COMPLEX ABELIAN VARIETIES OF WEIL TYPE

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Abstract. This text is an introduction to so-called complex abelian varieties of Weil type. It is heavily based on [vG94].

1. Complex abelian varieties

1.1. **Cohomology, algebraic and Hodge classes.** A complex abelian variety A of dimension g can be written as a quotient $\mathbf{C}^g/\Gamma$, where $\Gamma \subset \mathbf{C}^g$ is a lattice (of rank $2g$). We have $\Gamma = H_1(A, \mathbf{Z})$. Set

$$V \coloneqq H_1(A, \mathbf{Q}) = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Note that

$$H_p(A, \mathbf{Q}) = \textstyle\bigwedge^p V.$$

Then $\mathbf{C}^g$, which can be identified the tangent space $T_{A,0}$ to A at the origin, is the real $2g$-dimensional vector space $V_{\mathbf{R}} = H_1(A, \mathbf{R})$ endowed with the structure of a complex vector space, defined by multiplication by i, which is an $\mathbf{R}$-linear endomorphism J such that $J^2 = -\operatorname{Id}$. We denote by the same letter the induced endomorphism on the dual space $V_{\mathbf{R}}^\vee = H^1(A, \mathbf{R})$, which defines the complex structure on $T_{A,0}^\vee$.

One has $H^1(A, \mathbf{C}) = \operatorname{Hom}_{\mathbf{R}}(V_{\mathbf{R}}, \mathbf{C}) = V_{\mathbf{R}}^\vee \otimes \mathbf{C}$. In the Hodge decomposition

$$H^1(A, \mathbf{C}) = H^{1,0}(A) \oplus H^{0,1}(A), \tag{1}$$

the subspace $H^{1,0}(A) = T_{A,0}^\vee$ is the eigenspace of $J_{\mathbf{C}}$ for the eigenvalue i, and $H^{0,1}(A) = \overline{H^{1,0}(A)}$ is the eigenspace for the eigenvalue $-i$. Both have complex dimension g. One has

$$H^{p,q}(A) = \textstyle\bigwedge^p H^{1,0}(A) \otimes \bigwedge^q H^{0,1}(A) \subset \bigwedge^{p+q} H^1(A, \mathbf{C}).$$

Given $0 \le p \le g$, we denote by $\mathsf{N}^p(A)$ the finite dimensional $\mathbf{Q}$-vector space of numerical equivalence classes of codimension p algebraic cycles on A with rational coefficients. Since homological and numerical equivalences coincide on A, it is a vector subspace

$$\mathsf{N}^p(A) \subseteq \mathsf{Hdg}^p(A) \coloneqq H^{p,p}(A) \cap H^{2p}(A, \mathbf{Q}) \subset H^{2p}(A, \mathbf{C})$$

of the $\mathbf{Q}$-vector space $\mathsf{Hdg}^p(A)$ of rational Hodge classes of degree p. We also define the subspace

$$\mathsf{D}^p(A) \ \subseteq \ \mathsf{N}^p(A)$$

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generated by products of p divisor classes. One has[1]

$$\mathsf{D}^1(A)=\mathsf{N}^1(A)=\mathsf{Hdg}^1(A)\quad,\quad \mathsf{D}^{g-1}(A)=\mathsf{N}^{g-1}(A)=\mathsf{Hdg}^{g-1}(A). \tag{2}$$

1.2. **The Hodge conjecture.** The *Hodge (p,p)-conjecture* for A says

$$\mathsf{N}^p(A)=\mathsf{Hdg}^p(A).$$

It holds for $p\le 1$ or $p\ge g-1$ by (2), and also if $\mathsf{D}^p(A)=\mathsf{Hdg}^p(A)$.

Any isogeny $B\to A$ induces isomorphisms $H^\bullet(A,\mathbf{Q})\xrightarrow{\sim} H^\bullet(B,\mathbf{Q})$ of Hodge structures, and isomorphisms $\mathsf{N}^p(A)\xrightarrow{\sim}\mathsf{N}^p(B)$, $\mathsf{Hdg}^p(A)\xrightarrow{\sim}\mathsf{Hdg}^p(B)$, and $\mathsf{D}^p(A)\xrightarrow{\sim}\mathsf{D}^p(B)$ for all p. Therefore, the Hodge (p,p)-conjecture holds for A if and only if it holds for B.

Here are a couple of results on the Hodge conjecture for complex abelian varieties (see [Ma58], [Tt65], [T82], [R83]).

Theorem 1 (Mattuck)**.** *For a very general polarized abelian variety A of dimension g, one has*

$$\mathsf{D}^p(A)=\mathsf{Hdg}^p(A)=\mathbf{Q}$$

for all $p\in\{0,\dots,g\}$. In particular, the Hodge (p,p)-conjecture holds for A for all p.

Theorem 2 (Tate)**.** *For an abelian variety A which is isogeneous to a product of g elliptic curves, one has*

$$\mathsf{D}^p(A)=\mathsf{Hdg}^p(A)$$

for all $p\in\{0,\dots,g\}$. In particular, the Hodge (p,p)-conjecture holds for A for all p.

Theorem 3 (Tankeev, Ribet)**.** *For a simple abelian variety A of prime dimension g, one has*

$$\mathsf{D}^p(A^m)=\mathsf{Hdg}^p(A^m)$$

for all $m\ge 1$ and all $p\in\{0,\dots,mg\}$. In particular, the Hodge (p,p)-conjecture holds for A^m for all m and all p.

