# NONSMOOTHABLE CYCLES ON ALGEBRAIC VARIETIES

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ABSTRACT. In 1961, Borel and Haefliger asked whether, in the integral cohomology of a smooth projective algebraic variety of dimension $n$, the class of every algebraic subvariety of dimension $d$ is a linear combination, with integral coefficients, of classes of smooth subvarieties. Kollár and Voisin recently proved that this is true (in characteristic zero) in the Whitney dimension range, that is, whenever $2d < n$. Outside this range, counterexamples were first produced by Hartshorne, Rees, and Thomas in 1974. I will present some new examples. One example has $2d = n$; another example has $n = 6$ (the smallest possible dimension for a counterexample) and $d = 4$. This is joint work with Olivier Benoist.

## 1. INTRODUCTION

The story starts with the famous 1954 article "Quelques propriétés globales des variétés différentiables," where René Thom, then a professor in Strasbourg, studied which integral homology classes on a compact differentiable orientable real manifold can be represented as the homology class of an (orientable) submanifold. Among other things, he proved that

- any integral homology class has a nonzero multiple which is represented by a submanifold;
- some integral homology classes cannot be realized as the image by some continuous map of the fundamental class of another manifold.

Let us try to transpose the story in complex algebraic geometry. So let $X$ be a smooth projective complex algebraic variety of dimension $n$. Algebraic subvarieties (even singular ones) of $X$ have integral cohomology classes. Characterizing these "algebraic" cohomology classes is a very difficult problem in general and we will not address it here. Among these algebraic cohomology classes, one can ask which ones are represented by *smooth* subvarieties of $X$.[1]

When $X$ is the smooth $rs$-dimensional Grassmannian $G_{r,s}$ which parametrizes all vector subspaces of dimension $r$ of $\mathbf{C}^{r+s}$, where all integral cohomology classes are algebraic, there are, when $r, s \geq 4$, cohomology classes on $X$, no nonzero multiple of which can be represented by a smooth algebraic subvariety (Jaehyun Dong, 2005).

So a more reasonable question (formulated in 1961 by Borel and Haefliger) would be to ask whether algebraic classes of dimension $d$ are *integral linear combinations* of classes of smooth subvarieties of $X$. This question has a positive answer when $d \in \{n-1, n\}$ (Bertini

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[1] By Hironaka's resolution of singularities, any algebraic class on $X$ can be represented as the image of by some regular map of the class of a smooth variety.

and Serre), when $d \leq 3$ and $n > 2d$ (Hironaka, 1968), and when $n \leq 5$ (Kleiman, 1969) but also a strong negative answer when the codimension is 2 (Hartshorne–Rees–Thomas, 1974, using Thom's topological techniques): on the Grassmannian $G_{r,s}$ when $r, s \geq 3$, there are codimension 2 classes which are not integral linear combinations of classes of submanifolds (algebraic or not).

A complete positive answer was recently given by Kollár–Voisin in the case $2d < n$ and we will instead concentrate on further counter-examples. We set $c := n - d$ (the codimension). For any integer $m$, we let $\alpha(m)$ be the number of 1's in the binary expansion of $m$.

**Theorem** (Benoist–Debarre). *If $\alpha(c+\alpha(c)) > \alpha(c)$ and $n \geq 4c-2$, there exist a smooth projective complex algebraic variety $X$ of dimension $n$ and an algebraic cohomology class on $X$ of codimension $c$ which is not an integral linear combination of classes of smooth algebraic subvarieties of $X$.*

The weird condition on the codimension $c$ stems from the proof and holds for $c \in \{2, 4, 5, 8, 9, 12, 16, 17 \ldots\}$. In particular, the theorem gives a negative answer to the Borel–Haefliger question for $c = 2$ and $n \geq 6$—earlier examples that also give negative answers were constructed by Benoist whenever $\alpha(c+1) \geq 3$ and $n \geq 2c$. The first unknown case is $n = 6$ and $c = 3$.

