BÉZOUT'S THEOREM FOR ABELIAN VARIETIES

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Abstract. This is joint work with Ben Moonen. The main result is that if A is an absolutely simple abelian variety over some field and X, Y are subvarieties of A, the dimension of their sum $X+Y$ is the expected one: it is equal to the minimum of $\dim(A)$ and $\dim(X)+\dim(Y)$. In characteristic 0, there is a very elementary geometric proof, but that argument breaks down in positive characteristic. Instead, we prove this result as a consequence of a theorem about perverse sheaves, building upon work of Krämer and Weissauer.

1. Bézout's theorem

We work over an arbitrary field $\mathbf{k}$ of any characteristic. Subvarieties are closed.

Theorem (Bézout)**.** *Let $X, Y \subseteq \mathbf{P}^g_{\mathbf{k}}$ be subvarieties of respective dimensions d and e. If $d+e \geq g$, then $X \cap Y \neq \varnothing$.*

This classical theorem is not very hard to prove (one only needs Krull's Hauptidealsatz).

Our main theorem deals with the analogous question for subvarieties of an abelian variety.

Theorem 1 (Debarre–Moonen)**.** *Let A be an absolutely simple abelian variety of dimension g over $\mathbf{k}$ and let $X, Y \subseteq A$ be subvarieties of respective dimensions d and e. We have*

$$\dim(X+Y) = \min\{d+e, g\}.$$

In particular, if $d+e \geq g$, then $X \cap Y \neq \varnothing$.

The simplicity of A is clearly necessary for the theorem to hold, but one actually does need the absolute simplicity. The theorem generalizes to the sum of any number of subvarieties $X_1, \ldots, X_r \subseteq A$.

There is a very simple argument in characteristic 0 (Prasad, Debarre) that goes as follows. The essential case is when $d+e \leq g$ and we want to show that the addition map $\sigma\colon X \times Y \twoheadrightarrow Z \coloneqq X+Y$ is generically finite. Suppose it is not. Choose $z \in Z^{\mathrm{reg}}$ and a

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curve $F \subseteq \sigma^{-1}(z)$. Set $C \coloneqq \mathrm{pr}_X(F)$; then $F \xrightarrow{\sim} C$ (the inverse is $x \mapsto (x, z-x)$) and, for all $x \in C$,

$$T_{C,x} \subseteq T_{X,x} \subseteq T_{X,x} + T_{Y,z-x} \subseteq T_{Z,z} \subsetneq T_{A,0}.$$

because $\dim(Z) < d + e \leq g$.

The contradiction then comes from the following elementary result (whose proof uses generic smoothness).

Lemma. *Assume* $\mathrm{char}(\mathbf{k}) = 0$ *and* A *simple. Let* $C \subseteq A$ *be a curve. Then* $\mathrm{Span}\{T_{C,x} \mid x \in C\} = T_{A,0}$.

In positive characteristic, the lemma is wrong (Abramovich): take a purely inseparable isogeny $f\colon A' \to A$ and a regular curve $C' \subseteq A'$ such that $T_{C',0}$ is not contained in $\mathrm{Ker}(T_{f,0})$. Then, $C \coloneqq f(C')$ is a generically regular curve such that, for all $x \in C^{\mathrm{reg}}$,

$$T_{C,x} \subseteq \mathrm{Im}(T_{f,0}) \subsetneq T_{A,0}.$$

2. Semismall morphisms

We were unable to prove Theorem 1 in positive characteristic in general by elementary methods. Instead, we will obtain it as a consequence of a more general result.

Definition. A proper morphism $f\colon S \to T$ is *semismall* if for all $n \geq 0$,

$$\dim\big(\{t \in T \mid \dim(f^{-1}(t))\} \geq n\}\big) \leq \dim(S) - 2n.$$

If S is irreducible and f is semismall, f is generically finite onto its image (take $n = 1$).

Theorem 2 (Debarre–Moonen)**.** *Let* A *be an absolutely simple abelian variety of dimension* g *over* $\mathbf{k}$ *and let* $X, Y \subseteq A$ *be irreducible subvarieties of respective dimensions* d *and* e*, with* $d + e \leq g$*. The addition map* $X \times Y \to A$ *is semismall.*

This second theorem obviously implies Theorem 1 when $d+e \leq g$ and it is not difficult to reduce the general case to this case.

The theorem also generalizes to the addition map of any number of subvarieties $X_1, \ldots, X_r \subseteq A$ such that the sum of their dimensions is $\leq g$.

