# SUBVARIETIES OF ABELIAN VARIETIES

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We work over the field $\mathbf{C}$ of complex numbers.

## 1. Introduction

Any (complex) torus can be written as $X = V/\Gamma$, where $V$, the universal cover of $X$, is a complex vector space of dimension $g$ and $\Gamma \simeq \mathbf{Z}^{2g}$ is a lattice in $V$. One has

$$H_i(X, \mathbf{Z}) \simeq \bigwedge^i \Gamma,$$

for all $i \in \{0, \dots, 2g\}$.

If $L$ is a line bundle on $X$, its first Chern class $\ell$ is an element of $H^2(X, \mathbf{Z}) \simeq \bigwedge^2 \Gamma^\vee$ that can be seen as a skew-symmetric form on $\Gamma$. When $L$ is ample, this form is nondegenerate, hence it can be written, in a suitable $\mathbf{Z}$-basis $(x_1, \dots, x_{2g})$ of $\Gamma$, as

$$\ell = \delta_1 \, \mathrm{d}x_1 \wedge \mathrm{d}x_{g+1} + \cdots + \delta_g \, \mathrm{d}x_g \wedge \mathrm{d}x_{2g},$$

where $\delta_1, \dots, \delta_g$ are uniquely determined positive integers such that $\delta_1 \mid \cdots \mid \delta_g$. For future reference, note that for each $m \in \{1, \dots, g\}$, the class

$$\ell^m_{\min} \coloneqq \frac{\ell^m}{\delta_1 \cdots \delta_m \, m!} \in H^{2m}(X, \mathbf{Z})$$

is integral and nondivisible.

The pair $(X, \ell)$ is called a polarized abelian variety of type $\mathbf{d} \coloneqq (\delta_1 \mid \cdots \mid \delta_g)$. When $\delta_1 = \cdots = \delta_g = 1$, the polarization is called *principal.* There is an irreducible quasi-projective coarse moduli space of dimension $g(g+1)/2$ for polarized abelian varieties of fixed dimension $g$ and type $\mathbf{d}$. In particular, one can talk about general, or very general, polarized abelian varieties of given dimension and type.

For each $m \in \{1, \dots, g\}$, we denote by

$$\mathrm{Hdg}^m(X) \coloneqq H^{2m}(X, \mathbf{Z}) \cap H^{m,m}(X)$$

the group of codimension $m$ Hodge classes on $X$. It contains the minimal class $\ell^m_{\min}$.

We are interested in the following problem.

**Question 1.1.** On a polarized abelian variety $X$, which Hodge classes are classes of algebraic subvarieties of $X$, smooth or not, or integral linear combinations of products of Chern classes of vector bundles on $X$?

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For very general polarized abelian varieties, the groups of Hodge classes are as simple as they can be.[1]

**Theorem 1.2** (Comessatti–Mattuck). *A very general polarized abelian variety $(X, \ell)$ of dimension $g$ and type $\mathbf{d}$ satisfies the property*

$$\text{(P)} \qquad \forall m \in \{1, \dots, g\} \qquad \mathrm{Hdg}^m(X) = \mathbf{Z}\ell^m_{\min}.$$

We will mostly work with polarized abelian varieties that satisfy Property (P).

## 2. SMOOTHABILITY OF CYCLES

We begin by quickly reviewing recent results of Kollár–Voisin about smoothability of cycles on any smooth projective variety $X$ of dimension $g$, which vastly improve results of Kleiman and Hironaka from the 60s. Given $m \in \{0, \dots, g\}$, we introduce the subgroups

$$H^{2m}(X, \mathbf{Z})_{\mathrm{smalg}} \subseteq H^{2m}(X, \mathbf{Z})_{\mathrm{alg}} \subseteq \mathrm{Hdg}^m(X),$$

where $H^{2m}(X, \mathbf{Z})_{\mathrm{alg}}$ is the subgroup generated by classes of algebraic subvarieties of $X$ and $H^{2m}(X, \mathbf{Z})_{\mathrm{smalg}}$ is the subgroup generated by classes of smooth subvarieties.

