ON RATIONALITY PROBLEMS

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ABSTRACT. We discuss new and old techniques used for, and recent progress obtained on, the problem of detecting rationality, stable rationality, or unirationality of smooth projective complex varieties.

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1. INTRODUCTION

The rationality problem for a (smooth projective) variety X defined over a field $\mathbf{k}$ is to measure how close it is to the projective space $\mathbf{P}^n_{\mathbf{k}}$ of the same dimension n. There are several variants of this problem; we say that

(R) X is $\mathbf{k}$-*rational* if there is a birational isomorphism $\mathbf{P}^n_{\mathbf{k}} \overset{\sim}{\dashrightarrow} X$ (equivalenty, $\mathbf{k}(X)$ is a purely transcendental extension of $\mathbf{k}$);

(SR) X is *stably* $\mathbf{k}$-*rational* if there is a nonnegative integer m such that $X \times \mathbf{P}^m_{\mathbf{k}}$ is $\mathbf{k}$-rational (equivalenty, $\mathbf{k}(X)(t_1, \dots, t_m)$ is a purely transcendental extension of $\mathbf{k}$);

(UR) X is $\mathbf{k}$-*unirational* if there is a nonnegative integer m and a rational dominant map $\mathbf{P}^m_{\mathbf{k}} \dashrightarrow X$ (equivalenty, if $\mathbf{k}(X)$ is contained in a purely transcendental extension of $\mathbf{k}$);

(RC) X is $\mathbf{k}$-*rationally connected* if any two general geometric points of X can be joined by a rational curve.[1]

Note that each of these notions is invariant under birational isomorphisms; in other words, they only depend on the function field $\mathbf{k}(X)$.

Comments on the base field $\mathbf{k}$. In all rights, one should indicate the base field $\mathbf{k}$ in the notation: the properties (R), (SR), and (UR) strongly depend on $\mathbf{k}$. Also, the definitions (UR) and (RC) given above are actually not the "right ones" when the characteristic of $\mathbf{k}$ is positive (one should require that the unirationality map is separable—the property is then called separable unirationality; a similar adjustment can be made to define separable rational connectedness). It is therefore easier to assume that $\mathbf{k}$ is an algebraically closed field of characteristic zero. By the Lefschetz principle, we might as well take $\mathbf{k} = \mathbf{C}$.

Note that there is no need to define stably $\mathbf{k}$-unirational or stably $\mathbf{k}$-rationally connected and that in (UR), one can take $m = n$ (restrict the dominant rational map to a general linear subspace of dimension n). One obviously has

$$\text{(R)} \Longrightarrow \text{(SR)} \Longrightarrow \text{(UR)} \Longrightarrow \text{(RC)}. \tag{1}$$

All these implications are equivalences in dimensions ≤ 2 (see Section 2.3). The reverse implication (SR) $\Rightarrow$ (R) is known to be false in dimensions ≥ 3 (see Section 5) and so is the implication (UR) $\Rightarrow$ (SR) (see Theorem 6.7 and Corollary 7.11) but, embarrassingly, the nature of the reverse implication (RC) $\Rightarrow$ (UR) is *not* known, although it is certainly expected to be false.

The plan of these lectures is as follows. In Section 2, we briefly review what is known for hypersurfaces of the projective space and give a characterization of rationally connected varieties that shows that simple-minded topological or cohomological invariants are often unable to distinguish between the various notions defined above. In Section 3, we explain their behavior in smooth families, with a brief account of the beautiful results of Nicaise–Shinder and Kontsevich–Tschinkel on (stable) rationality in smooth families.

In Section 4, we turn to the more classical *Lüroth problem* of distinguishing between rationality and unirationality and present the classical counter-examples given in the seventies by Clemens–Griffiths, Iskovskikh–Manin, and Artin–Mumford. We emphasize the Clemens–Griffiths criterion of irrationality for Fano threefolds, which is based on the properties of their intermediate Jacobians, and its consequences. We briefly present in Section 5 the

[1] This is the correct definition only when $\mathbf{k}$ is uncountable. In general, the property should hold after passing to any algebraically closed extension $\mathbf{L}/\mathbf{k}$.

stably rational but not rational threefolds constructed by Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer in 1985.

