

Gushel–Mukai varieties

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Abstract. Gushel–Mukai varieties are smooth complex dimensionally transverse intersections of a cone over the Grassmannian $\mathrm{Gr}(2, 5)$ with a linear space and a quadratic hypersurface. The aim of this survey is to discuss the geometry, moduli, Hodge structures, and categorical aspects of these varieties. It is based on joint work with Alexander Kuznetsov and earlier work of Logachev, Iliev, Manivel, O’Grady, and others.

1. Introduction

Gushel–Mukai (or GM for short) *varieties* are smooth complex dimensionally transverse intersections of a cone over the Grassmannian $\mathrm{Gr}(2, 5)$ with a linear space and a quadratic hypersurface. They occur in each dimension 1 through 6 and they are Fano varieties (their anticanonical bundle is ample) in dimensions 3, 4, 5, and 6 (in this text, we mostly restrict ourselves to these dimensions).

These varieties first appeared in the classification of complex Fano threefolds: Gushel showed that any smooth Fano threefold of index 1, degree 10, and Picard number 1 is a GM threefold. Mukai later extended Gushel’s results and proved that all Fano varieties of coindex 3, degree 10, and Picard number 1, all Brill–Noether-general polarized K3 surfaces of degree 10, and all Clifford-general curves of genus 6 are GM varieties.

One of the reasons why their geometry is so rich is that, to any GM variety is canonically attached a sextic hypersurface in \mathbf{P}^5 , called an *Eisenbud–Popescu–Walter* (or EPW for short) *sextic* and a canonical double cover thereof, a hyper-Kähler fourfold called a *double EPW sextic*. In some sense, the pair consisting of a GM variety and its associated double EPW sextic behaves very much (but with some complications and differences) like a cubic hypersurface in \mathbf{P}^5 and its (hyper-Kähler) variety of lines. Not many examples of these pairs—a Fano variety and a hyper-Kähler variety—are known (another example is given by the hyperplane sections of the Grassmannian $\mathrm{Gr}(3, 10)$ and their associated Debarre–Voisin hyper-Kähler fourfold), so it is worth looking in depth into one of these.

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The first main difference with cubic fourfolds is that there is a positive-dimensional family of GM varieties attached to the same EPW sextic; this family can be precisely described. Another distinguishing feature is that each EPW sextic has a dual EPW sextic (its projective dual). GM varieties associated with isomorphic or dual EPW sextics are all birationally isomorphic.

A feature shared with cubic fourfolds is that the middle Hodge structure of a GM variety of *even* dimension is isomorphic (up to a Tate twist) to the second cohomology of its associated double EPW sextic. Together with the Verbitsky–Torelli theorem for hyper-Kähler fourfolds, this leads to a complete description of the period map for GM varieties of dimensions 4 or 6.

In *odd* dimensions 3 or 5, a GM variety has a 10 dimensional intermediate Jacobian and we show that it is isomorphic to the Albanese variety of a surface of general type canonically attached to the EPW sextic. So again, the Hodge structure of the GM variety is determined by the associated EPW sextic.

Another aspect of GM varieties that makes them very close to cubic fourfolds is the rationality problem. Whereas general (and conjecturally all) GM varieties of dimensions at most 3 are not rational, and GM varieties of dimensions at least 5 are all rational, the situation for GM fourfolds is still mysterious: some rational examples are known but not a single irrational example is known, although one expects, as for cubic fourfolds, that a very general GM fourfold should be irrational.

Finally, the derived categories of GM varieties were studied by Kuznetsov and Perry with this rationality problem in mind. They show that each of these categories contains a special subcategory, which is a K3 or Enriques category according to whether the dimension of the variety is even or odd.

Most of my contributions are joint work with Alexander Kuznetsov.

2. Definition of Gushel–Mukai varieties

We always work over \mathbf{C} and U_k, V_k, W_k all denote complex vector spaces of dimension k .

Let X be a (smooth complex) Fano variety of dimension n and Picard number 1, so that $\text{Pic}(X) = \mathbf{Z}H$, with H ample, and $-K_X \equiv_{\text{lin}} rH$ for some positive integer r called the *index* of X . The integer $d := H^n$ is the *degree*. For example, a smooth hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \leq n+1$ is a Fano variety of index $r = n+2-d$ and degree d .

The index satisfies $r \leq n+1$ and moreover,

- if $r = n+1$, then $X \simeq \mathbf{P}^n$ (Kobayashi–Ochiai, 1973; see [KO, Corollary to Theorem 1.1]);
- if $r = n$, then X is a quadric in \mathbf{P}^{n+1} (Kobayashi–Ochiai, 1973; see [KO, Corollary to Theorem 2.1]);
- if $r = n-1$ (del Pezzo varieties), there is a classification by Fujita and Iskovskikh (1977–1988; see [IP, Chapter 3]);

- if $r = n - 2$, there is a classification by Mukai (1989–1995; see [Mu1, Mu2, Mu3]),¹ modulo a result proved later by Shokurov in dimension 3, and Mella in all dimensions ([M]).

Mukai used the *vector bundle method* (initiated by Gushel in [G]) to prove that in the last case, $d \in \{2, 4, 6, 8, 10, 12, 14, 16, 18, 22\}$.

In this article, we restrict ourselves to the case $d = 10$. We call the corresponding (smooth Fano) varieties *Gushel–Mukai* (or GM for short) varieties.

Theorem 2.1 (Mukai). *Any smooth Fano n -fold X_n with Picard number 1, index $n - 2$, and degree 10 has dimension $n \in \{3, 4, 5, 6\}$ and can be obtained as follows:*

- either $n \in \{3, 4, 5\}$ and $X_n = \text{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subseteq \mathbf{P}(\wedge^2 V_5)$, where Q is a quadric and $W_{n+5} \subseteq \wedge^2 V_5$ is a vector subspace of dimension $n + 5$ (ordinary case);
- or $X_n \rightarrow \text{Gr}(2, V_5) \cap \mathbf{P}(W_{n+4})$ is a double cover branched along an X_{n-1} (special case).

In fact, Mukai had to make the extra assumption that the linear system $|H|$ contains a smooth element, a fact that was proved only later on by Mella (who actually showed that H is very ample).

In both cases, the morphism $\gamma: X_n \rightarrow \text{Gr}(2, V_5) \subseteq \mathbf{P}(\wedge^2 V_5)$ (called the *Gushel morphism*) is defined by a linear subsystem of $|H|$.

When $n = 1$, the constructions in the theorem provide genus 6 curves which are general in the sense that they are neither hyperelliptic, nor trigonal, nor plane quintics (this is also termed *Clifford general* because the Clifford index has the maximal value 2); when $n = 2$, these constructions provide K3 surfaces of degree 10 which are general in the sense that their general hyperplane sections are Clifford general genus 6 curves (they are also called *Brill–Noether general* K3 surfaces; see [Mu3, Definition 3.8]).

Sketch of proof. Set $X := X_n$. In order to construct the Gushel morphism $\gamma: X \rightarrow \text{Gr}(2, V_5)$ with $\gamma^* \mathcal{O}_{\text{Gr}(2, V_5)}(1) = H$, one needs to construct a rank 2 globally generated vector bundle \mathcal{E} on X with 5 independent sections and $c_1(\mathcal{E}) = H$; if \mathcal{U} is the tautological rank 2 vector bundle on $\text{Gr}(2, V_5)$, one then has $\mathcal{E} = \gamma^* \mathcal{U}^\vee$. There are several ways to construct \mathcal{E} .

We give in [DK2] a construction of \mathcal{E} based on the excess conormal sheaf that goes as follows. By Mella’s result, the line bundle H is very ample hence defines an embedding $\varphi_H: X \hookrightarrow \mathbf{P}(W_{n+5})$, where $W_{n+5} := H^0(X, H)^\vee$. We prove that $X \subseteq \mathbf{P}(W_{n+5})$ is the base locus of the space of quadrics $V_6 := H^0(\mathbf{P}(W_{n+5}), \mathcal{I}_X(2))$ and we define \mathcal{E} as the dual of the (rank 2) kernel of the canonical surjection

$$V_6 \otimes \mathcal{O}_X \twoheadrightarrow (\mathcal{I}_X / \mathcal{I}_X^2)(2) = N_{X/\mathbf{P}(W_{n+5})}^\vee(2).$$

Mukai had a different construction that went roughly as follows ([Mu1, Mu3]). The intersection of $n - 2$ general elements of $|H|$ is a smooth K3 surface S and the restriction \mathcal{E}_S of \mathcal{E} to S must satisfy $c_1(\mathcal{E}_S) = H_S$ and $c_2(\mathcal{E}_S) = 4$.

¹Mukai’s argument is in fact incomplete (see footnote 2). For a complete proof of a more general result, see [BKM].

So we start out by constructing such a vector bundle on the genus 6 K3 surface $S \subseteq \mathbf{P}^6$ using Serre's construction. A general hyperplane section of S is a canonical curve C of genus 6, hence has a g_4^1 . Let $Z \subseteq C$ be an element of this pencil, so that

$$\begin{aligned} h^0(S, \mathcal{I}_Z(H_S)) &= 1 + h^0(C, H_S - Z) \\ &= 1 + h^1(C, g_4^1) \\ &= 1 + h^0(C, g_4^1) - (4 + 1 - 6) = 4. \end{aligned}$$

By Serre's construction, there is a rank 2 vector bundle \mathcal{E}_S on S that fits into an exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{E}_S \rightarrow \mathcal{I}_Z(H_S) \rightarrow 0.$$

Assume $n = 3$. One proves that \mathcal{E}_S extends to a locally free sheaf \mathcal{E} on X .² Then one checks that $V_5 := H^0(X, \mathcal{E})^\vee$ has dimension 5 and that \mathcal{E} is generated by global sections hence defines a morphism $\gamma: X \rightarrow \text{Gr}(2, V_5)$.

We have a linear map

$$\eta: \wedge^2 H^0(X, \mathcal{E}) \rightarrow H^0(X, \wedge^2 \mathcal{E}) \simeq H^0(X, H)$$

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \text{Gr}(2, V_5) \\ \varphi_H \downarrow & & \downarrow \\ \mathbf{P}(H^0(X, H)^\vee) & \xrightarrow{\eta^\vee} & \mathbf{P}(\wedge^2 V_5). \end{array}$$

If η is surjective, the map η^\vee in the diagram is a morphism, which is moreover injective. The image of η^\vee is a $\mathbf{P}(W_8) \subseteq \mathbf{P}(\wedge^2 V_5)$. The intersection $M_X := \text{Gr}(2, V_5) \cap \mathbf{P}(W_8)$ has dimension 4 (because all hyperplane sections of $\text{Gr}(2, V_5)$ are irreducible) and degree 5, and $\varphi_H(X) \subseteq M_X$ is a smooth hypersurface. It follows that M_X must be smooth (otherwise, one easily sees that its singular locus would have positive dimension hence would meet $\varphi_H(X)$, which is impossible). Thus, $\varphi_H(X)$ is a Cartier divisor, of degree 10, hence is the intersection of M_X with a quadric.

If the corank of η is 1, consider the cone $\text{CGr} \subseteq \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$, with vertex $v = \mathbf{P}(\mathbf{C})$, over $\text{Gr}(2, V_5)$. In the above diagram, we may view $H^0(X, H)^\vee$ as embedded in $\mathbf{C} \oplus \wedge^2 V_5$ by mapping $(\text{Im}(\eta))^\perp$ to \mathbf{C} , and η^\vee as the projection from v , with image a $\mathbf{P}(W_7) \subseteq \mathbf{P}(\wedge^2 V_5)$. Again, the intersection $M_X := \text{Gr}(2, V_5) \cap \mathbf{P}(W_7) = \text{Im}(\psi)$ is a smooth threefold and $\varphi_H(X)$ is the intersection of the (smooth locus of the) degree 5 cone over M_X with a quadric.

²The argument given by Mukai for this step is incorrect, since it relies on an extension theorem of Fujita that does not apply to surfaces. It was only recently fixed in [BKM].

If the corank of η is ≥ 2 , one shows that $\varphi_H(X)$ must be singular, which is impossible. So this proves the theorem when $n = 3$.

For $n \geq 4$, one extends \mathcal{E} to successive hyperplane sections of X all the way up to X and proceed similarly (see [IP, Proposition 5.2.7]). ■

Why are people interested in GM varieties? As the rest of this article will, I hope, show, for several reasons:

- they have interesting geometries (Section 3);
- they have intriguing rationality properties in dimension 4 similar to that of cubic four-folds (most X_3 are irrational, all X_5 and X_6 are rational) (Section 4);
- they have interesting period maps (Section 5);
- they have interesting derived categories (Section 6).

3. Gushel–Mukai varieties, EPW sextics, and moduli

In this section, we explain an intricate (but elementary) connection (discovered by O’Grady and Iliev–Manivel) between Gushel–Mukai varieties and Lagrangian subspaces in a 20 dimensional symplectic complex vector space.

Let

$$X = \mathrm{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subseteq \mathbf{P}(\wedge^2 V_5)$$

be an ordinary³ GM n -fold, with $n \in \{3, 4, 5\}$. We set

$$M := \mathrm{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}),$$

a variety of degree 5 and dimension $n + 1$, so that $X = M \cap Q$.

Lemma 3.1. *The variety M is smooth and each nonzero element of the orthogonal complement $W_{n+5}^\perp \subseteq \wedge^2 V_5^\vee$, viewed as a skew-symmetric form on V_5 , has maximal rank 4.*

Proof. Let $W := W_{n+5}$. One has (see [PV, Corollary 1.6])

$$\mathrm{Sing}(M) = \mathbf{P}(W) \cap \bigcup_{\omega \in W^\perp \setminus \{0\}} \mathrm{Gr}(2, \mathrm{Ker}(\omega)). \quad (1)$$

Since X is smooth, $\mathrm{Sing}(M)$ must be finite.

