducibility of $\mathsf{Z}^{\geq 1}_{A,V_5}$ follows as in Theorem 2.5. To finish the proof, we need to show that $\mathsf{Z}^{\geq 2}_{A,V_5}$ is irreducible. We prove it by a degeneration argument, as in Corollary 4.20, using the fact, proved in Proposition 5.7, that $\mathsf{Z}^{\geq 2}_{A,V_5}$ is irreducible for V_5 general.

Let $(T, 0)$ be a smooth pointed connected curve parametrizing hyperplanes $V_t \subset V_6$, such that V_t is general in the sense of Proposition 5.7 for $t \neq 0$, and $V_0 = V_5$. Consider

$$\mathcal{G} \coloneqq \{(t, [U_3]) \in T \times \mathsf{Gr}(3, V_6) \mid [U_3] \in \mathsf{Gr}(3, V_t)\}.$$

The projection $\mathcal{G} \to T$ is smooth with 6-dimensional fibers, hence $\mathcal{G}$ is smooth of dimension 7. Consider the subvariety

$$\mathscr{Z}^{\geq 2}_{A,T} \coloneqq \{(t, [U_3]) \in \mathcal{G} \mid [U_3] \in \mathsf{Z}^{\geq 2}_{A,V_t}\}$$

of $\mathcal{G}$. It is a second determinantal locus, therefore it has codimension at most 3 everywhere in $\mathcal{G}$. Hence each irreducible component of $\mathscr{Z}^{\geq 2}_{A,T}$ has dimension at least 4. Since the scheme $\mathsf{Z}^{\geq 2}_{A,V_5}$ has dimension 3, it follows that every irreducible component of $\mathsf{Z}^{\geq 2}_{A,V_5}$ deforms in a neighborhood of 0. Since $\mathsf{Z}^{\geq 2}_{A,V_t}$ is irreducible for $t \neq 0$ by Proposition 5.7, we conclude that $\mathsf{Z}^{\geq 2}_{A,V_5}$ is irreducible as well. □

CHAPTER 6

Linear spaces and Hodge correspondence

Let X be a Gushel–Mukai manifold with Gushel map $\gamma_X\colon X \to \mathsf{Gr}(2, V_5)$, as defined in (81).

In [DK19, Section 4], the authors describe the Hilbert schemes $F_k(X)$ of linearly embedded $\mathbf{P}^k$ contained in X. For any such space P, the image $\gamma_X(P)$ is a linear space contained in $\mathsf{Gr}(2, V_5)$. This, together with the description of linear spaces contained in $\mathsf{Gr}(2, V_5)$, leads to the following definitions.

Definition 6.1. Let X be a Gushel–Mukai manifold and let $P \subset X$ be a linear space.

- P is a *σ-space* if $\gamma_X(P)$ is contained in $\mathbf{P}(V_1 \wedge V_5)$ for some $[V_1] \in \mathbf{P}(V_5)$;
- P is a *τ-space* if $\gamma_X(P)$ is contained in $\mathbf{P}(\bigwedge^2 U_3)$ for some $[U_3] \in \mathsf{Gr}(3, V_5)$;

We now restrict to the case of (2-dimensional) planes. By [DK19, Section 4.1], such a plane P cannot be both a σ-space and a τ-space, and the Hilbert scheme $F_2(X)$ of planes contained in X decomposes as

$$F_2(X) = F_2^\sigma(X) \sqcup F_2^\tau(X),$$

where $F_2^\sigma(X)$ is the subscheme of σ-planes and $F_2^\tau(X)$ is the subscheme of τ-planes. Moreover, there are well-defined morphisms

$$\sigma_2\colon F_2^\sigma(X) \longrightarrow \mathbf{P}(V_5), \qquad \tau_2\colon F_2^\tau(X) \longrightarrow \mathsf{Gr}(3, V_5), \tag{97}$$

such that $\sigma_2([P])$ is the unique point $[V_1] \in \mathbf{P}(V_5)$ such that $\gamma_X(P)$ is contained in $\mathbf{P}(V_1 \wedge V_5)$, and $\tau_2([P])$ is the unique point $[U_3] \in \mathsf{Gr}(3, V_5)$ such that $\gamma_X(P)$ is contained in $\mathbf{P}(\bigwedge^2 U_3)$.

The geometry of σ-planes is studied in [DK19, Section 4.3], where the authors relate the scheme $F_2^\sigma(X)$ to the EPW sextic stratification $\mathsf{Y}_A^{\geq k}$ associated with the Gushel–Mukai manifold, to the double covers constructed in [DK20a, Theorem 5.2]. In particular, they show that, when X is a GM sixfold, the scheme $F_2^\sigma(X)$ has dimension 4 and, for $\mathsf{Y}^3_{A,V_5} = \varnothing$, the morphism σ_2 factors as

$$F_2^\sigma(X) \xrightarrow{\widetilde{\sigma}} \widetilde{\mathsf{Y}}_A \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5) \xrightarrow{f_{A,5}} \mathsf{Y}_A^{\geq 1} \cap \mathbf{P}(V_5) \subset \mathbf{P}(V_5),$$

where $\widetilde{\sigma}$ is a $\mathbf{P}^1$-fibration and $f_{A,5}$ is the pullback to $\mathsf{Y}_A^{\geq 1} \cap \mathbf{P}(V_5)$ of the double EPW sextic $f_A\colon \widetilde{\mathsf{Y}}_A^{\geq 1} \to \mathsf{Y}_A$ defined in Section 2.2. Moreover, they give conditions for the smoothness of $F_2^\sigma(X)$ (see [DK19, Theorem 4.3(a)]).

In the smooth case, they then prove in [DK19, Section 5.2] that the morphism $\widetilde{\sigma}$ induces a Hodge isometry between the vanishing middle integral cohomology of X and the primitive integral cohomology of the double EPW sextic.

In this chapter, we study the component $F_2^\tau(X)$ that parametrizes τ-planes, starting from the description given in [DK19, Theorem 4.5].

In Section 6.1, we study the geometry of $F_2^\tau(X)$ for Gushel–Mukai manifolds X of dimension 5 or 6. When X is a general GM of dimension 5, the scheme $F_2^\tau(X)$ is finite, while it is smooth irreducible of dimension 3 for a general GM of dimension 6.

We then focus on Gushel–Mukai sixfolds. [**DK19**, Proposition 4.10] shows that $F_2^\tau(X)$ parametrizes planes contained in the fibers of the quadric fibration $\varrho\colon \mathcal{Q} \to \mathsf{Gr}(3, V_5)$, studied in Section 5.2. Moreover, by [**DK19**, Theorem 4.5] the morphism τ_2 induces a double cover

$$\tilde{\tau}\colon F_2^\tau(X) \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5}.$$

Using Theorem 5.9 and Corollary 5.4, we show that for *any* GM sixfold X, the scheme $F_2^\tau(X)$ is a connected locally complete intersection of dimension 3. Moreover, under mild hypotheses on the Lagrangian data associated to X, we prove that the scheme $F_2^\tau(X)$ is normal and the double cover $\tilde{\tau}$ is the restriction

$$\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5}$$

of the EPW cube $\widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$.

In Section 6.2, we study the incidence correspondence induced by the universal family of τ-planes on a GM sixfold X. When $F_2^\tau(X)$ is smooth, we show that this correspondence induces a Hodge isomorphism of rational cohomology groups

$$H^6(X, \mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(F_2^\tau(X), \mathbf{Q})^-,$$

between the vanishing cohomology of X (defined in Equation (82)) and the anti-invariant cohomology of $F_2^\tau(X)$ with respect to the involution associated with the double cover $\tilde{\tau}$. The main idea is to factor the above map through the variable cohomology $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$ of the quadric fibration $\mathcal{Q}$. Proposition 5.5 shows that $H^6(X, \mathbf{Q})_{\mathrm{van}}$ is isomorphic to $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$. By studying the Leray spectral sequence associated to ϱ and the cohomology of the higher direct image sheaves $R^k\varrho_*\mathbf{Q}$, we then show that $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$ is isomorphic to $H^2(F_2^\tau(X), \mathbf{Q})^-$.

Finally, in Section 6.3, we relate the rational vanishing cohomology of X to the rational primitive cohomology of the associated EPW cube $\widetilde{\mathsf{Z}}_A$. By the above discussion, for X very general, the scheme $F_2^\tau(X)$ embeds in the EPW cube $\widetilde{\mathsf{Z}}_A$. The pullback map gives an inclusion of the primitive cohomology of $\widetilde{\mathsf{Z}}_A$ in the anti-invariant cohomology of $F_2^\tau(X)$. In Theorem 6.20 we show that this is actually an isomorphism of rational cohomology groups, and induces a rational Hodge correspondence

$$H^6(X, \mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(\widetilde{\mathsf{Z}}_A, \mathbf{Q})_{\mathrm{prim}},$$

which is a rational multiple of an isometry.

Parts of the contents of this chapter will appear in [**Riz26**].

6.1. τ-planes in Gushel–Mukai manifolds

Let X be Gushel–Mukai manifold with associated Lagrangian data set (V_6, V_5, A). The following theorem relates the Hilbert scheme $F_2^\tau(X)$ of τ-planes contained in X to the stratification $\mathsf{Z}^{\geq k}_{A,V_5} \subset \mathsf{Gr}(3, V_5)$ (see (90)).

Theorem 6.2 ([DK19, Theorem 4.5])**.** *Let X be a GM manifold of dimension $n \in \{5,6\}$ with associated Lagrangian data set (V_6, V_5, A), and let $\tau_2 \colon F_2^\tau(X) \to \mathsf{Gr}(3,V_5)$ be the morphism introduced in Equation (97).*

(a) *When $n=6$, the map τ_2 factors as*

$$F_2^\tau(X) \xrightarrow{\tilde{\tau}} \mathsf{Z}^{\geq 2}_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5),$$

where $\tilde{\tau}$ is a double cover branched along $\mathsf{Z}^{\geq 3}_{A,V_5}$.

(b) *When $n=5$ and X is ordinary, the map τ_2 induces an isomorphism*

$$F_2^\tau(X) \xrightarrow{\sim} \mathsf{Z}^{\geq 3}_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5).$$

Remark 6.3. When $n=6$, the involution of $F_2^\tau(X)$ defined by the double cover $\tilde{\tau}$ is induced by the involution σ of X defined by the double cover $\gamma_X \colon X \to \mathsf{Gr}(2,V_5)$ and its fixed locus $\mathsf{Z}^{\geq 3}_{A,V_5}$ is the scheme of τ-planes contained in the branch locus $X' := Q \cap \mathsf{Gr}(2,V_5)$ of γ_X (a smooth ordinary GM manifold of dimension 5).