Equality for the inclusion on the right is the integral Hodge conjecture, for which counter-examples are known. We will see soon (Theorem 3.4) that the inclusion on the left may be strict, but is always an equality when $m > g/2$ (Theorem 2.1).

We also let $H^\bullet(X, \mathbf{Z})_{\mathrm{Ch}} \subseteq H^\bullet(X, \mathbf{Z})$ be the subring generated by Chern classes of vector bundles (or, equivalently, coherent sheaves) on $X$ and consider the subgroups

$$H^{2m}(X, \mathbf{Z})_{\mathrm{Ch}} \subseteq H^{2m}(X, \mathbf{Z})_{\mathrm{alg}}.$$

Again, we will see in Proposition 5.1 examples (with $m = g$) where this inclusion is strict. Hiwever, we always have, whenever $1 \le m \le g$,

$$\text{(1)} \qquad (m-1)!H^{2m}(X, \mathbf{Z})_{\mathrm{alg}} \subseteq H^{2m}(X, \mathbf{Z})_{\mathrm{Ch}}.$$

Indeed, this is a consequence of the Grothendieck–Riemann–Roch formula, which gives

$$(m-1)![Z] = (-1)^{m-1}c_m(\mathscr{O}_Z)$$

for all subvarieties $Z \subseteq X$ of codimension $m$.

**Theorem 2.1** (Kollár–Voisin). *Let $X$ be a smooth projective variety of dimension $g$. For $g/2 < m \le g$, one has*

$$H^{2m}(X, \mathbf{Z})_{\mathrm{smalg}} = H^{2m}(X, \mathbf{Z})_{\mathrm{alg}}.$$

## 3. JACOBIANS OF CURVES

Let $C$ be a (smooth connected projective) curve of genus $g \ge 2$ and let $JC$ be its Jacobian, endowed with its canonical principal polarization $\theta$. One can embed the curve $C$ in $JC$ by fixing a point $c$ of $C$ and sending a point $x$ of $C$ to the isomorphism class of $\mathscr{O}_C(x - c)$. For $i \in \{0, \dots, g\}$, we define $W_i \subseteq JC$ as the $i$-fold sum $C + \cdots + C$, with the convention $W_0 = \{o\}$. Its cohomology class is the minimal class $\theta^{g-i}_{\min}$, which is therefore algebraic.

[1] This is a theorem published by Mattuck in 1958; he says: "the present paper should essentially be regarded as a resuscitation, expansion, and interpretation of a little-known paper of Comessatti" [from 1934].

**Theorem 3.1** (Mattuck, 1961[2]). *Let $C$ be a curve of genus $g$ and let $(JC,\theta)$ be its Jacobian. There is a rank $g$ vector bundle $\mathscr{F}$ on $JC$ (the* Picard bundle*) such that*

$$\forall m \in \{1,\dots,g\} \qquad c_m(\mathscr{F}) = [W_{g-m}] = \theta_{\min}^m \in H^{2m}(JC,\mathbf{Z}).$$

**Question 3.2.** The theorem says in particular that when $(X,\theta)$ is the Jacobian of a curve (or a product of such), the inclusion

$$H^{2g}(X,\mathbf{Z})_{\mathrm{Ch}} \subseteq H^{2g}(X,\mathbf{Z}) = \mathbf{Z}$$

is an equality. So this holds in particular for all principally polarized abelian varieties when $g \le 3$. What happens when for (very) general principally polarized abelian varieties of dimension $g \ge 4$?

Using the fact that the Jacobian of a very general curve of genus $g$ satisfies Property (P), we obtain the following.