Contrary to what these results suggest, it is not always true that $\mathsf{D}^p(A)=\mathsf{Hdg}^p(A)$ for all p (although this holds for $g\le 3$ by (2)). A 4-dimensional abelian variety A for which $\mathsf{D}^2(A)\subsetneq\mathsf{Hdg}^2(A)$ was first constructed by Mumford. His example was later generalized by Weil and our aim is to explain Weil's construction.

More precisely, Weil constructed, for abelian varieties A of even dimension $2n$ satisfying certain conditions (see Theorem 16), elements of $\mathsf{Hdg}^n(A)$ that are not in $\mathsf{D}^n(A)$. We call them Weil–Hodge classes.

Moonen–Zarhin proved in [MZ99a, Theorem (0.1)] that for any four-dimensional abelian variety A, the Hodge ring is generated by divisor classes and Weil–Hodge classes[2] (unless A is isogeneous to a product of two abelian surfaces, in which case the Hodge conjecture was verified in [RM08, Theorem 4.11]). To prove the Hodge conjecture for four-dimensional abelian

[1] The first equalities follow from the Lefschetz Hyperplane Theorem. For the second equalities, let $h\in\mathsf{N}^1(A)$ be an ample class on A. By the Hard Lefschetz theorem, the cup-product map $\cdot\, h^{g-2}\colon H^2(A,\mathbf{Q})\to H^{2g-2}(A,\mathbf{Q})$ is an isomorphism of Hodge structures. It sends $\mathsf{D}^1(A)=\mathsf{N}^1(A)$ into $\mathsf{D}^{g-1}(A)\subset\mathsf{N}^{g-1}(A)$ and $\mathsf{Hdg}^1(A)$ onto $\mathsf{Hdg}^{g-1}(A)$. This proves what we want.

[2] For all possible structures of Weil type on A; see Definition 10.

varieties, it is therefore enough to prove that Weil–Hodge classes are algebraic (see Section 2.8 for further comments).

1.3. **Polarizations.** Divisor classes on A lie in the Néron–Severi group

$$\mathsf{NS}(A) \coloneqq H^2(A,\mathbf{Z}) \cap H^{1,1}(A),$$

a free abelian group of finite rank, called the Picard number of A.

We view such a class as an element of $H^2(A,\mathbf{Q}) = \bigwedge^2 V^\vee$, that is, as a skew-symmetric form E on V, which we extend to $V_\mathbf{R}$. The fact that E is of type $(1,1)$ means that $E(J(x),J(y)) = E(x,y)$ for all $x,y \in V_\mathbf{R}$.

A *polarization* on A is the class of an ample divisor. In terms of the associated form E, this means

$$\forall x \in V_\mathbf{R} \smallsetminus \{0\} \qquad E(x,J(x)) > 0.$$

This implies that E is nondegenerate.

1.4. **The endomorphism algebra.** The *endomorphism algebra* of A is the finite dimensional semisimple $\mathbf{Q}$-algebra

$$\mathrm{End}^0(A) \coloneqq \mathrm{End}(A) \otimes_\mathbf{Z} \mathbf{Q}.$$

It acts on $H^\bullet(A,\mathbf{Q})$ by pullback and preserves the subspaces $\mathsf{D}^p(A)$, $\mathsf{N}^p(A)$, and $\mathsf{Hdg}^p(A)$. Any isogeny $f\colon B \to A$ induces an isomorphism $\mathrm{End}^0(A) \xrightarrow{\sim} \mathrm{End}^0(B)$ given by $u \mapsto f^{-1}uf$.[3]

We consider

- the analytic (faithful) representation $\rho_a\colon \mathrm{End}^0(A) \hookrightarrow \mathrm{End}_\mathbf{C}(T_{A,0})$;
- the rational (faithful) representation $\rho_r\colon \mathrm{End}^0(A) \hookrightarrow \mathrm{End}_\mathbf{Q}(V)$, which is equivalent to the representation $\rho_a \oplus \overline{\rho_a}$ ([BL04, Proposition 5.1.2.a)]).

We will suppress ρ_a and ρ_r from the notation when it causes no confusion.

2. Abelian varieties of Weil type

2.1. **Definition.** Let $\mathbf{K} \coloneqq \mathbf{Q}(\sqrt{-d})$ be a quadratic imaginary number field (with d a positive integer). We let $e_0 \coloneqq \sqrt{-d} \in \mathbf{K}$.

Definition 4. An abelian variety *with multiplication by* $\mathbf{K}$ is an abelian variety A together with an embedding $\mathbf{K} \hookrightarrow \mathrm{End}^0(A)$.

Since its minimal polynomial t^2+d is separable, the endomorphism $\rho_a(e_0)$ of $T_{A,0}$ is diagonalizable and we write (recall $V = H_1(A,\mathbf{Q})$)

$$T_{A,0} = V_\mathbf{R} = V_+ \oplus V_-,$$

[3] There is an integer $m>0$ such that the kernel of f is contained in the kernel of the multiplication m_B by m on B. There is then an isogeny $g\colon A \to B$ such that $gf = m_B$ and we set $f^{-1} \coloneqq \frac{1}{m}g$. This is independent of the choice of m.

where the eigenspaces V_+ and V_- for the eigenvalues e_0 and $\overline{e_0}$ are complex vector subspaces of $T_{A,0}$ of respective dimensions n_+ and n_- (or J-stable $\mathbf{R}$-vector spaces of dimensions $2n_+$ and $2n_-$). The endomorphism $\rho_a(e_0)$ acts on $V_\pm$ as $\pm\sqrt{d}J$; it commutes with J.

