CYCLES AND VECTOR BUNDLES ON ABELIAN VARIETIES

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Abstract. This is a complement to the article [D].

1. Non-principally polarized abelian varieties

Let (X, ℓ) be a polarized abelian variety of dimension g and type $(\delta_1 \mid \cdots \mid \delta_g)$ ([BL], §3.1). For each $m \in \{1, \dots, g\}$, we denote by

$$\mathrm{Hdg}^m(X) \coloneqq H^{2m}(X, \mathbf{Z}) \cap H^{m,m}(X)$$

the group of codimension m Hodge classes on X and we set

$$\ell^m_{\min} \coloneqq \frac{\ell^m}{\delta_1 \cdots \delta_m \, m!} \in \mathrm{Hdg}^m(X).$$

This is a nondivisible class. We will make the assumption

$$\forall m \in \{1, \dots, g\} \qquad \mathrm{Hdg}^m(X) = \mathbf{Z}\ell^m_{\min}. \tag{1}$$

By Mattuck's theorem ([BL], Theorem 17.4.1), this holds when (X, ℓ) is very general. Given a vector bundle $\mathscr{E}$ on X, we write its Chern classes in cohomology as $c_i(\mathscr{E}) = a_i \ell^i_{\min}$, where $a_0, \dots, a_g$ are integers.

The following computation is [D, Proposition 1.1], but we provide more details in the proof.

Lemma 1.1. *Assume* (1) *holds. For every $m \in \{1, \dots, g\}$, the Chern character $\mathrm{ch}_m(\mathscr{E})$ can be written, in $H^{2m}(X, \mathbf{Q})$, as*

$$\mathrm{ch}_m(\mathscr{E}) = \Big((-1)^{m-1} \frac{a_m}{(m-1)!} + \frac{\delta_m}{\delta_1} \frac{b_m}{(m-1)!} \Big) \ell^m_{\min}, \tag{2}$$

for some integer b_m.

Proof. By [Mc, p. 28], we have

$$\mathrm{ch}_m(\mathscr{E}) = \frac{1}{m!} \begin{vmatrix} \frac{a_1}{\delta_1}\,\ell & 1 & 0 & \cdots & \cdots & 0 \\ 2\,\frac{a_2}{\delta_1\delta_2 2!}\,\ell^2 & \frac{a_1}{\delta_1}\,\ell & 1 & \ddots & & \vdots \\ 3\,\frac{a_3}{\delta_1\delta_2\delta_3 3!}\,\ell^3 & \frac{a_2}{\delta_1\delta_2 2!}\,\ell^2 & \frac{a_1}{\delta_1}\,\ell & \ddots & \ddots & \vdots \\ 4\,\frac{a_4}{\delta_1\delta_2\delta_3\delta_4 4!}\,\ell^3 & \frac{a_3}{\delta_1\delta_2\delta_3 3!}\,\ell^3 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ m\,\frac{a_m}{\delta_1\cdots\delta_m m!}\,\ell^m & \frac{a_{m-1}}{\delta_1\cdots\delta_{m-1}(m-1)!}\,\ell^{m-1} & \cdots & \frac{a_3}{\delta_1\delta_2\delta_3 3!}\,\ell^3 & \frac{a_2}{\delta_1\delta_2 2!}\,\ell^2 & \frac{a_1}{\delta_1}\,\ell \end{vmatrix}$$

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The expansion of this determinant shows that $\mathrm{ch}_m(\mathscr{E})$ is a sum of terms of the form

$$\pm\frac{1}{m!}\,i_1\,\frac{a_{i_1}}{\delta_1\cdots\delta_{i_1}i_1!}\,\ell^{i_1}\,\frac{a_{i_2}}{\delta_1\cdots\delta_{i_2}i_2!}\,\ell^{i_2}\cdots\frac{a_{i_m}}{\delta_1\cdots\delta_{i_m}i_m!}\,\ell^{i_m},$$

where $i_1,\dots,i_m$ are nonnegative integers such that $i_1+\cdots+i_m=m$ and $\{1,\dots,m\}=\{i_1,i_2+1,\dots,i_m+m-1\}$. Since $\ell^m=\delta_1\cdots\delta_m\,m!\,\ell_{\min}^m$, this term can be rewritten, up to sign, as

$$a_{i_1}\cdots a_{i_m}\,\frac{\delta_1\cdots\delta_m}{\delta_1\cdots\delta_{i_1}(i_1-1)!\,\delta_1\cdots\delta_{i_2}i_2!\cdots\delta_1\cdots\delta_{i_m}i_m!}\,\ell_{\min}^m$$

or as

$$a_{i_1}\cdots a_{i_m}\,\frac{(m-1)!}{(i_1-1)!\,i_2!\cdots i_m!}\,\frac{\delta_1\cdots\delta_m}{\delta_1\cdots\delta_{i_1}\delta_1\cdots\delta_{i_2}\cdots\delta_1\cdots\delta_{i_m}(m-1)!}\,\ell_{\min}^m.$$