The image by γ_X of a τ-plane $\Pi \subset X$ is a τ-plane $\mathsf{Gr}(2,U_3) \subset \mathsf{Gr}(2,V_5)$ which is such that $\gamma_X^{-1}(\gamma_X(\Pi)) = \Pi \cup \sigma(\Pi)$. This means that the τ-plane $\mathsf{Gr}(2,U_3)$ is everywhere tangent to the quadric Q (in other words, $\mathsf{Gr}(2,U_3) \cap Q$ is twice a line) or contained in Q (in which case it is contained in X' and corresponds to a branch point of $\widetilde{\tau}$).

We will compare in Section 6.1.1 the double cover of (a) with the double cover (90).

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, a general hyperplane $V_5 \subset V_6$ defines an ordinary GM manifold of dimension 5. The following statement therefore follows from Item (b) above and Proposition 5.7(b).

Corollary 6.4. *Let X be a general GM manifold of dimension 5. The scheme $F_2^\tau(X)$ is smooth finite of cardinality 40.*

The analogous statement for general GM manifolds of dimension 6 does not follow directly from Proposition 5.7(a) because we do not know whether the double cover $\tilde{\tau}$ in Theorem 6.2(a) is the same as the double cover (90) (we will prove that in Section 6.1.1). So we give a direct (elementary) proof.

Theorem 6.5 ([Riz26, Theorem 3.4])**.** *Let X be a general GM manifold of dimension 6. The scheme $F_2^\tau(X)$ is smooth irreducible of dimension 3.*

Proof. A GM manifold of dimension 6 is a smooth transverse intersection

$$X_Q = \mathsf{CGr}(2,V_5) \cap Q \subset \mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 V_5), \tag{98}$$

where Q is a quadric hypersurface in $\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ that does not contain the vertex $\nu = [\mathbf{C}]$ of the cone $\mathsf{CGr}(2,V_5)$.The Gushel map

$$\gamma_{X_Q} \colon X_Q \longrightarrow \mathsf{Gr}(2,V_5)$$

is a double cover, induced by the projection $\pi \colon \mathbf{C} \oplus \bigwedge^2 V_5 \to \bigwedge^2 V_5$.

By definition, a plane $\mathbf{P}(W_3) \subset \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ that does not contain ν is a τ-plane if and only if there exists a 3-dimensional subspace $U_3 \subset V_5$ such that W_3 is a hyperplane in $\mathbf{C} \oplus \bigwedge^2 U_3$ that does not contain $\mathbf{C}$. The space of all W_3 satisfying this condition is therefore

$$W = \{[W_3] \in \mathsf{Gr}(3, \mathbf{C} \oplus \textstyle\bigwedge^2 V_5) \mid \ \exists U_3 \subset V_5 \quad \pi(W_3) = \bigwedge^2 U_3 \subset \bigwedge^2 V_5\}.$$

Consider the morphism $\varpi\colon W \to \mathsf{Gr}(3, V_5)$ that sends $[W_3]$ to $[U_3]$. Its fiber over $[U_3]$ is given by

$$W_{[U_3]} = \mathbf{P}((\mathbf{C} \oplus \textstyle\bigwedge^2 U_3)^\vee) \smallsetminus \{[W_3] \mid \mathbf{C} \subset W_3\} \simeq \mathbf{A}^3.$$

Therefore, W is a 9-dimensional irreducible smooth variety.

Consider now the incidence variety

$$I = \{([W_3], [Q]) \in W \times \mathbf{P} \mid \mathbf{P}(W_3) \subset Q\},$$

where $\mathbf{P}$ is the linear system $|\mathcal{O}_{\mathbf{P}(\mathbf{C}\oplus\bigwedge^2 V_5)}(2)|$, with its projections

$$\begin{array}{ccc} & I & \\ {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ W & & \mathbf{P}. \end{array}$$

The morphism pr_1 is a projective bundle over W of relative dimension $\dim(\mathbf{P})-6$: indeed, the fiber $\mathrm{pr}_1^{-1}([W_3])$ is isomorphic to $\{Q \in \mathbf{P} \mid \mathbf{P}(W_3) \subset Q\}$. Hence I is smooth irreducible of dimension

$$9 + \dim(\mathbf{P}) - 6 = 3 + \dim(\mathbf{P}).$$

If $Q \in \mathbf{P}$ is a general quadric that defines a GM sixfold X_Q, the fiber $\mathrm{pr}_2^{-1}(Q)$ is equal to $F_2^\tau(X_Q)$. Observe that $F_2^\tau(X_Q)$ is not empty. Indeed, Theorem 6.2 shows that $F_2^\tau(X_Q)$ is a double cover of $\mathsf{Z}^{\geq 2}_{A,V_5}$, which is nonempty by Remark 5.8.

The morphism pr_2 is therefore a dominant morphism from a smooth variety, hence it is generically smooth. This implies that for a general quadric Q, the scheme $F_2^\tau(X_Q)$ is smooth of dimension $\dim(I) - \dim(\mathbf{P}) = 3$.

Finally, from Theorem 6.2, we see that $F_2^\tau(X_Q)$ is a double cover of $\mathsf{Z}^{\geq 2}_{A,V_5}$ branched over $\mathsf{Z}^{\geq 3}_{A,V_5}$. Since, for X general, $\mathsf{Z}^{\geq 2}_{A,V_5}$ is irreducible and $\mathsf{Z}^{\geq 3}_{A,V_5}$ is nonempty (Proposition 5.7), it follows that $F_2^\tau(X)$ is connected. In particular, since $F_2^\tau(X)$ is smooth, it is irreducible. □

Corollary 6.6. *For any GM manifold X of dimension 6, the Hilbert scheme $F_2^\tau(X)$ of τ-planes contained in X is connected.*

Proof. With the notation of the proof of the theorem, the general fibers of pr_2 are connected. Since $\mathbf{P}$ is normal, the Stein factorization of pr_2 is trivial and all its fibers are connected. □

6.1.1. The schemes $F_2^\tau(X)$ and $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ for a GM sixfold X. In [DK19, Remark 4.6], the authors claim that, for a GM manifold X of dimension 6, the double cover $\tilde\tau\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5}$ of Theorem 6.2(a) should be induced by the pullback (still denoted by g)

$$g\colon \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5} \tag{99}$$

to $\mathsf{Z}^{\geq 2}_{A,V_5}$ of the EPW cube $g\colon \widetilde{\mathsf{Z}}^{\geq 2}_{A} \to \mathsf{Z}^{\geq 2}_{A}$. The aim of this section is to prove the following theorem, which implies this claim under a mild generality assumption.

Theorem 6.7. *Let X be a GM manifold of dimension 6 with associated Lagrangian data set (V_6, V_5, A). The scheme $F_2^\tau(X)$ has (expected) dimension 3 and is a local*

complete intersection. When the loci $\mathsf{Y}^{\geq 3}_{A,V_5}$ *and* $\mathsf{Z}^{\geq 4}_{A,V_5}$ *are empty, the schemes* $F_2^\tau(X)$ *and* $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ *are normal and irreducible, and the double covers*

$$\tilde{\tau}\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5} \qquad \text{and} \qquad g\colon \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \to \mathsf{Z}^{\geq 2}_{A,V_5}$$

are isomorphic.

We start by showing the following result, whose proof follows the strategy of [**DK20a**, Section 5.4].

Theorem 6.8 ([**Riz26**, Theorem 3.5])**.** *Let* X *be a GM manifold of dimension* 6 *with associated Lagrangian data set* (V_6, V_5, A)*. The schemes* $F_2^\tau(X)$ *and* $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ *have the same normalization.*

Proof. We know from [**DK19**, Proposition 4.1] that the scheme $F_2^\tau(X)$ is the Hilbert scheme $F_2(\mathcal{Q}/\mathsf{Gr}(3,V_5))$ parametrizing projective planes in the fibers of the morphism ϱ defined in Equation (85). Consider the double covers

$$\tilde{\tau}\colon F_2^\tau(X) \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5},$$

introduced in Theorem 6.2, and

$$g\colon \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5},$$

introduced in Equation (99).

We use results from [**DK20a**, Section 3.3], applied to the family of quadrics $\mathcal{Q} \subset \mathbf{P}_{\mathsf{Gr}(3,V_5)}(\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3)$. In the notation of [**DK20a**, Section 3.3], the rank m of $\mathbf{C} \oplus \bigwedge^2 \mathcal{U}_3$ is equal to 4, and $k = 2$ and $d = 3$.

Corollary 5.4 shows that the degeneracy loci of ϱ are the schemes $\mathsf{Z}^{\geq k}_{A,V_5} \subset \mathsf{Gr}(3,V_5)$. Therefore, the morphism $\tilde{\tau}$ is the one constructed in [**DK20a**, Proposition 3.10]. Moreover, the proof of [**DK20a**, Proposition 3.10] gives an isomorphism

$$F_2^\tau(X) \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} \mathsf{Z}^2_{A,V_5} \simeq \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} \mathsf{Z}^2_{A,V_5}. \tag{100}$$

In other words, the double covers $\tilde{\tau}$ and g coincide over Z^2_{A,V_5}. This implies that $F_2^\tau(X)$ and $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ have the same normalizations (both are the normalization of $\mathsf{Z}^{\geq 2}_{A,V_5}$ in the field $\mathbf{C}(F_2^\tau(X)) = \mathbf{C}(\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5})$). □

Using the local study of Section 5.3.2, we can now prove Theorem 6.7.

Proof of Theorem 6.7. Since $\tilde{\tau}$ is finite, Theorem 5.9 shows that $F_2^\tau(X)$ has dimension 3.

We show that it is a local complete intersection. It was proved in [**DK24**, Lemma 3.4] that the scheme $\mathring{F}_2^\tau(\mathsf{CGr}(2,V_5))$ that parametrizes τ-planes in $\mathsf{CGr}(2,V_5)$ not containing the vertex is smooth irreducible of dimension 9. The locus of τ-planes contained in the quadric Q defining X (see (98)) is then the zero-locus of the section

$$[U_3] \longmapsto Q|_{\mathsf{Gr}(2,U_3)} = Q|_{\mathbf{P}(U_3^\vee))}$$

of the rank-6 bundle $\mathrm{Sym}^2(\mathcal{U}_3)$, where $\mathcal{U}_3$ is the rank-3 tautological subbundle on the scheme $\mathring{F}_2^\tau(\mathsf{CGr}(2,V_5)) \subset \mathsf{Gr}(3, \mathbf{C} \oplus \bigwedge^2 V_5)$. This implies that when $F_2^\tau(X)$ has the expected dimension $9 - 6 = 3$, it is a local complete intersection.