For any $e \in \mathbf{K}$, the characteristic polynomial of the $\mathbf{C}$-linear endomorphism $\rho_a(e)$ is $P_e(t) = (t-e)^{n_+}(t-\bar{e})^{n_-}$ and the characteristic polynomial of the $\mathbf{Q}$-linear endomorphism $\rho_r(e)$ of V is $P_e(t)\overline{P_e}(t) = (t-e)^g(t-\bar{e})^g$.

The $\mathbf{K}$-vector space $H^1(A,\mathbf{K}) = \mathrm{Hom}_{\mathbf{Q}}(V,\mathbf{K})$ decomposes as

$$H^1(A,\mathbf{K}) = W \oplus \overline{W}, \tag{3}$$

where W and $\overline{W}$ are conjugate g-dimensional $\mathbf{K}$-vector subspaces and e acts on W by multiplication by e and on $\overline{W}$ by multiplication by $\bar{e}$.

The Hodge decomposition (1) splits as

$$H^1(A,\mathbf{C}) = \underbrace{V_+^\vee \oplus V_-^\vee}_{H^{1,0}(A)} \oplus \underbrace{\overline{V_+^\vee} \oplus \overline{V_-^\vee}}_{H^{0,1}(A)},$$

where $V_\pm^\vee := \mathrm{Hom}_{\mathbf{C}}(V_\pm,\mathbf{C})$ and e acts by multiplication by e on $V_+^\vee \oplus \overline{V_-^\vee}$ and by multiplication by $\bar{e}$ on $V_-^\vee \oplus \overline{V_+^\vee}$. Therefore,

$$V_+^\vee \oplus \overline{V_-^\vee} = W \otimes_{\mathbf{K}} \mathbf{C} \quad , \quad V_-^\vee \oplus \overline{V_+^\vee} = \overline{W} \otimes_{\mathbf{K}} \mathbf{C} \tag{4}$$

are both defined over $\mathbf{K}$.

2.2. **Polarizations.** We prove the existence, on any abelian variety with multiplication by $\mathbf{K}$, of a "compatible" polarization.

Lemma 5. *Let $(A,\mathbf{K})$ be an abelian variety with multiplication by $\mathbf{K}$. There exists a polarization E on A such that*

$$\forall e \in \mathbf{K} \quad \forall x,y \in V_{\mathbf{R}} \qquad E(e(x),e(y)) = e\bar{e}\,E(x,y). \tag{5}$$

The triple $(A,\mathbf{K},E)$ is then called a polarized abelian variety with multiplication by $\mathbf{K}$.

The relation (5) is equivalent to

$$\forall e \in \mathbf{K} \quad \forall x,y \in V_{\mathbf{R}} \qquad E(e(x),y) = E(x,\bar{e}(y)).$$

Proof. The divisor class e_0^*E defined by $e_0^*E(x,y) = E(e_0(x),e_0(y))$ is ample, hence so is $dE + e_0^*E$. It satisfies

$$e_0^*(dE + e_0^*E) = de_0^*E + d^2E = d(dE + e_0^*E).$$

In other words, the equality (5) then holds for $dE + e_0^*E$ and $e = e_0$, and one checks that it then holds for all e in $\mathbf{K}$. $\square$

2.3. **Discriminant.** The embedding $\mathbf{K} \hookrightarrow \mathrm{End}^0(A)$ and the rational representation ρ_r make V into a $\mathbf{K}$-vector space of dimension g. We construct a $\mathbf{K}$-Hermitian form ($\mathbf{K}$-linear in the second variable) on V.

We let $\mathrm{Nm}\colon \mathbf{K} \to \mathbf{Q}_{\geq 0},\ a \mapsto a\bar{a}$, be the norm map. The multiplicative group $\mathbf{Q}^\times / \mathrm{Nm}(\mathbf{K}^\times)$ is an infinite 2-torsion group.[4]

Lemma 6. *Let $(A, \mathbf{K}, E)$ be a polarized abelian variety with multiplication by $\mathbf{K}$.*

(a) *The map*

$$H\colon V \times V \longrightarrow \mathbf{K}, \qquad (x, y) \longmapsto E(x, e_0 y) + e_0 E(x, y)$$

defines a nondegenerate $\mathbf{K}$-Hermitian form H on the $\mathbf{K}$-vector space V with signature (n_+, n_-). Its $\mathbf{R}$-linear extension to $V_{\mathbf{R}}$ is positive definite on V_+ and negative definite on V_- and these two subspaces are H-orthogonal.

(b) *Let $M \in \mathscr{M}_{2n}(\mathbf{K})$ be the Hermitian matrix of H in a $\mathbf{K}$-basis of V. Then*[5]

$$(-1)^{n_-} \det(M) \in \mathbf{Q}^\times / \mathrm{Nm}(\mathbf{K}^\times)$$

does not depend on the choice of the basis. It is positive and is an isogeny invariant of $(A, \mathbf{K}, E)$ and will be called its discriminant, *denoted by* $\mathrm{disc}(A, \mathbf{K}, E)$.