## 2. Proof of the Hartshorne–Rees–Thomas result on $G_{3,4}$

We prove a weaker form of the Hartshorne–Rees–Thomas result, with different techniques. When $r, s \geq 2$, one has

$$H^2(G_{r,s}, \mathbf{Z}) = \mathbf{Z}\sigma_1 \qquad , \qquad H^4(G_{r,s}, \mathbf{Z}) = \mathbf{Z}\sigma_1^2 \oplus \mathbf{Z}\sigma_2,$$

where $\sigma_1$ and $\sigma_2$ are Schubert classes.

**Theorem** (Debarre–Han). *Let $Y \subseteq G_{r,s}$, with $r \geq 3$, $s \geq 4$, be a smooth algebraic subvariety of codimension 2 and write its cohomology class as $a\sigma_1^2 + b\sigma_2$, with $a, b \in \mathbf{Z}$. The integer $b$ is even.*

*In particular, the class $\sigma_2$ is not an integral linear combination of classes of smooth algebraic subvarieties of $G_{r,s}$.*

This is the Hartshorne–Rees–Thomas result, except that it assumes $r \geq 3$, $s \geq 4$, and that the smooth subvarieties of $G_{r,s}$ are algebraic, not just (differentiable) submanifolds. The smallest dimension $n$ that we get for the ambiant smooth variety is 12.

*Steps of proof.* (a) A Lefschetz type theorem shows that the restriction $\mathrm{Pic}(G_{r,s}) \to \mathrm{Pic}(Y)$ is an isomorphism (this is where we need $Y$ smooth and $r \geq 3$, $s \geq 4$).

(b) A classical construction of Serre then produces a rank 2 vector bundle $\mathscr{E}$ on $G_{r,s}$ with a section with zero-locus $Y$. In particular, $c_2(\mathscr{E}) = [Y] = a\sigma_1^2 + b\sigma_2 \in H^4(G_{r,s}, \mathbf{Z})$.

(c) We restrict $\mathscr{E}$ to $G_{3,3} \subseteq G_{r,s}$ and compute

$$\int_{G_{3,3}} \mathrm{td}(G_{3,3})\,\mathrm{ch}(\mathscr{E}|_{G_{3,3}}).$$

By the Hirzebruch–Riemann–Roch formula, this is an integer (equal to $\chi(G_{3,3}, \mathscr{E}|_{G_{3,3}})$). With the help of Macaulay 2, we prove that the integer $b$ must be even.[2] ☐

[2] This is what the calculations look like. Upon twisting $\mathscr{E}$ by a line bundle, we can assume $c_1(\mathscr{E}) \in \{0, \sigma_1\}$, without changing the parity of $b$.

Note that step (b) above only works in codimension 2, but works also when $Y$ is a local complete intersection; however, we don't know how to prove step (a) in that case, although we suspect that it still holds (it does when $r = 1$, that is, for the projective space $G_{1,s} = \mathbf{P}^s$).

*Remark* (The Hartshorne conjecture). Hartshorne conjectured in 1974 that when $n \geq 7$, any smooth subvariety of $\mathbf{P}^n$ of codimension 2 is a complete intersection. Voisin showed, by a simple geometric argument, that Hartshorne's conjecture would imply that, when $r, s \geq 7$, any smooth subvariety of $G_{r,s}$ of codimension 2 should be a complete intersection; in particular, its class should be a multiple of $\sigma_1^2$. So, in the notation of the theorem, one should actually have $b = 0$!

## 3. Proof of the main theorem when $c = 2$ and $n \geq 6$

We now work on a complex torus $X = V/\Gamma$, where $V$, the universal cover of $X$, is a complex vector space of dimension $n$ and $\Gamma \simeq \mathbf{Z}^{2n}$ is a lattice in $V$. One has

$$H_i(X, \mathbf{Z}) \simeq \textstyle\bigwedge^i \Gamma \qquad , \qquad H^i(X, \mathbf{Z}) \simeq \textstyle\bigwedge^i \Gamma^\vee$$

for all $i \in \{0, \dots, 2n\}$.

