NUMERICAL INVARIANTS OF HYPER-KÄHLER MANIFOLDS

OLIVIER DEBARRE, WITH AN APPENDIX BY CHEN JIANG

Abstract. We study various constraints on the Beauville quadratic form and the Huybrechts–Riemann–Roch polynomial for hyper-Kähler manifolds, mostly in dimension 6 and in the presence of an isotropic class.

In an appendix, Chen Jiang proves that in general, the Huybrechts–Riemann–Roch polynomial can always be written as a linear combination with nonnegative coefficients of certain explicit polynomials with positive coefficients. This implies that the Huybrechts–Riemann–Roch polynomial satisfies a curious symmetry property.

1. Introduction

A *hyper-Kähler manifold* is a simply connected compact Kähler manifold X whose space of holomorphic 2-forms is spanned by a symplectic form. Its dimension is necessarily an even number $2n$. A fundamental tool in the study of hyper-Kähler manifolds is the *Beauville* form, a canonical integral nondivisible nondegenerate quadratic form q_X on the free abelian group $H^2(X,\mathbf{Z})$ ([B, th. 5]). Its signature is $(3, b_2(X)-3)$ and there is a positive rational number c_X (the *Fujiki constant*) such that ([F, Theorem 4.7])

$$\forall\alpha\in H^2(X,\mathbf{Z})\qquad \int_X \alpha^{2n} = c_X q_X(\alpha)^n. \tag{1}$$

There exists a polynomial $P_{RR,X}(T)$ (the *Huybrechts–Riemann–Roch polynomial*) with rational coefficients, leading term $\frac{c_X}{(2n)!}T^n$ and constant term $n+1$, such that, for every line bundle L on X, one has ([H2, Corollary 3.18])

$$\chi(X,L) = P_{RR,X}(q_X(c_1(L))). \tag{2}$$

The objects q_X, c_X, and $P_{RR,X}(T)$ only depend on the topology of X and are in particular deformation invariant (see Table (13) for the values of c_X and $P_{RR,X}(T)$ for all known examples of hyper-Kähler manifolds X).

In this note, we first prove in Section 2 a curious symmetry property for the polynomial $P_{RR,X}(T)$ (Proposition 2.1). This property also follows from a strengthening of [J, Theorem 1.1] (which says that the polynomial $P_{RR,X}(T)$ has positive coefficients) proved in the appendix by Chen Jiang.

We then study a conjecture made in [DHMV, Conjecture 1.4] (and proved in [DHMV, Theorem 1.5] when $n=2$) about the possible values of $P_{RR,X}(T)$ when the quadratic form q_X represents 0 (this is the case for all known X). There exists then a nonzero class $\mathsf{l}\in H^2(X,\mathbf{Z})$ such that $\int_X \mathsf{l}^{2n}=0$ and, for any $\mathsf{m}\in H^2(X,\mathbf{Z})$, if one writes $\int_X \mathsf{l}^n\mathsf{m}^n = an!$, the number a is

2020 *Mathematics Subject Classification.* 14J42.

Key words and phrases. Hyper-Kähler manifolds, Riemann–Roch polynomial, sixfolds.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Project HyperK — grant agreement 854361).

necessarily an integer ([DHMV, Lemma 2.2]). The conjecture deals with the case $a = 1$ (which happens for nth punctual Hilbert schemes of K3 surfaces and hyper-Kähler manifolds of OG10 deformation type).

Conjecture 1.1 (Debarre–Huybrechts–Macrì–Voisin). Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that

$$\int_X \mathsf{l}^{2n} = 0 \quad \text{and} \quad \int_X \mathsf{l}^n \mathsf{m}^n = n!.$$

Then $c_X = (2n-1)!!$ and the Huybrechts–Riemann–Roch polynomial of X is

$$P_{RR,X}(T) = \binom{\frac{1}{2}T + 1 + n}{n}. \tag{3}$$

Our main result is the following result (Proposition 4.3) which almost proves the conjecture (one would need to additionally prove that the case $n_X = 2$ does not happen) in dimension 6 ($n = 3$).

Proposition 1.2. *Let X be a hyper-Kähler manifold of dimension 6 with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^6 = 0$ and $\int_X \mathsf{l}^3 \mathsf{m}^3 = 3!$. We have $q_X(\mathsf{l}, \mathsf{m}) = 1$, the quadratic form q_X is even, the Fujiki constant c_X is 15, and*

$$P_{RR,X}(T) = \binom{\frac{T}{2} + 4}{3} - \frac{6 - n_X}{16} T^2,$$

where $n_X \in \{2, 6\}$.

One may make the following more ambitious conjecture for small positive values of a (it is verified for all known examples of hyper-Kähler manifolds and proved in general when $n = 2$ in [DHMV, Theorem 9.3 and Theorem 1.5]).

Conjecture 1.3. Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that

$$\int_X \mathsf{l}^{2n} = 0 \quad \text{and} \quad \int_X \mathsf{l}^n \mathsf{m}^n = a n!\,, \quad \text{with} \ \ a \in \{1, \dots, n\}.$$

Then $a = 1$ and X is of $\mathrm{K3}^{[n]}$ or OG10 deformation type.

Again when $n = 3$, we get in Proposition 4.4 a much weaker result in the case $a = 2$ (which, according to Conjecture 1.3, should not occur at all).