If $\dim(W) = 10$, then $M = \mathrm{Gr}(2, V_5)$ is smooth.

If $\dim(W) = 9$, either a generator ω of W^\perp has rank 2 and $\mathrm{Sing}(M)$ is a plane, which is impossible, or else M is smooth.

If $\dim(W) = 8$, either some $\omega \in W^\perp$ has rank 2, in which case $\mathrm{Gr}(2, \mathrm{Ker}(\omega))$ is a plane contained in the hyperplane ω^\perp , whose intersection with $\mathbf{P}(W)$ therefore contains a line along which M is singular, which is impossible, or else M is smooth. ■

³We make this assumption for the sake of simplicity, but our constructions also work for special GM varieties, with some modifications (see Remark 3.4). The theory also works in dimensions $n \in \{1, 2\}$, again with some modifications. We refer to [DK2] for details.

3.1. GM data sets

Note that M is cut out in $\mathbf{P}(W)$ by the traces of the Plücker quadrics

$$\mathbf{q}(v)(w, w) := v \wedge w \wedge w,$$

for $v \in V_5$ and $w \in W \subseteq \wedge^2 V_5$ (we choose an isomorphism $\wedge^5 V_5 \simeq \mathbf{C}$). More precisely, there is a (not quite canonical) identification

$$V_5 \simeq H^0(\mathbf{P}(W), \mathcal{I}_M(2)).$$

One then has

$$V_6 := H^0(\mathbf{P}(W), \mathcal{I}_X(2)) \simeq V_5 \oplus \mathbf{C}Q.$$

We formalize this by defining a *GM data set* $(W_{n+5}, V_6, V_5, \mathbf{q})$ (of dimension $n \leq 5$) by

- V_6 is a 6 dimensional vector space;
- $V_5 \subseteq V_6$ is hyperplane;
- $W_{n+5} \subseteq \wedge^2 V_5$ is a vector subspace of dimension $n + 5$;
- $\mathbf{q}: V_6 \rightarrow \text{Sym}^2 W_{n+5}^\vee$ is a linear map such that

$$\forall v \in V_5 \quad \forall w \in W_{n+5} \quad \mathbf{q}(v)(w, w) = v \wedge w \wedge w.$$

Given such a data set, we define

$$X := \bigcap_{v \in V_6} Q(v) \subseteq \mathbf{P}(W_{n+5}),$$

where $Q(v) \subseteq \mathbf{P}(W_{n+5})$ is the quadric hypersurface defined by $\mathbf{q}(v)$. When X is smooth of the expected dimension n , it is an ordinary GM variety.

3.2. Lagrangian data sets

We now set up a more complicated correspondance. We endow $\wedge^3 V_6$ with the (conformal) symplectic form given by wedge product. A *Lagrangian data set* is a triple (V_6, V_5, A) , where $V_5 \subseteq V_6$ is a hyperplane and $A \subseteq \wedge^3 V_6$ is a (10 dimensional) Lagrangian subspace.

Theorem 3.2. *There exists a bijection, explained in the proof, between GM data sets $(W, V_6, V_5, \mathbf{q})$ and Lagrangian data sets (V_6, V_5, A) .*

One has the following relations

$$\begin{aligned} \dim(W) &= 10 - \dim(A \cap \wedge^3 V_5), \\ \forall v \in V_6 \setminus V_5 \quad \text{Ker}(\mathbf{q}(v)) &\simeq A \cap (v \wedge \wedge^2 V_6) = A \cap (v \wedge \wedge^2 V_5) \subseteq W. \end{aligned}$$

Proof. This construction is explained in great details and more generality in the proof of [DK2, Theorem 3.6]. We present a simplified version.

For one direction, let us start with a Lagrangian data set (V_6, V_5, A) . Let $\lambda \in V_6^\vee$ be a linear form with kernel V_5 . By the Leibniz rule, it induces $\lambda_p: \wedge^p V_6 \rightarrow \wedge^{p-1} V_5$, with kernel $\wedge^p V_5$. Define $\mathbf{q}: V_6 \otimes \text{Sym}^2 A \rightarrow \mathbf{C}$ by

$$\forall v \in V_6 \quad \forall \xi, \xi' \in A \quad \mathbf{q}(v \otimes (\xi, \xi')) := -\lambda_4(v \wedge \xi) \wedge \lambda_3(\xi') \in \wedge^5 V_5.$$

The fact that A is Lagrangian implies $\xi \wedge \xi' = 0$, hence, by the Leibniz rule,

$$0 = \lambda_6(\xi \wedge \xi') = \lambda_3(\xi) \wedge \xi' - \xi \wedge \lambda_3(\xi').$$

Therefore, we have, by the Leibniz rule again,

$$\mathbf{q}(v \otimes (\xi, \xi')) := -\lambda_1(v) \wedge \xi \wedge \lambda_3(\xi') + v \wedge \lambda_3(\xi) \wedge \lambda_3(\xi'),$$

which is symmetric in ξ and ξ' .

Moreover, if $\xi' \in A \cap \wedge^3 V_5$, one has $\lambda_3(\xi') = 0$, hence $\mathbf{q}(v \otimes (\xi, \xi')) = 0$, so that \mathbf{q} descends to

$$\mathbf{q}: V_6 \otimes \text{Sym}^2 W \rightarrow \mathbf{C},$$

where $W := A/(A \cap \wedge^3 V_5)$.

Finally, if $v \in V_5$, one has $\lambda(v) = 0$, hence $\mathbf{q}(v \otimes (\xi, \xi')) = v \wedge \lambda_3(\xi) \wedge \lambda_3(\xi')$ by the calculation above. The canonical factorization of the composition

$$A \subseteq \wedge^3 V_6 \xrightarrow{\lambda_3} \wedge^2 V_5$$

induces an injection $W \hookrightarrow \wedge^2 V_5$ and all the conditions are met for $(W, V_6, V_5, \mathbf{q})$ to be a GM data set.

For the other direction, we start from a GM data set $(W, V_6, V_5, \mathbf{q})$. Choose $v_0 \in V_6 \setminus V_5$. We define A by the short exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & \wedge^3 V_5 \oplus W & \longrightarrow & W^\vee \\ & & & & (\xi, w) & \longmapsto & (w' \mapsto \xi \wedge w' + \mathbf{q}(v_0)(w, w')), \end{array} \quad (2)$$

where the middle term is seen inside $\wedge^3 V_5 \oplus (v_0 \wedge \wedge^2 V_5) = \wedge^3 V_6$.

Let us check that A is Lagrangian. If $(\xi, w), (\xi', w') \in A$, we have

$$\begin{aligned} (\xi, w) \wedge (\xi', w') &= (\xi + v_0 \wedge w) \wedge (\xi' + v_0 \wedge w') \\ &= \xi \wedge v_0 \wedge w' + v_0 \wedge w \wedge \xi' \\ &= v_0 \wedge (-\xi \wedge w' + \xi' \wedge w). \end{aligned}$$

But this vanishes because (ξ, w) goes to 0 in W^\vee , hence $\xi \wedge w' + \mathbf{q}(v_0)(w, w') = 0$, and similarly $\xi \wedge w + \mathbf{q}(v_0)(w', w) = 0$. So the fact that A is isotropic follows from the symmetry of $\mathbf{q}(v_0)$. Since A has dimension ≥ 10 by construction, it is Lagrangian (and the rightmost map in (2) is surjective).

One checks that A is independent of the choice of v_0 and that the two constructions are inverse of each other.

The formula for $\dim(W)$ was given in the course of the proof. By construction of A , the kernel of $\mathbf{q}(v_0)$ is the set of pairs $(0, w)$ that belong to A , which is isomorphic to the intersection $A \cap (v_0 \wedge \wedge^2 V_5)$. Since A is independent of the choice of v_0 , this remains valid for all $v_0 \in V_6 \setminus V_5$. ■

Putting everything together, we obtain a bijection between the set of isomorphism classes of (possibly singular) ordinary GM varieties and the set of isomorphism classes of certain Lagrangian data sets (V_6, V_5, A) . We need to worry about smoothness of the associated GM variety. It turns out, and this is what makes this construction so efficient, that there is a very simple criterion for smoothness that only involves the Lagrangian A and not the hyperplane V_5 .

Theorem 3.3. *Let $n \in \{3, 4, 5\}$. The bijections constructed above combine to give a bijection between the set of isomorphism classes of ordinary GM varieties of dimension n and the set of isomorphism classes of Lagrangian data sets (V_6, V_5, A) such that $\dim(A \cap \wedge^3 V_5) = 5 - n$ and A contains no decomposable vectors.*

The condition on A means $\mathbf{P}(A) \cap \text{Gr}(3, V_6) = \emptyset$ inside $\mathbf{P}(\wedge^3 V_6)$.

Proof. Let X be an ordinary GM variety of dimension n . Since X is smooth, by Lemma 3.1, $M = \text{Gr}(2, V_5) \cap \mathbf{P}(W)$ is also smooth and W^\perp contains no rank 2 forms, that is, $\mathbf{P}(W^\perp) \cap \text{Gr}(2, V_5^\vee) = \emptyset$.

If $[\xi] \in \mathbf{P}(A) \cap \text{Gr}(3, V_5)$, the exact sequence (2) implies that the linear form $w' \mapsto \xi \wedge w'$ is in W^\perp ; but it has rank 2, and this is absurd.

If $\xi \in A$ is decomposable, it is therefore not in $\text{Gr}(3, V_5)$, so we can write $\xi = v_0 \wedge v_1 \wedge v_2$ for some $v_0 \in V_6 \setminus V_5$ and $v_1, v_2 \in V_5$. From the exact sequence (2) defining A , we obtain that $v_1 \wedge v_2$ is in W and that $(0, v_1 \wedge v_2)$ goes to 0 in W^\vee . The bivector $v_1 \wedge v_2$ is therefore in the kernel of the quadratic form $\mathbf{q}(v_0)$. In other words, the quadric $Q(v_0)$ is singular at the point $[v_1 \wedge v_2]$ of $\mathbf{P}(W)$. But this point is on M , hence it is singular on $X = M \cap Q(v_0)$, which is absurd.

All this shows that A contains no decomposable vectors.

Assume for the converse that A contains no decomposable vectors. What we just showed implies that M is smooth of dimension $n + 1$ (because $A \cap \text{Gr}(3, V_5) = \emptyset$). If X is singular at a point $[w]$, where $w \in \text{Gr}(2, V_5)$, this therefore means that $\mathbf{T}_{Q(v), [w]} \supseteq \mathbf{T}_{M, [w]}$ for all $v \in V_6 \setminus V_5$. Since M is the intersection of Plücker quadrics, there exists $v' \in V_5$ such that $\mathbf{T}_{Q(v), [w]} = \mathbf{T}_{Q(v'), [w]}$. Replacing v by some $v + tv'$, we see that there exists $v \in V_6 \setminus V_5$ such that the quadric $Q(v)$ is singular at $[w]$. The second relation in Theorem 3.2 implies that $v \wedge w$ is in A , which is absurd since it is decomposable. ■

Remark 3.4 (Special GM varieties). Recall from Theorem 2.1 that a special GM variety X of dimension n is canonically a double cover of a linear section of the Grassman-

nian $\text{Gr}(2, V_5)$ branched along an ordinary GM variety X' of dimension $n - 1$. One defines the Lagrangian data set associated with X as the Lagrangian data set associated with X' by the procedure described above (see [DK2, Section 3]).

3.3. EPW sextics

Eisenbud–Popescu–Walter (EPW) sextics are special hypersurfaces of degree 6 in $\mathbf{P}(V_6)$ constructed from Lagrangian subspaces $A \subseteq \wedge^3 V_6$. More generally, given such a subspace A , we consider, for any integer ℓ , the locus

$$Y_A^{\geq \ell} := \{[v] \in \mathbf{P}(V_6) \mid \dim(A \cap (v \wedge \wedge^2 V_6)) \geq \ell\} \quad (3)$$

and endow it with a scheme structure as in [O1, Section 2] (see also Section 4.4). We also set $Y_A^\ell := Y_A^{\geq \ell} \setminus Y_A^{\geq \ell+1}$. We have inclusions

$$\mathbf{P}(V_6) = Y_A^{\geq 0} \supseteq Y_A^{\geq 1} \supseteq Y_A^{\geq 2} \supseteq \dots$$

When the scheme $Y_A := Y_A^{\geq 1}$ is not the whole space $\mathbf{P}(V_6)$, it is a sextic hypersurface ([O1, (1.8)]) called an *EPW sextic*. When A contains no decomposable vectors, the geometry of these loci was described by O’Grady ([O3]), as follows.

Theorem 3.5 (O’Grady). *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors.*

- (a) Y_A is an integral normal sextic hypersurface in $\mathbf{P}(V_6)$;
- (b) $Y_A^{\geq 2} = \text{Sing}(Y_A)$ is an integral normal Cohen–Macaulay surface of degree 40;
- (c) $Y_A^{\geq 3} = \text{Sing}(Y_A^{\geq 2})$ is finite and smooth, and is empty for A general;
- (d) $Y_A^{\geq 4}$ is empty.