Section 6 is devoted to the Artin–Mumford example of a unirational but not stably rational threefold. The proof we present uses basic properties of the Brauer group, which we explain. The last section, Section 7, we discuss how Chow groups (mostly of 0-cycles) can be used for rationality problems. The far-reaching idea of Voisin of using the Chow decomposition of the diagonal (pioneered by Boch–Srinivas in the eighties) has led to all kind of new results about rationality problems, which we briefly and partially survey.

However, despite all these progress, the most basic question of all remains unanswered: are there any irrational smooth cubic hypersurfaces in $\mathbf{P}^5_{\mathbf{C}}$? It seems that, after all, the simplest-looking examples are the hardest.

Conventions. A variety is an integral scheme of finite type over a field. Subvarieties are always closed. "General" means "outside a strict subvariety" and "very general" means "outside a countable union of strict subvarieties."

Acknowledgements. These notes are based on the beautiful text [B4] written by Arnaud Beauville in 2015 on the same subject. I have borrowed large parts of his notes and added a few improvements obtained in the last seven years. Far more competent authors have produced (far more complete) accounts of the subject which the reader interested in learning more is invited to consult: Jean-Louis Colliot-Thélène ([Co]), Stefan Schreieder ([S3]), and Claire Voisin ([Vo5, Vo7]).

2. Examples and first properties

2.1. **Fano varieties and hypersurfaces.** It is known (but difficult to prove) that any Fano variety (that is, any smooth projective variety whose anticanonical divisor is ample) is rationally connected. There are plenty of Fano varieties (although, once the dimension is fixed, there are only finitely many deformation types): for example, any smooth complete intersection in $\mathbf{P}^{n+c}$ of multidegree $(d_1, \dots, d_c)$ with $d_1 + \cdots + d_c \leq n + c$ is a Fano variety, hence is rationally connected. However, referring to the discussion in the introduction, *no examples of nonunirational Fano varieties are known!*

Example 2.1 (Unirationality of smooth cubic hypersurfaces)**.** Any smooth cubic complex hypersurface $X \subseteq \mathbf{P}^{n+1}$, with $n \geq 2$, contains a line ℓ. The space $\mathbf{P}(T_X|_\ell)$ is the set of lines tangent to X at a point of ℓ. Such a line meets X with mutiplicity (at least) 2 at its point of intersection with ℓ and, if not contained in X, at a third point. This defines a dominant rational map

$$f\colon \mathbf{P}(T_X|_\ell) \dashrightarrow X.$$

Since the space on the left is rational (any vector bundle on ℓ is trivial on a dense open subset of ℓ), *any smooth cubic hypersurface of dimension ≥ 2 is unirational.*

Let x be a general point of X. The intersection of the plane $\langle \ell, x\rangle$ with X is the union of ℓ and a conic that meets ℓ in two points x_1, x_2. The inverse image of x by f is the set of two lines $\langle x, x_1\rangle$ and $\langle x, x_2\rangle$, so that f has degree 2.

To insist on the difficulty of the problems we consider here and the lack of progress on even basic questions (despite tremendous recent advances), the question of the rationality of smooth cubic complex hypersurfaces (which are Fano varieties hence rationally connected in all dimensions $n \geq 2$) has only been answered when $n = 2$ (positively) or 3 (negatively) (see Section 4.1.2); the stable rationality of cubic threefolds is unknown.