Proposition 1.4. *Let X be a hyper-Kähler manifold of dimension 6 with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^6 = 0$ and $\int_X \mathsf{l}^3 \mathsf{m}^3 = 2 \cdot 3!$. We have $q_X(\mathsf{l}, \mathsf{m}) = 1$, the quadratic form q_X is even, the Fujiki constant c_X is 30, and*

$$P_{RR,X}(T) = \frac{1}{24} T^3 + \frac{n_X}{8} T^2 + \Big(\frac{4}{n_X} + \frac{n_X^2}{12}\Big) T + 4,$$

where $n_X \in \{1, 2, 3, 4\}$.

2. A symmetry property for the Huybrechts–Riemann–Roch polynomial

Let X be a hyper-Kähler manifold of dimension $2n$. In [N, Definition 17] (see also [J, Definition 2.2]), Nieper-Wißkirchen defined another quadratic form λ_X on $H^2(X,\mathbf{R})$ (which is *not* integral on $H^2(X,\mathbf{Z})$). It satisfies (see [N, (5.18)])

$$\forall\alpha\in H^2(X,\mathbf{Z})\qquad \frac{1}{(2n)!}\int_X\alpha^{2n}=A_X\lambda_X(\alpha)^n, \tag{4}$$

where $A_X\coloneqq\int_X \mathrm{td}^{1/2}(X)$. By [N, Proposition 10] and [J, Proposition 2.3], one can write

$$q_X=m_X\lambda_X,$$

where m_X is a positive rational number, so that (compare (1) and (4))

$$c_X=\frac{(2n)!A_X}{m_X^n}. \tag{5}$$

We will also set $n_X\coloneqq 2m_X$. When $n>1$, one has ([HS, Section 6], [J, Corollary 5.5])

$$0<A_X<1. \tag{6}$$

The Hirzebruch–Riemann–Roch theorem (2) takes the form

$$\chi(X,L)=\int_X\mathrm{td}(X)\exp(c_1(L))=Q_{RR,X}(\lambda_X(c_1(L))), \tag{7}$$

where $Q_{RR,X}(T)=P_{RR,X}(m_XT)$. The polynomial $Q_{RR,X}(T)$ was computed in [N, Theorem 5.2] in terms of the Chern numbers of X. The formula is

$$Q_{RR,X}(T)=\int_X\exp\Big(-\sum_{k=1}^{+\infty}\frac{B_{2k}}{2k}\,\mathrm{ch}_{2k}(X)T_{2k}\Big(\sqrt{\tfrac14T+1}\Big)\Big), \tag{8}$$

where

- the B_{2k} are the Bernoulli numbers;
- the $\mathrm{ch}_{2k}\in H^{2k,2k}(X)$ are the homogeneous components of the Chern character of X;
- the $T_{2k}(Y)$ are the (even) Chebyshev polynomials, defined by $T_{2k}(\cos\theta)=\cos 2k\theta$.

This formula implies curious symmetry relations for the polynomials $P_{RR,X}(T)$ and $Q_{RR,X}(T)$ for which we have no geometric explanations.

Proposition 2.1. *Let X be a hyper-Kähler manifold of dimension $2n$. The polynomial $Q_{RR,X}(T)$ satisfies the symmetry relation*

$$Q_{RR,X}(-T-4)=(-1)^nQ_{RR,X}(T). \tag{9}$$

Equivalently,

$$P_{RR,X}(-T-2n_X)=(-1)^nP_{RR,X}(T). \tag{10}$$

When n is odd, $-n_X$ is therefore a negative rational root of $P_{RR,X}(T)$. In all known examples, it is actually an integer (see also Lemma 4.2).

Proof. Let P_k be the degree k polynomial such that $P_k(T) = T_{2k}\Big(\sqrt{\frac14 T+1}\Big)$. Set $\cos\theta := \sqrt{\frac14 T+1}$, so that $T = 4(\cos^2\theta - 1) = -4\sin^2\theta$. We compute

$$\begin{aligned}P_k(-T-4) &= T_{2k}\Big(\sqrt{-\tfrac14 T}\Big) = T_{2k}(\sin\theta) = T_{2k}(\cos(\theta-\tfrac{\pi}{2})) = \cos(2k(\theta-\tfrac{\pi}{2}))\\ &= (-1)^k\cos 2k\theta = (-1)^k T_{2k}(\cos\theta) = (-1)^k T_{2k}\Big(\sqrt{\tfrac14 T+1}\Big) = (-1)^k P_k(T).\end{aligned}$$

By (8), the polynomial $Q_{RR,X}(T)$ is a $\mathbf{Q}$-linear combination of polynomials of the type

$$P_{k_1}(T)\cdots P_{k_r}(T)\int_X \mathrm{ch}_{2k_1}(X)\cdots\mathrm{ch}_{2k_r}(X)$$

for $k_1+\cdots+k_r = n$. The proposition therefore follows. □

Remark 2.2. The symmetry relation (10) implies that the polynomial $P_{RR,X}(T)$ is a linear combination with rational coefficients of the polynomials $(T+n_X)^{n-2j}$, for $0\le j\le n/2$. Since its leading coefficient is $\frac{c_X}{(2n)!}$, we can write

$$\begin{aligned}P_{RR,X}(T) &= \frac{c_X}{(2n)!}(T+n_X)^n + O(T^{n-2})\\ Q_{RR,X}(T) &= P_{RR,X}(m_X T) = A_X(T^n + 2nT^{n-1}) + O(T^{n-2}).\end{aligned} \tag{11}$$

The first two coefficients of $P_{RR,X}$ therefore determine m_X, A_X, and c_X (see also [J, Lemma 5.7]).

Chen Jiang proves in Appendix A that the polynomial $Q_{RR,X}(T)$ is a linear combination with *nonnegative* rational coefficients of the polynomials

$$Q_k(T) := \sum_{j=0}^{k}\binom{k+j+1}{2j+1}T^j$$

for $0\le k\le n$ and $n-k$ even. These polynomials satisfy the relation (9),[1] so this much stronger result implies Proposition 2.1.