Proof. One easily checks

$$H(x, e_0 y) = E(x, -dy) + e_0 E(x, e_0 y) = -dE(x, y) + e_0 E(x, e_0 y) = e_0 H(x, y),$$

so that H is $\mathbf{K}$-linear in the second variable. Then,

$$H(y, x) = E(y, e_0 x) + e_0 E(y, x) = E(-e_0 y, x) - e_0 E(x, y) = E(x, e_0 y) - e_0 E(x, y) = \overline{H(x, y)}.$$

If $x \in V_\pm$, we have

$$H(x, x) = E(x, e_0 x) = E(x, \pm\sqrt{d} J(x)) = \pm\sqrt{d}\, E(x, J(x))$$

and this has the sign $\pm$ by positivity of E. Finally, if $x \in V_+$ and $y \in V_-$, we have $H(e_0 x, e_0 y) = \bar{e}_0 e_0\, H(x, y) = d\, H(x, y)$ because H is Hermitian, but also

$$H(e_0 x, e_0 y) = H(\sqrt{d} J(x), -\sqrt{d} J(y)) = -dH(J(x), J(y)) = -d\, H(x, y).$$

Therefore, $H(x, y) = 0$ and (a) is proved.

For (b), note that changing bases transforms M into $\overline{P}^T M P$ for some $P \in \mathrm{GL}_{2n}(\mathbf{K})$, hence $\det(M)$ into $\det(M)\,\mathrm{Nm}(\det(P))$. Since H is positive definite on the n-dimensional subspace V_+, negative definite on the n-dimensional subspace V_-, and V_+ and V_- are H-orthogonal, we see that $(-1)^{n_-} \det(M) > 0$.

Finally, if $f\colon B \to A$ is an isogeny, $(B, \mathbf{K}, f^* E)$ is a polarized abelian variety with multiplication by $\mathbf{K}$ and the discriminants are the same. ∎

[4] If $a \in \mathbf{Q}^\times$, then $a^2 = \mathrm{Nm}(a) \in \mathrm{Nm}(\mathbf{K}^\times)$, so this group is 2-torsion. One checks that if p is prime and $pr \in \mathrm{Nm}(\mathbf{K}^\times)$ for some $r \in \mathbf{Z}_{>0}$, then $-d$ is a square modulo p. Assume for simplicity that d is a product $p_1 \cdots p_m$ of odd primes. Take $p \equiv 1 \pmod 4$. We want $\bigl(\frac{-p_1 \cdots p_m}{p}\bigr) = \bigl(\frac{p}{p_1}\bigr) \cdots \bigl(\frac{p}{p_m}\bigr)$ (quadratic reciprocity law) to be equal to -1. Dirichlet's theorem on arithmetic progressions implies that there are infinitely many such primes p, which give infinitely many different elements of $\mathbf{Q}^\times / \mathrm{Nm}(\mathbf{K}^\times)$.

[5] There are various normalizations for defining the discriminant. We chose the factor $(-1)^{n_-}$ in order to make it positive.

2.4. **Examples.** We construct three families of polarized abelian varieties with multiplication by a quadratic imaginary number field, all with $n_+ = n_-$ (this is the case of interest for what we want to do).

Example 7 (Polarized abelian varieties of dimension $2n$ with multiplication by $\mathbf{K} := \mathbf{Q}(\sqrt{-1})$ and discriminant 1). Principally polarized abelian varieties of dimension $2n$ can be described as $A := \mathbf{C}^{2n}/\Lambda$, with $\Lambda := \mathbf{Z}^{2n} \oplus \tau\mathbf{Z}^{2n}$, where $\tau \in \mathscr{M}_{2n}(\mathbf{C})$ is symmetric with $\mathrm{Im}(\tau) > 0$. The polarization $E\colon \Lambda \times \Lambda \to \mathbf{Z}$ is given by the matrix

$$\begin{pmatrix} 0_{2n} & I_{2n} \\ -I_{2n} & 0_{2n} \end{pmatrix},$$

that is,

$$E(x + \tau y, x' + \tau y') = x^T y' - y^T x'$$

(we write vectors as column matrices). An endomorphism of A is a $\mathbf{C}$-linear map $u\colon \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ such that $u(\Lambda) = \Lambda$. Choose u with matrix

$$\begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

with characteristic polynomial $(t^2 + 1)^n$. Then $u^2 = -I_{2n}$ and the condition $u(\Lambda) = \Lambda$ is equivalent to

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ -\tau_2 & \tau_1 \end{pmatrix}$$

(with τ_1 symmetric, τ_2 skew-symmetric, and $\mathrm{Im}(\tau) > 0$). Then, $u\tau = \tau u$ and this easily implies that u acts on A and preserves E. The triple $(A, \mathbf{Q}(\sqrt{-1}), E)$ is therefore a polarized abelian variety with multiplication by $\mathbf{Q}(\sqrt{-1})$, with $n_+ = n_- = n$.

We compute the discriminant in the $\mathbf{Q}(\sqrt{-1})$-basis

$$\left(\begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} e_n \\ 0 \end{pmatrix}, \tau\begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \dots, \tau\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right)$$

of $\mathbf{C}^{2n}$, where $e_1, \dots, e_n$ are the n first coordinate vectors in $\mathbf{Z}^{2n}$. We have

$$\begin{aligned} H\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau\begin{pmatrix} x'_1 \\ 0 \end{pmatrix}\right) &= E\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, u\tau\begin{pmatrix} x'_1 \\ 0 \end{pmatrix}\right) + i\, E\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau\begin{pmatrix} x'_1 \\ 0 \end{pmatrix}\right) \\ &= E\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau u\begin{pmatrix} x'_1 \\ 0 \end{pmatrix}\right) + i\, (x_1^T\ 0)\begin{pmatrix} x'_1 \\ 0 \end{pmatrix} \\ &= E\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \tau\begin{pmatrix} 0 \\ x'_1 \end{pmatrix}\right) + i\, x_1^T x'_1 = i\, x_1^T x'_1 \end{aligned}$$

and, with further analogous computations, we see that the matrix of H in this basis is

$$\begin{pmatrix} 0_n & iI_n \\ -iI_n & 0_n \end{pmatrix}$$

whose determinant is $(-1)^n$. The discriminant is thus 1. The entire construction depends on n^2 complex parameters ($n(n+1)/2$ for τ_1 and $n(n-1)/2$ for τ_2). See Section 2.7 for a generalization.