Given an ordinary GM variety X , with associated Lagrangian A , Iliev–Manivel described in [IM] a direct way to construct its associated EPW sextic Y_A . It goes as follows. The space V_6 of quadrics in $\mathbf{P}(W)$ containing X is 6 dimensional. Define the discriminant locus $\widetilde{\text{Disc}}(X)$ as the subscheme of $\mathbf{P}(V_6)$ of *singular* quadrics containing X . It is a hypersurface of degree $\dim(W) = n + 5$ in which the multiplicity of the hyperplane $\mathbf{P}(V_5)$ of Plücker quadrics is at least the corank of a general such quadric, which is at least $\dim(W) - 6 = n - 1$. We set

$$\text{Disc}(X) := \widetilde{\text{Disc}}(X) - (n - 1)\mathbf{P}(V_5). \quad (4)$$

This is an effective divisor of degree 6 in $\mathbf{P}(V_6)$.

Proposition 3.6. *Let X be an ordinary GM variety of dimension $n \in \{3, 4, 5\}$ with associated Lagrangian A . The subschemes Y_A and $\text{Disc}(X)$ of $\mathbf{P}(V_6)$ are equal.*

Proof. For all $v \in V_6 \setminus V_5$, we have $\text{Ker}(\mathbf{q}(v)) = A \cap (v \wedge \wedge^2 V_5)$ by Theorem 3.2 hence, Y_A and $\text{Disc}(X)$ coincide as sets on the complement of $\mathbf{P}(V_5)$. Since they are both sextics and Y_A is integral (Theorem 3.5(a)) and not contained (as a set) in $\mathbf{P}(V_5)$, they are equal. ■

3.4. Dual EPW sextics

If $A \subseteq \wedge^3 V_6$ is a Lagrangian subspace, so is its orthogonal complement $A^\perp \subseteq \wedge^3 V_6^\vee$ (which we also call its *dual*) and A contains decomposable vectors if and only if A^\perp does. The EPW loci (3) for A^\perp can be described in terms of A as

$$Y_{A^\perp}^{\geq \ell} = \{[V_5] \in \text{Gr}(5, V_6) = \mathbf{P}(V_6^\vee) \mid \dim(A \cap \wedge^3 V_5) \geq \ell\} \quad (5)$$

and the sextic hypersurface $Y_{A^\perp} \subseteq \mathbf{P}(V_6^\vee)$ is the projective dual of the hypersurface $Y_A \subseteq \mathbf{P}(V_6)$ ([DK2, Proposition B.3]).

In particular, for each $n \in \{3, 4, 5\}$, we can rewrite the bijection of Theorem 3.3 as a bijection between the set of isomorphism classes of ordinary GM varieties of dimension n and the set of isomorphism classes of Lagrangian data sets (V_6, V_5, A) such that A contains no decomposable vectors and $[V_5] \in Y_{A^\perp}^{5-n}$.

3.5. Moduli

A GIT moduli space for EPW sextics was constructed by O’Grady in [O4]: consider the natural action of the group $\text{PGL}(V_6)$ on the Lagrangian Grassmannian $\text{LGr}(\wedge^3 V_6)$ and its natural linearization in the line bundle $\mathcal{O}(2)$ (the line bundle $\mathcal{O}(1)$ has no such linearizations). The GIT quotient

$$\overline{\mathbf{M}}^{\text{EPW}} := \text{LGr}(\wedge^3 V_6) // \text{PGL}(V_6)$$

is a projective, irreducible, 20 dimensional, coarse moduli space for EPW sextics. The hypersurface

$$\Sigma := \{[A] \in \text{LGr}(\wedge^3 V_6) \mid A \text{ has decomposable vectors}\}$$

is $\text{PGL}(V_6)$ -invariant. Consider its complement: it is affine, consists of stable points ([O4, Corollary 2.5.1]), and its image \mathbf{M}^{EPW} in $\overline{\mathbf{M}}^{\text{EPW}}$ is affine and is a coarse moduli space for EPW sextics Y_A such that A has no decomposable vectors.

The duality map $A \mapsto A^\perp$ described in Section 3.4 induces a nontrivial involution r on $\overline{\mathbf{M}}^{\text{EPW}}$ that satisfies $r(\Sigma) = \Sigma$, hence also an involution on \mathbf{M}^{EPW} .

We do not know how to construct directly a moduli space for GM varieties. Instead, we extend the constructions of Section 3.2 to families of GM varieties (this is not an easy task, not least because we need to include special GM varieties). Here is the main result of [DK4].

Theorem 3.7. *Let $n \in \{3, 4, 5, 6\}$. There is a quasiprojective, irreducible, coarse moduli space \mathbf{M}_n^{GM} for GM varieties of dimension n . Its dimension is $25 - (5 - n)(6 - n)/2$, there is a surjective morphism*

$$\pi_n : \mathbf{M}_n^{\text{GM}} \longrightarrow \mathbf{M}^{\text{EPW}},$$

and the fiber $\pi_n^{-1}([A])$ of the class of a Lagrangian A with no decomposable vectors is isomorphic to $Y_{A^\perp}^{5-n} \cup Y_{A^\perp}^{6-n}$, modulo the finite group of linear automorphisms of V_6 that preserve A^\perp .

The group that appears in the last line of the statement is trivial for A very general ([DK2, Proposition B.9]). The piece $Y_{A^\pm}^{5-n}$ of the fiber corresponds to ordinary GM n -folds and the piece $Y_{A^\pm}^{6-n}$ to special GM n -folds. Also, the schemes \mathbf{M}_6^{GM} and \mathbf{M}_5^{GM} are affine⁴ and there is an open embedding $\mathbf{M}_6^{\text{GM}} \hookrightarrow \mathbf{M}_5^{\text{GM}}$ whose image is the moduli space for ordinary GM fivefolds.

4. Birationalities and rationality

In this section, we will show that all GM varieties of the same dimension $n \in \{3, 4, 5, 6\}$ with the same associated Lagrangian (that is, in the same fiber of the map π_n defined in Theorem 3.7), or with dual associated Lagrangians, are birationally isomorphic. Since all GM varieties of dimensions 5 or 6 are rational (see Proposition 4.3), this is meaningful only when $n \in \{3, 4\}$.

Let X be a GM variety of dimension $n \in \{3, 4, 5\}$, which we assume to be ordinary for simplicity, and let \mathcal{U}_X be the restriction to X of the tautological rank 2 vector bundle on $\text{Gr}(2, V_5)$. There are canonical maps

$$\begin{array}{ccc} & \mathbf{P}(\mathcal{U}_X) & \\ & \swarrow \quad \searrow & \\ X & & \mathbf{P}(V_5) \end{array} \quad \begin{array}{l} \rho \\ (6) \end{array}$$

Let

$$\Sigma_1(X) \subseteq \mathbf{P}(V_5)$$

be the union of the kernels of all nonzero elements of W^\perp , seen as rank 4 skew-symmetric forms on V_5 (see Lemma 3.1). It is

- empty for $n = 5$ ($W^\perp = 0$);
- a point for $n = 4$;
- a smooth conic for $n = 3$.

Proposition 4.1. *Let X be an ordinary GM variety of dimension $n \in \{3, 4, 5\}$. The map ρ is surjective and its fiber over a point $[v] \in \mathbf{P}(V_5)$ is*

- if $[v] \notin \Sigma_1(X)$, a quadric in a \mathbf{P}^{n-2} , whose corank is k if and only if $[v] \in Y_A^k$;
- if $[v] \in \Sigma_1(X)$, a quadric in a \mathbf{P}^{n-1} , whose corank is k if and only if $[v] \in Y_A^{k+1}$. In particular, $\Sigma_1(X) \subseteq Y_A$.

Proof. Choose $v_0 \in V_6 \setminus V_5$ and let $v \in V_5 \setminus \{0\}$. The fiber $\rho^{-1}([v])$ is the set of $V_2 \subseteq V_5$ containing v that correspond to points of X . This is the intersection $Q'(v)$ of the nonPlücker quadric $Q(v_0) \subseteq \mathbf{P}(W)$ defining X with the subspace $\mathbf{P}((v \wedge V_5) \cap W) \subseteq \mathbf{P}(W)$.

⁴It should be remarked that \mathbf{M}_6^{GM} can be directly constructed as a standard GIT quotient of an affine space ([DK4, Proposition 5.16]).

If $[v] \notin \Sigma_1(X)$, it is not in the kernel of any 2 form defining W , hence $\mathbf{P}((v \wedge V_5) \cap W)$ has codimension $5 - n$ in $\mathbf{P}(v \wedge V_5) \simeq \mathbf{P}^3$: it is a linear space of dimension $n - 2$.

If $[v] \in \Sigma_1(X)$, it is in the kernel of a unique 2 form defining W , hence $\mathbf{P}((v \wedge V_5) \cap W)$ has codimension $4 - n$ in $\mathbf{P}(v \wedge V_5) \simeq \mathbf{P}^3$: it is a linear space of dimension $n - 1$.

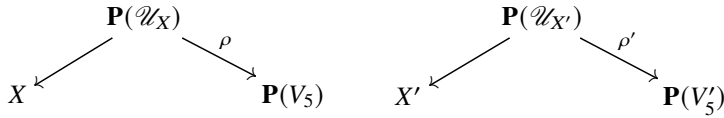
The statements about the rank of $Q'(v)$ are not so obvious and we skip their proofs (see [DK2, Proposition 4.5]). ■

4.1. Birationalities

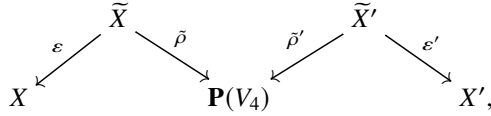
Our main result is the following ([DK2, Corollary 4.16 and Theorem 4.20]).

Theorem 4.2. *GM varieties of the same dimension $n \geq 3$ whose associated Lagrangians are either the same or dual one to another are birationally isomorphic.*

Proof. Because of Proposition 4.3 (proved below), the only interesting cases are when $n \in \{3, 4\}$. We will only sketch the proof in the case when X and X' are ordinary GM varieties associated with the same Lagrangian subspace A and different hyperplanes $[V_5], [V'_5] \in Y_{A^\perp}^{5-n}$ (the other cases are more difficult). Set $V_4 := V_5 \cap V'_5$ and restrict the diagrams



to



where $\tilde{X} := \rho^{-1}(\mathbf{P}(V_4))$ and $\tilde{X}' := \rho'^{-1}(\mathbf{P}(V_4))$. One can show that ε and ε' are birational maps.

Assume first $n = 3$. By Proposition 4.1, $\tilde{\rho}$ is, outside of $(\Sigma_1(X) \cup Y_A^{\geq 2}) \cap \mathbf{P}(V_4)$,⁵ a double cover branched along the sextic surface $Y_A \cap \mathbf{P}(V_4)$. It follows that above the complement of $(\Sigma_1(X) \cup \Sigma_1(X') \cup Y_A^{\geq 2}) \cap \mathbf{P}(V_4)$, the morphisms $\tilde{\rho}$ and $\tilde{\rho}'$ are double covers branched along *the same* sextic surface. The schemes \tilde{X} and \tilde{X}' are therefore birationally isomorphic, and so are the varieties X and X' .

A similar proof works when $n = 4$. The morphisms $\tilde{\rho}$ and $\tilde{\rho}'$ are then, outside a subvariety of $\mathbf{P}(V_4)$ of dimension ≤ 1 , conic bundles with the same discriminant $Y_A \cap \mathbf{P}(V_4)$. Over a general point of every component of that surface, the fiber is the union of two distinct lines, and the double cover of $Y_A \cap \mathbf{P}(V_4)$ that this defines is the same for X and for X' : one can show it is induced by restriction of the canonical double cover $\tilde{Y}_A \rightarrow Y_A$. A general argument based on Brauer groups (see [C-T]) then shows that the fields of functions of \tilde{X}

⁵By [DK2, Lemma B.6], this set has dimension ≤ 1 .

and \widetilde{X}' are the same extension of the field of functions of $\mathbf{P}(V_4)$. Therefore, \widetilde{X} and \widetilde{X}' are birationally isomorphic, and so are X and X' . ■

In [DIM1, Section 7], we describe explicit birational maps between ordinary GM threefolds associated with the same Lagrangian (“conic transformations”) or with dual Lagrangians (“line transformations”).

4.2. Rationality

We first consider GM varieties of dimension $n \in \{5, 6\}$.

Proposition 4.3. *All GM varieties of dimension $n \in \{5, 6\}$ are rational.*

Proof. Let us do the proof for an ordinary GM fivefold X , corresponding to a Lagrangian $A \subseteq \wedge^3 V_6$ with no decomposable vectors and a hyperplane $[V_5] \in \mathbf{P}(V_6) \setminus Y_{A^\perp}$. First, one can show that the surface $Y_A^{\geq 2}$ is not contained in a hyperplane, so we can choose $[v] \in Y_A^2 \setminus \mathbf{P}(V_5)$. By Theorem 3.2, the corresponding 8 dimensional nonPlücker quadric $Q(v)$ has corank 2 and $X = \text{Gr}(2, V_5) \cap Q(v)$. One shows that for a general 6 dimensional $\mathfrak{q}(v)$ -isotropic subspace I , the intersection $S := \mathbf{P}(I) \cap X = \mathbf{P}(I) \cap \text{Gr}(2, V_5)$ is a smooth quintic del Pezzo surface. Consider the linear projection

$$\pi_S: X \dashrightarrow \mathbf{P}^3$$

from $\mathbf{P}(I)$. A general fiber is also a quintic del Pezzo surface: indeed, the intersection of a general \mathbf{P}^6 containing $\mathbf{P}(I) = \mathbf{P}^5$ with $Q(v)$ is a union $\mathbf{P}(I) \cup \mathbf{P}(I')$, where I' is another Lagrangian subspace, and its intersection with X is the union of S and another smooth quintic del Pezzo surface $\mathbf{P}(I') \cap \text{Gr}(2, V_5)$.