Corollary 2.3. *When $n=3$, one has*

$$P_{RR,X}(T) = \frac{c_X}{720}(T+n_X)^3 + \Big(\frac{4}{n_X} - \frac{c_X}{720}n_X^2\Big)(T+n_X). \tag{12}$$

Proof. By Remark 2.2, we can write

$$P_{RR,X}(T) = \frac{c_X}{720}(T+n_X)^3 + b(T+n_X),$$

where b satisfies

$$\frac{c_X}{720}n_X^3 + bn_X = P_{RR,X}(0) = 4,$$

which gives the desired value for b. □

[1]One has $Q_k\big(T+\frac1T-2\big) = \sum_{j=0}^{k} T^{2j-k}$. In particular, the polynomials $Q_k(T)$ satisfy (9) (change T into $-T$) and the roots of $Q_k(T)$ are the k negative real numbers $-4\sin^2\frac{j\pi}{2(k+1)}$ for $1\le j\le k$, so that

$$Q_k(T) = \prod_{1\le j\le k}\Big(T + 4\sin^2\frac{j\pi}{2(k+1)}\Big).$$

Remark 2.4 (Known examples). The following table displays the values of the various objects we are considering for all known examples of hyper-Kähler manifolds.

(13)

	$\mathrm{K3}^{[n]}$ or OG10 ($n=5$) deformation type	Kum_n or OG6 ($n=3$) deformation type
$P_{RR,X}(T)$	$\binom{\frac{1}{2}T+1+n}{n}$	$(n+1)\binom{\frac{1}{2}T+n}{n}$
roots	$-4,-6,\dots,-2n-2$	$-2,-4,\dots,-2n$
c_X	$(2n-1)!!$	$(n+1)(2n-1)!!$
n_X	$n+3$	$n+1$
A_X	$\frac{(n+3)^n}{2^{2n}n!}$	$\frac{(n+1)^{n+1}}{2^{2n}n!}$

The roots of the polynomials displayed in the above table are negative integers (this was conjectured to hold for all hyper-Kähler manifolds in [J, Conjecture 1.3]). In the next two remarks, we discuss what can be said in general about the reality of the roots of the polynomial $P_{RR,X}(T)$ (or, equivalenty, of $Q_{RR,X}(T)$) in dimensions 4 and 6 (when real, the roots are negative, since both polynomials have positive coefficients).

Remark 2.5 (Real roots, $n=2$). When $n=2$, by (11), we have

$$Q_{RR,X}(T)=A_X(T^2+4T)+3.$$

Easy computations ([DHMV, Lemma 4.1]) based on [G, Main Theorem] give that

- either $b_2(X)=23$ and $b_3(X)=0$, in which case $A_X=\frac{25}{32}$,
- or $b_2(X)\le 8$, in which case $\frac{5}{6}\le A_X\le\frac{131}{144}$.

In particular, the discriminant $4A_X(4A_X-3)$ of the polynomial $Q_{RR,X}(T)$ is positive, hence its roots are real. Note that by [JL, Proposition 4.3] (also based on Guan's results), these roots are rational if and only if the Huybrechts–Riemann–Roch polynomial of X is one of the two known such poylnomials (see (13)); the roots of $P_{RR,X}(T)$ are then negative integers.

Remark 2.6 (Real roots, $n=3$). When $n=3$, we have by Remark 2.2

$$Q_{RR,X}(T)=(T+2)(A_X(T^2+4T)+2).$$

The roots of this polynomial are all real if and only if the discriminant

$$16A_X^2-8A_X=8A_X(2A_X-1)$$

is nonnegative, that is, if and only if $A_X\ge\frac{1}{2}$. The inequality $A_X>\frac{1}{2}$ is equivalent to the inequality (2) in [BS]. It implies an upper bound on $b_2(X)$. If $A_X\le\frac{1}{2}$, the class $c_2(X)$ is not in the image of the morphism $\mathrm{Sym}^2H^2(X,\mathbf{Q})\to H^4(X,\mathbf{Q})$ (the Verbitsky component).

3. Coefficients of the Huybrechts–Riemann–Roch polynomial

For each positive integer n, we define the positive integer

$$C_n := \gcd_{r_0,\dots,r_n\in\mathbf{Z}}\prod_{0\le j<k\le n}(r_j^2-r_k^2).$$

One computes easily $C_1 = 1$, $C_2 = 12$, and, with a computer,[2]

$$\begin{aligned} C_3 &= 2^5 \cdot 3^3 \cdot 5, \\ C_4 &= 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7, \\ C_5 &= 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^2, \\ C_6 &= 2^{27} \cdot 3^{14} \cdot 5^6 \cdot 7^3 \cdot 11, \\ C_7 &= 2^{37} \cdot 3^{19} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 13. \end{aligned}$$

Let X be a hyper-Kähler manifold of dimension $2n$. We write the Huybrechts–Riemann–Roch polynomial as

$$P_{RR,X}(T) =: a_n T^n + \cdots + a_1 T + a_0,$$

where $a_n = \frac{c_X}{(2n)!}$ and $a_0 = n + 1$. The proof of the following proposition uses the fact that the polynomial $P_{RR,X}(T)$ takes integral values on every integer represented by q_X: this is because of the relation (2) and the fact that, for every $\alpha \in H^2(X, \mathbf{Z})$, there is a deformation of X on which α becomes the first Chern class of a line bundle.