Suppose now $\mathsf{Y}^3_{A,V_5} = \varnothing$. In this case, $\mathsf{Z}^{\geq 2}_{A,V_5}$ is normal and the branch locus $\mathsf{Z}^{\geq 3}_{A,V_5}$ of $\tilde{\tau}\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5}$ has codimension at least 2, by Theorem 5.9 (and its proof).

Therefore, $F_2^\tau(X)$ is also nonsingular in codimension 1 and, being a local complete intersection, it is normal. Since it is connected by Corollary 6.6, it is also irreducible.

Finally we show that $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ is normal as well. Since $\mathsf{Z}^{\geq 4}_{A,V_5} = \varnothing$, the scheme $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ is contained in $\widetilde{\mathsf{Z}}^{\geq 2}_{A} \smallsetminus g^{-1}(\mathsf{Z}^{\geq 4}_{A})$, which is smooth by Theorem 2.14. Therefore, as $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ has the expected codimension, it is a locally complete intersection. The normality of $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ follows using that its singular locus is contained in $g^{-1}(\mathsf{Z}^{\geq 3}_{A,V_5})$, which has codimension at least 2. Theorem 6.8 then implies that the double covers $\tilde\tau\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5}$ and $g\colon \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \to \mathsf{Z}^{\geq 2}_{A,V_5}$ are isomorphic. □

When X is general, the schemes $F_2^\tau(X)$ and $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ are smooth by Theorem 6.5 and Proposition 5.7(b). Using in addition Proposition 5.7(c), we obtain the following corollary.

Corollary 6.9. *Let X be a general GM manifold of dimension 6. The scheme $F_2^\tau(X)$ is a smooth irreducible threefold and the double cover*

$$\tilde\tau\colon F_2^\tau(X) \longrightarrow \mathsf{Z}^{\geq 2}_{A,V_5}$$

is the pullback of the EPW cube. It is branched over the 40 points of $\mathsf{Z}^{\geq 3}_{A,V_5}$.

6.2. Incidence correspondence for τ-planes on Gushel–Mukai sixfolds

Let X be a GM sixfold with its canonical double covering $\gamma_X\colon X \to \mathsf{Gr}(2,V_5)$ and let $g\colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$ be the associated EPW cube. By Theorem 6.7, the Hilbert scheme $F_2^\tau(X)$ of τ-planes contained in X has dimension 3, and there is a commutative diagram

(101)
$$\begin{array}{ccccc} & & I = \{(x,\Pi) \in X \times F_2^\tau(X) \mid x \in \Pi\} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & F_2^\tau(X) \\ {\scriptstyle\gamma_X}\downarrow & & & & \downarrow{\scriptstyle\tilde\tau} \\ \mathsf{Gr}(2,V_5) & & & & \mathsf{Z}^{\geq 2}_{A,V_5}, \end{array}$$

where q is a $\mathbf{P}^2$-bundle and $\tilde\tau$ is the double cover of Theorem 6.2(a). In this section, we suppose that the scheme $F_2^\tau(X)$ is smooth (this is the case when X is general). In this case, I is smooth irreducible of dimension 5 and Diagram (101) induces a morphism

$$q_*p^*\colon H^6(X,\mathbf{Z}) \longrightarrow H^2(F_2^\tau(X),\mathbf{Z})$$

of Hodge structures.

Let σ be the involution of X associated with the double cover γ_X and let ι_τ be the involution of $F_2^\tau(X)$ associated with the double cover $\tilde\tau\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5}$. By Remark 6.3, the involution induced by σ on $F_2^\tau(X)$ is ι_τ, that is,

(102)
$$\forall\ [\Pi] \in F_2^\tau(X) \qquad [\sigma(\Pi)] = \iota_\tau([\Pi]).$$

We will use both notations depending on the context. These two involutions define a $\mathbf{Z}/2\mathbf{Z}$-action on I, sending a pair $(x,[\Pi])$ to $(\sigma(x),\iota_\tau([\Pi]))$, such that p and q are equivariant.

We restrict to the vanishing cohomology of X, defined in Equation (82).

Proposition 6.10. *Let X be a GM manifold of dimension 6 such that the scheme $F_2^\tau(X)$ is smooth. Let $(q_*p^*)_{\mathrm{van}}$ be the restriction of q_*p^* to $H^6(X,\mathbf{Z})_{\mathrm{van}}$. The image of $(q_*p^*)_{\mathrm{van}}$ is contained in the ι^*-antiinvariant cohomology*

$$H^2(F_2^\tau(X),\mathbf{Z})^- \coloneqq \{x \in H^2(F_2^\tau(X),\mathbf{Z}) \mid \iota_\tau^*(x) = -x\}.$$

Proof. The morphism q_*p^* is compatible with the $\mathbf{Z}/2\mathbf{Z}$-actions given by σ^* and ι_τ^*. In particular, the antiinvariant cohomology group $H^6(X,\mathbf{Z})^-$ maps to $H^2(F_2^\tau(X),\mathbf{Z})^-$. Finally, there is an equality

$$H^6(X,\mathbf{Z})^- = H^6(X,\mathbf{Z})_{\mathrm{van}}, \tag{103}$$

since both are the orthogonal complement to $H^6(X,\mathbf{Z})^+ = \gamma_X^* H^6(\mathsf{Gr}(2,V_5),\mathbf{Z})$ in $H^6(X,\mathbf{Z})$. Indeed, this is by definition for $H^6(X,\mathbf{Z})_{\mathrm{van}}$, and $H^6(X,\mathbf{Z})^-$ is contained in this orthogonal complement, has the same rank, and both lattices are saturated, hence they are equal. □

Therefore, the map q_*p^* induces a morphism

$$(q_*p^*)_{\mathrm{van}} \colon H^6(X,\mathbf{Z})_{\mathrm{van}} \longrightarrow H^2(F_2^\tau(X),\mathbf{Z})^- \tag{104}$$

of Hodge structures. The aim of this section is to study this morphism.

6.2.1. Relation with the second quadric fibration. In analogy with what happens for σ-planes in [**DK19**, Section 5], we relate Diagram (101) to the *second quadric fibration*

$$\varrho \colon \mathcal{Q} \longrightarrow \mathsf{Gr}(3,V_5)$$

defined in Equation (85). Recall from Equation (86) that $\mathcal{Q}$ is at the same time a $\mathbf{P}^2$-bundle π over X and a quadric surface bundle in a $\mathbf{P}^3$-bundle:

$$\begin{array}{ccccc} & & \mathcal{Q} & \xrightarrow{\ j\ } & \mathbf{P}(\mathbf{C}\oplus\bigwedge^2\mathcal{U}_3) \\ & \swarrow^{\pi} & & \searrow^{\varrho} & \downarrow{\scriptstyle \varphi} \\ X & & & & \mathsf{Gr}(3,V_5). \end{array} \tag{105}$$

In Section 5.2.1, we showed that the pullback map $\pi^*\colon H^6(X,\mathbf{Z}) \to H^6(\mathcal{Q},\mathbf{Z})$ induces an isomorphism between the rational vanishing cohomology of X and the rational *variable* cohomology of $\mathcal{Q}$, defined in Equation (87). In the following, we relate the variable cohomology of $\mathcal{Q}$ with the cohomology of the sheaves

$$(R^q\varrho_*\mathbf{Z})_{\mathrm{var}} \coloneqq \mathrm{Coker}(j^*\colon R^q\varphi_*\mathbf{Z} \longrightarrow R^q\varrho_*\mathbf{Z}) \tag{106}$$

on $\mathsf{Gr}(3,V_5)$ encoding the variable cohomology of the fibers of ϱ.

By [**Nag07**, Proposition 2.3.6], the Leray spectral sequence of ϱ induces a spectral sequence

$$E_2^{p,q} = H^p(\mathsf{Gr}(3,V_5),(R^q\varrho_*\mathbf{Z})_{\mathrm{var}}) \implies H^{p+q}(\mathcal{Q},\mathbf{Z})_{\mathrm{var}} \tag{107}$$

which gives a decreasing filtration $F^\bullet$ on $H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}}$ such that $F^p/F^{p+1} \simeq E_\infty^{p,6-p}$.

In order to obtain information on the variable cohomology of $\mathcal{Q}$, we study the sheaves $(R^q\varrho_*\mathbf{Z})_{\mathrm{var}}$ on $\mathsf{Gr}(3,V_5)$. The stalk of $(R^q\varrho_*\mathbf{Z})_{\mathrm{var}}$ at $[U_3]$ is the variable cohomology of the quadric surface $\varrho^{-1}([U_3]) \subset \mathbf{P}^3$, namely the cokernel of the restriction morphism $H^q(\mathbf{P}^3,\mathbf{Z}) \to H^q(\varrho^{-1}([U_3]),\mathbf{Z})$.