**Corollary 3.3.** *Let $C$ be a very general curve of genus $g$ and let $(JC,\theta)$ be its Jacobian. For every $m \in \{1,\dots,g\}$, one has*

$$H^{2m}(JC,\mathbf{Z})_{\mathrm{Ch}} = H^{2m}(JC,\mathbf{Z})_{\mathrm{alg}} = \mathrm{Hdg}^m(JC) = \mathbf{Z}\theta_{\min}^m \subseteq H^{2m}(JC,\mathbf{Z}). \tag{2}$$

Under the hypotheses of the corollary, the variety $W_{g-m}$ is smooth if and only if $m \ge (g-1)/2$, and in that range, the group $H^{2m}(JC,\mathbf{Z})_{\mathrm{smalg}}$ is also equal to the groups in (2) (compare with Theorem 2.1). However, outside that range of dimensions, the next theorem shows that the minimal class $\theta_{\min}^m$ is, in many cases, *not* in $H^{2m}(JC,\mathbf{Z})_{\mathrm{smalg}}$. For any positive integer $n$, we let $\alpha(n)$ be the number of ones in the binary expansion of $n$.

**Theorem 3.4** (Benoist, Debarre). *Let $C$ be a very general curve of genus $g$. Let $m$ be a positive integer such that $\alpha(m+\alpha(m)) > \alpha(m)$ and $m \le (g+2)/4$. Then,*

$$H^{2m}(JC,\mathbf{Z})_{\mathrm{smalg}} \subseteq 2\mathbf{Z}\theta_{\min}^m \subsetneq \mathbf{Z}\theta_{\min}^m = H^{2m}(JC,\mathbf{Z})_{\mathrm{alg}}.$$

The weird condition $\alpha(m+\alpha(m)) > \alpha(m)$ holds for $m \in \{2,4,5,8,9,12,16,17,\dots\}$. We also prove $H^8(JC,\mathbf{Z})_{\mathrm{alg}} \subseteq 4\mathbf{Z}\theta_{\min}^4$ when $g \ge 14$. The first unknown case is $g=6$ and $m=3$.

*Sketch of proof when $m=2$.* Given a smooth subvariety $Z \subseteq JC$ of codimension 2, with class $b\theta_{\min}^2$, a Lefschetz-type theorem (here, one needs $g \ge 6$) and the Serre construction produce a rank 2 vector bundle $\mathscr{E}$ on $JC$ with second Chern class $[Z]$. Write $c_1(\mathscr{E}) = a\theta$. In 1995, I used the Hirzebruch–Grothendieck–Riemann–Roch theorem, which says that $\chi(X,\mathscr{E}) = \int_{JC} \mathrm{ch}_g(\mathscr{E})$ is an integer (the tangent bundle to $X$ is trivial). This integer is

$$\int_{JC} \frac{1}{2^{g-1}g!} \sum_{0\le 2k \le g} (a\theta)^{g-2k}((a\theta)^2 - 2b\theta^2)^k \binom{g}{2k} = \frac{1}{2^{g-1}} \sum_{0 \le 2k \le g} a^{g-2k}(a^2-2b)^k \binom{g}{2k}.$$

Just to illustrate how the calculation works, when $g=8$, this sum is

$$a^8 - 4a^6b + 5a^4b^2 - 2a^2b^3 + \frac{1}{8}b^4,$$

so $b$ must be even. This argument works when $4 \mid g$. For other values of $g \ge 7$, one needs to use the fact that $\int_{JC} \mathrm{ch}_g(\mathscr{E}(r\theta))$ is an integer for all integers $r$ and choose $r$ suitably. But it does not work when $g=6$.

However, there is a simpler and more general argument that works for all $g \ge 6$. It is based on the rather deep fact (whose proof uses the Künneth product formula for topological

[2]Mukai reproved this result—and many more about Picard bundles—in the 1981 article where he introduced the Fourier–Mukai transform on abelian varieties.

