PARAMETRIZING SUBVARIETIES OF AN ALGEBRAIC VARIETY

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ABSTRACT. This talk has two parts. The first part (Sections 1–4) is a colloquium style discussion on the problem of parametrizing subvarieties (like lines or conics) of a given complex algebraic variety X in a projective space. We mostly concentrate on the case where X is a smooth cubic hypersurface and introduce at the end intermediate Jacobians and Abel–Jacobi maps.

In the second part, we switch to another family of Fano manifolds: Gushel–Mukai manifolds. We describe in Section 6 their families of lines and conics. In the last section, we discuss derived categories and how Bridgeland stability conditions and moduli spaces of stable objects can be used to describe moduli spaces of polarized hyper-Kähler manifolds of infinitely many different dimensions.

CONTENTS

1. LINES AND CONICS IN THE PROJECTIVE SPACE

A complex projective algebraic variety is a subset X of the complex projective space defined by homegeneous polynomial equations. We will always assume that X is a smooth

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(a complex manifold). One can then consider families of subvarieties of X of a given type (lines, conics, planes...). These families are themselves algebraic varieties which sometimes have interesting geometries.

All varieties are complex algebraic, and often projective (that is, subvarieties of some projective space) and in particular compact.

Examples 1.1. Take $X = \mathbf{P}^n = \mathbf{P}(V_{n+1})$.

(1) Projective lines in $\mathbf{P}^n$ are parametrized by the Grassmannian variety $\mathsf{Gr}(2, V_{n+1})$ (they correspond to 2-dimensional subspaces in V_{n+1}), itself a smooth irreducible projective variety of dimension $2(n-1)$, realized via the Plücker embedding

$$\mathsf{Gr}(2, V_{n+1}) \longrightarrow \mathbf{P}(\textstyle\bigwedge^2 V_{n+1}), \qquad [V_2] \longmapsto [\textstyle\bigwedge^2 V_2]$$

as the locus in $\mathbf{P}(\bigwedge^2 V_{n+1})$ of "bivectors" of rank 2. One defines similarly the Grassmannian $\mathsf{Gr}(k+1, V_{n+1})$, which parametrizes linear subspaces of $\mathbf{P}^n$ of dimension k; it is a smooth irreducible projective algebraic variety of dimension $(k+1)(n-k)$.

(2) For conics in $\mathbf{P}^n$, there is a morphism

$$\{\text{conics } C \subseteq \mathbf{P}^n\} \longrightarrow \{\mathbf{P}^2 \subseteq \mathbf{P}^n\} = \mathsf{Gr}(3, V_{n+1}), \qquad [C] \longmapsto \langle C \rangle$$

that sends a conic C to its linear span $\langle C \rangle$. It is a $\mathbf{P}^5$-bundle (conics contained in a given plane are parametrized by $\mathbf{P}^5$). So the family of conics in $\mathbf{P}^n$ is again a smooth irreducible algebraic variety of dimension $3(n-2)+5 = 3n-1$. Note that these "conics" also include intersecting pairs of lines and "double" (plane) lines.

In general, local deformations of a smooth projective subvariety $Y \subseteq X$ are governed by (algebraic) sections of its normal bundle $N_{Y/X}$. More exactly, the local deformation space is defined by $h^1(Y, N_{Y/X})$ equations in a smooth variety of dimension $h^0(Y, N_{Y/X})$, hence

- it has dimension $\geq h^0(Y, N_{Y/X}) - h^1(Y, N_{Y/X})$ (this number is called the "expected dimension");
- it is smooth when $H^1(Y, N_{Y/X}) = 0$.

Examples 1.2. (1) Let $L \subseteq \mathbf{P}^n$ be a line. Its normal bundle is

$$N_{L/\mathbf{P}^n} \simeq \mathscr{O}_L(1)^{\oplus(n-1)},$$

hence $h^0(L, N_{L/\mathbf{P}^n}) = 2n-2$ and $h^1(L, N_{L/\mathbf{P}^n}) = 0$: we recover the fact that the Grassmannian variety $\mathsf{Gr}(2, V_{n+1})$ is everywhere smooth of the expected dimension $2n-2$.

