COMPLETE FAMILIES OF SMOOTH PROJECTIVE MANIFOLDS

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ABSTRACT. We discuss nontrivial families of smooth projective manifolds over a smooth projective curve. Their existence depends on the nature of the fibers and the genus of the base curve. We will state some of the main theorems that give restrictions on the existence of such families and explain a couple of constructions.

CONTENTS

1. INTRODUCTION

We work over the field of complex numbers. All varieties are connected. We study *smooth* projective morphisms $f\colon X \to B$ with connected fibers, where B is a smooth *complete* variety which, for the sake of simplicity, will be assumed to have dimension one (so a smooth connected projective curve). The first example that comes to mind is the case when f is trivial, that is, F is a smooth projective variety and $f\colon F \times B \to B$ is the second projection. This case is not interesting, so we will look for morphisms f as above that are nontrivial.

In this talk, we will, after first discussing various notions of triviality (including isotriviality), try to answer the following two questions for various classes $\mathscr{F}$ of smooth projective varieties:

(Q1) does there exist nonisotrivial smooth fibrations with fibers in $\mathscr{F}$?
(Q2) does there exist nonisotrivial smooth fibrations with fibers in $\mathscr{F}$ and base a rational or elliptic curve?

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2. Triviality of smooth fibrations

Consider a smooth projective fibration $f\colon X \to B$. By Ehresmann's theorem, f is $\mathscr{C}^\infty$-locally trivial and in particular, its fibers are all diffeomorphic. So their Betti numbers are constant and, by a standard argument, its Hodge numbers $h^{p,q}$ are also constant (by semicontinuity, they can only go up by specialization, but the sum $\sum_{p+q=k} h^{p,q}$ is the Betti number b_k, so is constant). A difficult theorem of Siu says that the plurigenera (hence also the Kodaira dimensions) of the fibers are constant.

Obviously, when f is trivial, its fibers are all isomorphic. The converse is however false, as shown by the following example.

Example 2.1. Consider a Hirzebruch surface $f\colon \mathbf{F}_n := \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(n)) \to \mathbf{P}^1$, with $n > 0$. All fibers of f are isomorphic to $\mathbf{P}^1$ but f is not trivial, because the rank-2 vector bundles $\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(n)$ and $\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}$ do not differ by twisting by a rank-1 vector bundle. The same reasoning shows that f remains nontrivial even after pulling back by any finite morphism $B \to \mathbf{P}^1$.

The situation improves when one assumes that the fibers are not uniruled (that is, not covered by rational curves; this is the case if their Kodaira dimension is nonnegative).

Theorem 2.2 (Matsusaka–Mumford)**.** *Let $f\colon X \to B$ be a smooth projective fibration whose fibers are all projectively isomorphic to a fixed nonuniruled (smooth projective) variety F. There is a finite étale cover of B over which f becomes trivial.*

Sketch of proof. For tautological reasons, f trivializes over the base change $\mathrm{Isom}_B(X, B\times F) \to B$ (we are considering here isomorphisms of polarized varieties). This base change is surjective because all fibers are isomorphic to F; it is affine because we are considering isomorphisms of polarized varieties; it is proper by a theorem of Matsusaka–Mumford (this is where nonuniruledness is used); it is unramified because, in characteristic zero, all algebraic groups are smooth; and it is flat because all fibers are isomorphic to $\mathrm{Aut}(F)$. So the base change is finite étale. □

In the rest of the lecture, we will say that a smooth projective fibration $f\colon X \to B$ is *isotrivial* if its fibers are all projectively isomorphic.

3. Moduli stacks and moduli spaces

One way to look at our setup is to introduce the moduli stack $\mathscr{M}$ and (coarse) moduli space M for the fibers of a smooth fibration $f\colon X \to B$ (in most cases, they exist). Then there are moduli morphisms

$$B \longrightarrow \mathscr{M} \qquad \text{and} \qquad B \longrightarrow \mathsf{M}$$

The fibration f is trivial if the first map is constant with image a "nonstacky point." Its fibers are all isomorphic if the second map is constant. These two properties are not equivalent: a map $B \to \mathsf{M}$ might not even lift to a map $B \to \mathscr{M}$.