K-theory, Bott periodicity, and the fact that $X$ is homeomorphic to $(\mathbf{S}^1)^{2g}$) that on any abelian variety $X$, the whole Chern character takes values in $H^\bullet(X,\mathbf{Z})$. So we simply compute

$$\mathrm{ch}_4(\mathscr{E}) = \frac{1}{24}\big(c_1(\mathscr{E})^4 - 4c_1(\mathscr{E})^2c_2(\mathscr{E}) + 2c_2(\mathscr{E})^2\big) = \big(a^4 - 2a^2b + \frac{1}{2}b^2\big)\theta_{\min}^4.$$

So $b$ must be even.

When $m > 2$, this approach fails because the denominators are hard to control and one must replace K-theory with complex cobordism. □

**Remark 3.5** (Intermediate Jacobian of cubic threefolds)**.** There is another known family of principally polarized abelian varieties for which a minimal class is algebraic: if $Y \subseteq \mathbf{P}^4$ is a smooth cubic threefold, its intermediate Jacobian $(JY,\theta)$ is a principally polarized indecomposable abelian fivefold. The image by the Abel–Jacobi map of the surface of lines contained in $Y$ is a smooth surface in $JY$ with minimal class $\theta_{\min}^3$. So we have

$$\theta_{\min}^3 \in H^6(JY,\mathbf{Z})_{\mathrm{smalg}}$$

and, when $Y$ is very general (in which case $(JY,\theta)$ satisfies Property (P)), again

$$H^6(JC,\mathbf{Z})_{\mathrm{smalg}} = H^6(JC,\mathbf{Z})_{\mathrm{alg}} = \mathrm{Hdg}^3(X) = \mathbf{Z}\theta_{\min}^3,$$

but I do not know whether the class $\theta_{\min}^3$ is in $H^6(JY,\mathbf{Z})_{\mathrm{Ch}}$ (by Theorem 2.1, twice this class is in $H^6(JY,\mathbf{Z})_{\mathrm{Ch}}$).

**Remark 3.6** (Grassmannians)**.** Historically, the first examples of integral algebraic classes that are not integral linear combinations of classes of smooth subvarieties were found by Hartshorne, Rees, and Thomas in 1974 on the Grassmannian $\mathsf{Gr}(3,6)$ (but these examples immediately extend to all Grasmannians $G_{r,s} := \mathsf{Gr}(r,r+s)$ with $r,s \geq 3$ by taking sections with general sub-Grassmannians $\mathsf{Gr}(3,6)$). The cohomology ring of the Grassmannian $G_{r,s}$ is generated by the Chern classes $\sigma_1,\dots,\sigma_r$ of the rank $r$ tautological subbundle. In particular, one has

$$H^{2\bullet}(G_{r,s},\mathbf{Z})_{\mathrm{Ch}} = H^{2\bullet}(G_{r,s},\mathbf{Z})_{\mathrm{alg}} = \mathrm{Hdg}^\bullet(G_{r,s}) = H^{2\bullet}(G_{r,s},\mathbf{Z}).$$

However, the subring $H^{2\bullet}(G_{r,s},\mathbf{Z})_{\mathrm{smalg}}$ can be smaller: Hartshorne, Rees, and Thomas show that on $G_{3,3}$, if the class $a\sigma_1^2 + b\sigma_2$ is the class of a smooth submanifold (not necessarily algebraic), $b$ is even, so that

$$H^4(G_{r,s},\mathbf{Z})_{\mathrm{smalg}} \subsetneqq H^4(G_{r,s},\mathbf{Z})_{\mathrm{alg}}.$$

They use topological methods, so that their results is only valid over $\mathbf{C}$ (although they sketch a complicated argument valid over all fields of characteristic other than 2). Note that when $r=2$, the class of the smooth subvariety $G_{2,s-1} \subseteq G_{2,s}$ is $\sigma_{1,1} = \sigma_1^2 - \sigma_2$, hence

$$H^4(G_{2,s},\mathbf{Z})_{\mathrm{smalg}} = H^4(G_{2,s},\mathbf{Z})_{\mathrm{alg}}.$$