In particular, $R^q\varrho_*\mathbf{Z}$ and $(R^q\varrho_*\mathbf{Z})_{\mathrm{var}}$ are zero for q odd and $q \geq 6$. The stalks of the sheaves $(R^q\varrho_*\mathbf{Z})_{\mathrm{var}}$ for $q \in \{0,2,4\}$ can be computed using [**Nag07**, Proposition 2.2.6] and the description of the fibers of ϱ given in Section 5.2. The results are summarized in the following table:

	$(R^0\varrho_*\mathbf{Z})_{\mathrm{var},[U_3]}$	$(R^2\varrho_*\mathbf{Z})_{\mathrm{var},[U_3]}$	$(R^4\varrho_*\mathbf{Z})_{\mathrm{var},[U_3]}$
$[U_3] \in \mathsf{Z}^0_{A,V_5}$	0	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
$[U_3] \in \mathsf{Z}^1_{A,V_5}$	0	0	$\mathbf{Z}/2\mathbf{Z}$
$[U_3] \in \mathsf{Z}^2_{A,V_5}$	0	0	$\mathbf{Z}$
$[U_3] \in \mathsf{Z}^3_{A,V_5}$	0	0	0

The sheaf $(R^0\varrho_*\mathbf{Z})_{\mathrm{var}}$ is therefore also 0. Moreover, by [**Nag07**, Proposition 2.3.4], the sheaves $(R^2\varrho_*\mathbf{Z})_{\mathrm{var}}$ and $(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$ are constructible and are described as follows:

(a) $(R^2\varrho_*\mathbf{Z})_{\mathrm{var}} = \beta_{0,!}\mathcal{L}$, where $\mathcal{L}$ is a rank 1 local system supported on the affine open subset $\mathsf{Z}^0_{A,V_5} := \mathsf{Gr}(3,V_5) \smallsetminus \mathsf{Z}^{\geq 1}_{A,V_5}$, and $\beta_0 \colon \mathsf{Z}^0_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5)$ is the open embedding;

(b) $(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$ lies in a short exact sequence

$$0 \longrightarrow \beta_{2,!}(\mathbf{Z}/2\mathbf{Z}) \longrightarrow (R^4\varrho_*\mathbf{Z})_{\mathrm{var}} \longrightarrow \alpha_{2,*}\mathscr{M} \longrightarrow 0, \tag{108}$$

where $\beta_2 \colon \mathsf{Gr}(3,V_5) \smallsetminus \mathsf{Z}^2_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5)$ and $\alpha_2 \colon \mathsf{Z}^2_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5)$ are the inclusions, and the sheaf $\mathcal{L}_2$ is a rank 1 local system on Z^2_{A,V_5}. Note that $\mathscr{M}$ is the pullback (by α_2) of $(R^2\varrho_*\mathbf{Z})_{\mathrm{var}}$ to Z^2_{A,V_5}.

Things simplify over $\mathbf{Q}$ as

$$(R^2\varrho_*\mathbf{Q})_{\mathrm{var}} = \beta_{0,!}\mathcal{L}_{\mathbf{Q}} \quad , \quad (R^4\varrho_*\mathbf{Q})_{\mathrm{var}} = \alpha_{2,*}\mathscr{M}_{\mathbf{Q}}. \tag{109}$$

We go back to the spectral sequence.

Lemma 6.11. *The spectral sequence* (107) *induces isomorphisms*

$$E_2^{k-4,4} = H^{k-4}(\mathsf{Gr}(3,V_5), (R^4\varrho_*\mathbf{Z})_{\mathrm{var}}) \simeq H^k(\mathcal{Q},\mathbf{Z})_{\mathrm{var}}$$

for all $k \leq 6$.

Recall that the groups $H^k(\mathcal{Q},\mathbf{Z})$ vanish when k is odd.

Proof. Consider the spectral sequence $E_2^{p,q} \Rightarrow H^{p+q}(\mathcal{Q},\mathbf{Z})_{\mathrm{var}}$ described in (107).

The analysis above shows that $E_2^{p,q}$ is zero except perhaps when $q=2$ or $q=4$. In particular, all the d_2 differentials are zero and $E_2^{p,q} = E_3^{p,q}$ for all p,q. The only possibly nonzero d_3 differentials are then $d_3^{p,4} \colon E_2^{p,4} \to E_2^{p+3,2}$.

We saw above that the sheaf $(R^2\varrho_*\mathbf{Z})_{\mathrm{var}}$ is the proper pushforward of the local system $\mathcal{L}$ on Z^0_{A,V_5}, hence $E_2^{p,2} = H^p_c(\mathsf{Z}^0_{A,V_5},\mathcal{L})$, and this is zero when $p<6$ because Z^0_{A,V_5} is affine (see for example [**Nag07**, Proposition 2.4.1]).

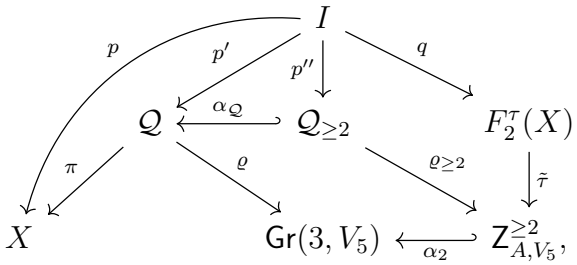
The conclusion is that $d_3^{p,q}$ vanishes, except possibly for $q=4$ and $p \geq 3$, and so do all further differentials in this range. In particular, $E_\infty^{p,q}$ is equal to $E_2^{p,q}$ for $p+q \leq 6$ and the filtration on $H^{p+q}(\mathcal{Q},\mathbf{Z})_{\mathrm{var}}$ has only the one nonzero term $E^{k-4,4}$. □

In particular, we obtain an isomorphism

$$\psi \colon H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}} \xrightarrow{\sim} H^2(\mathsf{Gr}(3,V_5), (R^4\varrho_*\mathbf{Z})_{\mathrm{var}}). \tag{110}$$

6.2.1.1. *Cohomology of the sheaf* $(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$. Set $\mathcal{Q}_{\geq 2} := \mathcal{Q} \times_{\mathsf{Gr}(3,V_5)} \mathsf{Z}^{\geq 2}_{A,V_5}$ and $\varrho_{\geq 2} := \varrho|_{\mathcal{Q}_{\geq 2}}$. By Remark 6.3, the fibers of $\varrho_{\geq 2}\colon \mathcal{Q}_{\geq 2} \to \mathsf{Z}^{\geq 2}_{A,V_5}$ are either two τ-planes meeting along a line (over the dense open subset Z^2_{A,V_5}) or a double τ-plane (over the finite set $\mathsf{Z}^{\geq 3}_{A,V_5}$). The double cover $g\colon F_2^\tau(X) \to \mathsf{Z}^{\geq 2}_{A,V_5}$ parametrizes the family of planes contained in the fibers of $\varrho_{\geq 2}$. Diagram (126) therefore factors as

(111)



where π is a $\mathbf{P}^2$-bundle and ϱ and $\varrho_{\geq 2}$ are quadric surface bundles.

In this section we study the cohomology group $H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Z})_{\mathrm{var}})$.

Lemma 6.12. *The pullback by* $\alpha_2\colon \mathsf{Z}^{\geq 2}_{A,V_5} \hookrightarrow \mathsf{Gr}(3,V_5)$ *induces a surjective morphism*

$$H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}) \twoheadrightarrow H^2(\mathsf{Z}^{\geq 2}_{A,V_5},(R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}})$$

which is an isomorphism over $\mathbf{Q}$.

Proof. The sheaf $(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$ was described in Equation (108) as an extension of $\alpha_{2,*}\mathscr{M}$, where $\mathscr{M}$ is the restriction $\alpha_2^*(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$ to $\mathsf{Z}^{\geq 2}_{A,V_5}$, by the sheaf $\beta_{2,!}\mathbf{Z}/2\mathbf{Z}$.

It follows from the definition (106) of $(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}$ that $\mathscr{M}$ is the cokernel of the morphism

$$j^*\colon \alpha_2^*R^4\varphi_*\mathbf{Z} \longrightarrow \alpha_2^*R^4\varrho_*\mathbf{Z}$$

(j is defined in Diagram (105)) which, by proper base change ([Mil08, Theorem 17.3]), is the same as the morphism

$$j^*\colon R^4(\varphi|_{\varphi^{-1}(\mathsf{Z}^{\geq 2}_{A,V_5})})_*\mathbf{Z} \longrightarrow R^4\varrho_{\geq 2,*}\mathbf{Z}.$$

In particular, $\mathscr{M}$ is isomorphic to $(R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}}$ and this already implies that $(R^4\varrho_*\mathbf{Q})_{\mathrm{var}}$ is isomorphic to $\alpha_{2,*}(R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}}$, hence the result over $\mathbf{Q}$.

Over $\mathbf{Z}$, the short exact sequence (108) becomes

$$0 \longrightarrow \beta_{2,!}\mathbf{Z}/2\mathbf{Z} \longrightarrow (R^4\varrho_*\mathbf{Z})_{\mathrm{var}} \longrightarrow \alpha_{2*}(R^4\varrho_{\geq 4,*}\mathbf{Z})_{\mathrm{var}} \longrightarrow 0 \tag{112}$$

and the associated long exact sequence in cohomology gives

$$H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}) \to H^2(\mathsf{Z}^{\geq 2}_{A,V_5},(R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}}) \to H^3_c(\mathsf{Gr}(3,V_5)\smallsetminus \mathsf{Z}^3_{A,V_5},\mathbf{Z}/2\mathbf{Z}).$$

By Proposition 5.7, Z^3_{A,V_5} is a finite set, hence

$$H^3_c(\mathsf{Gr}(3,V_5)\smallsetminus \mathsf{Z}^3_{A,V_5},\mathbf{Z}/2\mathbf{Z}) \simeq H^3(\mathsf{Gr}(3,V_5),\mathbf{Z}/2\mathbf{Z}) = 0$$

and we have the desired surjectivity. □

The above lemma shows that we can reduce to studying the second cohomology of the sheaf $(R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}}$ on $\mathsf{Z}^{\geq 2}_{A,V_5}$. Before studying it, we show some additional properties of the morphisms and varieties appearing in Diagram (111).

Proposition 6.13. *Let X be a GM manifold of dimension 6 such that the scheme $F_2^\tau(X)$ is smooth. The morphism p'' is the normalization of $\mathcal{Q}_{\geq 2}$ and I is an irreducible component of the fiber product $\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$.*

Proof. We start by showing that p'' is the the normalization of $\mathcal{Q}_{\geq 2}$. Since I is smooth, as it is a $\mathbf{P}^2$-bundle over the smooth scheme $F_2^\tau(X)$, it is enough to show that p'' is finite and birational. The fiber of I over a point $[U_3] \in \mathsf{Z}^{\geq 2}_{A,V_5}$ is either the disjoint union of two τ-planes (for $[U_3] \in \mathsf{Z}^2_{A,V_5}$), or a single τ-plane (for $[U_3] \in \mathsf{Z}^3_{A,V_5}$). In particular, the restriction of p'' to $I_{[U_3]}$ is the normalization of the fiber $\varrho^*_{\geq 2}([U_3])$ described above. This show that p'' is finite and birational.

We now prove that I is an irreducible component of $\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$. Since the diagram

$$\begin{array}{ccc} I & \xrightarrow{q} & F_2^\tau(X) \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle \tilde{\tau}} \\ \mathcal{Q}_{\geq 2} & \xrightarrow{\varrho_{\geq 2}} & \mathsf{Z}^{\geq 2}_{A,V_5}, \end{array}$$

is commutative, there exists a morphism $I \to \mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$. We show that it is injective by comparing the fibers over $[U_3] \in \mathsf{Z}^{\geq 2}_{A,V_5}$.