Therefore, the field of rational functions of X is the field of rational functions of a smooth quintic del Pezzo surface defined over the field of rational functions on \mathbf{P}^3 . But a smooth quintic del Pezzo surface is rational over any field by a theorem of Enriques (see [S-B]). ■

The situation for GM threefolds is the following. First, any GM threefold X is unirational (there is a degree 2 rational map $\mathbf{P}^3 \dashrightarrow X$). However, using the Clemens–Griffiths criterion and a degeneration argument, one obtains that a general GM threefold is *not rational*.⁶ People believe that this should be true for *all* GM threefolds but, lacking a description of the singular locus of the theta divisor of the intermediate Jacobian, this question remains open.

Similarly, any GM fourfold X is unirational (there is a degree 2 rational map $\mathbf{P}^4 \dashrightarrow X$; see [DIM2, Proposition 3.1]). The question of their rationality is not settled and is very much like the case of cubic fourfolds in \mathbf{P}^5 : some rational examples are known (see below) and it is expected that a very general GM fourfold is not rational, but not a single example

⁶An alternate argument can be found in [DMo], where an explicit irrational GM threefold is constructed.

is known. Note that by Theorem 4.2, the rationality of a GM fourfold only depends on its associated Lagrangian.⁷

Example 4.4 (GM fourfolds containing a quintic del Pezzo surface). Let X be a GM fourfold. If $Y_A^3 \not\subset \mathbf{P}(V_5)$, the same argument as in the proof of Proposition 4.3 shows that X contains a quintic del Pezzo surface and is rational. More generally, using Theorem 4.2, one can show that X is rational as soon as $Y_A^3 \neq \emptyset$ (see [KP, Lemma 4.7] for more details). This is a codimension 1 condition in the moduli space \mathbf{M}^{EPW} for Lagrangians with no decomposable vectors: the set

$$\Delta := \{[A] \in \text{LGr}(\wedge^3 V_6) \mid Y_A^3 \neq \emptyset\} \quad (7)$$

is a $\text{PGL}(V_6)$ -invariant irreducible hypersurface. Likewise, this is a codimension 1 condition in the moduli space \mathbf{M}_4^{GM} for GM fourfolds.

Example 4.5 (GM fourfolds containing a σ -plane). A σ -plane is a plane of the type $P = \mathbf{P}(V_1 \wedge V_4) \subseteq \mathbf{P}(\wedge^2 V_5)$; it is contained in $\text{Gr}(2, V_5)$. Let X be a GM fourfold, corresponding to a Lagrangian $A \subseteq \wedge^3 V_6$ with no decomposable vectors, and a hyperplane $[V_5] \in Y_{A^\perp}^1$. One can show that X contains a σ -plane if and only if $Y_A^3 \cap \mathbf{P}(V_5) \neq \emptyset$. In particular, $[A]$ is in the hypersurface Δ defined in (7); if $[A]$ is general in Δ , the scheme $Y_A^{\geq 3}$ consists of a single point $[v]$ and one needs $[v] \in \mathbf{P}(V_5)$, or equivalently, $[V_5] \in Y_{A^\perp}^1 \cap \mathbf{P}(v^\perp)$. So this is a codimension 1 condition on A in \mathbf{M}^{EPW} and a codimension 2 condition on X in \mathbf{M}_4^{GM} .

It is easy to see directly that X is rational: the image of the linear projection $\pi_P: X \dashrightarrow \mathbf{P}^5$ from P is a smooth quadric $Q \subseteq \mathbf{P}^5$ and the induced morphism $\text{Bl}_P X \rightarrow Q$ is the blowup of a smooth degree 9 surface $\tilde{S} \subseteq Q$ (itself the blowup of a degree 10 K3 surface $S \subseteq \mathbf{P}^6$ at a point). This was known to Roth ([R, Section 4]) and Prokhorov ([Pr, Section 3]).

Example 4.6 (Other examples of rational GM fourfolds). The image of \mathbf{P}^2 by the linear system of quartic curves through three simple points and one double point in general position is a smooth surface of degree 9 in \mathbf{P}^8 . It was proved in [HS] that any GM fourfold containing such a surface is rational. This is a codimension 1 condition in \mathbf{M}_4^{GM} .

4.3. Linear spaces contained in GM varieties

For any partial flag $V_1 \subseteq V_{r+2} \subseteq V_5$, we consider the r dimensional linear space $\mathbf{P}(V_1 \wedge V_{r+2}) \subseteq \text{Gr}(r, V_5)$; it is said to be of σ -type. For $r \in \{1, 3\}$, all \mathbf{P}^r contained in $\text{Gr}(2, V_5)$ are of σ -type; for $r = 2$, all planes contained in $\text{Gr}(2, V_5)$ are either of σ -type (called σ -planes, and already considered in Example 4.5) or of the form $\text{Gr}(2, V_3)$ (called τ -planes).

Let X be an ordinary GM variety and let $F_r(X)$ be the Hilbert scheme of r dimensional linear spaces of σ -type contained in X . There is a well defined morphism

$$\begin{aligned} \sigma: F_r(X) &\longrightarrow \mathbf{P}(V_5) \\ [\mathbf{P}(V_1 \wedge V_{r+2})] &\longmapsto [V_1]. \end{aligned}$$

⁷Laurent Manivel, Vladimiro Benedetti, and Nicolas Perrin have announced a proof of the irrationality of very general GM fourfolds based on recent work of Iritani and Katzarkov–Kontsevich–Pantev–Yu.

We will relate these schemes $F_r(X)$ to the fibration $\rho: \mathbf{P}(\mathcal{U}_X) \rightarrow \mathbf{P}(V_5)$ defined in (6).

Proposition 4.7. *Let X be an ordinary GM variety of dimension $n \geq 3$, with associated Lagrangian data set (V_6, V_5, A) . The map $\sigma: F_r(X) \rightarrow \mathbf{P}(V_5)$ lifts to an isomorphism*

$$F_r(X) \simeq \text{Hilb}^{\mathbf{P}^r}(\mathbf{P}(\mathcal{U}_X)/\mathbf{P}(V_5))$$

with the relative Hilbert schemes of r dimensional linear spaces contained in the fibers of the morphism ρ .

Proof. Assume that $P = \mathbf{P}(V_1 \wedge V_{r+2})$ is contained in X . The inclusion $P \hookrightarrow X$ lifts to an inclusion $P \hookrightarrow \mathbf{P}(\mathcal{U}_X)$ by mapping $x \in P$ to $(x, [V_1])$, and the image is contained in the fiber $\rho^{-1}([V_1])$. Conversely, any \mathbf{P}^r contained in a fiber $\rho^{-1}([V_1])$ projects isomorphically onto a $\mathbf{P}^r = \mathbf{P}(V_1 \wedge V_{r+2})$ of σ -type contained in X . ■

Corollary 4.8. *Let X be an ordinary GM n -fold. One has $F_3(X) = \emptyset$ and, if $n \leq 3$, one has $F_2(X) = \emptyset$.*

Proof. By Proposition 4.7, any \mathbf{P}^3 contained in X is contained in a fiber $\rho^{-1}([v])$, which, by Proposition 4.1, is a quadric of corank ≤ 3 in \mathbf{P}^{n-2} if $v \notin \Sigma_1(X)$, or a quadric of corank ≤ 2 in \mathbf{P}^{n-1} if $v \in \Sigma_1(X)$. But no such quadric contains a \mathbf{P}^3 , hence $F_3(X) = \emptyset$. The reasoning for $F_2(X)$ is analogous. ■

One could also deduce Corollary 4.8 from Hodge theory and the Lefschetz theorem (see Section 5.1). The next results are more interesting; we assume $Y_A^3 = \emptyset$ to make the statement cleaner (but then we miss the case of GM fourfolds containing a σ -plane, as explained in Example 4.5).⁸

Theorem 4.9 (σ -planes on a GM variety). *Let X be a GM variety of dimension $n \in \{4, 5, 6\}$, with associated Lagrangian data set (V_6, V_5, A) . We assume $Y_A^3 = \emptyset$.*

- If $n = 6$, there is a factorization

$$\sigma: F_2(X) \xrightarrow{\tilde{\sigma}} \tilde{Y}_{A, V_5} \xrightarrow{f_{A, V_5}} Y_A \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{\sigma}$ is a \mathbf{P}^1 -bundle and f_{A, V_5} is an irreducible ramified double cover between threefolds. In particular, $F_2(X)$ is irreducible of dimension 4.

- If $n = 5$ and X is ordinary, there is a factorization

$$\sigma: F_2(X) \xrightarrow{g_{A, V_5}} Y_A^2 \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$

where g_{A, V_5} is a connected étale double cover. In particular, $F_2(X)$ is a connected curve.

⁸For a complete statement, see [DK3, Theorem 4.3]. For the description of the scheme of τ -planes contained in a GM variety, see [DK3, Theorem 4.5].

- If $n = 4$, one has $F_2(X) = \emptyset$.

Proof. This is again a direct consequence of Propositions 4.7 and 4.1. Let us do the case $n = 5$ and X ordinary, where $\Sigma_1(X) = \emptyset$. We have to look at planes contained in quadrics of corank $k \leq 2$ in \mathbf{P}^3 . They only exist if $k = 2$ and there are exactly 2 of them; the conclusion follows (except for the connectedness of $F_2(X)$, which has to be shown by other means). ■

The next theorem is proved similarly.

Theorem 4.10 (Lines on a GM variety). *Let X be a GM variety of dimension $n \in \{3, 4\}$, with associated Lagrangian data set (V_6, V_5, A) . We assume $Y_A^3 = \emptyset$.*

- If $n = 4$, the scheme $F_1(X)$ is integral of dimension 3 and there is a factorization

$$\sigma: F_1(X) \xrightarrow{\tilde{\sigma}} \tilde{Y}_{A, V_5} \xrightarrow{f_{A, V_5}} Y_A \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{\sigma}$ is an isomorphism over the complement of the inverse image by f_{A, V_5} of the point $\Sigma_1(X)$ of $Y_A \cap \mathbf{P}(V_5)$ and has \mathbf{P}^1 fibers over that set, and f_{A, V_5} is an irreducible ramified double cover.⁹

- If $n = 3$, the scheme $F_1(X)$ is reduced of pure dimension 1 and there is a factorization

$$\sigma: F_1(X) \xrightarrow{\tilde{\sigma}} Y_A^2 \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{\sigma}$ is an isomorphism over the complement of $Y_A^2 \cap \Sigma_1(X)$ and a double étale cover over that set.

In the next section, we will identify the double covers $g_{A, V_5}: F_2(X) \rightarrow Y_A^2 \cap \mathbf{P}(V_5)$ and $f_{A, V_5}: \tilde{Y}_{A, V_5} \rightarrow Y_A \cap \mathbf{P}(V_5)$ occurring in the theorems: they are induced by restriction to $\mathbf{P}(V_5)$ of canonical double covers $g_A: \tilde{Y}_A^2 \rightarrow Y_A^2$ and $f_A: \tilde{Y}_A \rightarrow Y_A$.

Remark 4.11. All GM varieties of dimension ≥ 3 contain lines ([DK3, Theorem 4.7]). All GM varieties of dimension ≥ 5 contain σ -planes and τ -planes ([DK3, Theorems 4.3 and 4.5]).

4.4. Double EPW sextics

We start from a Lagrangian $A \subseteq \wedge^3 V_6$ with no decomposable vectors and its associated EPW sextic $Y_A \subseteq \mathbf{P}(V_6)$ as defined by (3). O'Grady ([O1]) constructed a double cover

$$f_A: \tilde{Y}_A := \mathbf{Spec}(\mathcal{O}_{Y_A} \oplus \mathcal{R}) \longrightarrow Y_A, \quad (8)$$

where \mathcal{R} is a certain reflexive self-dual rank 1 sheaf on Y_A , as follows.

In the trivial symplectic vector bundle $\mathcal{V} := \wedge^3 V_6 \otimes \mathcal{O}_{\mathbf{P}(V_6)}$, one considers the Lagrangian subbundles \mathcal{A}_1 , with constant fiber $A \subseteq \wedge^3 V_6$, and \mathcal{A}_2 , with fiber $v \wedge \wedge^2 V_6$ at

⁹The same as in Theorem 4.9.

$[v] \in \mathbf{P}(V_6)$. With the notation of (5), the schemes $Y_A^{\geq \ell}$ are the *Lagrangian degeneration schemes*

$$D^\ell(\mathcal{A}_1, \mathcal{A}_2) := \{[v] \in \mathbf{P}(V_6) \mid \dim(\mathcal{A}_{1,[v]} \cap \mathcal{A}_{2,[v]}) \geq \ell\}.$$

More precisely, $D^\ell(\mathcal{A}_1, \mathcal{A}_2)$ is defined as the corank ℓ degeneracy locus of the morphism

$$\omega_{\mathcal{A}_1, \mathcal{A}_2}: \mathcal{A}_1 \hookrightarrow \mathcal{V} \xrightarrow{\sim} \mathcal{V}^\vee \twoheadrightarrow \mathcal{A}_2^\vee,$$

where the middle isomorphism is induced by the symplectic form on \mathcal{V} . For example, $D^1(\mathcal{A}_1, \mathcal{A}_2) = Y_A^{\geq 1} = Y_A$ is the scheme-theoretic support of the sheaf $\mathcal{C} := \text{Coker}(\omega_{\mathcal{A}_1, \mathcal{A}_2})$. At a point $[v] \in Y_A^1$, the morphism $\omega_{\mathcal{A}_1, \mathcal{A}_2}$ induces an isomorphism

$$\mathcal{A}_{1,[v]}/(\mathcal{A}_{1,[v]} \cap \mathcal{A}_{2,[v]}) \xrightarrow{\sim} (\mathcal{A}_{1,[v]} \cap \mathcal{A}_{2,[v]})^\perp \subseteq \mathcal{A}_{2,[v]}^\vee \quad (9)$$

hence the sheaf \mathcal{C} has rank 1 on Y_A^1 , with fiber $(\mathcal{A}_{1,[v]} \cap \mathcal{A}_{2,[v]})^\vee$ at $[v]$.