Proposition 3.1. *Let X be a hyper-Kähler manifold of dimension $2n$. For each $i \in \{0, \ldots, n\}$, the coefficient a_i of the polynomial $P_{RR,X}(T)$ belongs to $\frac{1}{2^i C_n}\mathbf{Z}$ (and to $\frac{1}{C_n}\mathbf{Z}$ if the quadratic form q_X is not even). In particular, the Fujiki constant c_X is in $\frac{(2n)!}{2^n C_n}\mathbf{Z}$.*

Proof. Let q be an integer represented by q_X. For all $r_0, \ldots, r_n \in \mathbf{Z}$, the integers $r_0^2 q, \ldots, r_n^2 q$ are also represented by q_X, so that $P_{RR,X}(r_j^2 q) = \sum_{i=0}^n a_i r_j^{2i} q^i$ is an integer for all $j \in \{0, \ldots, n\}$. The corresponding linear system with unknowns $a_0 q^0, \ldots, a_n q^n$ has a Vandermonde matrix (r_j^{2i}), so we get

$$a_i q^i \prod_{0 \le j < k \le n} (r_j^2 - r_k^2) \in \mathbf{Z}$$

for all $i \in \{0, \ldots, n\}$, which implies $a_i q^i C_n \in \mathbf{Z}$. Since the integral bilinear form associated with q_X is not divisible, the gcd of all integers q represented by q_X is either 2 (if the form q_X is even) or 1 (if it is not) and the proposition follows. ∎

In particular, we get $c_X \in \frac{1}{2}\mathbf{Z}$ when $n = 2$, and $c_X \in \frac{1}{48}\mathbf{Z}$ when $n = 3$. For any n, Proposition 3.1 gives the lower bound $c_X \geq \frac{(2n)!}{2^n C_n}$, but what would be really interesting, in order to prove boundedness properties for hyper-Kähler manifolds, would be to find an upper bound on c_X (see [H1]).

Remark 3.2. Assume that q_X represents all large enough even numbers (this is the case for all known examples). Then $P_{RR,X}(T)$ takes integral values on all large enough even numbers and this implies that its leading coefficient is in $\frac{1}{n!2^n}\mathbf{Z}$, hence $c_X \in (2n-1)!!\mathbf{Z}$.

[2] Many thanks to Jieao Song for making these computations. For any positive integer n, the primes p that divide C_n are exactly those such that $p \leq 2n - 1$ (this is because one can find $n + 1$ distinct squares modulo p if and only if $p > 2n$).

4. The Huybrechts–Riemann–Roch polynomial in the presence of an isotropic class

Let X be a hyper-Kähler manifold of dimension $2n$. Assume that there is an isotropic class $\mathsf{l} \in H^2(X, \mathbf{Z})$, that is, $q_X(\mathsf{l}) = 0$. For any $\mathsf{m} \in H^2(X, \mathbf{Z})$,

$$a(\mathsf{m}) := \frac{1}{n!} \int_X \mathsf{l}^n \mathsf{m}^n$$

is an integer ([DHMV, Lemma 2.2]) and

$$c_X q_X(\mathsf{l}, \mathsf{m})^n = a(\mathsf{m}) \frac{(2n)!}{2^n n!} = a(\mathsf{m})(2n-1)!!. \tag{14}$$

From now on, we assume $q_X(\mathsf{l}, \mathsf{m}) > 0$. Using (5) and (6), we obtain

$$m_X^n = \frac{(2n)! A_X}{c_X} < \frac{(2n)!}{c_X} = \frac{2^n n! q_X(\mathsf{l}, \mathsf{m})^n}{a(\mathsf{m})}$$

hence

$$m_X < 2 q_X(\mathsf{l}, \mathsf{m}) \sqrt[n]{\frac{n!}{a(\mathsf{m})}}. \tag{15}$$

Using the bound $c_X \geq \frac{(2n)!}{2^n C_n}$, we also get $m_X < 2\sqrt[n]{C_n}$.

Lemma 4.1. *We have*

$$n! q_X(\mathsf{l}, \mathsf{m})^n \mid a(\mathsf{m}) C_n$$

and, if q_X is not even,

$$n! 2^n q_X(\mathsf{l}, \mathsf{m})^n \mid a(\mathsf{m}) C_n.$$

Proof. Using (14), we get

$$a_n = \frac{c_X}{(2n)!} = \frac{a(\mathsf{m})}{2^n n! q_X(\mathsf{l}, \mathsf{m})^n}.$$

Then use Proposition 3.1. ☐

Lemma 4.2. *We have*

$$a(\mathsf{m}) \Big(\frac{q_X(\mathsf{m}) + n_X}{2 q_X(\mathsf{l}, \mathsf{m})} - \frac{n-1}{2} \Big) \in \mathbf{Z}.$$

In particular,

$$n_X \in \mathbf{Z} + \frac{2 q_X(\mathsf{l}, \mathsf{m})}{a(\mathsf{m})} \mathbf{Z}$$

so that n_X is an integer when $a(\mathsf{m}) \in \{1, 2\}$.