Example 8 (Prym varieties of cyclic triple étale coverings of curves). Let C be a smooth connected projective curve. Let $\widetilde{C} \to C$ be a connected étale cyclic triple cover, with $\sigma \in \mathrm{Aut}(\widetilde{C})$ a generator of the covering group. There is a norm map $\mathrm{Jac}(\widetilde{C}) \to \mathrm{Jac}(C)$ whose kernel has 3 connected components, each of dimension $2g(C) - 2 =: 2n$. The neutral component $P \subset \mathrm{Jac}(\widetilde{C})$ (the *generalized Prym variety* of the covering) has an automorphism σ^* of order 3, which gives rise to an embedding $\mathbf{Q}(\sqrt{-3}) \subset \mathrm{End}^0(P)$ with $n_+ = n_- = n$. One checks that the polarization E on P induced by the canonical polarization of $\mathrm{Jac}(\widetilde{C})$ is compatible and that the discriminant is 1. The construction depends on $3n$ complex parameters (for the choice of the triple cover).

Example 9 (Products). Let A be an abelian variety of dimension n. The element $e_0 = \sqrt{-d}$ of $\mathbf{K} = \mathbf{Q}(\sqrt{-d})$ acts on the product $A \times A$ by the matrix $\begin{pmatrix} 0 & -d_A \\ 1_A & 0 \end{pmatrix}$ and this endows $A \times A$ with multiplication by $\mathbf{K}$. The characteristic polynomial of $\rho_a(e_0)$ is $(t^2 + d)^n$, so we have $n_+ = n_- = n$. The construction depends on $n(n+1)/2$ complex parameters (for the choice of A).

2.5. **Abelian varieties of Weil type and Weil–Hodge classes.** Let A be an abelian variety of dimension g with multiplication by $\mathbf{K}$ and consider the decomposition

$$H^1(A, \mathbf{K}) = W \oplus \overline{W}$$

from (3) into a direct sum of two g-dimensional $\mathbf{K}$-subspaces. The 2-dimensional $\mathbf{K}$-subspace

$$\bigwedge^g W \oplus \bigwedge^g \overline{W} \subset \bigwedge^g H^1(A, \mathbf{K}) = H^g(A, \mathbf{K})$$

is stable by conjugation hence is defined over $\mathbf{Q}$. In other words, there is 2-dimensional $\mathbf{Q}$-subspace

$$\mathsf{WH}(A, \mathbf{K}) \subset H^g(A, \mathbf{Q})$$

such that $\bigwedge^g W \oplus \bigwedge^g \overline{W} = \mathsf{WH}(A, \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{K}$.

From (4), we obtain

$$\Big(\bigwedge^g W\Big) \otimes_{\mathbf{K}} \mathbf{C} = \bigwedge^g \big(V_+^\vee \oplus \overline{V_-^\vee}\big) = \bigwedge^{n_+} V_+^\vee \otimes \bigwedge^{n_-} \overline{V_-^\vee}. \tag{6}$$

This subspace of $H^g(V, \mathbf{C})$ is therefore of Hodge type (n_+, n_-). Similarly, the subspace $\bigwedge^g \overline{W}$ of $H^g(V, \mathbf{C})$ is of Hodge type (n_-, n_+).

The case $n_+ = n_- = g/2$ is particularly interesting, because the space $\mathsf{WH}(A, \mathbf{K})$ is then contained in $\mathsf{Hdg}^{g/2}(A)$. This motivates the following definition, central to these notes (the concept was first introduced by Weil in [W77]).

Definition 10. An *abelian variety of Weil type* is an abelian variety A of even dimension $2n$ together with an embedding $\mathbf{K} \hookrightarrow \mathrm{End}^0(A)$ such that for all $e \in \mathbf{K}$, the $\mathbf{C}$-linear endomorphism $\rho_a(e)$ has characteristic polynomial $(t - e)^n (t - \bar{e})^n$.

In other words, we have $n_+ = n_- = n$ in our notation. The construction above produces a 2-dimensional $\mathbf{Q}$-subspace

$$\mathsf{WH}(A, \mathbf{K}) \subset \mathsf{Hdg}^n(A)$$

which we call the space of *Weil–Hodge classes.*[6]

If E is a compatible polarization on A in the sense of Lemma 5, we will say that the triple $(A, \mathbf{K}, E)$ is a *polarized abelian variety of Weil type.*

Remark 11. The space $\mathsf{WH}(A,\mathbf{K}) \subset H^{2n}(A,\mathbf{Q}) = \bigwedge^{2n} V^\vee$ is the space of alternating $2n$-$\mathbf{Q}$-multilinear forms ψ on V such that, for all $v_1, \dots, v_{2n} \in V$ and all $e \in \mathbf{K}$, one has

$$\psi(ev_1, v_2, \dots, v_{2n}) = \psi(v_1, ev_2, \dots, v_{2n}) = \cdots = \psi(v_1, v_2, \dots, ev_{2n}). \tag{7}$$