However, it is often the case that a map $B \to \mathsf{M}$ lifts, after taking a suitable finite cover $B' \to B$, to a map $B' \to \mathscr{M}$ and in this case, the existence of nonisotrivial smooth fibrations (that is, an affirmative answer to question (Q1)) is equivalent to the existence of complete curves contained in M.

The situation is different for question (Q2): an affirmative answer implies that M contains complete curves of genus 0 or 1, but the converse is false: we will see very simple examples where the answer to question (Q2) is negative, although M contains complete rational curves.

4. Nontrivial smooth fibrations: existence and nonexistence results

4.1. **Smooth fibrations in curves or abelian varieties.** We first answer (negatively) question (Q2) when the fibers of the smooth fibration are curves or polarized abelian varieties.

Theorem 4.1 (Shafarevich)**.** *Let B be a smooth projective curve of genus 0 or 1 (a rational or elliptic curve). Any smooth projective fibration $f\colon X \to B$ in curves or in polarized abelian varieties is isotrivial.*

Proof. Because of the Torelli theorem, it is enough to do the case of polarized abelian varieties (of dimension $g > 0$). The local system $R^1 f_* \mathbf{Z}$ trivializes on the universal cover $\widetilde{B} \to B$. Choosing a symplectic basis defines a holomorphic map $\mu\colon \widetilde{B} \to \mathsf{H}_g$ to the Siegel upper half-space. Note that

- $\widetilde{B}$ is either $\mathbf{P}^1$ (if B is rational) or $\mathbf{C}$ (if B is elliptic);
- H_g is a bounded domain: it is biholomorphic to a bounded domain in $\mathbf{C}^{g(g+1)/2}$.

By Liouville's theorem (any bounded holomorphic map $\mathbf{C} \to \mathbf{C}$ is constant), the map μ is constant, hence f is isotrivial. □

Let us now consider question (Q2) for curves of genus g.

The moduli space M_2 is affine, so it contains no complete curves. When $g \geq 3$, there is a projective compactification of M_g (the Satake compactification) whose boundary has codimension 2, so M_g does contain complete curves (passing through any point). As explained (here, there is a finite cover$\widetilde{\ } \to$ over which there is a universal curve So there exist non-isotrivial complete families of smooth curves of any genus $g \geq 3$.

Such explicit families were constructed by Kodaira. Briefly, he proceeds as follows: start from a morphism $h\colon B \to C$ between curves of genus ≥ 2, assume that a finite group G acts without fixed points on C and that there is a cyclic covering $X \to B \times C$ branched along the smooth curve $\bigsqcup_{g \in G} \{(b, gh(b)) \mid b \in B\}$. The surface X is smooth and the fiber at $b \in B$ of the fibration $f\colon X \to B$ is a cyclic covering of C branched along the (smooth) orbit $G \cdot h(b)$. These fibers are all smooth, but not all isomorphic (they are coverings of the fixed curve C, of genus ≥ 2, branched along a varying divisor, so they cannot be all isomorphic by a theorem of de Franchis).

Oort refined Kodaira's construction, starting from a curve B with a morphism $p\colon B \to \mathbf{P}^1$ and constructing the fibration $f\colon X \to B$ in such a way that the associated moduli map $B \to \mathsf{M}_{g(F)}$ factors through p, therefore proving that M_g actually contains complete rational curves for infinitely many values of g (this does not contradict Theorem 4.1!). Note that for $g \geq 22$, the compactification $\overline{\mathsf{M}}_g$ is not uniruled, so these rational curves have to be "special" (when $g \leq 16$, the projective variety $\overline{\mathsf{M}}_g$ *is* uniruled, but this says nothing about *complete* rational curves contained in M_g).

