ON THE SUM MAP FOR SUBVARIETIES OF SIMPLE ABELIAN VARIETIES

OLIVIER DEBARRE

Abstract. Let X and Y be subvarieties of a simple abelian variety A defined over an algebraically closed field and let $Z \coloneqq X + Y \subseteq A$. If $\dim(X) + \dim(Y) \leq \dim(A)$, we prove that the addition map $X \times Y \to Z$ is semismall. In particular, Z has the expected dimension $\dim(X) + \dim(Y)$. Over the field of complex numbers, the latter statement was proved in 1982 by Barth, and Prasad gave in 1993 a very simple proof over an algebraically closed field of characteristic zero. Surprisingly enough, it did not seem to be known in general in positive characteristics. In the above situation, we prove the semismallness of the addition map in all characteristics using perverse sheaves. This is joint work with Ben Moonen.

Dedicated to Lucia Caporaso, on the occasion of her sixtieth birthday

1. Bézout's theorem in characteristic zero

We work over an algebraically closed field $\mathbf{k}$ of any characteristic. It is not very hard to prove (Hartshorne does it in Section I.7 of his book using only Krull's Hauptidealsatz) that subvarieties of the projective space $\mathbf{P}^n$ whose dimensions add up to at least n must meet.

We can ask the same question about subvarieties of an abelian variety, but it is clear that it needs to be simple. The corresponding statement was then first proved (in the analytic setting) by Barth in 1982. I present Prasad's simple proof (of what he calls Bézout's theorem).

Theorem 1 (Prasad '93, Debarre '95)**.** *Let X and Y be subvarieties of a simple abelian variety A defined over an algebraically closed field of characteristic zero. We have*

$$\dim(X+Y) = \min\{\dim(X) + \dim(Y), \dim(A)\}. \tag{1}$$

This immediately implies Bézout's theorem: if $\dim(X) + \dim(Y) \geq \dim(A)$, then $X - Y = A$ hence $0 \in X - Y$ and $X \cap Y \neq \varnothing$.

Proof. It is easy to reduce to the case $\dim(X) + \dim(Y) \leq \dim(A)$ and, if we set $Z \coloneqq X + Y$, we must prove that the addition map $\sigma\colon X \times Y \to Z$ is generically finite.

Let $z \in Z_{\mathrm{reg}}$ and, for all $(x, y) \in \sigma^{-1}(z)$, consider the tangent map

$$T_{\sigma,(x,y)}\colon T_{X,x} \times T_{Y,y} \longrightarrow T_{Z,z}$$

(we can view, by translation, all these tangent spaces inside $T_{A,0}$, and the tangent map as the addition map).

Date: June 6, 2025.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC-2020-SyG-854361-HyperK).

Assume by contradiction that σ is not generically finite. Then $Z \neq A$, and $\sigma^{-1}(z)$ and its projection C_z in X have everywhere positive dimension. For all $x \in C_z$, we have

$$T_{C_z,x} \subseteq T_{X,x} \subseteq T_{Z,z} \subsetneq T_{A,0}.$$

We claim that, in characteristic zero, this is not possible in a simple abelian variety. Indeed, let $C \subseteq C_z$ be a (positive-dimensional) irreducible component of C_z and consider the map

$$\phi\colon C^{2m} \longrightarrow A, \quad (x_1, \dots, x_{2m}) \longmapsto x_1 + \cdots + x_m - x_{m+1} - \cdots - x_{2m}.$$

For $m \gg 0$, its image is an abelian subvariety B of A. But, by generic smoothness, we have, for $(x_1, \dots, x_{2m}) \in C^{2m}$ general,

$$T_{B,0} = T_{B,\phi(x_1,\dots,x_{2m})} = T_{\phi,(x_1,\dots,x_{2m})}(T_{C^{2m},(x_1,\dots,x_{2m})}) = T_{C,x_1} + \cdots + T_{C,x_{2m}} \subseteq T_{Z,z} \subsetneq T_{A,0},$$

hence $B \subsetneq A$. Since A is simple, we must have $B = 0$, which is absurd since B contains a translate of C. □