A structure of $\mathbf{K}$-vector space (of dimension 1) is given by setting

$$(e\psi)(v_1, v_2, \dots, v_{2n}) = \psi(ev_1, v_2, \dots, v_{2n}).$$

Remark 12. The restriction of the intersection form to $\mathsf{WH}(A,\mathbf{K})$ is nondegenerate. Indeed, in the decomposition $\mathsf{WH}(A,\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{K} = \bigwedge^g W \oplus \bigwedge^g \overline{W}$, one sees that both subspaces are isotropic for the intersection form and that their wedge product $\bigwedge^g W \otimes \bigwedge^g \overline{W}$ is nonzero, because $H^1(A,\mathbf{K}) = W \oplus \overline{W}$, hence $H^{2g}(A,\mathbf{K}) = \bigwedge^g W \otimes \bigwedge^g \overline{W}$.

Proposition 13. *Let $(A, \mathbf{K}, E)$ be a polarized abelian variety of Weil type of dimension $2n$. We have $\mathbf{Q}E^n \cap \mathsf{WH}(A,\mathbf{K}) = \{0\}$ and $E^n \cdot \mathsf{WH}(A,\mathbf{K}) = 0$.*

Proof. We use a trick from the proof of [MZ99b, Theorem 4.4]. Let $e \in \mathbf{K}$ be such that $e\bar{e} = 1$ (there are infinitely many such units in $\mathbf{K}$), but e is not a $2n^{\text{th}}$-root of unity (there are only finitely many such roots in $\mathbf{K}$). By (5), we have

$$\forall x, y \in V \qquad E(ex, ey) = E(x, y).$$

Therefore, the $2n$-multilinear form $E^n = \bigwedge^n E$ in $\mathsf{D}^n(A)$ satisfies

$$\forall x_1, \dots, x_{2n} \in V \qquad E^n(ex_1, \dots, ex_{2n}) = E^n(x_1, \dots, x_{2n}).$$

If E^n were a Weil–Hodge class, we would have, by (7),

$$E^n(ex_1, \dots, ex_{2n}) = E^n(e^{2n}x_1, x_2, \dots, x_{2n})$$

hence,

$$E^n((e^{2n} - 1)x_1, x_2, \dots, x_{2n}) = 0.$$

But $e^{2n} - 1$ is nonzero, hence is invertible in $\mathbf{K}$, and we deduce $E^n = 0$, which is absurd.

We use the same trick to prove the second statement: the space $H^{4n}(A,\mathbf{Q}) = \bigwedge^{4n} V^\vee$ is spanned by E^{2n}, which satisfies $E^{2n}(ex_1, \dots, ex_{4n}) = E^{2n}(x_1, \dots, x_{4n})$. However, if $\psi \in \mathsf{WH}(A,\mathbf{K})$, the form $E^n \wedge \psi$ satisfies $(E^n \wedge \psi)(ex_1, \dots, ex_{4n}) = (E^n \wedge e^{2n}\psi)(x_1, \dots, x_{4n})$. Since it is proportional to E^{2n}, it must also satisfy $(E^n \wedge \psi)(ex_1, \dots, ex_{4n}) = (E^n \wedge \psi)(x_1, \dots, x_{4n})$, so that $E^n \wedge e^{2n}\psi = E^n \wedge \psi$, or $E^n \wedge (e^{2n} - 1)\psi = 0$. Since $e^{2n} - 1$ is invertible in $\mathbf{K}$, this implies $E^n \wedge \psi = 0$ for all $\psi \in \mathsf{WH}(A,\mathbf{K})$. ☐

Remark 14. Assume $\mathbf{K} = \mathrm{End}^0(A)$ (this holds when $(A, \mathbf{K}, E)$ is "very general" of dimension $2n \geq 4$ by Theorem 16). The polarization E defines an injective $\mathbf{Q}$-linear map

$$\mathsf{N}^1(A) \hookrightarrow \mathrm{End}^0(A)$$

[6] There are many other ways to define this subspace (see for example [W77], [D82, Section 4], [ZM99, Section 2], [vG94, Section 4.9]).

whose image is (in general) the vector subspace of elements invariant by the so-called Rosati involution. In our case, the Rosati involution is the Galois involution of $\mathbf{K}$ over $\mathbf{Q}$, hence $\mathsf{N}^1(A)$ has rank 1: all divisor classes are proportional to E. The proposition therefore implies $\mathsf{D}^n(A) \cap \mathsf{WH}(A, \mathbf{K}) = \{0\}$. The latter equality does not always hold: some abelian varieties A of Weil type are isogeneous to a product of elliptic curves, in which case $\mathsf{D}^n(A) = \mathsf{Hdg}^n(A)$ by Theorem 2.

2.6. **Products.** Let $(A', \mathbf{K}, E')$ and $(A'', \mathbf{K}, E'')$ be polarized abelian varieties of Weil type of respective dimensions n' and n''. The diagonal embedding

$$\mathbf{K} \hookrightarrow \operatorname{End}^0(A') \times \operatorname{End}^0(A'') \subset \operatorname{End}^0(A' \times A'')$$

makes $A := A' \times A''$ into an abelian variety of Weil type. The product polarization $E := E' \boxtimes E''$ is compatible and

$$\operatorname{disc}(A, \mathbf{K}, E) = \operatorname{disc}(A', \mathbf{K}, E') \cdot \operatorname{disc}(A'', \mathbf{K}, E'').$$

If $\psi'' \in H^{n'',n''}(A'')$ and $\psi \in H^{n'+n'',n'+n''}(A)$ are Weil–Hodge classes, then

$$\operatorname{pr}_{A'*}(\psi \cdot \operatorname{pr}_{A''}^*(\psi'')) \in H^{n',n'}(A')$$

is also a Weil–Hodge class.