Proof. For every $t \in \mathbf{Z}$, the number

$$P(t) := P_{RR,X}(q_X(t\mathsf{l} + \mathsf{m})) = P_{RR,X}(2t q_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m}))$$

is an integer. We have, using (11) and (14),

$$\begin{aligned} P(t) &= \frac{c_X}{(2n)!}(2tq_X(\mathsf{l},\mathsf{m})+q_X(\mathsf{m})+n_X)^n+O(t^{n-2})\\ &= \frac{a(\mathsf{m})}{q_X(\mathsf{l},\mathsf{m})^n2^nn!}\big(2^nq_X(\mathsf{l},\mathsf{m})^nt^n+n2^{n-1}q_X(\mathsf{l},\mathsf{m})^{n-1}(q_X(\mathsf{m})+n_X)t^{n-1}\big)+O(t^{n-2})\\ &= \frac{a(\mathsf{m})}{n!}t^n+\frac{a(\mathsf{m})}{q_X(\mathsf{l},\mathsf{m})2(n-1)!}(q_X(\mathsf{m})+n_X)t^{n-1}+O(t^{n-2}). \end{aligned}$$

This is an integer for all $t\in\mathbf{Z}$, hence so is

$$\begin{aligned} P(t)-a(\mathsf{m})\binom{t+n-1}{n} &= P(t)-a(\mathsf{m})\,\frac{t^n+\frac{n(n-1)}{2}t^{n-1}}{n!}+O(t^{n-2})\\ &= \Big(\frac{q_X(\mathsf{m})+n_X}{2q_X(\mathsf{l},\mathsf{m})}-\frac{n-1}{2}\Big)\frac{a(\mathsf{m})}{(n-1)!}t^{n-1}+O(t^{n-2}). \end{aligned}$$

This implies the lemma. □

4.1. **Case $a(\mathsf{m})=1$.** We know from [DHMV, Theorem 1.5] that in dimension 4, this case only occurs when X is of $\mathrm{K3}^{[2]}$ deformation type. In particular, $P_{RR,X}(T)$ is then given by (3). We believe (Conjecture 1.1) that the same should happen for any $n\geq 2$ (one would then have $c_n=(2n-1)!!$ and $n_X=n+3$). We study the case $n=3$.

Proposition 4.3. *Assume $n=3$ and $a(\mathsf{m})=1$. Then $q_X(\mathsf{l},\mathsf{m})=1$, $c_X=15$, $n_X\in\{2,6\}$, and the quadratic form q_X is even. One also has*

$$P_{RR,X}(T)=\frac{1}{48}T^3+\frac{n_X}{16}T^2+\frac{13}{6}T+4=\binom{\frac{T}{2}+4}{3}-\frac{6-n_X}{16}T^2$$

and the sublattice $\mathbf{Z}\mathsf{l}\oplus\mathbf{Z}\mathsf{m}$ of $(H^2(X,\mathbf{Z}),q_X)$ is a hyperbolic plane.

Proof. We have $C_3=2^5\cdot 3^3\cdot 5$ and we obtain from Lemma 4.1

$$q_X(\mathsf{l},\mathsf{m})^3\mid 2^4\cdot 3^2\cdot 5\qquad(\text{and } q_X(\mathsf{l},\mathsf{m})^3\mid 2\cdot 3^2\cdot 5 \text{ if } q_X \text{ is not even}),$$

so that $q_X(\mathsf{l},\mathsf{m})\in\{1,2\}$ (and $q_X(\mathsf{l},\mathsf{m})=1$ if q_X is not even).

Assume $q_X(\mathsf{l},\mathsf{m})=1$. We have $c_X=15$ from (14), Lemma 4.2 gives $q_X(\mathsf{m})+n_X\in 2\mathbf{Z}$, and (15) gives $m_X<2\sqrt[3]{6}\sim 3.6$, so that $n_X\in\{1,2,3,4,5,6,7\}$. Furthermore, we have, by Corollary 2.3,

$$P_{RR,X}(T)=\frac{1}{48}(T+n_X)^3+\Big(\frac{4}{n_X}-\frac{1}{48}n_X^2\Big)(T+n_X).$$

For all values q taken by q_X, this must be an integer when $T=q$, so that

$$48n_X\mid n_X(q+n_X)^3+(192-n_X^3)(q+n_X). \tag{16}$$

In particular, $n_X\mid 192q$. If $n_X\in\{5,7\}$, this implies $n_X\mid q$, which is impossible because the gcd of all integers q represented by q_X is either 1 or 2. Otherwise, $16n_X\mid 192$, hence we obtain

$$16\mid (q+n_X)^3-n_X^2(q+n_X)=q(q+n_X)(q+2n_X). \tag{17}$$

- When $n_X=1$, the relation (17) is equivalent to $q\equiv 0,6,8,14,15 \pmod{16}$. The case $q\equiv 15\pmod{16}$ is impossible since $4q$ is also represented but not in this list, hence $q\equiv 0,6,8,14\pmod{16}$ and q_X is even. This contradicts the fact that $q_X(\mathsf{m})+n_X$ is even.

- When $n_X = 2$, the relation (17) is equivalent to q even.
- When $n_X = 3$, the only possible odd value is $q \equiv 13 \pmod{16}$. This implies that $4q \equiv 4 \pmod{16}$ should also be represented, but 4 does not satisfy the relation (17). So q_X is even, which contradicts the fact that $q_X(\mathsf{m}) + n_X$ is even.
- When $n_X = 4$, the relation (17) is equivalent to $4 \mid q$, which is impossible because the gcd of all integers q represented by q_X is either 1 or 2.
- When $n_X = 6$, the relation (17) is equivalent to q even.

All in all, we get $n_X \in \{2, 6\}$ and q_X even.