For $[U_3] \in \mathsf{Z}^3_{A,V_5}$, both fibers are equal to the unique τ-plane $\Pi \subset X$ such that $\tilde{\tau}(\Pi) = [U_3]$. For $[U_3] \in \mathsf{Z}^2_{A,V_5}$, the fiber of $\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$ over $[U_3]$ is the product

$$\varrho^{-1}_{\geq 2}([U_3]) \times \{[\Pi], [\sigma(\Pi)]\},$$

where Π is a τ-plane satisfying $\varrho^{-1}_{\geq 2}([U_3]) = \Pi \cup \sigma(\Pi)$, while the fiber of I over $[U_3]$ is the disjoint union $\Pi \times \{[\Pi]\} \sqcup \sigma(\Pi) \times \{[\sigma(\Pi)]\}$. Therefore, the morphism

$$I \longrightarrow \mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$$

is a closed embedding which identifies the irreducible variety I with an irreducible component of $\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}^{\geq 2}_{A,V_5}} F_2^\tau(X)$. □

As a first consequence, we obtain that the image of the morphism $p\colon I \to X$ is a hypersurface in X.

Corollary 6.14. *Let X be a GM manifold of dimension 6 such that the scheme $F_2^\tau(X)$ is smooth. The morphism p is generically finite onto its image.*

Proof. We first consider the morphism $p'\colon I \to \mathcal{Q}$. Proposition 6.13 shows that the image of p' is equal to $\mathcal{Q}_{\geq 2}$. Therefore, using Remark 5.10, we obtain an equality

$$p'_*([I]) = \varrho^*([\mathsf{Z}^{\geq 2}_{A,V_5}]) = 20\varrho^*(\sigma_{2,1}) \tag{113}$$

of classes in $H^6(\mathcal{Q}, \mathbf{Q})$ (up to multiplication by a positive constant if there are multiplicities; these constants are not important for our purpose so we disregard them here and in the rest of the proof). Recall that the Schubert cycle $\sigma_{2,1}$ is the class of the subvariety

$$\Sigma_{2,1}(V_1, V_3) = \{[U_3] \in \mathsf{Gr}(3, V_5) \mid V_1 \subset U_3,\ \dim(U_3 \cap V_3) \geq 2\} \subset \mathsf{Gr}(3, V_5),$$

for any choice of subspaces $V_1 \subset V_3$ of V_5 of respective dimensions 1 and 3. We show that

$$\varrho^{-1}(\Sigma_{2,1}) = \{([U_3], x) \in \mathsf{Gr}(3, V_5) \times X \mid \gamma_X(x) \subset U_3,\ V_1 \subset U_3,\ \dim(U_3 \cap V_3) \geq 2\}$$

is a subvariety of $\mathcal{Q}$ of dimension 5, so that the class $\varrho^*(\sigma_{2,1})$ is equal to $[\varrho^{-1}(\Sigma_{2,1})]$.

Note that for any point $([U_3], x) \in \varrho^{-1}(\Sigma_{2,1})$, the point $\pi([U_3], x)$ lies in the hypersurface $H = \gamma_X^{-1}(\{[U_2] \in \mathsf{Gr}(2, V_5) \mid U_2 \cap V_3 \neq 0\})$. Moreover, for a general $x \in H$ such that V_1 does not lie in $\gamma_X(x)$, the fiber of the morphism

$$\pi|_{\varrho^{-1}(\Sigma_{2,1})} \colon \varrho^{-1}(\Sigma_{2,1}) \longrightarrow H$$

over x is the point $([\gamma_X(x) \oplus V_1], x)$. Therefore, the map $\pi|_{\varrho^{-1}(\Sigma_{2,1})}$ is generically finite and $\varrho^{-1}(\Sigma_{2,1})$ has dimension 5. By the above discussion, we have

$$p_*([I]) = \pi_*(p'_*([I])) = 20\pi_*[\varrho(\Sigma_{2,1})]$$

in $H^2(X, \mathbf{Q})$. In particular, this shows that $p(I)$ is a hypersurface in X. □

We go back to the diagram

$$\begin{array}{ccc} I & \xrightarrow{q} & F_2^\tau(X) \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle \tilde{\tau}} \\ \mathcal{Q}_{\geq 2} & \xrightarrow{\varrho_{\geq 2}} & \mathsf{Z}_{A,V_5}^{\geq 2}, \end{array} \tag{114}$$

where the varieties I and $F_2^\tau(X)$ are smooth, q is a $\mathbf{P}^2$-bundle, p'' is the normalization morphism by Proposition 6.13, and $\tilde{\tau}$ is a double cover branched over $\mathsf{Z}_{A,V_5}^{\geq 3}$.

Lemma 6.15. *The smooth incidence variety I defines an isomorphism*

$$I_* \colon R^4\varrho_{\geq 2,*}\mathbf{Z} \xrightarrow{\sim} \tilde{\tau}_*\mathbf{Z}$$

of sheaves on $\mathsf{Z}_{A,V_5}^{\geq 2}$, which induces an isomorphism

$$\bar{I}_* \colon (R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}} \xrightarrow{\sim} \mathrm{Coker}(\mathbf{Z} \hookrightarrow \tilde{\tau}_*\mathbf{Z}).$$

Proof. We showed in Proposition 6.13 that I is an irreducible component of the fiber product $\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}_{A,V_5}^{\geq 2}} F_2^\tau(X)$. By [**Ful97**, Appendix B, Lemma 4], the component I defines, via the cycle class map, an element in the Borel–Moore homology group $H_{10}^{\mathrm{BM}}(\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}_{A,V_5}^{\geq 2}} F_2^\tau(X), \mathbf{Z})$. Since $F_2^\tau(X)$ is smooth, by [**CH00**, Lemma 2.21] (the proof goes the same with $\mathbf{Z}$ coefficients, as remarked in [**Nag07**, Lemma 3.1.3]), Diagram (114) induces an isomorphism

$$H_{10}^{\mathrm{BM}}(\mathcal{Q}_{\geq 2} \times_{\mathsf{Z}_{A,V_5}^{\geq 2}} F_2^\tau(X), \mathbf{Z}) \simeq \mathrm{Hom}_{\mathsf{Z}_{A,V_5}^{\geq 2}}(R\varrho_{\geq 2,*}\mathbf{Z}, R\tilde{\tau}_*\mathbf{Z}[-4]).$$

Thus the irreducible component I, induces a morphism $R\varrho_{\geq 2,*}\mathbf{Z} \to R\tilde{\tau}_*\mathbf{Z}[-4]$. Since $\tilde{\tau}$ is a finite morphism, the sheaf $R\tilde{\tau}_*\mathbf{Z}[-4] = \tilde{\tau}_*\mathbf{Z}[-4]$ is concentrated in degree 4, thus taking cohomology in degree 4 gives a morphism

$$I_* \colon R^4\varrho_{\geq 2,*}\mathbf{Z} \longrightarrow \tilde{\tau}_*\mathbf{Z} \tag{115}$$

between sheaves on $\mathsf{Z}_{A,V_5}^{\geq 2}$.

We show that I_* is an isomorphism. Let $[U_3]$ be a point in $\mathsf{Z}^{\geq 2}_{A,V_5}$ and write $\tilde{\tau}^{-1}([U_3]) = \{[\Pi], [\sigma(\Pi)]\}$. The stalk of $R^4\varrho_{\geq 2,*}\mathbf{Z}$ at $[U_3]$ is the cohomology group

$$(R^4\varrho_{\geq 2,*}\mathbf{Z})_{[U_3]} = H^4(\Pi \cup \sigma(\Pi), \mathbf{Z})$$

hence it is generated by $[x]$ and $[\sigma(x)]$, where x is a point in Π. Analogously, the stalk of $\tilde{\tau}_*\mathbf{Z}$ at $[U_3]$ is the cohomology group

$$(\tilde{\tau}_*\mathbf{Z})_{[U_3]} = H^0(\{[\Pi], [\sigma(\Pi)]\}, \mathbf{Z}),$$

and is generated by the classes $[\Pi]$ and $[\sigma(\Pi)]$. Finally, the map I_* at $[U_3]$ is given on stalks by the map

$$I_{*,[U_3]} = q_*(p'')^* \colon H^4(\Pi \cup \sigma(\Pi), \mathbf{Z}) \longrightarrow H^0(\{[\Pi], [\sigma(\Pi)]\}, \mathbf{Z}).$$

Since $q(p''^{-1}(x)) = [\Pi]$ and $q(p''^{-1}(\sigma(x)) = [\sigma(\Pi)]$, we obtain that $q_*(p'')^*$ is an isomorphism. This shows that I_* is an isomorphism.

We now prove the second statement of the lemma. T show that The sheaf $(R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}}$ is the cokernel of the morphism $j^* \colon R^4\varphi_*\mathbf{Z}|_{\mathsf{Z}^{\geq 2}_{A,V_5}} \to R^4\varrho_{\geq 2,*}\mathbf{Z}$, where the morphisms j and φ are defined in Diagram (105) (see the proof of Lemma 6.15 for more details). Note that the local system $R^4\varphi_*\mathbf{Z}$ is trivial, as $\mathsf{Gr}(3, V_5)$ is simply connected. Since I_* is an isomorphism, the diagram

$$\begin{array}{ccc} \mathbf{Z} \simeq R^4\varphi_*\mathbf{Z}|_{\mathsf{Z}^{\geq 2}_{A,V_5}} & & \\ \downarrow{\scriptstyle j^*} & \searrow^{\eta} & \\ R^4\varrho_{\geq 2,*}\mathbf{Z} & \xrightarrow[\sim]{I_*} & \tilde{\tau}_*\mathbf{Z} \end{array}$$

of sheaves over $\mathsf{Z}^{\geq 2}_{A,V_5}$, induces an isomorphism

$$\bar{I}_* \colon (R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}} \xrightarrow{\sim} \mathrm{Coker}(\eta). \tag{116}$$

To conclude the proof of the lemma we need to show that the composition

$$\eta \colon \mathbf{Z} = R^4\varphi_*\mathbf{Z}|_{\mathsf{Z}^{\geq 2}_{A,V_5}} \xrightarrow{j^*} R^4\varrho_{\geq 2,*}\mathbf{Z} \xrightarrow[\sim]{I_*} \tilde{\tau}_*\mathbf{Z} \tag{117}$$

is the inclusion map given by $\tilde{\tau}^*$—namely, it is the unit map of the adjunction $(\tilde{\tau}^*, \tilde{\tau}_*)$. Since $\tilde{\tau}$ is surjective, this morphism is in particular injective.