If $L$ is a line bundle on $X$, its first Chern class $\ell$ is an element of $H^2(X, \mathbf{Z}) \simeq \bigwedge^2 \Gamma^\vee$ that can be seen as a skew-symmetric form on $\Gamma$. When $L$ is ample, this form is nondegenerate, hence it can be written, in a suitable $\mathbf{Z}$-basis $(x_1, \dots, x_{2n})$ of $\Gamma$, as

$$\ell = \delta_1 \, \mathrm{d}x_1 \wedge \mathrm{d}x_{n+1} + \cdots + \delta_n \, \mathrm{d}x_n \wedge \mathrm{d}x_{2n},$$

where $\delta_1, \dots, \delta_n$ are uniquely determined positive integers such that $\delta_1 \mid \cdots \mid \delta_n$. We say that $\ell$ is a principal polarization (and the pair $(X, \ell)$ is a principally polarized abelian variety) if $\delta_1 = \cdots = \delta_n = 1$. Note that for each $m \in \{1, \dots, n\}$, the class

$$\ell^m_{\min} := \frac{\ell^m}{m!} \in H^{2m}(X, \mathbf{Z})$$

is integral and nondivisible.

Let $C$ be a (smooth connected projective) curve of genus $n \geq 2$. Its Jacobian $JC$, which parametrizes isomorphism classes of degree 0 line bundles on $C$, has a canonical principal polarization $\theta$. When the curve $C$ is very general (of genus $n$), the class of any algebraic subvariety of codimension $c$ of $JC$ is an integral multiple of $\theta^c_{\min}$.

We will examine which integral multiples of the minimal codimension 2 class $\theta^2_{\min}$ are classes of smooth algebraic subvarieties of $JC$. Note that one can embed the curve $C$ in $JC$ by fixing a point $x_0$ of $C$ and sending a point $x$ of $C$ to the isomorphism class of $\mathscr{O}_C(x - x_0)$. The class of the $(n-2)$-fold sum $W := C + \cdots + C$ in $JC$ is then the minimal class $\theta^2_{\min}$. However, $W$ is singular when $n \geq 6$.

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If $c_1(\mathscr{E}) = 0$, we obtain that the integral is

$$\frac{1}{6720}\big(42a^4 + 84a^3b + 66a^2b^2 + 24ab^3 + 3b^4 - 3332a^3 - 4998a^2b - 2618ab^2 - 476b^3 + 39018a^2 + 39018ab + 10227b^2 - 78568a - 39284b + 13440\big).$$

The integer in parentheses must therefore be divisible by 6720, hence by 8. This implies $4 \mid b$.

When $c_1(\mathscr{E}) = \sigma_1$, one finds that

$$\frac{1}{17280}\big(126a^4 + 252a^3b + 198a^2b^2 + 72ab^3 + 9b^4 - 13944a^3 - 20916a^2b - 10956ab^2 - 1992b^3 + 239736a^2 + 239736ab + 62812b^2 - 794304a - 397152b + 362880\big)$$

is an integer. This implies $6 \mid b$.

I will begin with an old result of mine which is a particular case of the main theorem and whose proof is very similar to the proof sketched above, and then explain how to extend the argument to the general case.

**Theorem** (Debarre, 1995). *Let $C$ be a* very general *smooth projective curve of genus $n \geq 7$. The class of any* smooth *subvariety $Y$ of $JC$ of codimension 2 is an* even *multiple of $\theta^2_{\min} = \theta^2/2$.*

*Remark.* When $C$ degenerates to a tree of elliptic curves $E_1, \dots, E_n$, the Jacobian $JC$ degenerates to $E_1 \times \cdots \times E_n$, and one sees that the class $\theta^c_{\min}$ is the sum of classes of all (smooth) products $F_1 \times \cdots \times F_n$, where $F_i = E_i$ except for $c$ values of $i \in \{1, \dots, n\}$, for which $F_i$ is a point. When $C$ is any curve, the differentiable structure of $JC$ remains the same: these products are no longer algebraic, but they persist as (differentiable) submanifolds of $JC$. So the class $\theta^c_{\min}$ is always the sum of classes of (nonalgebraic) submanifolds of $JC$.

*Steps of proof of the theorem.* (a) A Lefschetz type theorem shows that the restriction $\mathrm{Pic}(JC) \to \mathrm{Pic}(Y)$ is an isomorphism (for this, we need $Y$ smooth, but only $n \geq 6$).