The fiber of the dual \mathcal{C}^\vee at a point $[v] \in Y_A^1$ is therefore $\mathcal{A}_{1,[v]} \cap \mathcal{A}_{2,[v]}$, and the map (9) induces an isomorphism between $\mathcal{A}_1/\mathcal{C}^\vee$ and $(\mathcal{A}_2/\mathcal{C}^\vee)^\vee$ on Y_A^1 , hence between their determinants, which are $\mathcal{C}^{\vee\vee}$ and $\mathcal{C}^\vee(-6)$ (because \mathcal{A}_1 is trivial and $\det(\mathcal{A}_2) \simeq \mathcal{O}(6)$). If we set

$$\mathcal{R} := \mathcal{C}^{\vee\vee}(-3),$$

this gives an isomorphism between the invertible sheaf $\mathcal{R}|_{Y_A^1}$ and its dual. Since \mathcal{R} and \mathcal{R}^\vee are reflexive and $Y_A^{\geq 2}$ has codimension 2 in Y_A (Theorem 3.5), this self-duality extends uniquely to a self-duality on Y_A , which we see as a symmetric multiplication map

$$\mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{O}_{Y_A}$$

that defines an \mathcal{O}_{Y_A} -algebra structure on $\mathcal{O}_{Y_A} \oplus \mathcal{R}$ hence the double cover f_A in (8).

The singularities of Y_A and \tilde{Y}_A were described in [O3].

Theorem 4.12 (O’Grady). *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian with no decomposable vectors. The morphism $f_A: \tilde{Y}_A \rightarrow Y_A$ is branched along the integral normal surface $Y_A^{\geq 2}$, which is the singular locus of Y_A . The fourfold \tilde{Y}_A , called a double EPW sextic, is irreducible and its singular locus is the finite set $f_A^{-1}(Y_A^3)$.*

Using similar ideas (see [DK5]), one can also construct a canonical irreducible double cover

$$g_A: \tilde{Y}_A^{\geq 2} \longrightarrow Y_A^{\geq 2} \quad (10)$$

between surfaces, branched along the finite set Y_A^3 , such that

$$(g_A)_* \mathcal{O}_{\tilde{Y}_A^{\geq 2}} \simeq \mathcal{O}_{Y_A^{\geq 2}} \oplus \omega_{Y_A^{\geq 2}}(-3).$$

The singular locus of the surface $\tilde{Y}_A^{\geq 2}$ is the finite set $f_A^{-1}(Y_A^3)$. When $Y_A^3 = \emptyset$, the surface $\tilde{Y}_A^{\geq 2}$ is smooth of general type of irregularity $h^1(\tilde{Y}_A^2, \mathcal{O}_{\tilde{Y}_A^2}) = 10$.

Finally, one can prove (see [DK5]) that the double covers $g_{A, V_5}: F_2(X) \rightarrow Y_A^2 \cap \mathbf{P}(V_5)$ and $f_{A, V_5}: \tilde{Y}_{A, V_5} \rightarrow Y_A \cap \mathbf{P}(V_5)$ that appear in Theorems 4.9 and 4.10 are the restrictions to $\mathbf{P}(V_5)$ of the canonical double covers $g_A: \tilde{Y}_A^2 \rightarrow Y_A^2$ and $f_A: \tilde{Y}_A \rightarrow Y_A$.

Corollary 4.13. *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3 = \emptyset$. Let X be the ordinary GM fourfold associated with a general $[V_5]$ in Y_{A^\perp} .*

- *The threefold \tilde{Y}_{A, V_5} has two singular points.*
- *The threefold $F_1(X)$ is smooth irreducible and $\tilde{\sigma}: F_1(X) \rightarrow \tilde{Y}_{A, V_5}$ is a small resolution.*

Proof. We already mentioned that Y_{A^\perp} is the projective dual of Y_A . Projective duality implies that the inverse image by $\tilde{Y}_A \rightarrow Y_A \subseteq \mathbf{P}(V_6)$ of a general $[V_5]$ in Y_{A^\perp} has at least two singular points. More precisely, the single point $[v]$ of $\Sigma_1(X)$ is in $Y_A^1 \cap \mathbf{P}(V_5)$ (see Proposition 4.1) and the singular locus of \tilde{Y}_{A, V_5} consists of its two preimages in \tilde{Y}_A . The conclusion then follows from Theorem 4.10 and the verification via a tangent space computation that $F_1(X)$ is smooth. ■

Corollary 4.14. *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3 = \emptyset$. Let X be the ordinary GM fivefold associated with a general hyperplane $V_5 \subseteq V_6$.*

The curve $F_2(X)$ is smooth connected of genus 161 and $\tilde{\sigma}: F_2(X) \rightarrow Y_A^2 \cap \mathbf{P}(V_5)$ is an étale double cover.

Proof. Since V_5 is general and Y_A^2 is smooth, so is $Y_A^2 \cap \mathbf{P}(V_5)$ by Bertini's theorem. The conclusion then follows from Theorem 4.9 and a direct computation of the genus using the numerical invariants of the smooth surface Y_A^2 . ■

Corollary 4.15. *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3 = \emptyset$. Let X be the GM sixfold associated with a general hyperplane $V_5 \subseteq V_6$.*

The threefold $F_2(X)$ is smooth irreducible and $\tilde{\sigma}: F_2(X) \rightarrow f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ is a \mathbf{P}^1 -fibration.

Proof. Since V_5 is general and \tilde{Y}_A is smooth, so is $\tilde{Y}_{A, V_5} = f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ by Bertini's theorem. The conclusion then follows from Theorem 4.9. ■

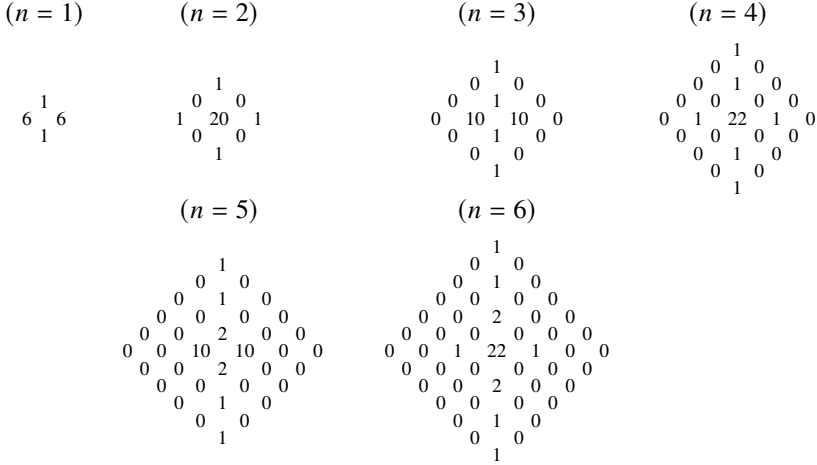
Remark 4.16. When X is a general GM threefold, $F_1(X)$ is a smooth connected curve of genus 71 ([IP, Theorem 4.2.7]), $Y_A^2 \cap \mathbf{P}(V_5)$ is an irreducible curve with 10 nodes as singularities, and $\tilde{\sigma}: F_1(X) \rightarrow Y_A^2 \cap \mathbf{P}(V_5)$ is its normalization.

5. Periods of Gushel–Mukai varieties

5.1. Cohomology and period maps of GM varieties

With some work, the Hodge numbers of GM varieties can be computed. The references for the next proposition are [DK1] and [DK3, Propositions 3.1 and 3.4].

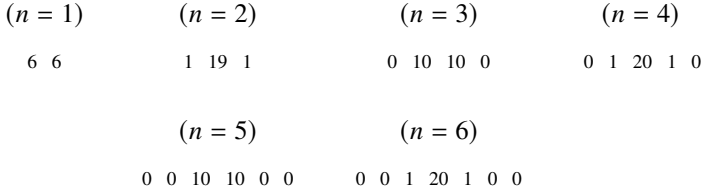
Proposition 5.1. *The integral cohomology of a GM variety of dimension n is torsionfree and its Hodge diamond is*



In particular, only the middle cohomology $H^n(X; \mathbf{Z})$ is interesting: in other degrees, it is induced from the cohomology of $\text{Gr}(2, V_5)$ via the Gushel map $\gamma: X \rightarrow \text{Gr}(2, V_5)$. We define the *vanishing cohomology* as

$$H^n(X; \mathbf{Z})_{\text{van}} := (\gamma^* H^n(\text{Gr}(2, V_5); \mathbf{Z}))^\perp \subseteq H^n(X; \mathbf{Z}).$$

The Hodge numbers for the vanishing cohomology are therefore



In other words,

- when $n \in \{3, 5\}$, this Hodge structure has weight 1 and there is a 10 dimensional principally polarized intermediate Jacobian

$$J(X) := H^n(X, \mathbf{C}) / (H^{(n+1)/2, (n-1)/2}(X) + H^n(X; \mathbf{Z}))$$

and a period map

$$\begin{aligned} \wp_n: \mathbf{M}_n^{\text{GM}} &\longrightarrow \mathcal{A}_{10} \\ [X] &\longmapsto [J(X)], \end{aligned}$$

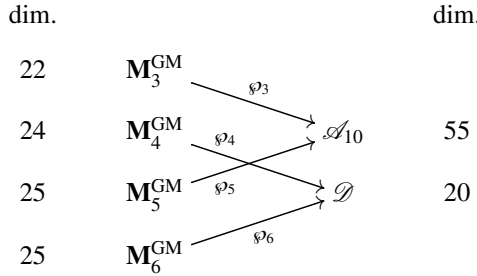
where \mathcal{A}_{10} is the moduli space for principally polarized abelian varieties of dimension 10, a 55 dimensional quasiprojective variety ;

- when $n \in \{4, 6\}$, this Hodge structure is of K3 type and there is a period map

$$\begin{aligned} \wp_n : \mathbf{M}_n^{\text{GM}} &\longrightarrow \mathcal{D} \\ [X] &\longmapsto [H^{n/2+1, n/2-1}(X)], \end{aligned}$$

where \mathcal{D} is a 20 dimensional quasiprojective variety (the same for both $n = 4$ and $n = 6$) which will be defined in Section 5.5.

None of these period maps



are injective. They are dominant when n is even. To analyze them further, we need to go back to the double EPW sextics introduced in Section 4.4.

Remark 5.2 (The Hodge conjectures for GM varieties). The *rational* Hodge conjecture holds for all GM varieties ([FM, Theorem 5.1]).

The *integral* Hodge conjecture holds for all GM varieties of dimension $n \in \{3, 5\}$, because the Hodge classes all come from the Grassmannian. It also holds for all GM fourfolds (see [P, Corollary 1.2], [PPZ, Remark 1.6]). It seems to be still unknown for $(3, 3)$ -classes on GM sixfolds (see [P, Corollary 8.4] for a partial result).

5.2. Double EPW sextics and hyper-Kähler fourfolds

The reason why O’Grady made the construction (explained in Section 4.4) of double EPW sextics is that they provide examples of hyper-Kähler fourfolds.

Definition 5.3. A hyper-Kähler variety is a smooth, compact, simply connected, Kähler variety whose space of holomorphic 2 forms is generated by a symplectic form.

Hyper-Kähler varieties of dimension 2 are K3 surfaces. If S is a K3 surface, it was shown by Beauville that the Douady space $S^{[n]}$ parametrizing length n subschemes of S is a hyper-Kähler variety of dimension $2n$. The main result of [O1] is the following (the double EPW sextic \tilde{Y}_A was defined in Theorem 4.12).

Theorem 5.4. *Let $A \subseteq \wedge^3 V_6$ be a Lagrangian with no decomposable vectors and such that $Y_A^3 = \emptyset$. The double EPW sextic \tilde{Y}_A is a hyper-Kähler fourfold.*

Moreover, O’Grady proved that \tilde{Y}_A is a deformation of the Douady square $S^{[2]}$ of a K3 surface S , so that its Hodge numbers are known by a computation of Beauville: for $H^2(\tilde{Y}_A)$,

they are

$$1 \quad 21 \quad 1.$$

Beauville also defined in [B, Théorème 5] an integral, nondegenerate quadratic form q_{BB} on $H^2(\tilde{Y}_A; \mathbf{Z})$. The class $h_A := f_A^* \mathcal{O}_{\mathbf{P}(V_6)}(1)$ is ample on \tilde{Y}_A with $q_{BB}(h_A) = 2$, and we define the *primitive cohomology* as the orthogonal complement

$$H^2(\tilde{Y}_A; \mathbf{Z})_{\text{prim}} := h_A^\perp \subseteq H^2(\tilde{Y}_A; \mathbf{Z}).$$

The primitive cohomology gives rise to a period map

$$\begin{aligned} \wp : \mathbf{M}^{\text{EPW}} &\longrightarrow \mathcal{D} \\ [A] &\longmapsto [H^{2,0}(\tilde{Y}_A)], \end{aligned}$$

where \mathcal{D} is the period domain already mentioned in Section 5.1 and which will be defined in Section 5.5 (strictly speaking, this map is only defined on the complement of the hypersurface Δ defined in (7)—where \tilde{Y}_A is smooth—but O’Grady showed in [O3] that it extends over Δ). By Verbitsky’s global Torelli Theorem, Markman’s monodromy results, and [O3, Section 5.7], *the map \wp is an open embedding*.