Assume $q_X(\mathsf{l}, \mathsf{m}) = 2$**.** The quadratic form q_X is even, we have $c_X = \frac{15}{8}$ from (14), Lemma 4.2 gives $\frac{1}{2} q_X(\mathsf{m}) + m_X \in 2\mathbf{Z}$, so that m_X is an integer, and (15) gives $m_X < 4\sqrt[3]{6} < 8$, so that $m_X \in \{1, 2, 3, 4, 5, 6, 7\}$. As above, we deduce from (12) that

$$\frac{1}{8 \cdot 48}(2q + 2m_X)^3 + \Big(\frac{2}{m_X} - \frac{1}{8 \cdot 48} 4m_X^2\Big)(2q + 2m_X)$$

is an integer for all values $2q$ taken by q_X, so that

$$48m_X \mid m_X(q + m_X)^3 + (192 - m_X^3)(q + m_X).$$

This is "the same" relation as (16) and the discussion above allows us to conclude that q must be even, so that all values taken by q_X are divisible by 4. This is impossible because the gcd of all values taken by q_X is 2. So this case does not occur. □

4.2. **Case** $a(\mathsf{m}) = 2$**.** We believe (Conjecture 1.3) this case should not occur for any $n \geq 2$ and we know from [DHMV, Theorem 9.3] that it does not when $n = 2$. We study the case $n = 3$.

Proposition 4.4. *Assume $n = 3$ and $a(\mathsf{m}) = 2$. Then, $q_X(\mathsf{l}, \mathsf{m}) = 1$, $c_X = 30$, $n_X \in \{1, 2, 3, 4\}$, and the quadratic form q_X is even. One also has $P_{RR,X}(T) = \frac{1}{24}T^3 + \frac{n_X}{8}T^2 + \big(\frac{4}{n_X} + \frac{n_X^2}{12}\big)T + 4$ and the sublattice $\mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$ of $(H^2(X, \mathbf{Z}), q_X)$ is a hyperbolic plane.*

Proof. We have $C_3 = 2^5 \cdot 3^3 \cdot 5$ and we obtain from Lemma 4.1

$$q_X(\mathsf{l}, \mathsf{m})^3 \mid 2^5 \cdot 3^2 \cdot 5 \qquad (\text{and } q_X(\mathsf{l}, \mathsf{m})^3 \mid 2^2 \cdot 3^2 \cdot 5 \text{ if } q_X \text{ is not even}),$$

so that $q_X(\mathsf{l}, \mathsf{m}) \in \{1, 2\}$ (and $q_X(\mathsf{l}, \mathsf{m}) = 1$ if q_X is not even).

Assume $q_X(\mathsf{l}, \mathsf{m}) = 1$**.** We have $c_X = 30$ from (14), Lemma 4.2 gives $n_X \in \mathbf{Z}$, and (15) gives $m_X < 2\sqrt[3]{3} \sim 2.9$, so that $n_X \in \{1, 2, 3, 4, 5\}$. Furthermore, we have, by (12),

$$P_{RR,X}(T) = \frac{1}{24}(T + n_X)^3 + \Big(\frac{4}{n_X} - \frac{1}{24}n_X^2\Big)(T + n_X),$$

For all values q taken by q_X, this must be an integer when $T = q$, so that

$$24n_X \mid n_X(q + n_X)^3 + (96 - n_X^3)(q + n_X).$$

In particular, $n_X \mid 96q$. If $n_X = 5$, this implies $n_X \mid q$, which is impossible because the gcd of all integers q represented by q_X is either 1 or 2. Otherwise, $8n_X \mid 96$, hence we obtain

$$8 \mid (q + n_X)^3 - n_X^2(q + n_X) = q(q + n_X)(q + 2n_X). \tag{18}$$

- When $n_X = 1$, the relation (18) is equivalent to $q \equiv 0, 2, 4, 6, 7 \pmod 8$; this means that every odd value taken by q_X is $\equiv 7 \pmod 8$. Assume there exists α such that

$q_X(\alpha) \equiv 7 \pmod 8$. Since $q_X(k\mathsf{l} + \mathsf{m}) = 2k + q_X(\mathsf{m})$, the integer $q_X(\mathsf{m})$ must be even and we may even assume $q_X(\mathsf{m}) = 0$. For all $t, u \in \mathbf{Z}$, the integer

$$q_X(t\mathsf{l} + u\mathsf{m} + \alpha) = 2tu + 2tq_X(\mathsf{l}, \alpha) + 2uq_X(\mathsf{m}, \alpha) + q_X(\alpha)$$

is odd, hence is $\equiv 7 \pmod 8$. This implies $tu + tq_X(\mathsf{l}, \alpha) + uq_X(\mathsf{m}, \alpha) \equiv 0 \pmod 4$. Taking $t = 1$ and $u = 0$, we obtain $q_X(\mathsf{l}, \alpha) \equiv 0 \pmod 4$; taking $t = 0$ and $u = 1$, we obtain $q_X(\mathsf{m}, \alpha) \equiv 0 \pmod 4$; taking $t = u = 1$, we obtain a contradiction. Hence $q \equiv 0, 2, 4, 6 \pmod 8$ and q_X is even.

- When $n_X = 2$, the relation (18) is equivalent to $q \equiv 0, 2, 4, 6 \pmod 8$ and q_X is even.
- When $n_X = 3$, the relation (18) is equivalent to $q \equiv 0, 2, 4, 5, 6 \pmod 8$. If the case $q \equiv 5 \pmod 8$ occurs, the same reasoning as in the case $n_X = 1$, $q \equiv 7 \pmod 8$, gives a contradiction, hence q_X is even.
- When $n_X = 4$, the relation (18) is equivalent to $q \equiv 0, 2, 4, 6 \pmod 8$ and q_X is even.