Example 2.2 (Rationality of some smooth cubic hypersurfaces)**.** Let P_1 and P_2 be disjoint m-dimensional linear spaces in $\mathbf{P}^{2m+1}$. A general cubic hypersurface $X \subseteq \mathbf{P}^{2m+1}$ containing P_1 and P_2 is smooth. I claimed that any such X is rational; indeed, there is a birational isomorphism $P_1 \times P_2 \overset{\sim}{\dashrightarrow} X$ obtained by sending a general pair of points $(p_1, p_2) \in P_1 \times P_2$ to the third point of intersection with X of the line $\langle p_1 p_2 \rangle$ spanned by p_1 and p_2 (given $x \in X$ general, its unique preimage is $(\langle P_2 x\rangle \cap P_1, \langle P_1 x\rangle \cap P_2)$). This gives examples of rational smooth cubic hypersurfaces in all *even* dimensions. *No odd-dimensional smooth cubic hypersurfaces are known* (they are known not to exist in dimension 3, as we will show in Section 4.1.2).

Example 2.3 (Other hypersurfaces)**.** Any smooth complex hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \le n+1$ is a Fano variety, hence is rationally connected. Moreover, fixing the degree d, any smooth degree-d complex hypersurface $X \subseteq \mathbf{P}^{n+1}$ is unirational when $n \ge 2^{d!}$ (see [BR, Theorem 1.4]; for quartics, $n \ge 6$ is enough; see [HMP, Corollary 3.8 and Remark 2.2]). At the other end, when $n \ge 3$, a very general complex hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \ge \log_2 n + 2$ is not stably rational (see [S1, S2]). It is expected that very general hypersurfaces of degree d not too much smaller than n should not be unirational. For example, Schreieder proved in [S4, Theorem 1.1] that if $X \subseteq \mathbf{P}^{n+1}$ is a very general complex hypersurface of degree $d \ge 4$, the degree of any dominant rational map $\mathbf{P}^n \dashrightarrow X$ is divisible by any integer $\le n - \log_2 n$.[2]

The following table roughly sums up what is known (for very general complex hypersurfaces of given degree d in $\mathbf{P}^{n+1}$). Parentheses indicate that the answer is conjectural.

d	2	3	$\cdots$	$d \ll n$	$\cdots$	$\log_2 n + 2$	$\cdots$	$n+1$
(R)	YES	(NO)	$\cdots$	$\cdots$	$\cdots$	NO	$\cdots$	NO
(SR)	YES	(NO)	$\cdots$	$\cdots$	(NO)	NO	$\cdots$	NO
(UR)	YES	YES	$\cdots$	YES	?	?	?	(NO)
(RC)	YES	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	YES

2.2. **Rationally connected varieties.** We now prove some easy properties of rationally connected smooth projective complex varieties. One consequence is that simple-minded topological invariants do not distinguish between the various properties in (1) (see Proposition 6.1 for a topological invariant that does).

An important player is the notion of *very free* rational curve on a smooth projective variety X: this is a rational curve $f\colon \mathbf{P}^1 \to X$ such that the vector bundle $f^*\Omega^1_X$ is a direct sum of line bundles of negative degrees. Surprisingly, the existence of a single such curve is sufficient to characterize rationally connected varieties.

Proposition 2.4. *A smooth projective complex variety X is rationally connected if and only if there is a very free rational curve on X.*

Sketch of proof. This follows from the deformation theory of rational curves on X. Assume that X is rationally connected; since a general point $x \in X$ can be joined to any other general point of X by a rational curve, there exist a smooth quasi-projective variety M and a

[2]As Schreieder points out, the strength of his result lies in its asymptotic behavior for large n. For instance, the degree of any unirational parametrization of a very general hypersurface of degree 100 in $\mathbf{P}^{101}$ is divisible by 718766754945489455304472257065075294400. It is tempting to think that no unirational parametrizations exist.

dominant morphism $g\colon \mathbf{P}^1 \times M \to X$ such that $g(\{0\} \times M) = \{x\}$ and the rational curve $g_m := g|_{\mathbf{P}^1 \times \{m\}}\colon \mathbf{P}^1 \to X$ is nonconstant for all $m \in M$. The differential of g is then surjective at a general point (t, m) (this is generic smoothness, which holds because we are over a field of characteristic 0) and one checks that this is equivalent to the fact that the vector bundle $(g_m^* T_X)(-1)$ on $\mathbf{P}^1$ is globally generated, that is, the rational curve g_m is very free.