5.3. Factorization of the period maps of GM varieties

We explain in this section that the period map $\wp_n : \mathbf{M}_n^{\text{GM}} \rightarrow (\mathcal{A}_{10} \text{ ou } \mathcal{D})$ for GM n -folds, defined in Section 5.1, factors through the surjective morphism

$$\pi_n : \mathbf{M}_n^{\text{GM}} \longrightarrow \mathbf{M}^{\text{EPW}}$$

defined in Theorem 3.7 that sends a GM variety X of dimension n to its associated Lagrangian $A \subseteq \wedge^3 V_6$.

For that, we relate the Hodge structure $H^n(X)_{\text{van}}$ to the Hodge structure $H^2(\tilde{Y}_A)_{\text{prim}}$ when $n \in \{4, 6\}$, and to the Hodge structure $H^1(\tilde{Y}_A^2)$ when $n \in \{3, 5\}$.

Theorem 5.5 ([DK3, DK6]). *Let X be a GM variety of dimension $n \in \{3, 4, 5, 6\}$ with associated Lagrangian A . We assume $Y_A^3 = \emptyset$.¹⁰*

(a) *When $n \in \{4, 6\}$, there is an isomorphism of polarized Hodge structures*

$$(H^n(X; \mathbf{Z})_{\text{van}}, \smile) \xrightarrow{\sim} (H^2(\tilde{Y}_A; \mathbf{Z})_{\text{prim}}, (-1)^{n/2-1} q_{BB}).$$

(b) *There is a canonical principal polarization on the abelian variety $\text{Alb}(\tilde{Y}_A^2)$ and, when $n \in \{3, 5\}$, there is an isomorphism*

$$J(X) \xrightarrow{\sim} \text{Alb}(\tilde{Y}_A^2)$$

of principally polarized abelian varieties.

¹⁰This hypothesis ensures that both \tilde{Y}_A and \tilde{Y}_A^2 are smooth.

When $n \in \{4, 6\}$, the period map $\varphi_n : \mathbf{M}_n^{\text{GM}} \rightarrow \mathcal{D}$ therefore factors as

$$\begin{array}{ccc}
 \text{dim.} & & \text{dim.} \\
 24 & \mathbf{M}_4^{\text{GM}} & \xrightarrow{\pi_4} \\
 & & \searrow \\
 & & \mathbf{M}^{\text{EPW}} \hookrightarrow \mathcal{D} \\
 & & \xrightarrow{\varphi} \\
 25 & \mathbf{M}_6^{\text{GM}} & \xrightarrow{\pi_6}
 \end{array} \quad (11)$$

where φ is the (extended) period map for double EPW sextics. When $n \in \{3, 5\}$, the period map $\varphi_n : \mathbf{M}_n^{\text{GM}} \rightarrow \mathcal{A}_{10}$ factors as

$$\begin{array}{ccc}
 \text{dim.} & & \text{dim.} \\
 22 & \mathbf{M}_3^{\text{GM}} & \xrightarrow{\pi_3} \\
 & & \searrow \\
 & & \mathbf{M}^{\text{EPW}} \longrightarrow \mathbf{M}^{\text{EPW}}/r \xrightarrow{q} \mathcal{A}_{10} \\
 & & \nearrow \\
 25 & \mathbf{M}_5^{\text{GM}} & \xrightarrow{\pi_5}
 \end{array} \quad (12)$$

where r is the (nontrivial) duality involution of \mathbf{M}^{EPW} (see [DK6, Lemma 6.1]). The morphism q is known to be unramified ([DIM1]) and is expected to be (generically) injective.

5.4. Sketches of proofs

The standard argument for this kind of results goes back to Clemens–Griffiths ([CG]) and Beauville–Donagi ([BD]), who treated the case of a smooth cubic fourfold $W \subseteq \mathbf{P}(V_6)$ and its (smooth) hyper-Kähler fourfold of lines $F_1(W) \subseteq \text{Gr}(2, V_6)$. In that case, there is an incidence diagram

$$\begin{array}{ccc}
 & I & \\
 q \swarrow & & \searrow p \\
 W & & F_1(W),
 \end{array}$$

where p is the universal line (a \mathbf{P}^1 -bundle) and q is dominant and generically finite. Beauville and Donagi proved that the Abel–Jacobi map $p_*q^* : H^4(W; \mathbf{Z}) \rightarrow H^2(F_1(W); \mathbf{Z})$ is an isomorphism of Hodge structures which induces an isometry between the primitive cohomologies $(H^4(W; \mathbf{Z})_{\text{prim}}, \smile)$ and $(H^2(F_1(W); \mathbf{Z})_{\text{prim}}, -q_{BB})$. The two ingredients used in this argument are:

- $F_1(W)$ parametrizes curves on W , so there is a correspondence between W and $F_1(W)$;
- $F_1(W)$ and I are smooth, so one can define p_* in singular cohomology.

We follow a similar approach, using Hilbert schemes of lines or σ -planes on GM varieties, depending on the dimension. In what follows, X is a GM n -fold with associated Lagrangian A . We assume $Y_A^3 = \emptyset$.

Dimension 4. By Corollary 4.13, for $[V_5]$ general in Y_{A^+} , the scheme of lines $F_1(X)$ is a smooth threefold and $\tilde{\sigma}: F_1(X) \rightarrow f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ is a small resolution. One shows that the induced composition

$$a: H^2(\tilde{Y}_A; \mathbf{Z}) \xrightarrow{\sim} H^2(f_A^{-1}(Y_A \cap \mathbf{P}(V_5)); \mathbf{Z}) \xrightarrow{\tilde{\sigma}^*} H^2(F_1(X); \mathbf{Z})$$

(where the first map is an isomorphism by the Lefschetz theorem) is injective.

The Abel–Jacobi map $p_*q^*: H^4(X; \mathbf{Z})_{\text{van}} \rightarrow H^2(F_1(X); \mathbf{Z})$ is also injective (as in [BD]) and induces an antiisometry between $H^4(X; \mathbf{Z})_{\text{van}}$ and the subspace $a(H^2(\tilde{Y}_A; \mathbf{Z})_{\text{prim}}) \subseteq H^2(F_1(X); \mathbf{Z})$ (see [DK3, Section 5.1] for more details).

Dimension 6. For a general GM sixfold X , the proof is similar: one uses instead the smooth fourfold $F_2(X)$ parametrizing σ -planes contained in X , and Corollary 4.15 (see [DK3, Section 5.2] for more details).

Dimension 5. By Corollary 4.14, for $[V_5]$ general in Y_{A^+} , the curve $F_2(X)$ is a connected double étale cover of a general genus 81 hyperplane section of the smooth surface $Y_A^2 \subseteq \mathbf{P}(V_6)$. A generalization of an old argument of Clemens (written by Tjurin in [T]) shows that the corresponding Abel–Jacobi map $q_*p^*: H_1(F_2(X); \mathbf{Z}) \rightarrow H_5(X; \mathbf{Z})$ in homology is surjective. It induces a surjective morphism

$$a: J(F_2(X)) \longrightarrow J(X)$$

with connected kernel. By the Lefschetz theorem, there is another surjective morphism

$$b: J(F_2(X)) \longrightarrow \text{Alb}(\tilde{Y}_A^2)$$

with connected kernel. We want to show that the morphisms a and b are the same.

We use a trick: it follows from Deligne–Picard–Lefschetz theory that since the surface Y_A^2 is regular, for a very general choice of hyperplane V_5 , the Jacobian $J(Y_A^2 \cap \mathbf{P}(V_5))$ is simple (of dimension 81). It is therefore contracted by both a and b , and this induces surjective morphisms

$$a': \text{Prym} \longrightarrow J(X) \quad \text{and} \quad b': \text{Prym} \longrightarrow \text{Alb}(\tilde{Y}_A^2)$$

with connected kernels.¹¹ Using again monodromy arguments, one shows that the kernel of b' is simple (of dimension 70). It is therefore contracted by a' , and this induces an isomorphism

$$\text{Alb}(\tilde{Y}_A^2) \xrightarrow{\sim} J(X)$$

(see [DK6, Section 5] for more details).

¹¹Here, $\text{Prym} := J(F_2(X))/J(Y_A^2 \cap \mathbf{P}(V_5))$ is the 80 dimensional Prym variety associated with the double étale cover of curves $F_2(X) \rightarrow Y_A^2 \cap \mathbf{P}(V_5)$.

Dimension 3. By Remark 4.16, lines on a general GM threefold X are parametrized by a smooth connected curve $F_1(X)$ of genus 71 which is the normalization of the singular curve $Y_A^2 \cap \mathbf{P}(V_5)$ (the hyperplane V_5 is not general anymore), but we were unable to relate this curve to the surface \widetilde{Y}_A^2 .

However, it was proved by Logachev in [L] that the Hilbert scheme of conics contained in a general GM threefold X is the blowup of the smooth surface $\widetilde{Y}_{A^\perp}^2$ at a point.¹² This gives an Abel–Jacobi map $\text{Alb}(\widetilde{Y}_{A^\perp}^2) \rightarrow J(X)$ which should be an isomorphism.¹³

We proceed differently in [DK6] and prove instead that the Abel–Jacobi map associated with a family of rational quartic curves parametrized by the surface \widetilde{Y}_A^2 gives the desired isomorphism $\text{Alb}(\widetilde{Y}_A^2) \xrightarrow{\sim} J(X)$ (see [DK6, Section 4] for more details).

5.5. Hodge special GM fourfolds

Let us first explain what the period domain \mathcal{D} is. For any GM fourfold X , the unimodular cohomology lattice $(H^4(X; \mathbf{Z}), \smile)$ is isomorphic to $(1)^{\oplus 22} \oplus (-1)^{\oplus 2}$ and the vanishing cohomology lattice $(H^4(X; \mathbf{Z})_{\text{van}}, \smile)$ to the even lattice

$$\Lambda := E_8^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

where E_8 is the rank 8 positive definite even unimodular lattice. It has signature $(20, 2)$.

Similarly, for any Lagrangian A , the cohomology lattice $(H^2(\widetilde{Y}_A; \mathbf{Z}), q_{BB})$ is isomorphic to the lattice

$$E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus (-2),$$

and the primitive cohomology lattice $(H^2(\widetilde{Y}_A; \mathbf{Z})_{\text{prim}}, q_{BB})$ is isomorphic to $\Lambda(-1)$ (in accordance with Theorem 5.5(a)).

The manifold

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0, (\omega \cdot \bar{\omega}) < 0\}$$

is a homogeneous space with two components, Ω^+ and Ω^- , both isomorphic to the 20 dimensional open complex manifold $\text{SO}_0(20, 2)/\text{SO}(20) \times \text{SO}(2)$, a bounded symmetric domain of type IV acted on by the isometry group $O(\Lambda)$ of the lattice Λ . The quotient

$$\mathcal{D} := \widetilde{O}(\Lambda) \backslash \Omega^+,$$

where $\widetilde{O}(\Lambda) \subseteq O(\Lambda)$ is a subgroup of index 2 called the *stable orthogonal group*, has the structure of an irreducible quasiprojective variety of dimension 20 (see [DIM2, Section 5] for more details). It is a period domain for both GM fourfolds and double EPW sextics.

¹²For a complete description of the Hilbert scheme of conics contained in any GM threefold, see [DK7, Theorems 1.1 and 7.3].

¹³Note that by [DK6, Theorem 1.1], the smooth surfaces \widetilde{Y}_A^2 and $\widetilde{Y}_{A^\perp}^2$ have isomorphic Albanese varieties, although they are in general not isomorphic.

The domain \mathcal{D} carries a nontrivial canonical involution $r_{\mathcal{D}}$ corresponding to the double cover $\mathcal{D} \rightarrow O(\Lambda) \setminus \Omega^+$. If $\wp: \mathbf{M}^{\text{EPW}} \rightarrow \mathcal{D}$ is the (extended) period map for double EPW sextics, it is related to the duality involution r of \mathbf{M}^{EPW} defined in Section 3.5 by the relation

$$\wp \circ r = r_{\mathcal{D}} \circ \wp$$

proved in [O2].

The fact that the period map \wp is dominant easily implies that the Picard group of a very general EPW sextic \tilde{Y}_A is generated by the class of the polarization $h_A = f_A^* \mathcal{O}_{\mathbf{P}(V_6)}(1)$. Indeed, for $H^{1,1}(\tilde{Y}_A) \cap H^2(\tilde{Y}_A; \mathbf{Z})_{\text{prim}}$ to be nonzero, the corresponding period must be in one of the (countably many) loci $\alpha^\perp \cap \mathcal{D}$, for some nonzero $\alpha \in \Lambda$. We label these loci as \mathcal{D}_d , where d is the discriminant of the lattice $\alpha^\perp \subseteq \Lambda$, called the *nonspecial lattice*. The Picard group of a double EPW sextic whose period point is a very general point of \mathcal{D}_d (more precisely, it is not in any other $\mathcal{D}_{d'}$, for $d' \neq d$) has rank 2: it is the saturation of the subgroup $\mathbf{Z}h_A \oplus \mathbf{Z}\alpha$ of $H^2(\tilde{Y}_A; \mathbf{Z})$.

The following result is [DIM2, Lemma 6.1 and Corollary 6.3].