Assume $q_X(\mathsf{l}, \mathsf{m}) = 2$. The quadratic form q_X is even, we have $c_X = \frac{15}{4}$ from (14), Lemma 4.2 gives $m_X \in \mathbf{Z}$, and (15) gives $m_X < 5.8$, so that $m_X \in \{1, 2, 3, 4, 5\}$. As above, we deduce from (12) that

$$\frac{1}{8 \cdot 24}(2q + 2m_X)^3 + \Big(\frac{2}{m_X} - \frac{1}{8 \cdot 24} 4m_X^2\Big)(2q + 2m_X)$$

must be an integer for all values $2q$ taken by q_X, so that

$$24m_X \mid m_X(q + m_X)^3 + (96 - m_X^3)(q + m_X).$$

We reason as above to conclude that the integer q must be even, so that all values taken by the quadratic form q_X are divisible by 4. This is impossible because the gcd of all values taken by q_X is 2. So this case does not occur. ☐

Appendix A. Positivity of the Huybrechts–Riemann–Roch polynomial

by Chen Jiang

Throughout this appendix, X is a hyper-Kähler manifold of dimension $2n$ and we fix a symplectic form $\sigma \in H^0(X, \Omega_X^2)$. The degree n Huybrechts–Riemann–Roch polynomial $P_{RR,X}(T)$ was defined in the introduction, and the polynomial $Q_{RR,X}(T) = P_{RR,X}(m_X T)$ in Section 2. These polynomials were proved in [J, Theorem 1.1] to have positive coefficients. The purpose of this appendix is to prove a refinement of this result. For every nonnegative integer k, we define a degree k monic polynomial with positive coefficients by

$$Q_k(T) \coloneqq \sum_{j=0}^{k} \binom{k+j+1}{2j+1} T^j = T^k + 2kT^{k-1} + \cdots + k + 1.$$

Our result is the following.

Proposition A.1. *Let X be a hyper-Kähler manifold of dimension $2n > 2$. There are non-negative rational numbers $b_0, b_1, \ldots, b_{\lfloor n/2 \rfloor}$ such that*

$$Q_{RR,X}(T) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i Q_{n-2i}(T). \tag{19}$$

Moreover, $b_0 = A_X = \int_X \mathrm{td}^{1/2}(X) > 0$ and $b_1 > 0$.

For any $\alpha \in H^2(X, \mathbf{R})$, we have

$$Q_{RR,X}(\lambda_X(\alpha)) = \int_X \mathrm{td}(X)\exp(\alpha),$$

where λ_X is the quadratic form on $H^2(X, \mathbf{R})$ discussed in Section 2. Indeed, by (7), this equality holds when α is the first Chern class of a line bundle on X. It then holds for each $\alpha \in H^2(X, \mathbf{Z})$ because there is a deformation of X on which α becomes the first Chern class of a line bundle. Finally, it holds for every $\alpha \in H^2(X, \mathbf{R})$ since both sides are polynomial functions of α.

Moreover, one has ([N, Definition 17], [J, Definition 2.2])

$$\lambda_X(\alpha) := \begin{cases} \frac{24n\int_X \exp(\alpha)}{\int_X c_2(X)\exp(\alpha)} & \text{if well-defined;} \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, we set $\lambda_\sigma := \lambda_X(\sigma + \overline{\sigma})$. We know that $\lambda_\sigma > 0$ (see [J, Lemma 2.4(2)]).

In [J, Definition 4.1], for any $0 \le k \le n/2$, we defined a class

$$\mathrm{tp}_{2k} := \sum_{i=0}^{k} \frac{(n-2k+1)!\,\mathrm{td}_{2i}^{1/2}\wedge(\sigma\overline{\sigma})^{k-i}}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} \in H^{4k}(X, \mathbf{R})$$

which is of Hodge type $(2k, 2k)$. One important fact is that, by [J, Corollary 4.4],

$$\int_X \mathrm{tp}_{2k}^2(\sigma\overline{\sigma})^{n-2k} \ge 0.$$

Lemma A.2. *The numbers*

$$C_k := \frac{\int_X \mathrm{tp}_{2k}^2(\sigma\overline{\sigma})^{n-2k}}{\lambda_\sigma^{n-2k}}$$

are deformation invariants of X. In particular, C_k is independent of the choice of σ.

Here we remark that we cannot directly apply [H2, Corollary 23.17] as tp_{2k} might no longer be of type $(2k, 2k)$ on deformations of X.

Proof. By definition of tp_{2k}, the number C_k can be written as

$$C_k = \sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij}\,\frac{\int_X \mathrm{td}_{2i}^{1/2}\,\mathrm{td}_{2j}^{1/2}(\sigma\overline{\sigma})^{n-i-j}}{\lambda_\sigma^{n-i-j}},$$

where the a_{ij} are constants depending only on n, k, i, j. By [H2, Corollary 23.17] and [J, Proposition 2.3],

$$\frac{\int_X \mathrm{td}_{2i}^{1/2}\,\mathrm{td}_{2j}^{1/2}(\sigma\overline{\sigma})^{n-i-j}}{\lambda_\sigma^{n-i-j}} = \frac{(n-i-j)!^2}{(2n-2i-2j)!}\,\frac{\int_X \mathrm{td}_{2i}^{1/2}\,\mathrm{td}_{2j}^{1/2}(\sigma+\overline{\sigma})^{2n-2i-2j}}{\lambda_\sigma^{n-i-j}}$$

only depends on $\mathrm{td}_{2i}^{1/2}\,\mathrm{td}_{2j}^{1/2}$, $c_2(X)$, and c_X, which implies that C_k is a deformation invariant of X. □