Lemma 15. *In the situation and notation above, the classes* $\operatorname{pr}_{A'*}(\psi \cdot \operatorname{pr}_{A''}^*(\psi''))$ *span the vector space* $\mathsf{WH}(A', \mathbf{K})$.

In particular, if all Weil–Hodge classes on A *and* A'' *are algebraic, so are all Weil–Hodge classes on* A'.

Proof. The first item follows from the fact that a basis of $\mathsf{WH}(A, \mathbf{K})$ is given by $(\operatorname{pr}_A^*\psi_1' \otimes \operatorname{pr}_{A''}^*\psi_1'', \operatorname{pr}_{A'}^*\psi_2' \otimes \operatorname{pr}_{A''}^*\psi_2'')$, where (ψ_1', ψ_1'') and (ψ_2', ψ_2'') are respective bases for $\mathsf{WH}(A', \mathbf{K})$ and $\mathsf{WH}(A'', \mathbf{K})$, and the fact that the intersection form is nondegenerate on $\mathsf{WH}(A'', \mathbf{K})$ (Remark 12).

The second item follows since the class $\operatorname{pr}_{A'*}(\psi \cdot \operatorname{pr}_{A''}^*(\psi''))$ is algebraic if both ψ and ψ'' are. □

This lemma can be useful in the following situation: fix a quadratic imaginary number field $\mathbf{K}$ and assume that one can prove that Weil–Hodge classes are algebraic for all polarized abelian varieties $(A, \mathbf{K}, E)$ of Weil type of dimension $2n$ and some fixed discriminant δ_0. Then Weil–Hodge classes are algebraic for all polarized abelian varieties $(A', \mathbf{K}, E')$ of Weil type of dimension $2n-2$ and any discriminant δ: just take in the construction above for $(A'', \mathbf{K}, E'')$ a polarized abelian surface of Weil type and discriminant $\delta\delta_0$.

2.7. **Moduli spaces.** Given a quadratic imaginary number field $\mathbf{K} := \mathbf{Q}(\sqrt{-d})$, a positive integer n, and a positive class $\delta \in \mathbf{Q}^\times / \operatorname{Nm}(\mathbf{K}^\times)$, we construct a family of polarized abelian varieties $(A, \mathbf{K}, E)$ of Weil type of dimension $2n$ and discriminant δ parametrized by a connected analytic complex manifold of dimension n^2, such that any polarized abelian variety $(A, \mathbf{K}, E)$ of Weil type of dimension $2n$ and discriminant δ is isogeneous to some member of this family.

We start from a $\mathbf{K}$-vector space $V_{\mathbf{K}}$ of dimension $2n$ with a $\mathbf{K}$-Hermitian form of signature (n,n). The classification theory of these forms tells us that any such form H with discriminant δ can be written, in a suitable basis of $V_{\mathbf{K}}$, as

$$H(z,w) = \delta \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n - \bar{z}_{n+1} w_{n+1} - \cdots - \bar{z}_{2n} w_{2n}.$$

Since $\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{C}$, the real vector space $V_{\mathbf{R}} \coloneqq V_{\mathbf{K}} \otimes_{\mathbf{Q}} \mathbf{R}$ is a complex vector space (of dimension $2n$) for the complex structure i given by multiplication by $\sqrt{-d} \otimes (1/\sqrt{d})$. The polarization $E\colon V_{\mathbf{K}} \times V_{\mathbf{K}} \to \mathbf{Q}$ is the skew-symmetric form $\operatorname{Im}(H)$ (here $V_{\mathbf{K}}$ is seen as a $\mathbf{Q}$-vector space of dimension $4n$). We fix a lattice $\Gamma \subset V_{\mathbf{K}}$ on which E takes integral values.

The abelian varieties that we construct are all $V_{\mathbf{R}}/\Gamma$, but the complex structure J on $V_{\mathbf{R}}$ varies. To define J, we choose a complex subspace $V_+ \subset (V_{\mathbf{R}}, i)$ of dimension n on which H (extended $\mathbf{R}$-linearly) is positive definite and we define $V_- \coloneqq V_+^{\perp}$. Then,

$$V_{\mathbf{R}} = V_+ \oplus V_-$$

and we define J to be multiplication by i on V_+ and multiplication by $-i$ on V_-.

The pair $(V_{\mathbf{R}}, J)$ is a complex vector space, with $V_\pm \subset V_{\mathbf{R}}$ complex subspaces. We let A be its quotient by Γ. To prove that it is an abelian variety, we show that E defines a polarization. Let $x, y \in V_{\mathbf{R}}$, which we write as $x = x_+ + x_-$ and $y = y_+ + y_-$, with $x_\pm, y_\pm \in V_\pm$. We have, using the fact that V_+ and V_- are H-orthogonal,

$$\begin{aligned} E(x,y) &= \operatorname{Im}(H(x_+ + x_-, y_+ + y_-)) = \operatorname{Im}(H(x_+, y_+) + H(x_-, y_-)),\\ E(J(x), J(y)) &= \operatorname{Im}(H(ix_+ - ix_-, iy_+ - iy_-)) = \operatorname{Im}(H(x_+, y_+) + H(x_-, y_-)), \end{aligned}$$

so that $E(J(x), J(y)) = E(x,y)$. Moreover,

$$\begin{aligned} E(x, J(x)) &= \operatorname{Im}(H(x_+ + x_-, ix_+ - ix_-))\\ &= \operatorname{Im}(iH(x_+, x_+) - iH(x_-, x_-))\\ &= H(x_+, x_+) - H(x_-, x_-) \qquad \text{(since } H(x_\pm, x_\pm) \text{ is real).} \end{aligned}$$

This is positive if $x \neq 0$, because $H(x_+, x_+) > 0$ (unless $x_+ = 0$) and $H(x_-, x_-) < 0$ (unless $x_- = 0$).