We show this on stalks. Let $[U_3]$ be a point in $\mathsf{Z}^{\geq 2}_{A,V_5}$ and set $\varrho^{-1}([U_3]) = \Pi \cup \sigma(\Pi)$, where Π and $\sigma(\Pi)$ are planes in $\varphi^{-1}([U_3]) = \mathbf{P}^3$. The morphism $\eta_{[U_3]}$ is the composition

$$\eta_{[U_3]} \colon H^4(\varphi^{-1}([U_3]), \mathbf{Z}) \xrightarrow{j^*_{[U_3]}} H^4(\varrho^*_{\geq 2}([U_3]), \mathbf{Z}) \xrightarrow{I_{*,[U_3]}} H^0(\tilde{\tau}^{-1}([U_3]), \mathbf{Z}),$$

where $j^*_{[U_3]} \colon H^4(\mathbf{P}^3, \mathbf{Z}) \to H^4(\Pi \cup \sigma(\Pi), \mathbf{Z})$ is the restriction morphism and the stalks of the second map have been described above.

- For $[U_3] \in \mathsf{Z}^2_{A,V_5}$, the morphism $j^*_{[U_3]}$ sends the class of a line ℓ in $\mathbf{P}^3$ to the class $[x] + [\sigma(x)]$, for some point $x \in \Pi$. Therefore, we obtain

  $$\eta_{[U_3]}([\ell]) = [\Pi] + [\sigma(\Pi)] \in H^0(\tilde{\tau}^{-1}([U_3]), \mathbf{Z}) = \mathbf{Z}[\Pi] \oplus \mathbf{Z}[\sigma(\Pi)].$$

- For $[U_3] \in \mathsf{Z}^3_{A,V_5}$, the morphism $\eta_{[U_3]}$ sends the class of a line ℓ in $\mathbf{P}^3$ to the class $[\Pi] \in H^0(\tilde{\tau}^{-1}([U_3]), \mathbf{Z}) = \mathbf{Z}[\Pi]$.

Therefore, in both cases, $\eta_{[U_3]}$ is the diagonal inclusion $\mathbf{Z} \hookrightarrow \bigoplus_{\pi_0(\tilde{\tau}^{-1}(U_3))} \mathbf{Z}$. This shows that η behaves on stalks as the inclusion $\mathbf{Z} \hookrightarrow \tilde{\tau}_*\mathbf{Z}$ induced by the double cover $\tilde{\tau}$, and therefore the two morphisms are the same. In particular, we have proved that Equation (116) gives an isomorphism

$$\bar{I}_*\colon (R^4\varrho_{\geq 2,*}\mathbf{Z})_{\mathrm{var}} \xrightarrow{\sim} \mathrm{Coker}(\mathbf{Z} \hookrightarrow \tilde{\tau}_*\mathbf{Z}) \tag{118}$$

of sheaves over $\mathsf{Z}^{\geq 2}_{A,V_5}$. □

As a corollary, we obtain the following characterization of the cohomology group $H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Q})_{\mathrm{var}})$.

Corollary 6.16. *Let X be a GM manifold of dimension 6 such that the scheme $F_2^\tau(X)$ is smooth. The isomorphism $\bar{I}_*$ constructed in Lemma 6.15 induces an isomorphism in cohomology*

$$H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) \simeq H^2(F_2^\tau(X),\mathbf{Q})^-$$

Proof. Lemma 6.15 shows that the incidence variety I induces an isomorphism of sheaves $(R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}} \simeq \mathrm{Coker}(\eta)$ over $\mathsf{Z}^{\geq 2}_{A,V_5}$, where

$$\eta\colon \mathbf{Q} \hookrightarrow \tilde{\tau}_*\mathbf{Q},$$

is the inclusion induced by $\tilde{\tau}$. This induces isomorphisms in cohomology

$$H^k(\bar{I}_*)\colon H^k(\mathsf{Z}^{\geq 2}_{A,V_5},(R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}}) \simeq H^k(\mathsf{Z}^{\geq 2}_{A,V_5},\mathrm{Coker}(\eta)) \tag{119}$$

Lemma 6.15 shows that the pullback of the inclusion $\alpha_2\colon \mathcal{Q}_{\geq 2} \hookrightarrow \mathcal{Q}$ induces an isomorphism $H^2(\mathsf{Gr}(3,V_5),(R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) \simeq H^2(\mathsf{Z}^{\geq 2}_{A,V_5},(R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}})$. Therefore, in order to prove the corollary, we need to show that $H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathrm{Coker}(\eta))$ is isomorphic to $H^2(F_2^\tau(X),\mathbf{Q})^-$.

Consider the long exact sequence in cohomology

$$\begin{aligned} H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathbf{Q}) \xrightarrow{H^2(\eta)} H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\tilde{\tau}_*\mathbf{Q}) \to H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathrm{Coker}(\eta)) \\ \to H^3(\mathsf{Z}^{\geq 2}_{A,V_5},\mathbf{Q}) \xrightarrow{H^3(\eta)} H^3(\mathsf{Z}^{\geq 2}_{A,V_5},\tilde{\tau}_*\mathbf{Q}) \end{aligned} \tag{120}$$

induced by the inclusion η. The Leray spectral sequence for the double cover $\tilde{\tau}$ gives an isomorphism

$$H^k(\mathsf{Z}^{\geq 2}_{A,V_5},\tilde{\tau}_*\mathbf{Q}) \xrightarrow{\sim} H^k(F_2^\tau(X),\mathbf{Q}),$$

whose composition with $H^k(\eta)$ give the pullback map $\tilde{\tau}^*\colon H^k(\mathsf{Z}^{\geq 2}_{A,V_5},\mathbf{Q}) \to H^k(F_2^\tau(X),\mathbf{Q})$. Moreover, since $\tilde{\tau}$ is a double cover, the pullback $\tilde{\tau}^*$ is the inclusion with image the invariant subspace with respect to the involution ι_τ^*. Putting everything together, the long exact sequence (120) reduces to

$$0 \to H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathbf{Q}) \xrightarrow{\tilde{\tau}^*} H^2(F_2^\tau(X),\mathbf{Q}) \to H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathrm{Coker}(\eta)) \to 0.$$

This shows that the group $H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\mathrm{Coker}(\eta))$ is the cokernel of $\tilde{\tau}^*$, which is isomorphic to $H^2(F_2^\tau(X),\mathbf{Q})^-$, and concludes the proof. □

6.2.2. The morphism $(q_*p^*)_{\mathrm{van}}$. We showed in Proposition 6.10 that Diagram (101) induces a Hodge morphism $(q_*p^*)_{\mathrm{van}}\colon H^6(X,\mathbf{Z})_{\mathrm{van}} \to H^2(F_2^\tau(X),\mathbf{Z})$ whose image is contained in the antiinvariant cohomology $H^2(F_2^\tau(X),\mathbf{Z})^-$.

Theorem 6.17 ([Riz26, Theorem 4.6])**.** *Let X be a Gushel–Mukai sixolfd such that the Hilbert scheme $F_2^\tau(X)$ of τ-planes contained in X is smooth. The morphism*

$$(q_*p^*)_{\mathrm{van}}\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(F_2^\tau(X),\mathbf{Q})^-$$

is an isomorphism.

Proof. Since $p=\pi p'$, the morphism $(q_*p^*)_{\mathrm{van}}$ factors as

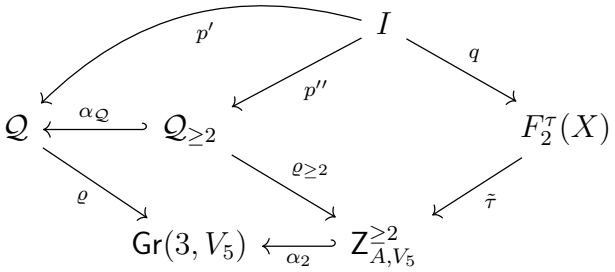
$$(q_*p^*)_{\mathrm{van}}\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \xrightarrow{\pi^*} H^6(\mathcal{Q},\mathbf{Q})_{\mathrm{var}} \xrightarrow{(q_*p'^*)_{\mathrm{var}}} H^2(F_2^\tau(X),\mathbf{Q})^-$$

and the map π^* is an isomorphism by Proposition 5.5. To show that the morphism $(q_*p'^*)_{\mathrm{van}}$ is also an isomorphism, we prove that the diagram

$$\begin{array}{ccc} H^6(\mathcal{Q},\mathbf{Q})_{\mathrm{var}} & \xrightarrow{(q_*p'^*)_{\mathrm{var}}} & H^2(F_2^\tau(X),\mathbf{Q})^- \\ \downarrow{\scriptstyle\psi}\wr & & \| \\ H^2(\mathsf{Gr}(3,V_6),(R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) & \xrightarrow{\sim} & H^2(F_2^\tau(X),\mathbf{Q})^- \end{array} \tag{121}$$

is commutative, where ψ is the isomorphism from (110) given by the spectral sequence associated to ϱ, and the lower map is the isomorphism proved in Corollary 6.16, and induced by the incidence correspondence I.