As for question (Q2) for principally polarized abelian varieties of dimension g, the moduli space A_g also has a Satake compactification, whose boundary has codimension g, so A_g contains complete subvarieties of dimension $g-1$ (passing through any point). Explicit examples of complete rational curves in A_2 given by suitable Shimura varieties.

4.2. **Smooth fibrations in Calabi–Yau varieties.** Question (Q2) for varieties with trivial canonical bundles also has a negative answer. This follows from a theorem of Griffiths from the '70s which has since been vastly generalized by many different authors. One very general recent version is an article of Brunebarbe–Cadorel about varieties carrying a complex polarized variation of Hodge structures ([BC]).

Theorem 4.2 (Griffiths). *Let B be a smooth projective curve of genus 0 or 1 (a rational or elliptic curve). Any smooth projective fibration $f\colon X \to B$ in varieties with trivial canonical bundles is isotrivial.*

Sketch of proof. The proof follows the same path as that of Theorem 4.1. Griffiths proved that there is a holomorphic period map $B \to D/\Gamma$, where D is an analytic domain and Γ is a properly discontinuous group of analytic automorphisms of D.

Griffiths and Schmid then proved that, if $\widetilde{B} \to B$ is the universal cover, its composition with the period map lifts to a holomorphic map $\mu\colon \widetilde{B} \to D$. Unfortunately, unlike H_g, the domain D is in general not bounded; however, Griffiths and Schmid proved that the map μ is "horizontal," that is, the image of its tangent map lies into a specific subbundle of the tangent bundle to D and that any horizontal holomorphic map $\mathbf{C} \to D$ is constant. This proves that the period map is constant ([GS, Corollary (9.7)]).

The second ingredient (due to Andreotti) is that varieties with trivial canonical bundles satisfy the infinitesimal Torelli property for their middle cohomology (the differential of their period map for their local universal deformation is injective). This implies that f is isotrivial. $\square$

Projectivity is essential here: twistor lines give analytic nonisotrivial families of K3 surfaces (or hyper-Kähler manifolds) over $\mathbf{P}^1$.

Concerning question (Q1), we explain now how to construct complete nonisotrivial families of smooth polarized K3 surfaces. The original idea comes from [BKPS]. Thanks to the Torelli theorem and the existence of the Baily–Borel compactification, there is a coarse moduli space F_{2e} for smooth polarized K3 surfaces of degree $2e > 0$ which is an open subset of a projective (19-dimensional) variety $\overline{\mathsf{F}}_{2e}$; unfortunately, the complement of F_{2e} in $\overline{\mathsf{F}}_{2e}$ is a divisor. When this divisor is ample (this happens when $e = 1$ for example), F_{2e} is affine hence contains no complete curves and any complete family of smooth polarized K3 surfaces of degree $2e$ is isotrivial. However, for all $e \gg 0$, there are complete nonisotrivial families of smooth polarized K3 surfaces of degree $2e$.

Theorem 4.3 (Debarre–Macrì). *For each integer $e \geq 62$ or in the set*

$$\{14, 18, 26, 28, 29, 32, 34, 36, 38, 40, 42, 44, 45, 46, 47, 49, 50, 53, 54, 56, 57, 59, 60\},$$

there exists a nonisotrivial complete family of smooth K3 surfaces with a relative polarization of degree $2e$.