Since J and i commute, so do J and the action of $\mathbf{K}$ (by multiplication on the left on $V_{\mathbf{R}} \coloneqq V_{\mathbf{K}} \otimes_{\mathbf{K}} \mathbf{R}$), so that $\mathbf{K} \subset \operatorname{End}^0(A)$. Finally, $e_0 = \sqrt{-d}$ acts by multiplication by e_0 on V_+ and by multiplication by $\overline{e_0}$ on V_-.

All in all, this proves that $(A, \mathbf{K}, E)$ is a polarized abelian variety of Weil type of dimension $2n$ and discriminant δ.

Therefore, we constructed an irreducible family of polarized abelian varieties $(A, \mathbf{K}, E)$ of Weil type of dimension $2n$ and discriminant δ and, by construction, any polarized abelian variety $(A, \mathbf{K}, E)$ of Weil type of dimension $2n$ and discriminant δ is isogeneous to some member of this family. This family is parametrized by an analytic open subset $\mathscr{H}_n$ of the Grassmannian $\operatorname{Gr}(n, (V_{\mathbf{R}}, i))$ corresponding to subspaces on which H is positive definite. It is therefore a complex analytic manifold of dimension n^2. One can show that

$$\mathscr{H}_n \simeq \operatorname{SU}(n,n)/(\operatorname{SU}(n,n) \cap (U(n) \times U(n))).$$

This manifold is acted on properly and discontinuously by the arithmetic group

$$G := \{g \in \mathrm{SU}(n,n) \mid g(\Lambda) \subset \Lambda\}$$

and the quotient $\mathscr{H}_n/G$ is a quasi-projective variety which is a coarse moduli space for abelian varieties of Weil type of dimension $2n$ and discriminant δ, with a polarization of a certain type.

In any event, given $\mathbf{K}$, our construction allows us to speak about "very general" polarized abelian varieties of Weil type with fixed dimension and discriminant. We will not give the proof of the next result, which is based on the theory of Mumford–Tate groups (see [W77], [vG94, Theorem 6.12]).

Theorem 16 (Weil)**.** *Let $(A, \mathbf{K}, E)$ be a very general polarized abelian variety of Weil type of dimension $2n \geq 4$. One has $\mathbf{K} = \mathrm{End}^0(A)$ and for $p \in \{0, \ldots, 2n\}$,*

$$\mathsf{Hdg}^p(A) = \begin{cases} \mathbf{Q}E^p & \text{for } p \neq n; \\ \mathbf{Q}E^n \oplus \mathsf{WH}(A, \mathbf{K}) & \text{for } p = n. \end{cases}$$

Recall from Proposition 13 that the two subspaces $\mathbf{Q}E^n$ and $\mathsf{WH}(A, \mathbf{K})$ of $H^{2n}(A, \mathbf{Q})$ are orthogonal for the intersection form.

Note the hypothesis $2n \geq 4$. In fact, there are no abelian surfaces A such that $\mathrm{End}^0(A)$ is a quadratic imaginary field: if $\mathbf{K} \subset \mathrm{End}^0(A)$, then $\mathrm{End}^0(A)$ is automatically larger (for abelian surfaces of Weil type, Weil–Hodge classes are divisor classes, hence the Picard number is at least 3).

2.8. **Final comments on algebraicity of Weil–Hodge classes.** Our original interest was to prove that Weil–Hodge classes are algebraic. It follows from general principles (see the end of the proof of [S98, Theorem (3.2)]) that this property is closed in a family of polarized abelian varieties of Weil type: if one can prove that general elements of an irreducible family satisfy this property, then all elements will.

This is the method employed by Schoen in [S88] and [S98]: he proves (by explicit geometric constructions) that Weil–Hodge classes are algebraic for Prym varieties of cyclic triple étale coverings of curves (see Example 8). When $n = 2$ or $n = 3$, these Prym varieties are *general* polarized abelian varieties of Weil type with field $\mathbf{K} = \mathbf{Q}(\sqrt{-3})$ and discriminant 1. It follows that for *all* polarized abelian varieties of dimension $2n \in \{4, 6\}$ of Weil type with field $\mathbf{K} = \mathbf{Q}(\sqrt{-3})$ and discriminant 1, Weil–Hodge classes are algebraic.

As explained at the end of Section 2.6, this implies that Weil–Hodge classes are algebraic for *all* polarized abelian varieties of dimension 4 of Weil type with field $\mathbf{K} = \mathbf{Q}(\sqrt{-3})$ and *any* discriminant.

Similar results were obtained in [S88] and [vG96] for polarized abelian varieties of dimension $2n$ of Weil type with field $\mathbf{K} = \mathbf{Q}(\sqrt{-1})$ and discriminant 1 (see Example 7).

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