The product $\delta_1\cdots\delta_{i_1}\delta_1\cdots\delta_{i_2}\cdots\delta_1\cdots\delta_{i_m}$ in the denominator clearly divides the numerator $\delta_1\cdots\delta_m$. More exactly,

- either δ_1 only occurs once in the product, which only happens when $i_1=m$ and $i_2=\cdots=i_m=0$, and the term is $(-1)^{m-1}\,\frac{a_m}{(m-1)!}\,\ell_{\min}^m$;
- or δ_1 occurs at least twice, δ_m does not occur, the product divides $\delta_1^2\delta_2\cdots\delta_{m-1}$, and the term is an integral multiple of $\frac{\delta_m}{\delta_1}\,\frac{1}{(m-1)!}\,\ell_{\min}^m$.

Therefore, one can write $\mathrm{ch}_m(\mathscr{E})$ as in (2). □

Proposition 1.2. *Let (X,ℓ) be a polarized abelian variety of dimension g and type $(\delta_1\mid\cdots\mid\delta_g)$ that satisfies* (1)*. Let $\mathscr{E}$ be a vector bundle on X, with Chern classes $c_i(\mathscr{E})=a_i\ell_{\min}^i$. For every $m\in\{1,\dots,g\}$, the integer a_m is divisible by* $\gcd(\frac{\delta_m}{\delta_1},(m-1)!)$.

Proof. This follows from Lemma 1.1 and the fact that, as explained in [BD, Lemma 4.1], the class $\mathrm{ch}_m(\mathscr{E})$ is integral. □

Following [KV, Section 1], we let $\mathrm{CH}(X)_{\mathrm{Ch}}$ be the subring of the Chow ring of X generated by Chern classes of vector bundles (or, equivalently, coherent sheaves) on X. In cohomology, we have inclusions

$$H^{2m}(X,\mathbf{Z})_{\mathrm{Ch}}\subset H^{2m}(X,\mathbf{Z})_{\mathrm{alg}}\subset \mathrm{Hdg}^m(X)\subset H^{2m}(X,\mathbf{Z})$$

of subgroups, where $H^{2m}(X,\mathbf{Z})_{\mathrm{alg}}$ is the image of $\mathrm{CH}^m(X)$ by the cycle class map, and $H^{2m}(X,\mathbf{Z})_{\mathrm{Ch}}$ is the image of $\mathrm{CH}^m(X)_{\mathrm{Ch}}$ by the same map.

For each $m\in\{1,\dots,g\}$, one has inclusions

(3) $$(m-1)!\,\mathrm{CH}^m(X)\subset\mathrm{CH}^m(X)_{\mathrm{Ch}}\quad,\quad(m-1)!H^{2m}(X,\mathbf{Z})_{\mathrm{alg}}\subset H^{2m}(X,\mathbf{Z})_{\mathrm{Ch}}$$

(they follow from the equality $(m-1)![Z]=(-1)^{m-1}c_m(\mathscr{O}_Z)$ in $\mathrm{CH}^m(X)$, valid for all subvarieties $Z\subset X$ of codimension m).

Corollary 1.3. *Let (X,ℓ) be a polarized abelian variety of dimension g and type $(\delta_1\mid\cdots\mid\delta_g)$ that satisfies* (1)*. For every $m\in\{1,\dots,g\}$, one has*

$$H^{2m}(X,\mathbf{Z})_{\mathrm{Ch}}\subset\gcd\Big(\frac{\delta_m}{\delta_1},(m-1)!\Big)\mathbf{Z}\ell_{\min}^m.$$

In particular, if $\delta_1(m-1)!\mid\delta_m$, one has

$$(m-1)!H^{2m}(X,\mathbf{Z})_{\mathrm{alg}}\subset H^{2m}(X,\mathbf{Z})_{\mathrm{Ch}}\subset(m-1)!\mathbf{Z}\ell_{\min}^m.$$

Proof. Let $\mathscr{E}_1, \dots, \mathscr{E}_r$ be vector bundles on X and consider the product $\Pi \coloneqq c_{i_1}(\mathscr{E}_1) \cdots c_{i_r}(\mathscr{E}_r)$ of Chern classes, where $i_1 + \cdots + i_r = m$, with $i_1 \geq \cdots \geq i_r$. We write

$$c_{i_k}(\mathscr{E}_k) = a_k \ell_{\min}^{i_k} = \frac{\ell^{i_k}}{\delta_1 \cdots \delta_{i_k}\, i_k!},$$

with $a_1, \dots, a_r \in \mathbf{Z}$, so that

$$\Pi = a_1 \cdots a_r \frac{\delta_1 \cdots \delta_m}{\delta_1 \cdots \delta_{i_1} \delta_1 \cdots \delta_{i_2} \cdots \delta_1 \cdots \delta_{i_r}} \, \frac{m!}{i_1! i_2! \cdots i_r!} \, \ell_{\min}^m.$$