Frédéric Han and I, using Macaulay 2, were able to show very simply that the integrality of the number $\chi(G_{3,3},\mathscr{E}) = \int_{G_{3,3}} \mathrm{td}(G_{3,3})\,\mathrm{ch}(\mathscr{E})$, for any rank 2 vector bundle $\mathscr{E}$ on $G_{3,3}$, shows that the integer $b$ in the decomposition $c_2(\mathscr{E}) = a\sigma_1^2 + b\sigma_2$ must be even. The arguments used above (using a Lefschetz-type theorem and the Serre construction) give the same conclusion (over any field), but only on $G_{3,4}$. This method cannot be used further, because the Chern character does *not* take integral values on Grassmannians.

**Remark 3.7** (Grassmannians, continued)**.** Hartshorne conjectured in 1974 that when $n \geq 7$, any smooth subvariety of $\mathbf{P}^n$ of codimension 2 is a complete intersection. Voisin showed, by a simple geometric argument, that Hartshorne's conjecture would imply that, when $r,s \geq 7$, any smooth subvariety of $G_{r,s}$ of codimension 2 should be a complete intersection; in particular, its class should be a multiple of $\sigma_1^2$ and this would imply

$$H^4(G_{r,s},\mathbf{Q})_{\rm smalg} = \mathbf{Q}\sigma_1^2 \subsetneq \mathbf{Q}\sigma_1^2 \oplus \mathbf{Q}\sigma_2 = H^4(G_{r,s},\mathbf{Q})_{\rm alg}.$$

## 4. Curves classes

As we saw in Section 3, on the Jacobian $(JC,\theta)$ of a curve $C$ of genus $g$, the minimal class $\theta_{\min}^{g-1}$ is the class of the curve $C$ embedded in $JC$; it is in particular algebraic. Whether this Hodge class is algebraic on any principally polarized abelian variety is a difficult question (when $g \geq 4$). The following result shows that this question is equivalent to the integral Hodge conjecture for curve classes.

**Theorem 4.1** (Beckman–de Gaay Fortman, 2023)**.** *Let $(X,\theta)$ be a principally polarized abelian variety of dimension $g$. One has*

$$(\star) \qquad H^{2g-2}(X,\mathbf{Z})_{\rm alg} = \mathrm{Hdg}^{g-1}(X) \iff \theta_{\min}^{g-1} \in H^{2g-2}(X,\mathbf{Z})_{\rm alg}.$$

In particular, (products of) Jacobians of curves satisfy the integral Hodge conjecture for curve classes. Voisin proved a nice complement to this result.

**Theorem 4.2** (Voisin, 2023)**.** *Let $(X,\theta)$ be a principally polarized abelian variety of dimension $g$. The equivalent conditions of $(\star)$ imply that $X$ is a direct summand in a product of Jacobians of curves.*

The reason why Voisin was interested in the algebraicity of the minimal curve class was the following earlier result of hers. In the situation of Remark 3.5, one can prove that the intermediate Jacobian $(JY,\theta)$ is not a product of Jacobians of curves (the Clemens–Griffiths criterion then implies that $Y$ is not rational). This implies (by the Matusaka criterion) that the minimal curve class $\theta_{\min}^4$ is not the class of a subvariety of $JY$. Voisin proved that if the class $\theta_{\min}^4$ is not algebraic, then $Y$ is not *stably rational.*[3]

## 5. Nonprincipally polarized abelian varieties

We show that for many types $\mathbf{d} = (\delta_1 \mid \cdots \mid \delta_g)$, for a very general polarized abelian variety $(X,\ell)$ of type $\mathbf{d}$, the groups $H^{2m}(X,\mathbf{Z})_{\rm Ch}$ are as small as they can be, taking into account the inclusion (1). We recall from Theorem 1.2 that a very general $(X,\ell)$ satisfies Property (P).