(2) Let $C \subseteq \mathbf{P}^n$ be a smooth conic. There is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{C/\langle C \rangle} & \longrightarrow & N_{C/\mathbf{P}^n} & \longrightarrow & N_{C/\mathbf{P}^n}|_C & \longrightarrow & 0 \\ & & \downarrow\wr & & & & \downarrow\wr & & \\ & & \mathscr{O}_C(4) & & & & \mathscr{O}_C(2)^{\oplus(n-2)} & & \end{array}$$

of normal bundles, from which we obtain $h^0(C, N_{C/\mathbf{P}^n}) = 3(n-2)+5 = 3n-1$ and $h^1(C, N_{C/\mathbf{P}^n}) = 0$: we recover the fact that the family of conics in $\mathbf{P}^n$ is everywhere smooth of the expected dimension $3n-1$.

2. Lines on smooth cubic hypersurfaces

Let $X \subseteq \mathbf{P}^{n+1} = \mathbf{P}(V_{n+2})$ be a smooth cubic hypersurface (that is, defined by a homogeneous polynomial of degree 3 or, more canonically, by a nonzero $f \in \operatorname{Sym}^3 V_{n+2}^\vee$). We let $F(X) \subseteq \mathsf{Gr}(2, V_{n+2})$ be the family of lines contained in X.

Proposition 2.1. (a) *The family $F(X)$ is smooth and*

- *it is empty if $n = 1$;*
- *it consists of 27 points if $n = 2$;*
- *it is connected and has dimension $2n-4$ if $n \geq 3$.*

(b) *Its canonical line bundle is $\omega_{F(X)} = \Omega^{\mathrm{top}}_{F(X)} \simeq \mathscr{O}_{F(X)}(4-n)$.*

Proof. If $L \subseteq X$ is a line, there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{L/X} & \longrightarrow & N_{L/\mathbf{P}^{n+1}} & \longrightarrow & N_{X/\mathbf{P}^{n+1}}|_L & \longrightarrow & 0 \\ & & & & \downarrow\wr & & \downarrow\wr & & \\ & & & & \mathscr{O}_L(1)^{\oplus n} & & \mathscr{O}_L(3) & & \end{array}$$

of normal bundles. The fact that X is smooth implies that the induced map $H^0(L, N_{L/\mathbf{P}^{n+1}}) \to H^0(L, N_{X/\mathbf{P}^{n+1}}|_L)$ between spaces of global sections is surjective. This implies

- $h^1(L, N_{L/X}) = 0$: the variety $F(X)$ is smooth at $[L]$;
- $h^0(L, N_{L/X}) = 2n-4$: the variety $F(X)$ has dimension $2n-4$ at $[L]$.

To obtain global statements about $F(X)$, one needs a different approach. We note that $F(X) \subseteq \mathsf{Gr}(2, V_{n+2})$ is the zero locus of a section σ of a vector bundle of rank 4 on $\mathsf{Gr}(2, V_{n+2})$: if $\mathscr{U}_2$ is the rank-2 tautological subbundle on $\mathsf{Gr}(2, V_{n+2})$ (its fiber at a point $[V_2]$ is just the vector space V_2), we consider the rank-4 vector bundle $\operatorname{Sym}^3 \mathscr{U}_2^\vee$. The map $\mathsf{Gr}(2, V_{n+2}) \to \operatorname{Sym}^3 \mathscr{U}_2^\vee$ that sends a point $[V_2] \in \mathsf{Gr}(2, V_{n+2})$ to the restriction of the equation $f \in \operatorname{Sym}^3 V_{n+2}^\vee$ of X to V_2 is a section of that vector bundle whose zero locus is $F(X)$ and we just proved (by the local study above) that it has the expected codimension $4 = \operatorname{rank}(\operatorname{Sym}^3 \mathscr{U}_2^\vee)$. This has the following consequences:

- the cohomology class of $F(X)$ in $H^\bullet(\mathsf{Gr}(2, V_{n+2}), \mathbf{Z})$ is the top Chern class of $\operatorname{Sym}^3 \mathscr{U}_2^\vee$ (this gives in particular the 27 points when $n = 2$);
- its canonical bundle is $\omega_{F(X)} = \omega_{\mathsf{Gr}(2, V_{n+2})} \otimes \det(\operatorname{Sym}^3 \mathscr{U}_2^\vee)|_{F(X)} = \mathscr{O}_{F(X)}(-n-2+6)$ (adjunction formula).