Proof of Proposition A.1. From [J, Proof of Theorem 5.1], for any $0 \le m \le n$, we have

$$\int_X \mathrm{td}_{2m}(\sigma\overline{\sigma})^{n-m} = \sum_{i=0}^{\lfloor m/2\rfloor} \frac{(n-m)!^2}{\lambda_\sigma^{m-2i}(n-2i)!^2}\binom{2n-2i-m+1}{m-2i}\int_X (\mathrm{tp}_{2i})^2(\sigma\overline{\sigma})^{n-2i}.$$

In other words,

$$\int_X \mathrm{td}_{2m}(\sigma+\overline{\sigma})^{2n-2m} = \sum_{i=0}^{\lfloor m/2\rfloor} \frac{(2n-2m)!}{\lambda_\sigma^{m-2i}(n-2i)!^2}\binom{2n-2i-m+1}{m-2i}\int_X (\mathrm{tp}_{2i})^2(\sigma\overline{\sigma})^{n-2i}.$$

Thus we have the following equalities:

$$\begin{aligned}
\int_X \mathrm{td}(X)\exp(\sigma+\overline{\sigma}) &= \sum_{m=0}^{n}\int_X \frac{1}{(2n-2m)!}\,\mathrm{td}_{2m}(X)(\sigma+\overline{\sigma})^{2n-2m}\\
&= \sum_{m=0}^{n}\sum_{i=0}^{\lfloor m/2\rfloor}\frac{1}{\lambda_\sigma^{m-2i}(n-2i)!^2}\binom{2n-2i-m+1}{m-2i}\int_X(\mathrm{tp}_{2i})^2(\sigma\overline{\sigma})^{n-2i}\\
&= \sum_{m=0}^{n}\sum_{i=0}^{\lfloor m/2\rfloor}\frac{1}{(n-2i)!^2}\binom{2n-2i-m+1}{m-2i}C_i\lambda_\sigma^{n-m}\\
&= \sum_{i=0}^{\lfloor n/2\rfloor}\frac{C_i}{(n-2i)!^2}\sum_{m=2i}^{n}\binom{2n-2i-m+1}{m-2i}\lambda_\sigma^{n-m}\\
&= \sum_{i=0}^{\lfloor n/2\rfloor}\frac{C_i}{(n-2i)!^2}\sum_{m=0}^{n-2i}\binom{2n-4i-m+1}{m}\lambda_\sigma^{n-m-2i}\\
&= \sum_{i=0}^{\lfloor n/2\rfloor}\frac{C_i}{(n-2i)!^2}\,Q_{n-2i}(\lambda_\sigma).
\end{aligned}$$

In other words,

$$Q_{RR,X}(\lambda_\sigma) = \sum_{i=0}^{\lfloor n/2\rfloor}\frac{C_i}{(n-2i)!^2}\,Q_{n-2i}(\lambda_\sigma).$$

Here $C_i \geq 0$ by [J, Corollary 4.4]. By Lemma A.2, C_i is independent of the choice of σ, so after replacing σ by $t\sigma$ for any $t\in\mathbf{C}^\times$, we can get an equality of polynomials

$$Q_{RR,X}(T) = \sum_{i=0}^{\lfloor n/2\rfloor}\frac{C_i}{(n-2i)!^2}\,Q_{n-2i}(T),$$

which gives the desired equation (19).

The last assertion is a consequence of [J, Corollary 5.2]. □

References

[B] Beauville, A., Variétés kählériennes dont la première classe de Chern est nulle, *J. Differential Geom.* **18** (1983),755–782. 1

[BS] Beckmann, T., Song, J., Second Chern class and Fujiki constants of hyperkähler manifolds, eprint `arXiv:2201.07767`. 5

[DHMV] Debarre, O., Huybrechts, D., Macrì, E., Voisin, C., Computing Riemann–Roch polynomials and classifying hyper-Kähler fourfolds, *J. Amer. Math. Soc.* **37** (2024), 151–185. 1, 2, 5, 7, 8, 9

[F] Fujiki, A., On the de Rham cohomology group of a compact Kähler symplectic manifold, in *Algebraic geometry, Sendai, 1985,* 105–165, Adv. Stud. Pure Math. **10** North-Holland Publishing Co., Amsterdam, 1987. 1

[G] Guan, D., On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four, *Math. Res. Lett.* **8** (2001), 663–669. 5

[HS] Hitchin, N., Sawon, J., Curvature and characteristic numbers of hyper-Kähler manifolds, *Duke Math. J.* **106** (2001), 599–615. 3

[H1] Huybrechts, D., Finiteness results for compact hyperkähler manifolds, *J. reine angew. Math.* **558** (2003), 15–22. 6

[H2] Huybrechts, D., Compact hyperkähler manifolds, in *Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001),* 161–225, Universitext, Springer, Berlin, 2003. 1, 11

[J] Jiang, C., Positivity of Riemann–Roch polynomials and Todd classes of hyperkähler manifolds, *J. Algebraic Geom.* **32** (2023), 239–269. 1, 3, 4, 5, 10, 11, 12

[JL] Jiang, C., Liu, W., On numerically trivial automorphisms of compact hyperkähler manifolds of dimension 4, *Math. Res. Lett.* **31** (2024), 1771–1784. 5

[N] Nieper-Wisskirchen, M., Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds, *J. Algebraic Geom.* **12** (2003), 715–739. 3, 11

Université Paris Cité, CNRS, Institut de Mathématiques de Jussieu-Paris rive gauche,
8 Place Aurélie Nemours, 75013 Paris, France

E-mail address: olivier.debarre@imj-prg.fr

Shanghai Center for Mathematical Sciences & School of Mathematical Sciences, Fudan University, Shanghai 200438, China

E-mail address: chenjiang@fudan.edu.cn