(b) The Serre construction produces a rank 2 vector bundle $\mathscr{E}$ on $JC$ with $c_2(\mathscr{E}) = [Y] = b\theta^2_{\min} \in H^4(JC, \mathbf{Z})$, with $b \in \mathbf{Z}$. Write $c_1(\mathscr{E}) = a\theta$, with $a \in \mathbf{Z}$ (upon twisting $\mathscr{E}$ by a line bundle, we can even assume $a \in \{0, 1\}$, without changing the parity of $b$).

(c) Since the tangent bundle to $JC$ is trivial, the Hirzebruch–Riemann–Roch formula says that

$$\int_{JC} \mathrm{ch}(\mathscr{E}) = \int_{JC} \frac{1}{2^{n-1} n!} \sum_{0 \leq 2k \leq n} (a\theta)^{n-2k} ((a\theta)^2 - 2b\theta^2)^k \binom{n}{2k} = \frac{1}{2^{n-1}} \sum_{0 \leq 2k \leq n} a^{n-2k} (a^2 - 2b)^k \binom{n}{2k}$$

is an integer. Just to illustrate how the calculation works, when $n = 8$, this sum is

$$a^8 - 4a^6 b + 5a^4 b^2 - 2a^2 b^3 + \frac{1}{8} b^4,$$

so $b$ must be even. A similar argument works whenever $4 \mid n$. For other values of $n \geq 7$, a variation of this argument still works. But this method does not work when $n = 6$ (one gets that $a^6 - 3a^4 b + \frac{9}{4} a^2 b^2 - \frac{1}{4} b^3$ is an integer, but this says nothing about the parity of $b$ when $a$ is odd). $\square$

A different argument works for (c) for all $n \geq 6$. It is based on the rather deep fact (whose proof uses the Künneth product formula for topological K-theory, Bott periodicity, and the fact that $X$ is homeomorphic to $(\mathbf{S}^1)^{2n}$) that on any abelian variety $X$, the whole Chern character takes values in $H^\bullet(X, \mathbf{Z})$.[3] So we simply compute

$$\mathrm{ch}_4(\mathscr{E}) = \frac{1}{24}\big(c_1(\mathscr{E})^4 - 4c_1(\mathscr{E})^2 c_2(\mathscr{E}) + 2c_2(\mathscr{E})^2\big) = \big(a^4 - 2a^2 b + \frac{1}{2} b^2\big)\theta^4_{\min}.$$

So $b$ must be even.

[3]This is proved as follows. The Chern character in algebraic $K$-theory factors through the Chern character in complex topological $K$-theory as

$$\mathrm{ch}\colon K^0_{\mathrm{alg}}(X) \longrightarrow K^0_{\mathrm{top}}(X) \xrightarrow{\mathrm{ch}_{\mathrm{top}}} H^{2\bullet}(X, \mathbf{Q}).$$

Complex topological $K$-theory extends to a $\mathbf{Z}/2$-graded cohomological theory $K^0_{\mathrm{top}} \oplus K^1_{\mathrm{top}}$. One computes (using suspension and Bott periodicity) that there is a factorization $\mathrm{ch}_{\mathrm{top}}\colon K^\bullet_{\mathrm{top}}(\mathbf{S}^1) \xrightarrow{\sim} H^\bullet(\mathbf{S}^1, \mathbf{Z}) \hookrightarrow H^\bullet(\mathbf{S}^1, \mathbf{Q})$ and, using Künneth's formula for both $K^\bullet_{\mathrm{top}}$ and $H^\bullet$, that the same factorization holds for $(\mathbf{S}^1)^{2n}$. In particular, the topological Chern character of $(\mathbf{S}^1)^{2n}$ takes values in $H^\bullet((\mathbf{S}^1)^{2n}, \mathbf{Z})$ and, since $X$ is homeomorphic to $(\mathbf{S}^1)^{2n}$, so does the algebraic Chern character of $X$.

When $c > 2$, this approach fails: first because step (b) does not work any more—but this can be circumvented—, but also because the denominators in the Chern character become hard to control and one must replace K-theory with complex cobordism.

*Remark.* On Grassmannians, the Chern character does *not* take integral values, hence this method cannot be used.

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