**Proposition 5.1.** *Let $(X,\ell)$ be a polarized abelian variety of dimension $g$ and type $\mathbf{d} = (\delta_1 \mid \cdots \mid \delta_g)$ that satisfies Property* (P)*. For every $m \in \{1,\dots,g\}$, one has*

$$H^{2m}(X,\mathbf{Z})_{\rm Ch} \subseteq \gcd\Big(\frac{\delta_m}{\delta_1},(m-1)!\Big)\mathbf{Z}\ell_{\min}^m.$$

*In particular, if $\delta_1(m-1)! \mid \delta_m$, one has*

$$(m-1)!H^{2m}(X,\mathbf{Z})_{\rm alg} \subseteq H^{2m}(X,\mathbf{Z})_{\rm Ch} \subseteq (m-1)!\mathbf{Z}\ell_{\min}^m$$

*and (when $m = g$)*

$$(3) \qquad H^{2g}(X,\mathbf{Z})_{\rm Ch} = (g-1)!H^{2g}(X,\mathbf{Z}).$$

[3]Note that, since $(JY,\theta)$ is a Prym variety, the class $2\theta_{\min}^4$ is algebraic.

*Proof.* Given a vector bundle $\mathscr{E}$ on $X$, we may, using Property (P), write its Chern classes as

$$c_i(\mathscr{E}) = a_i \ell_{\min}^i,$$

where $a_0, \dots, a_g$ are integers. Using a simple computation based on the expression of the Chern character as a determinant in the Chern classes, one checks that for every $m \in \{1, \dots, g\}$, the Chern character $\mathrm{ch}_m(\mathscr{E})$ can be written, in $H^{2m}(X, \mathbf{Q})$, as

$$\mathrm{ch}_m(\mathscr{E}) = \Big( (-1)^{m-1} \frac{a_m}{(m-1)!} + \frac{\delta_m}{\delta_1} \frac{b_m}{(m-1)!} \Big) \ell_{\min}^m,$$

for some integer $b_m$. Using the fact, mentioned earlier, that the class $\mathrm{ch}_m(\mathscr{E})$ is integral, we deduce that the integer $a_m$ is divisible by $\gcd(\frac{\delta_m}{\delta_1}, (m-1)!)$.

Consider now vector bundles $\mathscr{E}_1, \dots, \mathscr{E}_r$ on $X$ and the product $\Pi \coloneqq c_{i_1}(\mathscr{E}_1) \cdots c_{i_r}(\mathscr{E}_r)$ of Chern classes, where $i_1 + \cdots + i_r = m$, with $i_1 \geq \cdots \geq i_r$. One checks that either $i_1 = m$ and $\Pi = c_m(\mathscr{E}_1)$, or $r \geq 2$ and $\Pi$ is a multiple of $\frac{\delta_m}{\delta_1} \ell_{\min}^m$. The proposition follows. $\square$

## 6. Subvarieties of abelian varieties

We now turn our attention to subvarieties (as opposed to cycles) of very general principally polarized abelian varieties (I don't have anything to say on this subject about non-principally polarized abelian varieties). We define, for $m \in \{0, \dots, g\}$, integers

$$e_m(g) \coloneqq \min\{e \in \mathbf{Z}_{>0} \mid \forall [(X, \theta)] \in \mathscr{A}_g \quad \exists Z \subseteq X \quad [Z] = e\, \theta_{\min}^m\}.$$

One has $e_0(g) = e_1(g) = e_g(g) = 1$ and $e_m(g) \leq m!$.