This proves the proposition (except for the connectedness statement). □

Examples 2.2. (1) When $n = 3$, the variety $F(X)$ is a surface of general type. One computes its first Betti number $b_1(F(X)) = 10$ (all Betti numbers are known).

(2) When $n = 4$, the variety $F(X)$ is a fourfold with trivial canonical bundle. It is in fact a hyper-Kähler manifold: it is simply connected and carries a holomorphic symplectic 2-form ([BD]). One computes $b_1(F(X)) = 0$ and $b_2(F(X)) = 23$ (all Betti numbers are known).

3. Twisted cubics on a smooth cubic hypersurface

Conics on a cubic hypersurface are too much related to lines to produce anything interesting, so we go up one degree more to curves of degree 3. A *twisted cubic* is the image of the morphism given in homogeneous coordinates by

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^3, \qquad [s,t] \longmapsto [s^3, s^2t, st^2, t^3].$$

Unlike lines and conics, twisted cubics have complicated degenerations. However, the closure $H(\mathbf{P}^3)$ of the family $H(\mathbf{P}^3)^0$ of (smooth) twisted cubics in $\mathbf{P}^3$ is still a smooth projective irreducible variety of dimension 10. It contains a smooth hypersurface corresponding to singular plane cubic curves with an "embedded" point.

Given a smooth cubic hypersurface $X \subseteq \mathbf{P}^{n+1} = \mathbf{P}(V_{n+2})$, we let $H(X)^0$ be the scheme of (smooth) twisted cubics in X and we let $H(X)$ be its closure. There is a morphism

$$\begin{aligned} h\colon H(X) &\longrightarrow \mathsf{Gr}(4, V_{n+2}) \\ [C] &\longmapsto \langle C \rangle. \end{aligned} \tag{1}$$

The fiber of $[\mathbf{P}^3] \in \mathsf{Gr}(4, V_{n+2})$ is the family of twisted cubics (and their degenerations) contained in $S := \mathbf{P}^3 \cap X$. We face two potential problems here:

- the linear space $\mathbf{P}^3$ might be contained in X (this can happen when $n \geq 6$);
- even if S is a surface, it might be very singular.

3.1. **Twisted cubics on cubic threefolds.** Let $X \subseteq \mathbf{P}^4$ be a smooth cubic threefold. The scheme $H(X)^0$ of smooth twisted cubics in X was studied by Harris–Roth–Starr: they proved that it is smooth irreducible of the expected dimension 6 ([HRS, Theorem 4.4]). The threefold X contains no planes, so any hyperplane section $S := \mathbf{P}^3 \cap X$ is an irreducible cubic surface. One shows that the curve $C \subseteq S$ moves into a two-dimensional family of curves inside S. In fact, the morphism h in (1) factors as

$$H(X) \xrightarrow[]{\mathbf{P}^2\text{-fibration}} H'(X) \xrightarrow[\text{degree } 72]{\text{finite of}} \mathsf{Gr}(4, V_{n+2})$$

(see [HRS, Theorem 4.5]). It is likely that the later article [LLSvS] proves more (see next section).

3.2. **Twisted cubics on cubic fourfolds.** Let $X \subseteq \mathbf{P}^4$ be a smooth cubic threefold. Then X contains no $\mathbf{P}^3$ and, in general, no $\mathbf{P}^2$; we make that assumption. Any linear section $\mathbf{P}^3 \cap X$ is then an irreducible surface. The *Stein factorization* of the morphism h in (1) is then

$$\underset{\substack{\text{smooth proj.}\\ \text{of dim. } 10}}{H(X)} \xrightarrow{\mathbf{P}^2\text{-fibration}} \underset{\substack{\text{smooth proj.}\\ \text{of dim. } 8}}{H'(X)} \xrightarrow[\text{degree } 72]{\text{finite of}} \mathsf{Gr}(4, V_{n+2}).$$

There is a final step to this construction: the contraction $H'(X) \to H''(X)$ of the hypersurface coming from "strange" cubics. The final product $H''(X)$ is a *smooth hyper-Kähler manifold of dimension 8.* All these results appear in [LLSvS].