Remark. An old theorem of Martens (1967) says that if C is a smooth projective curve of genus g and $d \leq g$, we have

$$\dim(W^n_d(C)) \leq d - 2n$$

for any $n \geq 0$. This is equivalent to saying that the addition map $C^d \to \mathrm{Jac}^d(C)$ is semismall (note that Martens does not assume that $\mathrm{Jac}(C)$ is simple).

In the same vein, it is not difficult to show, using results of Debarre–Fahlaoui, that if a curve C generates an abelian variety A of dimension g and $d \leq g$, the sum map $C^d \to A$ is semismall.

3. Perverse sheaves

Theorem 2 will in turn be a consequence of a more general theorem about perverse sheaves on abelian varieties. We take a few minutes to discuss them.

Let ℓ be a prime number invertible in the field $\mathbf{k}$. Let X be a variety over $\mathbf{k}$. The derived category $\mathsf{D}^b_c(X, \overline{\mathbf{Q}_\ell})$ of bounded constructible[1] complexes of sheaves of $\overline{\mathbf{Q}_\ell}$-modules on X is only triangulated, not abelian. We would like to define a subcategory $\mathbf{P}(X)$ of so-called perverse sheaves which is abelian and stable by the so-called Verdier duality (an analog for complexes of Poincaré duality).

A perverse sheaf on a variety X is a complex of sheaves, or more correctly, an object K of the derived category $\mathsf{D}^b_c(X, \overline{\mathbf{Q}_\ell})$. Perversity means that there are restrictions on the supports of the cohomology sheaves $\mathscr{H}^m(K)$, namely:

(P1) the support of the constructible sheaf $\mathscr{H}^m(K)$ has dimension at most $-m$, for all m;
(P2) the Verdier dual of K has the same property.

Perverse sheaves live in nonpositive degrees: $\mathscr{H}^m(K) = 0$ when $m \notin [-\dim(X), 0]$. The support of K is the union of the supports of the $\mathscr{H}^m(K)$.

Perverse sheaves form an abelian category stable by Verdier duality, where every object has finite length (it has a finite filtration with simple quotients).

Intersection complexes. An important example of simple perverse sheaves is the following. Let X be an irreducible proper variety over $\mathbf{k}$. Assume first that X is *smooth* over $\bar{\mathbf{k}}$. Then,

$$\mathrm{IC}_X \coloneqq \overline{\mathbf{Q}_\ell}[\dim(X)]$$

is a simple perverse sheaf on X called the intersection complex of X.

A fundamental feature of the theory of perverse sheaves (which makes it a generalization of intersection cohomology) is that there is a natural way of defining IC_X as a simple perverse sheaf on X even when X is singular as

$$\mathrm{IC}_X \coloneqq j_{!*}\overline{\mathbf{Q}_\ell}[\dim(X)],$$

where $j\colon U \hookrightarrow X$ is a dense open subset such that $U_{\bar{\mathbf{k}}}$ is smooth over $\bar{\mathbf{k}}$. The difficulty is to define the so-called intermediate extension $j_{!*}$ and to prove that the resulting object is perverse and independent of the choice of U. When X is reducible, IC_X is just the direct sum of the intersection complexes of its irreducible components.

If X is a (closed) subvariety of an ambiant variety A, with inclusion $i_X\colon X \hookrightarrow A$, the push-forward $i_{X*}\,\mathrm{IC}_X$ is a simple perverse sheaf on A (with support in X) which we still denote by IC_X.

Convolution. We go back to the abelian variety A defined over $\mathbf{k}$, with addition map $\sigma_A\colon A \times A \to A$. We define the convolution product on the category $\mathsf{D}^b_c(A, \overline{\mathbf{Q}_\ell})$ by the formula

$$K \star L \coloneqq R\sigma_{A*}(K \boxtimes L).$$

[1] A sheaf on X is constructible if X can be written as a finite union of locally closed subschemes on which the sheaf is locally constant. A complex of sheaves is constructible if it has constructible cohomology sheaves.

Theorem 3 (Debarre–Moonen)**.** *Let A be an absolutely simple abelian variety of dimension g over $\mathbf{k}$ and let $X, Y \subseteq A$ be subvarieties of respective dimensions d and e, with $d+e \leq g$. Let K be a perverse sheaf supported on X and let L be a perverse sheaf supported on Y. Then, $K \star L$ is perverse.*

To be precise, when $\mathrm{char}(\mathbf{k}) > 0$, we need K and L to be defined over a finitely generated subfield of $\mathbf{k}$.