**Curve classes.** For $e_{g-1}(g)$, one has

- $e_2(3) = 1$ because every principally polarized abelian threefold is a product of Jacobians of curves;
- $e_{g-1}(g) \geq 2$ for all $g \geq 4$, because products of Jacobians of curves are characterized by the fact that they contain a curve with minimal class (Matsusaka's criterion);
- $e_3(4) = e_4(5) = 2$, because every principally polarized abelian variety is a Prym variety in these dimensions;
- $e_{g-1}(g) \geq 3$ for all $g \geq 6$, because of Welters' characterization of principally polarized abelian varieties containing a curve with twice the minimal class.

A degeneration argument proves that the sequence $(e_{g-1}(g))_{g \geq 2}$ is nonincreasing. In 1994, I gave a lower bound on $e_{g-1}(g)$ which, using the upper bound on the geometric genus of a curve in a principally polarized abelian variety in terms of its degree proved by Pareschi–Popa in 2008 and the lower bound on the genus proved by Pirola in 1993, one can improve to

$$e_{g-1}(g) > \sqrt{g-1} - \frac{1}{2},$$

giving the (not very good) lower bounds

| $g$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_{g-1}(g) \geq$ | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 |

I have an argument which I cannot quite make rigorous (yet) that would prove

$$e_{g-1}(g) > \tfrac{1}{\sqrt{2}}(g - \tfrac{5}{2}),$$

giving the better lower bounds

| $g$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_{g-1}(g) \geq$ | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 7 |

As we saw above, these bounds are sharp for $g \leq 5$.

As for upper bounds, very little is known for $g \geq 6$: apart from the trivial inequality $e_{g-1}(g) \leq (g-1)!$, the only one I know is $e_5(6) \leq 6$ (Alexeev–Donagi–Farkas–Izadi–Ortega).

**Higher dimensional subvarieties.** For $g \geq 4$ and $1 < m < g$, I proved in 1995 that the Jacobian locus is an irreducible component of the set of principally polarized abelian varieties with a subvariety of class $\theta^m_{\min}$. In particular, $e_m(g) > 1$ and $e_2(g) = 2$.

Let $(P, \theta)$ be the $g$ dimensional Prym variety associated with a double étale cover of a curve $C$ of genus $g+1$. When $C$ has a base-point-free $g^r_d$ with $0 < d < 2g+2$, Beauville's theory of special subvarieties constructs a subvariety of $P$ with class $2^{d-2r-1}\theta^{g-r}_{\min}$. In particular, since every curve of genus at most $6$ has a $g^2_6$, one has

$$e_2(4) = e_3(5) = 2$$

(in dimensions $4$ and $5$, every principally polarized abelian variety is a Prym variety) hence, in dimensions $4$ and $5$, there are always surfaces with twice the minimal class.

Finally, fixing $m$, a degeneration argument proves that both sequences $(e_m(g))_{g \geq m}$ and $(e_{g-m}(g))_{g \geq m}$ are nonincreasing.

**Subvarieties with minimal classes.** Although this is a slightly different line of research, I will finish by briefly discussing a conjecture that I made in 1995.

**Conjecture 6.1.** Let $(X, \theta)$ be an indecomposable principally polarized abelian variety of dimension $g$ that contains a subvariety with class $\theta^m_{\min}$, with $1 < m < g$. Then either $(X, \theta)$ is the Jacobian of a curve, or $g = 5$, $m = 3$, and $(X, \theta)$ is the intermediate Jacobian of a smooth cubic threefold.

When $m = g - 1$, this is Matsusaka's criterion. Despite the introduction by Pareschi–Popa of powerful techniques to attack it, this conjecture is still open almost 30 years later. Here is one easily stated result proved in 2018 by Casalaina-Martin–Popa–Schreieder in dimension 5.

**Theorem 6.2.** *Let $(X, \theta)$ be an indecomposable principally polarized abelian fivefold that contains surfaces $V, W$, both with class $\theta^3_{\min}$, such that $[V + W] = \theta$. Then $(X, \theta)$ is either the Jacobian of a curve, or the intermediate Jacobian of a smooth cubic threefold.*

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