Remark 3.1 (Curves of higher degrees)**.** Studying geometrically curves of higher degrees on X seems very difficult. It is however easy to compute the expected dimension of the space of smooth rational curves C of degree d on X. Indeed, one has

$$\deg(N_{C/X}) = \deg(T_X|_C) - \deg(T_C) = d(n-1) - 2.$$

The expected dimension of the local deformation space of C into X is

$$h^0(C, N_{C/X}) - h^1(C, N_{C/X}) = \chi(C, N_{C/X}),$$

which is equal to, by the Riemann–Roch formula,

$$\chi(C, N_{C/X}) = \deg(N_{C/X}) + \operatorname{rank}(N_{C/X})(1 - g(C)) = (d+1)(n-1) - 2.$$

The following table gives the expected dimensions for small values of $n = \dim(X)$ and $d = \deg(C)$.

$d{=}\deg(C)$ \ $n{=}\dim(X)$	2	3	4	5
1	0	2	4	6
2	1	4	7	10
3	2	6	10	14

4. Cohomological constructions

We take advantage of the Hodge decomposition of the singular cohomology of smooth compact Kähler manifolds to produce more constructions.

4.1. **Jacobians of curves.** Let C be a smooth projective curve of genus g and fix a point $p_0 \in C$. One would like to integrate holomorphic 1-forms on C from p_0 to another point p. This will depend on the choice of a path from p_0 to p. The way to make this construction work is the consider the map

$$H_1(C, \mathbf{Z}) \longrightarrow H^0(C, \Omega_C^1)^\vee, \qquad \gamma \longmapsto \Big(\omega \mapsto \int_\gamma \omega\Big).$$

This is an embedding of a rank-$2g$ free abelian group into a g-dimensional vector space. The quotient

$$\operatorname{Jac}(C) \coloneqq H^0(C, \Omega_C^1)^\vee / H_1(C, \mathbf{Z})$$

is a g-dimensional complex torus (homeomorphic to $(\mathbf{S}^1)^{2g}$) called the *Jacobian* of C. The Abel–Jacobi map

$$\mathsf{AJ}\colon C \longrightarrow \operatorname{Jac}(C), \qquad p \mapsto \Big(\omega \mapsto \int_{p_0}^p \omega\Big)$$

is well defined (it depends on the choice of the point p_0).

Alternatively, one may start from the Hodge decomposition

$$H^1(C, \mathbf{C}) = H^{0,1}(C) \oplus H^{1,0}(C)$$

into complex vector subspaces of dimension g, where $H^{1,0}(C) = \overline{H^{0,1}(C)} \simeq H^0(C, \Omega_C^1)$. Then,

$$\operatorname{Jac}(C) \coloneqq H^1(C, \mathbf{C}) / \big(H^1(C, \mathbf{Z}) + H^{1,0}(C)\big).$$

An important additional point is that the intersection form (cap product) on $H_1(C, \mathbf{Z})$ defines an algebraic structure on $\operatorname{Jac}(C)$ that makes it into a principally polarized abelian variety: it contains a *theta divisor* (well defined up to translation) that can be defined as the sum

$$\overbrace{\mathsf{AJ}(C) + \cdots + \mathsf{AJ}(C)}^{g-1 \text{ times}} \subseteq \operatorname{Jac}(C).$$

4.2. **Intermediate Jacobians.** More generally, let X be a smooth projective manifold of odd dimension $2m+1$. There is again a Hodge decomposition of the middle cohomology

$$H^{2m+1}(X,\mathbf{C}) = H^{0,2m+1}(X) \oplus \cdots \oplus H^{m,m+1}(X) \oplus H^{m+1,m}(X) \oplus \cdots \oplus H^{2m+1,0}(X)$$

into a direct sum of complex subspaces and Griffiths defined the *intermediate Jacobian*

$$\mathrm{Jac}(X) := H^{2m+1}(X,\mathbf{C})/\big(H^{2m+1}(X,\mathbf{Z}) + H^{m+1,m}(X) \oplus \cdots \oplus H^{2m+1,0}(X)\big)$$

of X. Given a family $(Z_t \subseteq X)_{t\in T}$ of subvarieties of X of dimension m parametrized by a connected algebraic variety T and a point $t_0 \in T$, there is also an Abel–Jacobi map

$$\mathsf{AJ}\colon T \longrightarrow \mathrm{Jac}(X), \qquad t \mapsto \Big(\omega \mapsto \int_\Gamma \omega\Big),$$

where Γ is any $(m+1)$-chain such that $\partial\Gamma = Z_t - Z_{t_0}$ (this chain Γ exists because all the subvarieties Z_t are cohomologous to Z_{t_0}). Griffiths proved that this map is holomorphic.