Sketch of proof. The idea is to use Kummer surfaces: if (A, L) is a polarized abelian surface, the quotient $A/\pm 1$ has 16 ordinary double points corresponding to the 2-torsion points of A, and resolving them gives a (smooth) K3 surface $K(A)$. It inherits a nef and big line bundle H coming from the polarization L of A and 16 (-2)-curves $E_1, \dots, E_{16}$. The trick is to concoct ample line bundles on $K(A)$ out of these. This is achieved by the following result: *On any Kummer surface, any rational class*

$$L = aH - \sum_{i=1}^{16} a_i E_i,$$

where $a_1 \geq \cdots \geq a_{16} > 0$ *and* $a > a_1+a_2+a_3+a_4$, *is ample.* In particular, the class $aH - \sum_{i=1}^{16} E_i$ is ample for all $a > 4$ and when the polarization L is principal (so that $H^2 = 4$), the self-intersection of the integral ample class $2e = 3H - \frac{1}{2}\sum_{i=1}^{16} E_i$ is $9 \cdot 4 + \frac{1}{4} \cdot 16 \cdot (-2) = 28$. This is the smallest degree $2e$ that we found; of course, other values of $a, a_1, \ldots, a_{16}$ give many more values of $2e$.

To construct the family, one starts from a nonisotrivial complete family of polarized abelian surfaces and perform the Kummer construction on this family. □

Using recent results on Bridgeland stability conditions for moduli spaces of stable objects on K3 surfaces and their behavior in families, one can construct nonisotrivial complete families of polarized hyper-Kähler varieties of any dimensions.

Starting from complete rational curves in A_2 (such as Shimura curves), it is not difficult (using the theorem) to construct complete rational curves in F_{2e} (or in the coarse moduli space of polarized hyper-Kähler varieties) for infinitely many values of e. These moduli spaces are known not to be uniruled for $e \gg 0$, so these curves have to be "special."

4.3. **Smooth fibrations in varieties of general type.** The following theorem is a particular case of a very general result (itself the termination of a long list of previous works by many different authors) of Wei–Wu ([WW]). It gives a negative answer to question (Q2) for varieties of general type.

Theorem 4.4. *Let B be a smooth projective curve of genus 0 or 1 (a rational or elliptic curve). Any smooth projective fibration $f \colon X \to B$ in varieties with big canonical bundles is isotrivial.*

Concerning question (Q1), I have already explained how to obtain complete non isotrivial families of smooth projective curves of genus 2; taking their relative self-product gives complete non isotrivial families of smooth projective varieties of general type of any dimensions.

4.4. **Smooth fibrations in Fano varieties.** A Fano variety is a (smooth projective) variety whose anticanonical bundle is ample. In dimension 1, there is just $\mathbf{P}^1$. Fano surfaces are more commonly called del Pezzo surfaces: they are $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ blown up at at most 8 distinct points "in general position" (no three on a line, no six on a conic, no eight on a cubic having a node at one of them).

When $d \in \{5, \ldots, 9\}$, all del Pezzo surfaces of degree d are isomorphic, so any smooth projective fibration is isotrivial according to our definition. When $d \in \{1, 2, 3, 4\}$, a del Pezzo surface is the blow up of $\mathbf{P}^2$ at $9-d$ distinct points in general position. In that case, the coarse moduli space of del Pezzo surfaces is open in an affine,[1] hence contains no complete curves. So in all cases, any smooth projective fibration in smooth del Pezzo surfaces is isotrivial.

[1] I could not find a reference for this fact. Here is a quick argument based on [I]: the open subset $U \subseteq \mathrm{Sym}^{9-d}(\mathbf{P}^2)$ of distinct points such that no three of them are colinear is the complement of an ample divisor, hence is affine. Morever ([I, Table (4)]), it consists of stable points for the action of $\mathrm{Aut}(\mathbf{P}^2)$, hence the GIT quotient $U/\!/\operatorname{Aut}(\mathbf{P}^2)$ is also affine. It contains as an open subset the coarse moduli space of del Pezzo surfaces of degree d.

Note that it does not mean that the fibration becomes trivial on a finite cover of the base because the Matsusaka–Mumford theorem does not apply (any nontrivial $\mathbf{P}^2$-bundle on $\mathbf{P}^1$ gives a counter-example).