Proposition 5.6. *The locus \mathcal{D}_d is nonempty if and only if $d > 0$ and*

- *either $d \equiv 0 \pmod{4}$, in which case $\mathcal{D}_d \subseteq \mathcal{D}$ is an irreducible hypersurface;*
- *or $d \equiv 2 \pmod{8}$, in which case $\mathcal{D}_d \subseteq \mathcal{D}$ is the union of two irreducible hypersurfaces \mathcal{D}'_d and \mathcal{D}''_d , which are interchanged by the involution $r_{\mathcal{D}}$.*

It is also known that

- *the image of the (extended) period map $\wp: \mathbf{M}^{\text{EPW}} \rightarrow \mathcal{D}$ does not meet the hypersurface $\mathcal{D}_2 \cup \mathcal{D}_4 \cup \mathcal{D}_8$ ([O5, Theorem 1.3], [DM, Example 6.3]) and is conjectured to be equal to its complement $\mathcal{D} \setminus (\mathcal{D}_2 \cup \mathcal{D}_4 \cup \mathcal{D}_8)$;*
- *the image of the hypersurface Δ defined in (7) is the hypersurface \mathcal{D}''_{10} (see Example 5.7 below for a proof);*
- *the rational map $\bar{\mathbf{M}}^{\text{EPW}} \dashrightarrow \mathcal{D}$ induced by \wp is defined at general points of the hypersurface $\bar{\mathbf{M}}^{\text{EPW}} \setminus \mathbf{M}^{\text{EPW}}$ corresponding to Lagrangians with decomposable vectors and the image of this hypersurface is the hypersurface \mathcal{D}_8 (see [DK3, Remark 5.29]).*

This whole discussion translates in terms of GM fourfolds.¹⁴ We say that a GM fourfold X is *Hodge-special* if its period is in $\bigcup_d \mathcal{D}_d$. For X nonHodge-special, we have

$$H^{2,2}(X) \cap H^4(X; \mathbf{Z}) = \gamma^* H^4(\text{Gr}(2, V_5); \mathbf{Z})$$

(a lattice of rank 2) and, if X is very general in a hypersurface $\wp_4^{-1}(\mathcal{D}_d)$, the lattice $H^{2,2}(X) \cap H^4(X; \mathbf{Z})$ of Hodge (2, 2) classes has rank 3 and discriminant d .

Because of diagram (11), the images of the period maps \wp and

$$\wp_4: \mathbf{M}_4^{\text{GM}} \longrightarrow \mathcal{D}$$

¹⁴It also applies to GM sixfolds, but we will not discuss that aspect here.

are the same. The examples below, which describe geometrically certain families of GM fourfolds whose periods dominate the hypersurfaces \mathcal{D}''_{10} , \mathcal{D}'_{10} , and \mathcal{D}_{20} of \mathcal{D} , are taken from [DIM2] and [DK3, Remark 5.29].

Example 5.7 (GM fourfolds containing a σ -plane). This is a continuation of Example 4.5. Let $P = \mathbf{P}(V_1 \wedge V_4)$ be a σ -plane contained in a GM fourfold X . Its class in $\mathrm{Gr}(2, V_5)$ is the Schubert class $\sigma_{3,1}$ and one computes (see [DIM2, Section 7.1]) the intersection numbers

$$(\gamma^* \sigma_{1,1} \cdot P)_X = (\sigma_{1,1} \cdot P)_{\mathrm{Gr}(2, V_5)} = 0, \quad (\gamma^* \sigma_2 \cdot P)_X = (\sigma_2 \cdot P)_{\mathrm{Gr}(2, V_5)} = 1, \quad (P \cdot P)_X = 3$$

of classes in $H^4(X; \mathbf{Z})$. It follows that the intersection form on the rank 3 sublattice

$$K := \langle \gamma^* H^4(\mathrm{Gr}(2, V_5); \mathbf{Z}), [P] \rangle = \langle \gamma^* \sigma_{1,1}, \gamma^* \sigma_2, [P] \rangle$$

is given by the matrix

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{pmatrix},$$

whose determinant is 10. The orthogonal complement of $[P]$ in $H^4(X; \mathbf{Z})_{\mathrm{van}}$ is the same as the orthogonal complement of the primitive sublattice K in the unimodular lattice $H^4(X; \mathbf{Z})$, and they have the same discriminant as K . The period of X therefore belongs to the hypersurface \mathcal{D}_{10} , and more precisely to the component \mathcal{D}''_{10} (the comment after [DIM2, Corollary 6.3] explains how we label the two components). Since the periods of these fourfolds are dense in the image of Δ in \mathcal{D} (see Example 4.5), we see that this image is \mathcal{D}''_{10} .

Example 5.8 (GM fourfolds containing a quintic del Pezzo surface). This is a continuation of Example 4.4. We consider quintic del Pezzo surfaces obtained as the intersection of $\mathrm{Gr}(2, V_5)$ with a \mathbf{P}^5 ; their class is $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$ in $\mathrm{Gr}(2, V_5)$. Let X be a GM fourfold containing such a surface S . One computes (see [DIM2, Section 7.5]) the intersection numbers

$$(\gamma^* \sigma_{1,1} \cdot S)_X = (\sigma_{1,1} \cdot \sigma_1^4)_{\mathrm{Gr}(2, V_5)} = 2, \quad (\gamma^* \sigma_2 \cdot S)_X = (\sigma_2 \cdot \sigma_1^4)_{\mathrm{Gr}(2, V_5)} = 3, \quad (S \cdot S)_X = 5$$

of classes in $H^4(X; \mathbf{Z})$. The intersection form on the rank 3 sublattice $\langle \gamma^* \sigma_{1,1}, \gamma^* \sigma_2, [S] \rangle$ is therefore given by the matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix},$$

whose determinant is 10. Again, the period of X belongs to the hypersurface \mathcal{D}''_{10} (this was expected since the associated Lagrangians are the same as the Lagrangians associated with GM fourfolds containing a σ -plane). More precisely, GM fourfolds containing a quintic del Pezzo surface form a dense subset of $\varphi_4^{-1}(\mathcal{D}''_{10})$.

Example 5.9 (GM fourfolds containing a τ -quadric surface). A τ -quadric surface Q in the Grassmannian $\text{Gr}(2, V_5)$ is a linear section of a $\text{Gr}(2, V_4)$ for some $V_4 \subseteq V_5$; its class in $\text{Gr}(2, V_5)$ is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$. If X is a GM fourfold containing Q , one computes as above (see [DIM2, Section 7.3]) that the intersection form on the rank 3 sublattice $\langle \gamma^* \sigma_{1,1}, \gamma^* \sigma_2, [Q] \rangle$ is given by the matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix},$$

whose determinant is 10. This time however, the period of X belongs to the hypersurface \mathcal{D}'_{10} and GM fourfolds containing a τ -quadric surface form a dense subset of $\wp_4^{-1}(\mathcal{D}'_{10})$. According to Theorem 4.2, these fourfolds, like those containing a σ -plane, are all rational; this can also be seen directly by projecting X from the \mathbf{P}^3 spanned by Q .

Example 5.10. The GM fourfolds constructed in Example 4.6 were shown in [HS] to fill out a dense subset of $\wp_4^{-1}(\mathcal{D}_{20})$.

5.6. K3 surfaces associated with GM fourfolds

For some integers d , the nonspecial lattice is isomorphic to the primitive lattice of a polarized K3 surface (with the sign of the intersection form reversed), necessarily of degree d . The precise lattice-theoretic result is the following ([DIM2, Proposition 6.6]).

Proposition 5.11. *The positive integers d for which the nonspecial lattice is isomorphic to the (opposite of the) primitive cohomology lattice of a degree d polarized K3 surface are the positive integers $d \equiv 2$ or $4 \pmod{8}$ such that the only odd primes that divide d are $\equiv 1 \pmod{4}$.*

The first values of d that satisfy the conditions of the proposition and for which the hypersurface \mathcal{D}_d meets the image of the period map are 10, 20, and 26.

Let X be a GM fourfold whose period point belongs to a hypersurface \mathcal{D}_d , for some positive integer d satisfying the conditions of Proposition 5.11. Since the period map for semipolarized K3 surfaces is bijective, the period point of X in \mathcal{D} is the period point of a unique K3 surface S with a nef (ample when the period of X is general in \mathcal{D}_d) algebraic class h such that $h^2 = d$, so that the polarized Hodge structures K^\perp and $H^2(S)_{\text{prim}}(-1)$ are isomorphic. We say that the K3 surface S is *associated* with X . By Theorem 5.5, this surface only depends on the Lagrangian associated with X .

Example 5.12. In some cases, the associated K3 surface can be recovered geometrically from the GM fourfold X . This is the case in Example 4.5, and here is another example: when $[X]$ is general in $\wp_4^{-1}(\mathcal{D}'_{10})$, it contains a τ -quadric surface Q (see Example 5.9) and it was shown in the proof of [DIM2, Proposition 7.3] that the projection $\text{Bl}_Q X \rightarrow \mathbf{P}^4$ from the \mathbf{P}^3 spanned by Q is the blowup of the surface in \mathbf{P}^4 obtained as the projection of a degree 10 K3 surface $S \subseteq \mathbf{P}^6$ from the line joining two points of S .

This surface S is the K3 surface associated with X . It can be constructed in several other ways: the dual A^\perp of the Lagrangian A associated with X is in the hypersurface Δ defined in (7), so that $Y_{A^\perp}^3 \neq \emptyset$. Since $[X]$ is general in $\wp_4^{-1}(\mathcal{D}'_{10})$, the set $Y_{A^\perp}^3$ consists of only one point and the associated GM variety has dimension 2 (see Theorem 3.2): this is the K3 surface S again; finally, a small resolution of the singular double EPW sextic \tilde{Y}_{A^\perp} is isomorphic to the Hilbert square $S^{[2]}$ (this is all explained in [O3]).

The reason why this strange notion of associated K3 surface is interesting lies in the following conjecture, originally envisioned for cubic fourfolds by Harris–Hassett.

Conjecture 5.13. A GM fourfold is rational if and only if it has an associated K3 surface.

This conjecture implies in particular that a very general GM fourfold is not rational. We will discuss it at the end of the next section.

6. Derived categories

For a very nice geometrically oriented introduction to derived categories, I refer to [Hu]. For very detailed lecture notes on the topics that concern us here, but in the case of cubic fourfolds, I refer to [K3]. I will only give a very brief and superficial overview of the situation for GM varieties.

6.1. Semiorthogonal decompositions

Let \mathcal{T} be a (\mathbf{C} -linear) triangulated category. We say that a pair $(\mathcal{A}_1, \mathcal{A}_2)$ of full triangulated subcategories of \mathcal{T} forms a (two-step) *semiorthogonal decomposition* (s.o.d. for short), and we write

$$\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle,$$

if

- $\text{Hom}(A_2, A_1) = 0$ for any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$;
- \mathcal{A}_1 and \mathcal{A}_2 generate \mathcal{T} , in the sense that for any $T \in \mathcal{T}$, there is a (unique and functorial) distinguished triangle

$$A_2 \rightarrow T \rightarrow A_1 \rightarrow A_2[1],$$

with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

In that case, one shows ([K3, Lemma 2.2]) that \mathcal{A}_2 is a *right admissible subcategory* of \mathcal{T} : the inclusion functor $\alpha_2: \mathcal{A}_2 \rightarrow \mathcal{T}$ has a right adjoint $\alpha_2^!: \mathcal{T} \rightarrow \mathcal{A}_2$, given by $T \mapsto A_2$, that satisfies $\alpha_2^! \circ \alpha_2 \simeq \text{Id}_{\mathcal{A}_2}$.

Conversely, any right admissible triangulated functor $\alpha: \mathcal{A} \rightarrow \mathcal{T}$ is fully faithful and there is an s.o.d.

$$\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle,$$

where $\mathcal{A}^\perp := \text{Ker}(\alpha^!)$ is the *right orthogonal* of \mathcal{A} .

This terminology can be generalized to several factors: the equality

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$$

means

- (a) $\text{Hom}(A_j, A_i) = 0$ for any $A_i \in \mathcal{A}_i$, $A_j \in \mathcal{A}_j$, and $i < j$ (no morphisms from right to left);
- (b) $\mathcal{A}_1, \dots, \mathcal{A}_r$ generate \mathcal{T} (this is defined by induction on r).

A useful example of right admissible subcategory is the following. Let E be an object of \mathcal{T} and let $\langle E \rangle$ be the subcategory of \mathcal{T} generated by E : this is the smallest full triangulated subcategory of \mathcal{T} containing E . If E is *exceptional*, that is, if

$$\text{Ext}^\bullet(E, E) = \mathbf{C},$$

(this means $\text{Hom}(E, E[m]) = \mathbf{C}$ if $m = 0$, and 0 otherwise), the category $\langle E \rangle$ is admissible ([Hu, Lemma 1.58]). In particular, we have an s.o.d. $\mathcal{T} = \langle \langle E \rangle^\perp, \langle E \rangle \rangle$ which we write simply as

$$\mathcal{T} = \langle E^\perp, E \rangle.$$

Finally, exceptional objects E_1, \dots, E_r form an *exceptional collection* if the subcategories that they generate satisfy condition (a) above, which means in this case

$$\text{Hom}(E_j, E_i[m]) = 0$$

for all m and all $i < j$. There is then an s.o.d.

$$\mathcal{T} = \langle \langle E_1, \dots, E_r \rangle^\perp, E_1, \dots, E_r \rangle.$$

6.2. Examples of semiorthogonal decompositions and exceptional collections

We will apply these concepts in the following categories: if X is a smooth projective (complex) variety, the derived category $D^b(X)$ of the category of bounded complexes of coherent sheaves on X is a triangulated category.