Let us explain why Theorem 3 implies Theorem 2. For simplicity, I will restrict to the case where $\mathbf{k} = \bar{\mathbf{k}}$ and X and Y are smooth irreducible. We apply Theorem 3 to the perverse sheaves

$$K = \mathrm{IC}_X = i_{X*}\overline{\mathbf{Q}_\ell}[d] \quad \text{and} \quad \mathrm{L} = \mathrm{IC_Y} = \mathrm{i_{Y*}}\overline{\mathbf{Q}_\ell}[\mathrm{e}]$$

on A. Theorem 3 says that $\mathrm{IC}_X \star \mathrm{IC}_Y$ (which is the (derived) direct image by the addition map $\sigma\colon X \times Y \to A$ of the intersection complex $I_{X\times Y}$) is perverse, and we want to deduce that σ is semismall. The argument is very classical

Let z be a geometric point z of Z and set $F_z \coloneqq \sigma^{-1}(z) \subseteq X \times Y$. By proper base change, the stalk of the sheaf $\mathscr{H}^m(\mathrm{IC}_X \star \mathrm{IC}_Y)$ at z is isomorphic to

$$\begin{aligned} &H^m(F_z, (\mathrm{IC}_X \star \mathrm{IC}_Y)|_{F_z}) \\ &= H^m(F_z, i_{X*}\overline{\mathbf{Q}_\ell}[d] \boxtimes i_{Y*}\overline{\mathbf{Q}_\ell}[e])|_{F_z}) \\ &= H^m(F_z, \overline{\mathbf{Q}_\ell}[d+e]) \\ &= H^{m+d+e}(F_z, \overline{\mathbf{Q}_\ell}). \end{aligned}$$

Smoothness of X and Y was used for the first equality (when they are singular, their intersection complexes are more complicated).

Perversity of $\mathrm{IC}_X \star \mathrm{IC}_Y$ says that the locus of z where this is nonzero has dimension $\leq -m$. If $\dim(F_z) = n$, take $m = 2n - d - e$, so that $m + d + e = 2\dim(F)$. Then it is standard that this group is nonzero, and the locus where this happens has dimension $\leq -m = d + e - 2n$. This is exactly the definition of semismallness for σ.

Theorem 3 is based on work of Krämer and Weissauer. For simplicity, we assume $\mathbf{k} = \bar{\mathbf{k}}$ of positive characteristic p and all perverse sheaves are defined over a finitely generated extension of $\mathbf{F}_p$. Then, on any abelian variety A,

(a) For all perverse sheaves P on A, one has $\chi(P) \geq 0$.[2]
(b) If P is a simple perverse sheaf on A with $\chi(P) = 0$, the group scheme $\mathrm{Stab}(P) \coloneqq \{a \in A \mid \tau_a^* P \simeq P\}$ has positive dimension.
(c) If P and Q are perverse sheaves on A, the perverse cohomology sheaf ${}^{\mathrm{p}}H^i(P \star Q)$ has $\chi = 0$ for $i \neq 0$.

Let us prove the theorem when $d+e < g$ (one needs an extra argument when $d+e = g$). Assume that $K \star L$ is not perverse. Then, some ${}^{\mathrm{p}}H^i(P \star Q)$ with $i \neq 0$ is nonzero. By (c), it has $\chi = 0$, hence, by (a), so has some simple subquotient, whose stabilizer then has positive dimension by (b). When A is simple, it must be the whole of A. This implies that

[2]We set $\chi(P) \coloneqq \sum_{i\in\mathbf{Z}} (-1)^i \dim\bigl(H^i(A, P)\bigr)$.

the support of $K \star L$ is A; since it is contained in $X + Y$, this implies $d + e = g$, contradicting our hypothesis.

4. Further results

One would like to extend these results to subvarieties of any (possibly nonsimple) abelian variety. In characteristic zero, I was able in an 1995 article to find a necessary and sufficient condition for the addition map $X \times Y \to A$ to be surjective: this condition is that for any abelian quotient $\pi\colon A \to B$, one has $\dim(\pi(X)) + \dim(\pi(Y)) \geq \dim(B)$. This we can also extend to all characteristics. However, we do not know how to characterize the semismallness of the addition map, even in characteristic 0.

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