Conversely, if there is a very free rational curve on X, one shows that the deformations of this curve pass through two general points of X (see [D2, Proposition 4.7] for a proof). □

Using these very free rational curves, we obtain cohomological and topological properties of rationally connected varieties.

Proposition 2.5. *Let X be a smooth projective rationally connected complex variety.*

(a) *The variety X is covered by* very free *rational curves.*
(b) *One has $H^0(X, (\Omega_X^p)^{\otimes m}) = 0$ for all positive integers m and p; in particular, $\chi(X, \mathscr{O}_X) = 1$.*
(c) *The variety X is simply connected.*

Sketch of proof. With the notation of the first part of the proof of Proposition 2.4, the rational curves g_m are very free for $m \in M$ general and they cover a dense open subset of X. To prove that very free rational curves cover the whole of X is much more difficult (see [KMM] or [D2, Corollary 4.28]) and will not be used in these notes.

For (b), note that any section of $(\Omega_X^p)^{\otimes m}$ must vanish on the image of any very free rational curve, hence on X by (a).

For (c), let $\pi\colon \widetilde{X} \to X$ be a connected finite étale cover. Since $\mathbf{P}^1$ is simply connected, any very free curve $\mathbf{P}^1 \to X$ lifts to a curve $\mathbf{P}^1 \to \widetilde{X}$ which is still very free. Proposition 2.4 then applies to prove that $\widetilde{X}$ is rationally connected. By (b), $H^0(\widetilde{X}, \Omega_{\widetilde{X}}^m)$ vanishes for all $m > 0$ and, by Hodge theory, so does $H^m(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. This implies $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 1$. But $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = \deg(\pi)\, \chi(X, \mathscr{O}_X)$ hence π is an isomorphism. This already proves that any finite étale cover of X is trivial.

To prove that $\pi_1(X)$ is trivial, we use the dominant morphism $g\colon \mathbf{P}^1 \times M \to X$ introduced in the proof of Proposition 2.4. The composition of g with the inclusion $\iota\colon \{0\} \times M \hookrightarrow \mathbf{P}^1 \times M$ is constant, hence

$$\pi_1(\iota) \circ \pi_1(g) = 0.$$

Since $\mathbf{P}^1$ is simply connected, $\pi_1(\iota)$ is bijective, hence $\pi_1(g) = 0$. Since g is dominant and X is normal, it is a general fact that the image of $\pi_1(g)$ has finite index (see [D2, Lemma 4.18] for a proof). Therefore, the group $\pi_1(X)$ is finite, hence trivial by what we saw earlier. □

Remark 2.6. The vanishing $H^0(X, (\Omega_X^1)^{\otimes m}) = 0$ for all $m > 0$ is conjectured to characterize rationally connected (smooth projective) varieties. This is known in dimensions ≤ 3.

2.3. **Curves and surfaces.** Lüroth proved in 1876 in [L] that a unirational smooth projective curve is rational. This is now easily proved using Proposition 2.5: for such a curve C, one has $H^0(C, \Omega_C^1) = 0$. Thus C has genus 0 and this implies $C \simeq \mathbf{P}^1$.

Castelnuovo then proved that any unirational smooth projective complex surface S is rational. He used the vanishings $H^0(S, \Omega_S^1) = H^0(S, (\Omega_S^2)^{\otimes 2}) = 0$ obtained as above from Proposition 2.5 and proved the difficult result that they characterize rational surfaces.

Using Proposition 2.5, one sees in fact that in dimensions 1 and 2, the implications in (1) are all equivalences (over $\mathbf{C}$).

3. BEHAVIOR IN FAMILIES

We will start from the oldest result ([KMM], [K, Theorem IV.3.11]).