In general, $\mathrm{Jac}(X)$ is only a complex torus of dimension $\frac{1}{2}b_{2m+1}(X)$ with no algebraic structure. However, when there are only two nonzero terms in the Hodge decomposition,

$$H^{2m+1}(X,\mathbf{C}) = H^{m,m+1}(X) \oplus H^{m+1,m}(X), \tag{2}$$

the cup product makes $\mathrm{Jac}(X)$ into a principally polarized abelian variety (as in the case of curves).

Examples 4.1. (1) A typical example is when $X \subseteq \mathbf{P}^4$ is a cubic threefold. The condition (2) is satisfied (because $H^{0,3}(X) = 0$) and $\mathrm{Jac}(X)$ is a 5-dimensional principally polarized abelian variety. The Abel–Jacobi map

$$\mathsf{AJ}\colon F(X) \longrightarrow \mathrm{Jac}(X)$$

for the family $F(X)$ of lines on X is a closed immersion whose image is a surface $S(X)$ such that the set of differences $S(X) - S(X) \subseteq \mathrm{Jac}(X)$ is a theta divisor (Mumford–Beauville, [B]).

Consider the *universal line*

$$\begin{array}{ccc} & \mathscr{L} := \{(x,L) \mid x \in L \subseteq X\} & \\ \swarrow^{\mathsf{pr}_1} & & \searrow^{\mathsf{pr}_2} \\ X & & F(X). \end{array}$$

The induced map

$$H^3(X,\mathbf{Z}) \xrightarrow{\sim} H^1(\mathrm{Jac}(X),\mathbf{Z}) \xrightarrow{\mathsf{AJ}^*} H^1(F(X),\mathbf{Z})$$

in cohomology is then $\mathsf{pr}_{2*}\,\mathsf{pr}_1^*$.

For the 6-dimensional family $H(X)$ of twisted cubics on X, it was shown in [HRS, Corollary 4.7] that the Abel–Jacobi map

$$\mathsf{AJ}\colon H(X) \longrightarrow \mathrm{Jac}(X)$$

is birationally a $\mathbf{P}^2$-bundle onto a theta divisor.

(2) When $X \subseteq \mathbf{P}^5$ is a cubic fourfold, there is no intermediate Jacobian, because the dimension of X is even. However, there is still an isomorphism

$$\mathsf{pr}_{2*}\,\mathsf{pr}_1^*\colon H^4(X,\mathbf{Z}) \xrightarrow{\sim} H^2(F(X),\mathbf{Z}).$$

These results have been generalized in any dimension (see [H, Chapter 2, Section 5.2]).

5. Gushel–Mukai manifolds

We now switch to different families of manifolds that present many similarities with cubic hypersurfaces. A *Gushel–Mukai* manifold is a smooth transverse intersection

$$X = \mathsf{CGr}(2, V_5) \cap Q \cap \mathbf{P}(W_{n+5}) \subseteq \mathbf{P}(\mathbf{C} \oplus \textstyle\bigwedge^2 V_5) = \mathbf{P}^{10},$$

where $\mathsf{CGr}(2, V_5)$ is the cone with vertex $\mathbf{v}$ over the Grassmannian $\mathsf{Gr}(2, V_5) \subseteq \mathbf{P}(\bigwedge^2 V_5)$ and Q is a quadric. It has dimension $n \le 6$ and its canonical bundle is $\omega_X = \mathscr{O}_X(2-n)$. There is a canonical projection map $\gamma\colon X \to \mathsf{Gr}(2, V_5)$ and we say that X is

- *ordinary* if $\mathbf{v} \notin \mathbf{P}(W_{n+5})$, in which case $n \le 5$ and γ is an embedding;
- *special* if $\mathbf{v} \in \mathbf{P}(W_{n+5})$, in which case γ is a double cover of its image.

We will assume $3 \le n \le 6$, so that ω_X is antiample (X is a genus-6 curve when $n = 1$, and a K3 surface when $n = 2$).