On the projective space \mathbf{P}^n , each line bundle $\mathcal{O}_{\mathbf{P}^n}(m)$ is exceptional and Beilinson proved that there is an s.o.d. ([Hu, Corollary 8.29])

$$D^b(X) = \langle \mathcal{O}_{\mathbf{P}^n}, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(n) \rangle.$$

Beilinson's result was extended by Kapranov to all Grassmannians.

In the case of $G := \text{Gr}(2, V_5)$ which interests us here, Kuznetsov produced the s.o.d.

$$D^b(G) = \langle \mathcal{O}_G, \mathcal{U}^\vee, \mathcal{O}_G(1), \mathcal{U}^\vee(1), \dots, \mathcal{O}_G(4), \mathcal{U}^\vee(4) \rangle$$

([K1, Section 6.1]), where \mathcal{U} is the rank 2 tautological subbundle on G . For smooth linear sections $M := \text{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5})$ of G of dimension $n + 1 \geq 4$, he showed

$$D^b(M) = \langle \mathcal{O}_M, \mathcal{U}_M^\vee, \mathcal{O}_M(1), \mathcal{U}_M^\vee(1), \dots, \mathcal{O}_M(n-1), \mathcal{U}_M^\vee(n-1) \rangle.$$

Finally, for any GM variety X of dimension $n \geq 3$, he obtained with Perry ([KP, Proposition 2.3]) that

$$\mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3)$$

form an exceptional collection in $D^b(X)$. If one defines

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3) \rangle^\perp,$$

the right orthogonal of this exceptional collection, we obtain an s.o.d.

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3) \rangle.$$

We say that the triangulated category \mathcal{A}_X is associated with the GM variety X .

6.3. Serre functors and K3/Enriques categories

A *Serre functor* for a triangulated category \mathcal{T} is an equivalence $S_{\mathcal{T}}: \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ with bifunctorial isomorphisms

$$\mathrm{Hom}(F, S_{\mathcal{T}}(G)) \xrightarrow{\sim} \mathrm{Hom}(G, F)^\vee$$

for all $F, G \in \mathcal{T}$. If a Serre functor exists, it is unique (up to functorial isomorphisms), so it is an invariant of the category.

If X is a smooth projective variety of dimension n , the category $D^b(X)$ has a Serre functor given by

$$S_{D^b(X)}(F) = (F \otimes \omega_X)[n],$$

where ω_X is the dualizing sheaf of X . In particular, if ω_X is trivial (one often says that X is a *Calabi–Yau variety*), the Serre functor is just the shift by the dimension of X .

Bondal and Kapranov proved that an admissible subcategory of a triangulated category which has a Serre functor also has a Serre functor and Kuznetsov computed it in several interesting examples. The following is [KP, Proposition 2.6].

Proposition 6.1. *Let X be a GM variety of dimension $n \geq 3$.*

- *If n is even, the associated category \mathcal{A}_X is a K3 category: it has a Serre functor given by the shift $[2]$.*
- *If n is odd, the associated category \mathcal{A}_X is an Enriques category: it has a Serre functor $\sigma \circ [2]$, where $\sigma: \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ is a nontrivial involution.*

6.4. The Kuznetsov–Perry conjecture

Given a GM variety of dimension $n \in \{4, 6\}$, we have constructed a K3 category \mathcal{A}_X . The question is now the following: can \mathcal{A}_X be equivalent to the derived category of an actual K3 surface? The answer is no for X very general ([KP, Proposition 2.29]).

Proposition 6.2. *If the category associated with a GM variety X of even dimension is equivalent to the derived category of a K3 surface, the variety X is Hodge special (see Section 5.5).*

When a GM variety X of dimension 4 has an associated K3 surface S in the Hodge theoretic sense of Section 5.6, its category \mathcal{A}_X is equivalent to the category $D^b(S)$. However, the converse does not hold.¹⁵

Therefore ([KP, Conjecture 3.12], [PPZ, Remark 1.10]), there are two competing conjectural conditions (existence of a categorical versus Hodge-theoretic K3 surface) for the rationality of a GM fourfold. This should be contrasted with the case of cubic fourfolds, where these conditions are known to be equivalent.

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References

- [AT] Addington, N., Thomas, R., Hodge theory and derived categories of cubic fourfolds, *Duke Math. J.* **163** (2014), 1886–1927.
- [BKM] Bayer, A., Kuznetsov, A., Macrì, E., Mukai models of Fano varieties, to appear in *J. reine angew. Math.*
- [B] Beauville, A., Variétés kählériennes dont la première classe de Chern est nulle, *J. Differential Geom.* **18** (1983), 755–782.
- [BD] Beauville, A., Donagi, R., La variété des droites d’une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 703–706.
- [CG] Clemens, C.H., Griffiths, P.A., The Intermediate Jacobian of the Cubic Threefold, *Ann. of Math.* **95** (1972), 281–356.
- [C-T] Colliot-Thélène, J.-L., Rationalité d’un fibré en coniques, *Manuscripta Math.* **147** (2015), 305–310.

¹⁵To prove these two statements, one has to use [PPZ, Theorem 1.9], which says that the category \mathcal{A}_X associated with a GM variety X of dimension 4 or 6 is equivalent to the derived category of a K3 surface if and only if a certain lattice $\tilde{H}^{1,1}(\mathcal{A}_X; \mathbf{Z})$ associated with \mathcal{A}_X contains a hyperbolic plane.

If X is a GM fourfold with a Hodge theoretically associated K3 surface S , the lattice $\tilde{H}^{1,1}(\mathcal{A}_X; \mathbf{Z})$ does contain a hyperbolic plane by [Pe, Theorem 1.2], and the categories \mathcal{A}_X and $D^b(S)$ are equivalent.

Then, Pertusi constructed in [Pe, Section 3.3] GM fourfolds X such that $\tilde{H}^{1,1}(\mathcal{A}_X; \mathbf{Z})$ does contain a hyperbolic plane (so that the category \mathcal{A}_X is equivalent to the derived category of a K3 surface), but with no Hodge theoretically associated K3 surface. Many thanks to Alex Perry for these explanations.

- [CM] Conte, A., Murre, J.P., The Hodge conjecture for fourfolds admitting a covering by rational curves, *Math. Ann.* **238** (1978), 79–88.
- [DIM1] Debarre, O., Iliev, A., Manivel, L., On the period map for prime Fano threefolds of degree 10, *J. Algebraic Geom.* **21** (2012), 21–59.
- [DIM2] Debarre, O., Iliev, A., Manivel, L., Special prime Fano fourfolds of degree 10 and index 2, in *Recent Advances in Algebraic Geometry*, 123–155, C. Hacon, M. Mustață, and M. Popa editors, London Mathematical Society Lecture Notes Series **417**, Cambridge University Press, 2014.
- [DIM3] Debarre, O., Iliev, A., Manivel, L., On nodal prime Fano threefolds of degree 10, *Sci. China Math.* **54** (2011), 1591–1609.
- [DK1] Debarre, O., Kuznetsov, A., On the cohomology of Gushel-Mukai sixfolds, eprint arXiv:1606.09384.
- [DK2] Debarre, O., Kuznetsov, A., Gushel-Mukai varieties: classification and birationalities, *Algebr. Geom.* **5** (2018), 15–76.
- [DK3] Debarre, O., Kuznetsov, A., Gushel-Mukai varieties: linear spaces and periods, *Kyoto J. Math.* **59** (2019), 857–953.
- [DK4] Debarre, O., Kuznetsov, A., Gushel-Mukai varieties: moduli, *Internat. J. Math.* **31** (2020), 2050013.
- [DK5] Debarre, O., Kuznetsov, A., Double covers of quadratic degeneracy and Lagrangian intersection loci, *Math. Ann.* **378** (2020), 1435–1469.
- [DK6] Debarre, O., Kuznetsov, A., Gushel-Mukai varieties: intermediate Jacobians, *Épjournal Géom. Algébrique* **4** (2020), <https://epiga.episciences.org/6987>.
- [DK7] Debarre, O., Kuznetsov, A., Quadrics on Gushel-Mukai varieties, eprint arXiv:2409.03528.
- [DM] Debarre, O., Macrì, E., On the period map for polarized hyperkähler fourfolds, *Int. Math. Res. Not. IMRN* **22** (2019), 6887–6923
- [DMo] Debarre, O., Mongardi, G., Gushel-Mukai varieties with many symmetries and an explicit irrational Gushel-Mukai threefold, *Boll. Unione Mat. Ital.* **15** (2022), 131–161.
- [F] Ferretti, A., Special subvarieties of EPW sextics, *Math. Z.* **272** (2012), 1137–1164.
- [FM] Fu, L., Moonen, B., Algebraic cycles on Gushel-Mukai varieties, *Épjournal Géom. Algébrique*, Volume spécial en l’honneur de C. Voisin (2024), <https://epiga.episciences.org/13857>.
- [G] Gushel, N.P., Fano varieties of genus 6 (in Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* **46** (1982), 1159–1174, 1343. English transl.: *Izv. Math.* **21** (1983), 445–459.
- [H] Hassett, B., Special cubic fourfolds, *Compos. Math.* **120** (2000), 1–23.
- [HS] Hoff, M., Staglianò, G., New examples of rational Gushel-Mukai fourfolds, *Math. Z.* **296** (2020), 1585–1591.
- [Hu] Huybrechts, D., *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.
- [IM] Iliev, A., Manivel, L., Fano manifolds of degree 10 and EPW sextics, *Ann. Sci. École Norm. Sup.* **44** (2011), 393–426.
- [IP] Iskovskikh, V.A., Prokhorov, Yu., Fano varieties, *Algebraic geometry*, V, 1–247, Encyclopaedia Math. Sci. **47**, Springer-Verlag, Berlin, 1999.
- [KO] Kobayashi, S., Ochiai, T., Characterizations of complex projective spaces and hyperquadrics, *J. Math. Kyoto Univ.* **13** (1973), 31–47.
- [K1] Kuznetsov, A., Hyperplane sections and derived categories (in Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* **70** (2006), 23–128; English transl.: *Izv. Math.* **70** (2006), 447–547.

- [K2] Kuznetsov, A., Derived categories of cubic fourfolds, in *Cohomological and geometric approaches to rationality problems*, 219–243, Progr. Math. **282**, Birkhäuser Boston Inc., Boston, MA, 2010.
- [K3] Kuznetsov, A., Derived categories view on rationality problems, in *Rationality problems in algebraic geometry*, 67–104, Lecture Notes in Math. **2172**, CIME Found. Subser., Springer, Cham, 2016.
- [KP] Kuznetsov, A., Perry, A., Derived categories of Gushel–Mukai varieties, *Compos. Math.* **154** (2018), 1362–1406.
- [L] Logachev, D., Fano threefolds of genus 6, *Asian J. Math.* **16** (2012), 515–559.
- [M] Mella, M., Existence of good divisors on Mukai varieties, *J. Algebraic Geom.* **8** (1999), 197–206.
- [Mu1] Mukai, S., Biregular Classification of Fano 3-Folds and Fano Manifolds of Coindex 3, *Proc. Natl. Acad. Sci. USA* **86** (1989), 3000–3002.
- [Mu2] Mukai, S., Curves and Grassmannians, *Algebraic geometry and related topics (Inchon, 1992)*, 19–40, Conf. Proc. Lecture Notes Algebraic Geom., I, International Press, Cambridge, MA, 1993.
- [Mu3] Mukai, S., New development of theory of Fano 3-folds: vector bundle method and moduli problem, (in Japanese) *Sūgaku* **47** (1995), 125–144. English transl.: *Sugaku Expositions* **11** (2002), 125–150.
- [O1] O’Grady, K., Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics, *Duke Math. J.* **134** (2006), 99–137. Extended version at arXiv:math/0507398.
- [O2] O’Grady, K., Dual double EPW-sextics and their periods, *Pure Appl. Math. Q.* **4** (2008), 427–468.
- [O3] O’Grady, K., Double covers of EPW-sextics, *Michigan Math. J.* **62** (2013), 143–184.
- [O4] O’Grady, K., Moduli of double EPW-sextics, *Mem. Amer. Math. Soc.* **240** (2016), no. 1136.
- [O5] O’Grady, K., Periods of double EPW-sextics, *Math. Z.* **280** (2015), 485–524.
- [P] Perry, A., The integral Hodge conjecture for two-dimensional Calabi-Yau categories, *Compos. Math.* **158** (2022), 287–333.
- [PPZ] Perry, A., Pertusi, L., Zhao, X., Stability conditions and moduli spaces for Kuznetsov components of Gushel–Mukai varieties, *Geom. Topol.* **26** (2022), 3055–3121.
- [Pe] Pertusi, L., On the double EPW sextic associated to a Gushel–Mukai fourfold, *J. London Math. Soc.* **100** (2019), 83–106.
- [PV] Piontkowski, J., Van de Ven, A., The automorphism group of linear sections of the Grassmannian $G(1, N)$, *Doc. Math.* **4** (1999), 623–664.
- [Pr] Prokhorov, Yu., Rationality constructions of some Fano fourfolds of index 2, *Moscow Univ. Math. Bull.* **48** (1993), 32–35.
- [R] Roth, L., Algebraic varieties with canonical curve sections, *Ann. Mat. Pura Appl. (4)* **29** (1949), 91–97.
- [S-B] Shepherd-Barron, N.I., The rationality of quintic del Pezzo surfaces—a short proof, *Bull. London Math. Soc.* **24** (1992), 249–250.
- [T] Tjurin, A.N., Five lectures on three-dimensional varieties, (in Russian) *Uspehi Mat. Nauk* **27** (1972), no. 5, (167), 3–50. English transl.: *Russian Math. Surveys* **27** (1972), 1–53.
- [V] Voisin, C., Some aspects of the Hodge conjecture, *Japan. J. Math.* **2** (2007), 261–296.

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