The characteristic zero hypothesis is used only once (for the generic smoothness of ϕ), but crucially. Indeed, in positive characteristic, the last argument definitely fails: there exist positive-dimensional subvarieties $C \subset A$, with A simple, such that all tangent spaces $T_{C,x}$, for $x \in C_{\mathrm{reg}}$, are contained in a fixed proper subspace of $T_{A,0}$ (one can find in Prasad's paper an example constructed by D. Abramovich).

2. Bézout's theorem in positive characteristic

We are at the moment unable to prove Theorem 1 in positive characteristic in general by elementary methods. Here are a couple of results when $\dim(X) \in \{1, 2\}$.

When $\dim(X) = 1$, this is very easy: if $\dim(X + Y) < \dim(X) + \dim(Y)$, we have $X + Y = x + Y$ for all $x \in X$, hence

$$X - x \subseteq \mathrm{Stab}(Y) := \{a \in A \mid a + Y = Y\}.$$

Since $\mathrm{Stab}(Y)$ is a closed subgroup of A and A is simple, this implies $\mathrm{Stab}(Y) = A$, hence $Y = A$.

Here is a sample of our results (we can do a little more, for instance the cases $\dim(X) = 2$ and $\dim(Y) \in \{2, 3\}$).

Theorem 2 (Debarre–Moonen). *Let X and Y be subvarieties of a simple abelian variety A defined over an algebraically closed field. Assume* $\dim(X) = 2$ *and* $\dim(A) > 2\dim(Y)$. *Then*

$$\dim(X + Y) = \dim(X) + \dim(Y).$$

Proof. We may assume $e := \dim(Y) \geq 2$. Assume $\dim(Z) = e+1$. We keep the same notation as above. For $z \in Z$, the subvariety $C_z \subset X$ is everywhere of dimension ≥ 1, hence the inclusion $C_z + Y \subseteq Z$ is an equality.

Consider the morphism

$$\psi\colon X \times Y \times Y \longrightarrow Z \times Z\ , \quad (x, y_1, y_2) \longmapsto (x + y_1, x + y_2)$$

and let $W \subset Z \times Z$ be its image. For $z \in Z$, we have $C_z + Y = Z$ (as explained above), hence $\{z\} \times Z \subset W$. It follows that ψ is surjective hence, for every $z_1, z_2 \in Z$, there exist points $x \in X$ and $y_1, y_2 \in Y$ with $z_1 = x + y_1$ and $z_2 = x + y_2$. Therefore, $Z - Z \subseteq Y - Y$; but $Z - Z = X - X + Y - Y$, so we get $X - X \subseteq \mathrm{Stab}(Y - Y)$. As explained above, this is possible only if $Y - Y = A$, which contradicts our hypothesis $\dim(A) > 2\dim(Y)$. □

3. Semismallness of the addition map

3.1. **The main result.** We keep the same setup: A is a simple abelian variety defined over an algebraically closed field $\mathbf{k}$ and $X, Y \subseteq A$ are subvarieties.

Recall that a morphism $f \colon S \to T$ is *semismall* if, for all $n \geq 0$,

$$\dim\bigl(\{t \in T \mid \dim(f^{-1}(t)) \geq n\}\bigr) \leq \dim(S) - 2n.$$

If f is semismall and surjective, it is generically finite.

Our main result is the following.

Theorem 3 (Debarre–Moonen). *Let $X_1, \ldots, X_r$ be subvarieties of a simple abelian variety A defined over an algebraically closed field. If $\dim(X_1) + \cdots + \dim(X_r) \leq \dim(A)$, the addition map*

$$X_1 \times \cdots \times X_r \longrightarrow X_1 + \cdots + X_r$$

is semismall.