As before, we study lines, conics... on X. As in Remark 3.1, the expected dimension of the family of degree-d rational curves $C \subseteq X$ is

$$\chi(C, N_{C/X}) = \deg(T_X|_C) - \deg(T_C) + \operatorname{rank}(N_{C/X})(1 - g(C)) = (d+1)(n-1) - 1.$$

The following table gives the expected dimensions for small values of $d = \deg(C)$.

$d{=}\deg(C)$ \ $n{=}\dim(X)$	3	4	5	6
1	1	3	5	7
2	2	5	8	11
3	3	7	11	15

6. Lines and conics on a Gushel–Mukai manifold

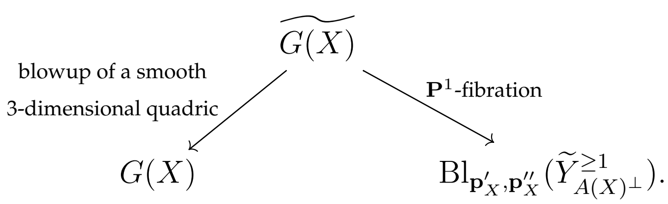
The scheme of lines on a Gushel–Mukai manifold of dimension n always has the expected dimension $2n-3$; it is connected, and smooth when X is general (we also have a precise description of its singularities in all cases).

For conics, the most interesting cases are $n \in \{3, 4\}$. The following theorem states our results under the simplifying generality assumptions that X is ordinary and $Y^{\ge 3}_{A(X)^\perp}$ is empty (the varieties $\widetilde{Y}^{\ge k}_{A(X)^\perp}$ will be "defined" after the statement of the theorem).

Theorem 6.1 (D.–Kuznetsov). *Let X be an ordinary Gushel–Mukai manifold of dimension n such that $Y^{\ge 3}_{A(X)^\perp} = \varnothing$ and let $G(X)$ be the scheme of conics on X.*

(a) *When $n = 3$, the scheme $G(X)$ is a smooth irreducible projective surface of general type which is the blow up of $\widetilde{Y}^{\ge 2}_{A(X)^\perp}$ at one point.*

(b) *When $n = 4$ and X contains no planes, the scheme $G(X)$ is a smooth irreducible projective fourfold and fits into a diagram*



The projective varieties $\widetilde{Y}^{\geq 1}_{A(X)^\perp}$, $\widetilde{Y}^{\geq 2}_{A(X)^\perp}$, and $Y^{\geq 3}_{A(X)^\perp}$ are important objects attached to any Gushel–Mukai manifold X:

- $Y^{\geq 3}_{A(X)^\perp}$ is a finite set which is empty when X is general;
- $\widetilde{Y}^{\geq 1}_{A(X)^\perp}$ is a hyper-Kähler fourfold (called a *double Eisenbud–Popescu–Walter sextic*) which is singular along $Y^{\geq 3}_{A(X)^\perp}$;
- $\widetilde{Y}^{\geq 2}_{A(X)^\perp}$ is a surface which is singular along $Y^{\geq 3}_{A(X)^\perp}$.

In dimensions 5 and 6, we have analogous results about the scheme $G_2(X)$ of quadric surfaces contained in X:

- when $n = 5$, the scheme $G_2(X)$ has a component which has a $\mathbf{P}^1$-fibration over the surface $\widetilde{Y}^{\geq 2}_{A(X)^\perp}$;
- when $n = 6$ and $Y^{\geq 3}_{A(X)^\perp} = \varnothing$, the scheme $G_2(X)$ has a component which has a $\mathbf{P}^3$-fibration over the fourfold $\widetilde{Y}^{\geq 1}_{A(X)^\perp}$.