As in the proof of Proposition 1.2, either $i_1 = m$ and $\Pi = c_m(\mathscr{E}_1)$, or $r \geq 2$ and Π is a multiple of $\frac{\delta_m}{\delta_1} \ell_{\min}^m$. Using Proposition 1.2, we obtain the corollary. ☐

Corollary 1.4. *Let (X, ℓ) be a polarized abelian variety of dimension g and type $(\delta_1 \mid \cdots \mid \delta_g)$ that satisfies* (1)*. Assume $\delta_1 (g-1)! \mid \delta_g$. One has*

$$\mathrm{CH}^g(X)_{\mathrm{Ch}} = (g-1)!\, \mathrm{CH}^g(X).$$

Proof. The inclusion $\supset$ is (3). Conversely, let $\eta \in \mathrm{CH}^g(X)_{\mathrm{Ch}}$. Since $\delta_1(g-1)! \mid \delta_g$, Corollary 1.3 implies that $\deg(\eta)$ is divisible by $(g-1)!$. Therefore, there exists $\eta' \in \mathrm{CH}^g(X)$ such that $\eta - (g-1)!\eta'$ has degree 0. Since the subgroup of $\mathrm{CH}^g(X)$ of 0-cycles of degree 0 is divisible, there exists $\eta'' \in \mathrm{CH}^g(X)$ such that $\eta - (g-1)!\eta' = (g-1)!\eta''$. This implies $\eta \in (g-1)!\, \mathrm{CH}^g(X)$. ☐

2. Principally polarized abelian varieties

Remark 2.1 (Jacobians of curves). Let C be a smooth connected projective curve of genus $g \geq 2$ and let JC be its Jacobian, endowed with its canonical principal polarization θ.

We fix a point c of C and embed the curve C in JC by sending a point x of C to the isomorphism class of $\mathscr{O}_C(x - c)$. For $i \in \{0, \dots, g\}$, we define $W_i \subset JC$ as the i-fold sum $C + \cdots + C$, with the convention $W_0 = \{o\}$. Its cohomology class is the minimal class $\theta_{\min}^{g-i}$.

Let $\mathscr{P}$ be the Poincaré line bundle on $C \times JC$, uniquely defined by the properties

$$\mathscr{P}|_{\{c\} \times JC} \simeq \mathscr{O}_{JC} \quad \text{and} \quad \mathscr{P}|_{C \times \{\xi\}} \simeq P_\xi|_C \quad \text{for all } \xi \in JC,$$

where P_ξ is the numerically trivial line bundle on JC defined by ξ. Following [S, §2, Definition] (see also [Mk], and [Mu, Definition 4.1]), we define the *Picard bundle* on JC by

$$\mathscr{F} \coloneqq R^1 q_*(\mathscr{P} \otimes p^* \mathscr{O}_C(-c))$$

where $p : C \times JC \to C$ and $q : C \times JC \to JC$ are the projections. By [S], *the sheaf $\mathscr{F}$ is locally free of rank g on JC.*

The Chern classes of $\mathscr{F}$ were computed by Mattuck in [Mk, §6, Corollary] (see also [S, §4] and [G, Corollary 3 to Theorem 4]); he obtains:

$$\forall m \in \{1, \dots, g\} \qquad c_m(\mathscr{F}) = [W_{g-m}] \in \mathrm{CH}^m(JC). \tag{4}$$

Considering translates of $\mathscr{F}$, we get

$$\mathrm{CH}^g(JC)_{\mathrm{Ch}} = \mathrm{CH}^g(JC).$$

Moreover, when C is very general, so that (JC, θ) satisfies (1), one has

$$\forall m \in \{1, \dots, g\} \qquad H^{2m}(JC, \mathbf{Z})_{\mathrm{Ch}} = H^{2m}(JC, \mathbf{Z})_{\mathrm{alg}} = \mathrm{Hdg}^m(JC) = \mathbf{Z}\theta_{\min}^m.$$

Question 2.2. What are the subgroups $H^{2m}(X, \mathbf{Z})_{\mathrm{Ch}} \subset \mathbf{Z}\theta_{\min}^m$ for a very general principally polarized abelian variety (X, θ) of dimension $g \geq 4$? The example of Jacobians of curves shows that there are no numerical obstructions. This question is already intriguing for $m = g$ (a case where all classes are trivially algebraic).

Question 2.3. The intermediate Jacobian (JX, θ) of a smooth cubic threefold $X \subset \mathbf{P}^4$ is a principally polarized abelian variety of dimension 5 which contains a surface with minimal class $\theta_{\min}^3$. Is there a vector bundle $\mathscr{E}$ on JX with $c_3(\mathscr{E}) = \theta_{\min}^3$?

Question 2.4. On a Prym variety (P, θ), are there vector bundles with "small" Chern classes?

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