Consider the diagram



We denote by $E_r^{p,q}(f)$ the Leray spectral sequence associated to a morphism f. By functoriality of Leray spectral sequences, we obtain morphisms

$$E_r^{p,q}(\varrho) \xrightarrow{\alpha_{\mathcal{Q}}^*} E_r^{p,q}(\varrho_{\geq 2}) \xrightarrow{q_*p''^*} E_r^{p,q-4}(\tilde\tau)$$

of spectral sequences. They induce morphisms on the filtrations associated to the abutment of the spectral sequences. In particular, since the fibers of ϱ and $\varrho_{\geq 2}$ are quadric surfaces, there are equalities $E^{0,6}(\varrho)=E^{1,5}(\varrho)=0$ and $E^{0,6}(\varrho_{\geq 2})=E^{1,5}(\varrho_{\geq 2})=0$. Moreover, since $\tilde\tau$ is a finite morphism, $E_r^{p,q}(\tilde\tau)$ is nonzero only for $q=0$. Hence, we obtain a commutative diagram

$$\begin{array}{ccccc} H^6(\mathcal{Q},\mathbf{Q}) & \xrightarrow{\alpha_{\mathcal{Q}}^*} & H^6(\mathcal{Q}_{\geq 2},\mathbf{Q}) & \xrightarrow{q_*p''^*} & H^2(F_2^\tau(X),\mathbf{Q}) \\ \downarrow & & \downarrow & & \| \\ H^2(\mathsf{Gr}(3,V_6),R^4\varrho_*\mathbf{Q}) & \xrightarrow{H^2(\alpha_2^*)} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5},R^4\varrho_{\geq 2,*}\mathbf{Q}) & \xrightarrow{H^2(I_*)} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5},\tilde\tau_*\mathbf{Q}). \end{array} \tag{122}$$

Consider now the morphism $j^*\colon \mathbf{Q} = R^4\varphi_*\mathbf{Q} \to R^4\varrho_*\mathbf{Q}$ introduced in Equation (106), whose cokernel is the sheaf $(R^4\varrho_*\mathbf{Q})_{\mathrm{var}}$, and the diagram

$$\begin{array}{ccccc} \mathbf{Q} = R^4\varphi_*\mathbf{Q} & \xrightarrow{\alpha_2^*} & \alpha_2^*R^4\varphi_*\mathbf{Q} & & \\ \downarrow{\scriptstyle j^*} & & \downarrow & \searrow^{\eta} & \\ R^4\varrho_*\mathbf{Q} & \xrightarrow[\alpha_2^*]{} & R^4\varrho_{\geq 2,*}\mathbf{Q} & \xrightarrow[I_*]{} & \tilde{\tau}_*\mathbf{Q}, \end{array} \tag{123}$$

where I_* is the isomorphism constructed in Lemma 6.15. The proof of Lemma 6.15 shows that the composition

$$\eta\colon \mathbf{Q} \simeq \alpha_2^*R^4\varphi_*\mathbf{Q} \xrightarrow{j^*} R^4\varrho_{\geq 2,*}\mathbf{Q} \xrightarrow{I_*} \tilde{\tau}_*\mathbf{Q}$$

is the inclusion $\mathbf{Q} \hookrightarrow \tilde{\tau}_*\mathbf{Q}$ induced by the double cover $\tilde{\tau}$. Finally, by proper base change, the cokernel of the morphism $\alpha_2^*R^4\varphi_*\mathbf{Q} \to R^4\varrho_{\geq 2,*}\mathbf{Q}$ is the sheaf $(R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}}$ (see the proof of Lemma 6.12 for more details).

Taking cohomology, the morphisms in (123) induce the diagram

$$\begin{array}{ccccc} H^2(\mathsf{Gr}(3,V_5), R^4\varphi_*\mathbf{Q}) & \xrightarrow{H^2(\alpha_2^*)} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, R^4\varphi_*\mathbf{Q}) & & \\ \downarrow{\scriptstyle H^2(j^*)} & & \downarrow{\scriptstyle H^2(j^*)} & \searrow & \\ H^2(\mathsf{Gr}(3,V_6), R^4\varrho_*\mathbf{Q}) & \xrightarrow{H^2(\alpha_2^*)} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, R^4\varrho_{\geq 2,*}\mathbf{Q}) & \xrightarrow{H^2(I_*)} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, \tilde{\tau}_*\mathbf{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\mathsf{Gr}(3,V_6), (R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) & \xrightarrow[H^2(\alpha_2^*)]{\sim} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, (R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}}) & \xrightarrow[H^2(\bar{I}_*)]{\sim} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, \mathrm{Coker}(\eta)), \end{array} \tag{124}$$

where the morphisms on the lower line are isomorphisms by Corollary 6.16, and $H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, \mathrm{Coker}(\eta))$ can be identified with $H^2(F_2^\tau(X), \mathbf{Q})^-$.

Putting together Diagrams (122) and (124), we obtain

$$\begin{array}{ccccc} H^6(\mathcal{Q}, \mathbf{Q}) & \xrightarrow{H^2(\alpha_2^*)} & H^6(\mathcal{Q}_{\geq 2}, \mathbf{Q}) & \xrightarrow{q_*p''^*} & H^2(F_2^\tau(X), \mathbf{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\mathsf{Gr}(3,V_6), (R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) & \xrightarrow[H^2(\alpha_2^*)]{\sim} & H^2(\mathsf{Z}^{\geq 2}_{A,V_5}, (R^4\varrho_{\geq 2,*}\mathbf{Q})_{\mathrm{var}}) & \xrightarrow[H^2(\bar{I}_*)]{\sim} & H^2(F_2^\tau(X), \mathbf{Q})^-. \end{array} \tag{125}$$

Consider finally the morphism $j^*\colon H^6(\mathbf{P}(\mathbf{C} \oplus \bigwedge^2\mathcal{U}_3), \mathbf{Q}) \to H^6(\mathcal{Q}, \mathbf{Q})$ whose cokernel defines the variable cohomology $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$. We identify $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$ with a subspace of $H^6(\mathcal{Q}, \mathbf{Q})$ that does not meet $\mathrm{Im}(j^*)$. The argument at the beginning of the proof shows that the morphism $q_*p''^* \circ \alpha_2^*$ send $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$ to $H^2(F_2^\tau(X), \mathbf{Q})^-$. Therefore, the restriction of Diagram (125) to $H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}}$ induces the wanted commutative diagram

$$\begin{array}{ccc} H^6(\mathcal{Q}, \mathbf{Q})_{\mathrm{var}} & \xrightarrow{(q_*p'^*)_{\mathrm{var}}} & H^2(F_2^\tau(X), \mathbf{Q})^- \\ \psi\downarrow\wr & & \| \\ H^2(\mathsf{Gr}(3,V_6), (R^4\varrho_*\mathbf{Q})_{\mathrm{var}}) & \xrightarrow{\sim} & H^2(F_2^\tau(X), \mathbf{Q})^-. \end{array}$$

□

Remark 6.18. Diagrams (122) and (124) also exist with $\mathbf{Z}$ coefficients.

The difference arises when we restrict to $H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}} \subset H^6(\mathcal{Q},\mathbf{Q})$. Indeed, by Remark 5.6, the pullback π^* induces an isomorphism $H^6(X,\mathbf{Z})/\operatorname{Im}(\gamma_X^*) \simeq H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}}$. Since $\operatorname{Im}(\gamma_X^*)$ is the σ-invariant part $H^6(X,\mathbf{Z})$, the argument of Proposition 6.10 shows that $q_*p^*(\operatorname{Im}(\gamma_X^*))$ is contained in $H^2(F_2^\tau(X),\mathbf{Z})^+$.

Therefore, the morphisms q_*p^* induces a map

$$H^6(X,\mathbf{Z})/\operatorname{Im}(\gamma_X^*) \xrightarrow{\pi^*} H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}} \longrightarrow H^2(F_2^\tau(X),\mathbf{Z})/H^2(F_2^\tau(X),\mathbf{Z})^+$$

where the last term is the cokernel of the map $\tilde{\tau}^*\colon H^2(\mathsf{Z}_{A,V_5}^{\geq 2},\mathbf{Z}) \to H^2(F_2^\tau(X),\mathbf{Z})$.

Finally, by the proof of Corollary 6.16, the cokernel of $\tilde{\tau}_*$ injects into the group $H^2(\mathsf{Z}_{A,V_5}^{\geq 2},\operatorname{Coker}(\eta))$. Hence, we obtain a diagram

$$\begin{array}{ccccc} H^6(X,\mathbf{Z})/\operatorname{Im}(\gamma_X^*) & \xrightarrow{\pi^*} & H^6(\mathcal{Q},\mathbf{Z})_{\mathrm{var}} & \longrightarrow & H^2(F_2^\tau(X),\mathbf{Z})/H^2(F_2^\tau(X),\mathbf{Z})^+ \\ & \searrow^{\sim} & \downarrow{\scriptstyle \psi} & & \downarrow \\ & & H^2(\mathsf{Gr}(3,V_6),(R^4\varrho_*\mathbf{Z})_{\mathrm{var}}) & \xrightarrow{H^2(\bar{I})\circ H^2(\alpha_2^*)} & H^2(\mathsf{Z}_{A,V_5}^{\geq 2},\operatorname{Coker}(\eta)), \end{array}$$

where π^* is an isomorphism by Remark 5.6, ψ is an isomorphism by (110), $H^2(\bar{I})$ is an isomorphism by Lemma 6.15, and $H^2(\alpha_2^*)$ is surjective by Lemma 6.12.

Therefore the inclusion $H^2(F_2^\tau(X),\mathbf{Z})/H^2(F_2^\tau(X),\mathbf{Z})^+ \hookrightarrow H^2(\mathsf{Z}_{A,V_5}^{\geq 2},\operatorname{Coker}(\eta))$ is actually an isomorphism, and the map

$$H^6(X,\mathbf{Z})/\operatorname{Im}(\gamma_X^*) \longrightarrow H^2(F_2^\tau(X),\mathbf{Z})/H^2(F_2^\tau(X),\mathbf{Z})^+$$

is surjective. Note that $(H^6(X,\mathbf{Z})/\operatorname{Im}(\gamma_X^*)) \otimes \mathbf{Q}$ is isomorphic to $H^6(X,\mathbf{Q})^-$ and $(H^2(F_2^\tau(X),\mathbf{Z})/H^2(F_2^\tau(X),\mathbf{Z}))\otimes\mathbf{Q}$ is isomorphic to $H^2(F_2^\tau(X),\mathbf{Q})^-$. Therefore Theorem 6.17 implies that the above map is an isomorphism.

6.3. The rational Hodge isometry

Let X be a general GM manifold of dimension 6 with its canonical double covering $\gamma_X\colon X \to \mathsf{Gr}(2,V_5)$ and let $g\colon \widetilde{\mathsf{Z}}_A \to \mathsf{Z}_A$ be the associated EPW cube. By Corollary 6.9, the Hilbert scheme $F_2^\tau(X)$ of τ-planes contained in X is a smooth irreducible threefold, and there is a commutative diagram

(126)

$$\begin{array}{ccccc} & & I=\{(x,\Pi)\in X\times F_2^\tau(X)\mid x\in\Pi\} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & F_2^\tau(X)\simeq \widetilde{\mathsf{Z}}_{A,V_5}^{\geq 2} \xrightarrow{\tilde{\tau}} \mathsf{Z}_{A,V_5}^{\geq 2} \\ \downarrow{\scriptstyle \gamma_X} & & & & \downarrow{\scriptstyle i} \quad\quad \downarrow \\ \mathsf{Gr}(2,V_5) & & & & \widetilde{\mathsf{Z}}_A \xrightarrow[g]{} \mathsf{Z}_A, \end{array}$$

where q is a $\mathbf{P}^2$-bundle, I is smooth irreducible of dimension 5, and the map p is generically finite onto its image by Corollary 6.14 (a canonically defined hypersurface in X).