Therefore,

- *When n is odd,* the important object is the surface $\widetilde{Y}^{\geq 2}_{A(X)^\perp}$. One can show that X has a 10-dimensional principally polarized intermediate Jacobian $\mathrm{Jac}(X)$ and that there is an isomorphism
$$H^n(X,\mathbf{Z}) \xrightarrow{\sim} H^1(\widetilde{Y}^{\geq 2}_{A(X)^\perp},\mathbf{Z})$$
which induces an isomorphism
$$\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)^\perp}) \xrightarrow{\sim} \mathrm{Jac}(X),$$
where $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)^\perp})$ is the *Albanese variety* of the surface $\widetilde{Y}^{\geq 2}_{A(X)^\perp}$ (compare with Examples 4.1).
- *When n is even,* the important object is the hyper-Kähler fourfold $\widetilde{Y}^{\geq 1}_{A(X)^\perp}$. There is also an isomorphism
$$H^n(X,\mathbf{Z})_0 \xrightarrow{\sim} H^2(\widetilde{Y}^{\geq 1}_{A(X)^\perp},\mathbf{Z})_0$$
where the indices 0 indicate that you have to take some kind of "primitive part."

In both cases, we expect the isomorphisms to be given by the Abel–Jacobi maps associated with families of conics or quadric surfaces, but we have not written down the proof yet.

So what is next? Studying twisted cubics on Gushel–Mukai manifolds? Their geometry seems very complicated. We describe in the next section a more fruitful approach based on derived categories.

7. Derived categories

Given a smooth projective variety X, we denote by $\mathbf{D}(X)$ the derived category of bounded complexes of coherent sheaves (see [Ku, Section 1] for a bit more details).

7.1. **Cubic hypersurfaces.** Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth cubic hypersurface. Kuznetsov proved (see [Ku, Example 2.11]) that there is a semiorthogonal decomposition
$$\mathbf{D}(X) = \langle \mathbf{Ku}(X), \mathscr{O}_X, \mathscr{O}_X(1), \dots, \mathscr{O}_X(n-2)\rangle,$$

where the full triangulated subcategory $\mathbf{Ku}(X)$ is the *Kuznetsov component*. The category $\mathbf{Ku}(X)$ has a Serre functor $S_{\mathbf{Ku}(X)}$ that satisfies $S_{\mathbf{Ku}(X)}^{3/c} \simeq \left[\frac{n+2}{c}\right]$; in particular, when $n = 4$, it is a K3 category on which Bayer–Lahoz–Macrì-Stellari constructed in [BLMS] a Bridgeland stability condition σ (this had been done in [BMMS] for cubic threefolds). In particular, one can then study moduli spaces of σ-stable objects in $\mathbf{Ku}(X)$.

More precisely, still when $n = 4$, there is an extended Mukai lattice

$$\widetilde{H}(\mathbf{Ku}(X), \mathbf{Z}) \simeq H^\bullet(K3, \mathbf{Z}) = H^0 \oplus H^2 \oplus H^4$$

with a weight-2 Hodge structure and a Mukai vector

$$\mathbf{v} \colon K_{\mathrm{top}}(\mathbf{Ku}(X)) \longrightarrow \widetilde{H}(\mathbf{Ku}(X), \mathbf{Z})$$

such that $\mathbf{v}(E) \cdot \mathbf{v}(F) = -\chi(E, F)$. We let $\widetilde{H}_{\mathrm{alg}}(\mathbf{Ku}(X), \mathbf{Z})$ be the image of $\mathbf{v}$ (the "algebraic classes").

The following result was proved in [BLMNPS].

Theorem 7.1. *Let X be a smooth cubic fourfold, let $v \in \widetilde{H}_{\mathrm{alg}}(\mathbf{Ku}(X), \mathbf{Z})$ be a nonzero primitive vector, and let σ be a stability condition which is general with respect to v. The moduli space $M_\sigma(\mathbf{Ku}(X), v)$ of σ-stable objects in $\mathbf{Ku}(X)$ with Mukai vector v is a smooth irreducible projective hyper-Kähler manifold of deformation type $K3^{[\frac{v^2}{2}+1]}$.*

This result is obtained by deformation from the previously known case where $\mathbf{Ku}(X) \simeq \mathbf{D}(K3)$.

The lattice $\widetilde{H}_{\mathrm{alg}}(\mathbf{Ku}(X), \mathbf{Z})$ contains a sublattice $\mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2$, with intersection matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$; if $L \subseteq X$ is a line and p is the left-adjoint to the inclusion $\mathbf{Ku}(X) \hookrightarrow \mathbf{D}(X)$, the class λ_1 is $\mathbf{v}(p(\mathscr{O}_L(1)))$ and the class λ_2 is $\mathbf{v}(p(\mathscr{O}_L(2)))$.