We end the lecture by explaining what we know for Gushel–Mukai (or GM) varieties. If $\mathsf{CGr}(2,\mathbf{C}^5) \subseteq \mathbf{P}(\mathbf{C}\oplus\bigwedge^2\mathbf{C}^5)$ is the cone over the Plücker embedded Grassmannian $\mathsf{Gr}(2,\mathbf{C}^5) \subseteq \mathbf{P}(\bigwedge^2\mathbf{C}^5)$, a GM variety is a smooth transverse intersection

$$X := \mathsf{CGr}(2,\mathbf{C}^5) \cap \mathbf{P}^{n+4} \cap Q,$$

where Q is a quadric hypersurface. Its dimension n is at most 6 and $\omega_X = \mathscr{O}_X(2-n)$, so that X is a Fano variety for $n \in \{3,4,5,6\}$ (GM curves are Brill–Noether general curves of genus 6; GM surfaces are Brill–Noether general polarized K3 surfaces of degree 10). There is a quasi-projective coarse moduli space $\mathsf{M}_n^{\mathrm{GM}}$ for GM n-folds; it is irreducible of dimension $25-(5-n)(6-n)/2$ and there is a map

$$\pi_n \colon \mathsf{M}_n^{\mathrm{GM}} \longrightarrow \mathsf{M}^{\mathrm{EPW}},$$

where $\mathsf{M}^{\mathrm{EPW}}$ is the irreducible 20-dimensional affine coarse moduli space for EPW sextics $Y \subseteq \mathbf{P}^5$: these are sextic hypersurfaces whose singular locus is a normal irreducible surface with finite singular locus, empty for $[Y] \in \mathsf{M}^{\mathrm{EPW}}$ general. When $n \in \{3,4,5,6\}$, the map π_n is surjective and its fibers are as follows:[2]

$$\pi_n^{-1}([Y]) = \begin{cases} \mathbf{P}^5 \smallsetminus Y & \text{when } n=6; \\ \mathbf{P}^5 \smallsetminus Y_{\text{sing}} & \text{when } n=5; \\ Y \smallsetminus (Y_{\text{sing}})_{\text{sing}} & \text{when } n=4; \\ Y_{\text{sing}} & \text{when } n=3. \end{cases}$$

The coarse moduli space $\mathsf{M}_6^{\mathrm{GM}}$ is affine, so it contains no complete curves and any complete family of GM sixfolds is isotrivial. It is clear from the description of π_n that $\mathsf{M}_3^{\mathrm{GM}}, \mathsf{M}_4^{\mathrm{GM}}, \mathsf{M}_5^{\mathrm{GM}}$ all contain complete curves (necessarily contained in the fibers of π_n) through any point. This answers question (Q1) for these Fano vareties.

Question (Q2) is more delicate. With Kuznetsov, we were able to prove that the following families exist:

- nonisotrivial families of GM 5folds parametrized by $\mathbf{P}^1$ (more exactly, by any bitangent line to Y);
- nonisotrivial families of GM 4folds parametrized by $\mathbf{P}^1$ (more exactly, by any rational curve contained in a canonical double cover of Y called a double EPW sextic);
- nonisotrivial families of GM 3folds parametrized by the (projective) surface Y_{sing} when that surface is smooth. However, these surfaces contain no rational curves.

The answer to the following subquestion of question (Q2) seems unknown (refering to the description above, it is not known whether the surface Y_{sing} contains rational curves when it is singular).

Question 4.5 (Javanpeykar)**.** Are there nonisotrivial families of Fano threefolds parametrized by $\mathbf{P}^1$?

Note that there are nonisotrivial families of Fano threefolds parametrized by an elliptic curve (these are blowups of a fixed smooth complete intersection of two quadrics in $\mathbf{P}^5$ along a varying line).

[2]To be precise, one needs to quotient out these quasiprojective varieties by the action of a finite group of automorphisms which is trivial for $[Y]$ general (see [DK, Section 2.3]).

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