Theorem 3.1. *Rational connectedness is an open and closed property: given a smooth projective morphism $\mathscr{X} \to B$ with B connected, if some fiber is rationally connected,* all *fibers are rationally connected.*

Sketch of proof. For openness, one uses Proposition 2.4: rational connectedness of a (smooth projective) variety X is equivalent to the existence of one very free rational curve on X. It is not difficult to prove that the existence of such a curve is an open property in a smooth family.

Closedness is harder to prove. A smooth projective degeneration of rationally connected smooth projective varieties is a priori only rationally chain connected: any two points can be joined by a chain of rational curves. One then applies a delicate smoothing argument of Kollár–Miyaoka–Mori to show that for smooth projective varieties, this a priori weaker property implies rational connectedness.

So, rational connectedness is a property that is both open and closed in smooth projective families (see [K, Theorem IV.3.11] for a full proof). □

It had long been suspected that things were not that simple for (stable) rationality. The following result, which settles that question, was only proved recently.

Theorem 3.2 (Nicaise–Shinder, Kontsevich–Tschinkel)**.** *Let $\mathscr{X} \to B$ be a smooth projective morphism between smooth complex varieties with $\dim(B) = 1$. If very general fibers are (stably) rational, all fibers are (stably) rational.*

The theorem was first proved by de Fernex–Fusi in [dFF] when the fibers have dimension 3. The stably rational case is [NS] and the general case is [KT] (see [NO1] for a unified treatment of both cases). I do not know of any analogous result for unirationality.

Corollary 3.3. *Let $\mathscr{X} \to B$ be a smooth projective morphism between smooth complex varieties. The set of points of B whose fiber is (stably) rational is a countable union of closed subsets of B.*

Sketch of proof. It follows from properties of Hilbert schemes (in particular that they have countably many components) that the subset of B under consideration is a countable union of *locally closed* subsets of B ([dFF, Proposition 2.3]). Theorem 3.2 implies that this set is stable under specialization and this implies the corollary. □

Example 3.4. Consider smooth hypersurfaces $X \subseteq \mathbf{P}^2 \times \mathbf{P}^3$ of bidegree $(2,2)$; they are parametrized by a dense open subset $B \subseteq \mathbf{P}(H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(2)) \otimes H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2)))$. They are Fano fourfolds and projection onto the first factor makes them into quadric surface bundles $X \to \mathbf{P}^2$. We have

(a) for every $b \in B$, the fourfold X_b is unirational ([M, Theorem 1.8]);
(b) for $b \in B$ very general, the fourfold X_b is not stably rational ([HPT1, Theorem 1]);
(c) the set of $b \in B$ for which the fourfold X_b is rational is dense in B for the Euclidean topology.

We will comment on (b) later (Section 7.3). For (c), one applies a criterion of Hassett ([H, Proposition 2.3]) that says that if $X \to \mathbf{P}^2$ is a quadric surface bundle such that there exists a Hodge class in $H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ that has odd intersection number with a fiber, then X is rational.

This gives an example of a family for which (stable) rationality is neither open nor closed, so we do need the adjective "countable" in Corollary 3.3.

The proof of Theorem 3.2 is based on constructions that are radically different from what was used before for this kind of problems, and which we now explain.

For any field $\mathbf{k}$ of characteristic 0 and any nonnegative integer n, we define the *Burnside ring* $\mathrm{Burn}_n(\mathbf{k})$ as the free abelian group on the isomorphism classes of field extensions of $\mathbf{k}$ of transcendence degree n (or, if you prefer, on birational isomorphism classes of (smooth) varieties of dimension n over $\mathbf{k}$). The main result is the following.