Corollary 7.2. *Let (a, b) be a pair of coprime integers and set $n := a^2 - ab + b^2$. There exists a unirational locally complete 20-dimensional family of smooth polarized hyper-Kähler manifolds of deformation type $K3^{[n+1]}$. The polarization has*

$$\begin{cases} \text{degree } 6n \text{ and divisibility } 2n \text{ if } 3 \nmid n; \\ \text{degree } \frac{2n}{3} \text{ and divisibility } \frac{2n}{3} \text{ if } 3 \mid n. \end{cases}$$

Proof. This is $M_\sigma(\mathbf{Ku}(X), a\lambda_1 + b\lambda_2)$. □

Examples 7.3. (1) When $a = b = 1$, we get the variety $F(X)$ of lines contained in X studied in Section 2 ([LPZ]).

(2) When $a = 2$ and $b = 1$ and X contains no planes, we get the variety $H''(X)$ studied in Section 3 ([LPZ]).

7.2. **Gushel–Mukai manifolds.** Let X be a Gushel–Mukai manifold of dimension $n \in \{3, 4, 5, 6\}$. Kuznetsov and Perry proved in [KP] that there is a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle \mathbf{Ku}(X), \mathscr{O}_X, \mathscr{U}_X^\vee, \mathscr{O}_X(1), \mathscr{U}_X^\vee(1), \dots, \mathscr{O}_X(n-3), \mathscr{U}_X^\vee(n-3) \rangle,$$

where $\mathscr{U}_X$ is the pullback by the map $\gamma \colon X \to \mathsf{Gr}(2, V_5)$ of the tautological rank-2 subbundle on $\mathsf{Gr}(2, V_5)$. The Serre functor $S_{\mathbf{Ku}(X)}$ is

- $[2]$ if n is even (so that $\mathbf{Ku}(X)$ is a K3 category);
- $\iota \circ [2]$ if n is odd, where ι is a nontrivial involutive autoequivalence.

Moreover, there are Bridgeland stability conditions on $\mathbf{Ku}(X)$ ([PPZ]). As in the case of cubic fourfolds, there is an extended Mukai lattice $\widetilde{H}(\mathbf{Ku}(X),\mathbf{Z})$, a Mukai vector

$$\mathbf{v}\colon K_{\mathrm{top}}(\mathbf{Ku}(X)) \longrightarrow \widetilde{H}(\mathbf{Ku}(X),\mathbf{Z}),$$

and two algebraic classes $\lambda_1,\lambda_2\in \widetilde{H}_{\mathrm{alg}}(\mathbf{Ku}(X),\mathbf{Z})$ with intersection matrix $\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$. The following result was proved in [PPZ].

Theorem 7.4. *Let X be a Gushel–Mukai manifold of dimension $n\in\{4,6\}$, let $v\in \widetilde{H}_{\mathrm{alg}}(\mathbf{Ku}(X),\mathbf{Z})$ be a nonzero primitive vector, and let σ be a stability condition which is general with respect to v. The moduli space $M_\sigma(\mathbf{Ku}(X),v)$ of σ-stable objects in $\mathbf{Ku}(X)$ with Mukai vector v is a smooth irreducible projective hyper-Kähler manifold of deformation type $K3^{[\frac{v^2}{2}+1]}$.*

Corollary 7.5. *Let (a,b) be a pair of coprime integers. There exists a unirational locally complete 20-dimensional family of smooth polarized hyper-Kähler manifolds of deformation type $K3^{[a^2+b^2+1]}$. The polarization has degree $2(a^2+b^2)$ and divisibility a^2+b^2.*

Proof. This is $M_\sigma(\mathbf{Ku}(X),a\lambda_1+b\lambda_2)$. □

Examples 7.6. Let X be a very general Gushel–Mukai fourfold.

(1) When $a=1$ and $b=0$, we get the hyper-Kähler fourfold $\widetilde{Y}^{\ge 1}_{A(X)^\perp}$ studied in Section 6, and when $a=0$ and $b=1$, we get the hyper-Kähler fourfold $\widetilde{Y}^{\ge 1}_{A(X)}$ ([PPZ, Proposition 5.17], [GLZ, Theorem 1.1]).

(2) When $a=b=1$, we get the hyper-Kähler sixfold constructed in [IKKR] ([KKM, Theorem 1.1]).

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