In Section 6.2, we studied the morphism $q_*p^*\colon H^6(X,\mathbf{Z}) \to H^2(F_2^\tau(X),\mathbf{Z})$. In particular, Theorem 6.17 shows that it induces a Hodge isomorphism

$$(q_*p^*)_{\mathrm{van}}\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(F_2^\tau(X),\mathbf{Q})^-.$$

Consider now the inclusion $i\colon F_2^\tau(X) \simeq \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \hookrightarrow \widetilde{\mathsf{Z}}_A$. By Theorem 6.9, the involution ι_τ^* associated with the double cover $\tilde{\tau}$ is the restriction of the involution ι associated with the EPW cube g. Moreover, there is an equality

$$H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z})^- = H^2(\widetilde{\mathsf{Z}},\mathbf{Z})_{\mathrm{prim}}$$

as the invariant second cohomology of $\widetilde{\mathsf{Z}}_A$ is given by $\mathbf{Z}h$ by Section 2.5.1. Therefore, the same argument as in Proposition 6.10 shows that the pullback map $i^*\colon H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z}) \to H^2(F_2^\tau(X)\mathbf{Z})$ restricts to a morphism

$$i^*_{\mathrm{prim}}\colon H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z})_{\mathrm{prim}} \longrightarrow H^2(F_2^\tau(X),\mathbf{Z})^-.$$

Lemma 6.19. *The map i^*_{prim} is injective.*

Proof. It is enough to prove the injectivity of i^*_{prim} when A is very general and, since $H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z})$ is torsion-free, we can work with $\mathbf{C}$ coefficients. In that case, the Hodge structure $H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}$ is simple (see Proposition 1.13) so that i^*_{prim} is either injective or 0.

Consider the class $[\sigma] \in H^2(\widetilde{\mathsf{Z}}_A,\mathbf{C})_{\mathrm{prim}}$ of a symplectic form σ on $\widetilde{\mathsf{Z}}_A$. Its restriction to $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ is nonzero, because $\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}$ is not Lagrangian. Indeed, by Lemma 2.24, its class is equal to

$$[\widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5}] = g^*\sigma_{1,1,1} = \frac{1}{24}(h^3 + hc_2) + 2\eta,$$

whereas the class of any Lagrangian submanifold is of the form $a(3h^3 - hc_2) + c\eta$ for some $a, c \in \mathbf{Q}$ (Proposition 1.19). Thus, $i^*_{\mathrm{prim},\mathbf{C}}([\sigma]) \neq 0$, hence i^*_{prim} is nonzero and therefore injective. □

Putting all together we obtain Hodge morphisms

$$\begin{array}{ccc} H^6(X,\mathbf{Q})_{\mathrm{van}} & \xrightarrow[(q_*p^*)_{\mathrm{van}}]{\sim} & H^2(F_2^\tau(X),\mathbf{Q})^- \\ & & \uparrow{\scriptstyle i^*_{\mathrm{prim}}} \\ & & H^2(\widetilde{\mathsf{Z}}^{\geq 2}_A,\mathbf{Q})_{\mathrm{prim}} \end{array} \tag{127}$$

between rational cohomology groups, where $(q^*p_*)_{\mathrm{van}}$ is an isomorphism by Theorem 6.17, and the map i^*_{prim} is injective by Lemma 6.19.

We show that the above diagram induces a Hodge isomorphism between the vanishing middle cohomology of X and the primitive second cohomology of $\widetilde{\mathsf{Z}}_A$.

Theorem 6.20 ([Riz26, Theorem 4.7])**.** *Let X be a general GM manifold of dimension 6 and let $\widetilde{\mathsf{Z}}_A$ be the associated EPW cube. The incidence correspondence*

$$\begin{array}{ccccc} & & I = \{(x,\Pi) \in X \times F_2^\tau(X) \mid x \in \Pi\} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & F_2^\tau(X) \simeq \widetilde{\mathsf{Z}}^{\geq 2}_{A,V_5} \xhookrightarrow{i} \widetilde{\mathsf{Z}}_A \end{array}$$

induces a Hodge isomorphism

$$\alpha\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}$$

which is a multiple of an isometry.

Proof. We can assume that X is very general. Consider Diagram (127). By [**DK19**, Theorem 5.19], the Hodge structure on $H^6(X,\mathbf{Q})_{\mathrm{van}}$ is isomorphic to the Hodge structure on the primitive cohomology of the double EPW sextic $\widetilde{\mathsf{Y}}_A$ associated to X. Since very general smooth double EPW sextics have Picard rank 1, Proposition 1.13 implies that the Hodge structure of $H^6(X,\mathbf{Q})_{\mathrm{van}}$ is simple and, since $(q_*p^*)_{\mathrm{van}}$ is an isomorphism of Hodge structures, the same holds true for $H^2(F_2^\tau(X),\mathbf{Q})^-$. This implies that the morphism i^*_{prim} is surjective.

It is therefore an isomorphism and Diagram (127) gives a Hodge isomorphism

$$\alpha\colon H^6(X,\mathbf{Q})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}.$$

To show that α is a constant multiple of an isometry, we follow [**Son22**, Lemma 4.5.2] and use the uniqueness of the quadratic form on $H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}$ of Proposition 1.12.

Consider the intersection form on $H^6(X,\mathbf{Q})_{\mathrm{van}}$. The subspace $H^{4,2}(X,\mathbf{C})$ is orthogonal to $H^{2,4}(X,\mathbf{C})\oplus H^{3,3}(X,\mathbf{C})_{\mathrm{van}}$. Since α is a morphism of Hodge structures, the intersection form on $H^6(X,\mathbf{Q})_{\mathrm{van}}$ transports to a second quadratic form q' on $H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}$ such that $H^{2,0}(\widetilde{\mathsf{Z}}_A,\mathbf{Q})$ is orthogonal to $H^{0,2}(\widetilde{\mathsf{Z}}_A,\mathbf{Q})\oplus H^{1,1}(\widetilde{\mathsf{Z}}_A,\mathbf{Q})_{\mathrm{prim}}$ with respect to q'.

Since q' does not change under deformation, by varying the smooth Gushel–Mukai sixfold X, and thus A, the above orthogonality remains true for the locally complete family of polarized EPW cubes hence, by Theorem 1.12, q' is a constant multiple of the Beauville–Bogomolov–Fujiki quadratic form on $H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Q})$. □

Remark 6.21. We expect the above result to be true also with $\mathbf{Z}$ coefficients, giving a Hodge *isometry*

$$H^6(X,\mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z})_{\mathrm{prim}}. \tag{128}$$

Together with the Hodge isometry of [**DK19**, Theorem 5.19] between $H^6(X,\mathbf{Z})_{\mathrm{van}}$ and $H^2(\widetilde{\mathsf{Y}}_A,\mathbf{Z})_{\mathrm{prim}}$, this would give a geometric realization of the Hodge isometry

$$H^2(\widetilde{\mathsf{Y}}_A,\mathbf{Z})_{\mathrm{prim}} \xrightarrow{\sim} H^2(\widetilde{\mathsf{Z}}_A,\mathbf{Z})_{\mathrm{prim}}$$

proved in [**KKM24**, Theorem 1.1].

Let X be a GM sixfold containing a linear space of dimension 3. [**DK19**, Remark 5.29] shows that in this case the Lagrangian $[A]$ associated to X is in the divisor Δ. By Proposition 3.5, the period of the EPW cube $\widetilde{\mathsf{Z}}_A^{\geq 2}$ is in the divisor ${}^{[3]}\mathscr{D}^{(2)}_{4,10}$, which means that the lattice $h^\perp = H^2(\widetilde{\mathsf{Z}}_A^{\geq 2},\mathbf{Z})_{\mathrm{prim}}$ contains a class with square -10. If the isomorphism (128) holds true, there should be a corresponding class in $H^6(X,\mathbf{Z})_{\mathrm{van}}$. The next lemma shows the existence of this class.

Lemma 6.22. *Let X be a very general GM manifold of dimension 6 containing a linear space P of dimension 3. The class of P in $H^6(X,\mathbf{Z})$ is given by*

$$[P] = \frac{1}{2}\gamma_X^*\sigma_3 + c,$$

where the class $2c = [P] - [\sigma(P)]$ lies in $H^6(X,\mathbf{Z})_{\mathrm{van}}$ and has square -10.

Proof. Write

$$[P] = a\gamma_X^*\sigma_3 + b\gamma_X^*\sigma_{2,1} + c,$$

with $a, b \in \mathbf{Q}$ and $c \in H^6(X, \mathbf{Q})_{\mathrm{van}}$. In $\mathsf{Gr}(2, V_5)$, we have $\sigma_3^2 = \sigma_{2,1}^2 = 1$ and $\sigma_3 \cdot \sigma_{2,1} = 0$. Therefore, the projection formula gives

$$2a = [P] \cdot \gamma_X^*\sigma_3 = \gamma_{X,*}[P] \cdot \sigma_3 = \sigma_3 \cdot \sigma_3 = 1$$

and

$$2b = [P] \cdot \gamma_X^*\sigma_{2,1} = \gamma_{X,*}[P] \cdot \sigma_{2,1} = \sigma_3 \cdot \sigma_{2,1} = 0.$$

It follows from the proof of [**DK24**, Lemma A.1] that

$$N_{P/\mathsf{CGr}(2,V_5)} \simeq \mathcal{O}_P(1) \oplus T_P(-1),$$

hence we have an exact sequence

$$0 \to N_{P/X} \to \mathcal{O}_P(1) \oplus T_P(-1) \to \mathcal{O}_P(2) \to 0.$$

The Euler sequence for P gives $c(T_P(-1)) = (1-h)^{-1} = 1 + h + h^2 + h^3$, where h is the class of a hyperplane in P. Therefore the above short exact sequence gives the equality of Chern polynomials

$$c(N_{P/X})(1+2h) = (1+h)(1+h+h^2+h^3).$$

In particular, we obtain $c_1(N_{P/X}) = 0$, $c_2(N_{P/X}) = 2h^2$ and

$$[P]^2 = c_3(N_{P/X}) = -2,$$

hence $2a^2 + c^2 = -2$, or $c^2 = -2 - 1/2 = -5/2$. Therefore the class $2c = 2[P] - \gamma_X^*\sigma_3$ is integral and has square -10. □

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