Theorem 3.5. *Let B be a smooth connected complex curve, with generic point η and function field $\mathbf{K} = \mathbf{C}(B)$. Given a nonnegative integer n and a closed point $b_0 \in B$ with local ring $R := \mathscr{O}_{B,b_0}$, there exist a "specialization" group morphism*

$$\rho_n \colon \mathrm{Burn}_n(\mathbf{K}) \longrightarrow \mathrm{Burn}_n(\mathbf{C})$$

such that, for any smooth proper morphism $\mathscr{X} \to \mathrm{Spec}(R)$ of relative dimension n, one has

$$\rho_n([\mathbf{K}(\mathscr{X}_\eta)/\mathbf{K}]) = [\mathbf{C}(\mathscr{X}_{b_0})/\mathbf{C}].$$

Before addressing the proof of this fundamental result, we briefly sketch how it implies Theorem 3.2. Given a smooth projective morphism $\mathscr{X} \to B$ as in the statement of that theorem, after a finite base change $B' \to B$, the generic fiber of $\mathscr{X}' := \mathscr{X} \times_B B' \to B'$ is rational (over the function field of B') (see the argument in [dFF, Proof of Theorem 3.1] involving again the Hilbert scheme and the uncountability of $\mathbf{C}$, or use the fact that the geometric generic fiber of $\mathscr{X} \to B$ is isomorphic, via a field isomorphism $\overline{\mathbf{K}} \simeq \mathbf{C}$, to a very general fiber; see [V, Lemma 2.1]).

Replacing B by B', we now have two smooth models of the extension $\mathbf{K}(\mathscr{X}_\eta)/\mathbf{K} \simeq \mathbf{K}(\mathbf{P}^n_{\mathbf{K}})/\mathbf{K}$: one is $\mathscr{X} \to B$ and the other is $\mathbf{P}^n_B \to B$. Given any $b_0 \in B$, we apply Theorem 3.5: the image by ρ_n of these isomorphic extensions is the common class of the extensions $\mathbf{C}(\mathscr{X}_{b_0})$ and $\mathbf{C}(\mathbf{P}^n)$, which are therefore isomorphic.

Sketch of proof of Theorem 3.5. [3] We define ρ_n on extensions $\mathbf{L}/\mathbf{K}$ and extend it by linearity. Given such an extension, we choose a smooth proper model $X \to \mathrm{Spec}(\mathbf{K})$ with $\mathbf{K}(X) \simeq \mathbf{L}$ and a simple normal crossing model $\mathscr{X} \to \mathrm{Spec}(R)$ with generic fiber X (semistable reduction). In other words,

$$(\mathscr{X}_{b_0})_{\mathrm{red}} = D = \sum_{i=1}^{r} D_i,$$

a simple normal crossing divisor (we do not care about possible multiplicities). For any nonempty $I \subseteq \{1, \dots, r\}$, we set

$$D_I := \bigcap_{i \in I} D_i,$$

a smooth irreducible subvariety of codimension $|I| - 1$ in $\mathscr{X}_{b_0}$ and

$$\mathbf{L}_I := \mathbf{C}(D_I)(x_1, \dots, x_{|I|-1}),$$

of transcendence degree n over $\mathbf{C}$. Then we define

$$\rho_n([\mathbf{L}/\mathbf{K}]) := \sum_{I \subseteq \{1,\dots,r\},\ I \neq \varnothing} (-1)^{|I|-1} [\mathbf{L}_I/\mathbf{C}] \in \mathrm{Burn}_n(\mathbf{C}). \tag{2}$$

[3] For more details on the proof, I recommend the excellent lecture "Rationality in families of varieties" given in 2021 by de Fernex for the Dipartimento di Matematica Tor Vergata (it can be found on YouTube).

Of course, for ρ_n to be well defined by this formula, one needs to check that this is independent of the choices of

(a) the smooth proper model $X \to \mathrm{Spec}(\mathbf{K})$;
(b) the simple normal crossing model $\mathscr{X} \to \mathrm{Spec}(R)$.

The main tool for this is the *Weak Factorization Theorem*, which says that any birational morphism between smooth proper varieties is a composition of blowups with smooth centers and their inverses.