Corollary 4. *Let $X_1, \ldots, X_r$ be subvarieties of a simple abelian variety A defined over an algebraically closed field. Then*

$$\dim(X_1 + \cdots + X_r) = \min\{\dim(X_1) + \cdots + \dim(X_r), \dim(A)\}.$$

Proof. The inequality $\leq$ is obvious. To prove the reverse inequality, we choose subvarieties $X'_i \subseteq X_i$ such that

$$\dim(X'_1) + \cdots + \dim(X'_r) = \min\{\dim(X_1) + \cdots + \dim(X_r), \dim(A)\}.$$

The theorem then applies to $X'_1, \ldots, X'_r$: the addition map is semismall, hence generically finite and

$$\dim(X_1 + \cdots + X_r) \geq \dim(X'_1 + \cdots + X'_r) = \dim(X'_1) + \cdots + \dim(X'_r).$$

This implies the inequality $\geq$. □

Remark 5. An old theorem of Martens (1967) says that if C is a smooth projective curve of genus g and $d \leq g$, we have

$$\dim(W^n_d(C)) \leq d - 2n$$

for any $n \geq 0$. This is equivalent to saying that the addition map $C^d \to W_d(C) \subset \mathrm{Jac}(C)$ is semismall. Note that we are not assuming that $\mathrm{Jac}(C)$ is simple; however, C generates $\mathrm{Jac}(C)$ and our theorem should generalize to "geometrically nondegenerate" subvarieties of any abelian variety (the corollary does).

3.2. **Perverse sheaves.** The proof of our theorem uses perverse sheaves. I will go through the basics of the theory.

Let X be a variety over a field $\mathbf{k}$. The derived category $\mathsf{D}^b_c(X, \overline{\mathbf{Q}_\ell})$ of bounded constructible[1] complexes of sheaves of $\overline{\mathbf{Q}_\ell}$-modules on X (where ℓ is a prime number invertible in $\mathbf{k}$) is only triangulated, not abelian. We would like to define a subcategory $\mathbf{P}(X)$ of so-called perverse sheaves which is abelian and stable by the so-called Verdier duality (an analog for complexes of Poincaré duality).

A perverse sheaf on a variety X is a complex of sheaves, or more correctly, an object of the derived category $\mathsf{D}^b_c(X, \overline{\mathbf{Q}_\ell})$. Perversity of a complex of sheaves K means that there are restrictions on the supports of the cohomology sheaves $\mathscr{H}^m(K)$, namely:

(P1) the support of the constructible sheaf $\mathscr{H}^m(K)$ has dimension at most $-m$, for all m;
(P2) if $\mathbf{D}(K)$ is the Verdier dual of K, the support of $\mathscr{H}^m(\mathbf{D}(K))$ also has dimension at most $-m$.

Perverse sheaves live in nonpositive degrees: $\mathscr{H}^m(K) = 0$ when $m \notin [-\dim(X), 0]$.

The simplest example is that if $i_X \colon Z \hookrightarrow X$ is a closed subvariety which is *smooth* over $\mathbf{k}$ of dimension n, and if E is a local system on Z, then $\mathrm{IC}_E \coloneqq i_{X*}(E)[n]$ is a perverse sheaf on X. Here $[n]$ means that you shift a complex n places to the left. When $E = \overline{\mathbf{Q}_\ell}$, the perverse sheaf $\mathrm{IC}_Z \coloneqq i_*(\overline{\mathbf{Q}_\ell})[n]$ is called the intersection complex of Z.

One of the important facts about perverse sheaves is that there is a natural way to define a perverse sheaf IC_Z, with support Z, for any closed subvariety $Z \subseteq X$.

3.3. **Convolution product.** Our argument uses that on an abelian variety A, there is a purely sheaf-theoretic analogue of what on the level of cycles would be the Pontryagin product. Namely, if $\sigma_A \colon A \times A \to A$ is the addition map, and if K_1 and K_2 are complexes of sheaves on A, we define

$$K_1 \star K_2 \coloneqq R\sigma_{A*}(\mathrm{pr}_1^*(K_1) \otimes^L \mathrm{pr}_2^*(K_2)).$$

3.4. **Idea of proof.** I will explain the idea of the proof of Theorem 3 when $r = 2$ and X_1 and X_2 are *smooth*, of respective dimensions d_1 and d_2 satisfying $d_1 + d_2 < g$ (so that $X_1 + X_2 \neq A$). The steps are as follows:

Step 1. *The convolution* $\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2}$ *is a perverse sheaf on* A.

It follows from results of Krämer and Weissauer that $\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2}$ is the direct sum of an element of $\mathbf{P}(A)$ and shifted simple perverse sheaves K with (ordinary) Euler characteristic 0; furthermore, these simple perverse sheaves K are of the form $L \otimes q^*Q[g]$, where L is a rank-1 local system on A and $Q \in \mathbf{P}(B)$, where $q \colon A \to B$ is an abelian quotient of A with $\dim(B) < \dim(A)$.

In our case, since A is simple, we have $B = 0$ and shifted simple perverse sheaves have support A. But $\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2}$ has support in $X_1 + X_2 \neq A$, hence $\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2} \in \mathbf{P}(A)$.

[1] A sheaf on X is constructible if X can be written as a finite union of locally closed subschemes on which the sheaf is locally constant. A complex of sheaves is constructible if it has constructible cohomology sheaves.

Step 2. *If $z \in X_1 + X_2$ and $F_z := \sigma^{-1}(z) \subset X_1 \times X_2$ is the fiber, the stalk of the sheaf $\mathscr{H}^m(\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2})$ at z is $H^{m+d_1+d_2}(F_z, \overline{\mathbf{Q}_\ell})$.*

Indeed, by proper base change, this stalk is isomorphic to

$$\begin{aligned} &H^m(F_z, \big(\mathrm{pr}_1^*(\mathrm{IC}_{X_1}) \otimes^L \mathrm{pr}_2^*(\mathrm{IC}_{X_2})\big)|_{F_z}) \\ &= H^m(F_z, \big(\mathrm{pr}_1^*(i_{X_1*}(\overline{\mathbf{Q}_\ell})[d_1]) \otimes \mathrm{pr}_2^*(i_{X_2*}(\overline{\mathbf{Q}_\ell})[d_2]\big)|_{F_z}) \\ &= H^m(F_z, i_{X_1 \times X_2 *}(\overline{\mathbf{Q}_\ell})[d_1 + d_2]|_{F_z}) \\ &= H^{m+d_1+d_2}(F_z, \overline{\mathbf{Q}_\ell}). \end{aligned}$$

Step 3. *If $\dim(F_z) = n$, then $H^{2n}(F_z, \overline{\mathbf{Q}_\ell}) \neq 0$.*

This is standard: for any scheme F of dimension n, one has $H_c^{2n}(F, \overline{\mathbf{Q}_\ell}) \neq 0$ (when F is smooth, this is Poincaré dual to $H^0(F, \overline{\mathbf{Q}_\ell})$, which is nonzero, and one easily reduces to this case).

Step 4. *Conclusion.*

Using Steps 3 and 2, we have, for $n \geq 0$,

$$\begin{aligned} \{z \in X_1 + X_2 \mid \dim(\sigma^{-1}(z)) = n\} &\subseteq \{z \in X_1 + X_2 \mid H^{2n}(F_z, \overline{\mathbf{Q}_\ell}) \neq 0\} \\ &= \{z \in X_1 + X_2 \mid \mathscr{H}^{2n-d_1-d_2}(\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2})_z \neq 0\}. \end{aligned}$$

By perversity of $\mathrm{IC}_{X_1} \star \mathrm{IC}_{X_2}$ (Step 1), this locus has dimension at most $-2n + d_1 + d_2$. This proves that σ is semismall.

Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France

E-mail address: olivier.debarre@imj-prg.fr