Once this is done, if we have a smooth proper morphism $\mathscr{X} \to \mathrm{Spec}(R)$ as in the theorem, it is its own simple normal crossing model with smooth central fiber $\mathscr{X}_{b_0}$. Therefore, we can take $r = 1$ in the proof above and the defining formula (2) gives $\rho_n([\mathbf{L}/\mathbf{K}]) = [\mathbf{C}(\mathscr{X}_{b_0})/\mathbf{C}]$ in $\mathrm{Burn}_n(\mathbf{C})$. □

Remark 3.6. Let $\mathscr{X} \to B$ be a smooth projective morphism between smooth complex varieties, with $\dim(B) = 1$. In the notation of the proof of Theorem 3.5, assume that one can find a model $\mathscr{X} \to \mathrm{Spec}(R)$ with generic fiber X and simple normal crossing central fiber $(\mathscr{X}_{b_0})_{\mathrm{red}} = \sum_{i=1}^r D_i$ such that

$$\sum_{I \neq \varnothing} (-1)^{|I|-1} [\mathbf{C}(D_I)(x_1, \dots, x_{|I|-1})/\mathbf{C}] \neq [\mathbf{C}(\mathbf{P}^n]/\mathbf{C}] \quad \text{in} \, \mathrm{Burn}_n(\mathbf{C})$$

(for example, all D_I are rational except for one which is not unirational). Then very general fibers are irrational. A much more elaborate version of this remark was used in [NO2] to prove many new stable irrationality results: for example, in $\mathbf{P}^6_{\mathbf{C}}$, a very general quartic or a very general intersection of a quadric and a cubic are both stably irrational ([NO2, Corollary 5.2 and Theorem 7.1]).

4. Rationality versus unirationality

We now go back in time to the implication (UR) ⇒ (R), known as the *Lüroth problem:* is a unirational variety rational? In other words, is every extension of $\mathbf{k}$ contained in $\mathbf{k}(t_1, \dots, t_n)$ purely transcendental?

We saw in Section 2.3 that the answer is affirmative when $n \leq 2$ and $\mathbf{k} = \mathbf{C}$. After many unsuccessful attemps by Enriques, Fano, and Roth during the first half of the twentieth century, three different counter-examples to the Lüroth problem in dimension 3 over $\mathbf{C}$ appeared in 1971–72. We briefly indicate here the authors, their examples, and the methods they use to prove irrationality (this table was borrowed from [B4]).

Authors	Example	Method
Clemens–Griffiths	all smooth cubic threefolds	intermediate Jacobian
Iskovskikh–Manin	all smooth quartic threefolds	birational automorphisms
Artin–Mumford	some quartic double solids	torsion of $H^3(\bullet, \mathbf{Z})$

More precisely:

• Clemens–Griffiths proved in [CG] the longstanding conjecture that all smooth cubic threefolds $X \subseteq \mathbf{P}^4_{\mathbf{C}}$ are irrational (although they are all unirational by Example 2.1). They showed that the *intermediate Jacobian* of X is not the Jacobian of a curve (*Clemens–Griffiths criterion*; see Theorem 4.2 below).

• Iskovskikh–Manin proved in [IM] that all smooth quartic threefolds $X \subseteq \mathbf{P}^4_{\mathbf{C}}$ are irrational. Some unirational quartic threefolds had been constructed by B. Segre in [Se2], so these provide counter-examples to the Lüroth problem. They showed that the group of birational automorphisms of X is finite, while the corresponding group for $\mathbf{P}^3_{\mathbf{C}}$ (hence for any rational variety) is huge.

• Artin–Mumford proved in [AM] that a particular double covering X of $\mathbf{P}^3_{\mathbf{C}}$, branched along a quartic surface in $\mathbf{P}^3_{\mathbf{C}}$ with 10 nodes, is unirational but not rational. They showed that the torsion subgroup of $H^3(X,\mathbf{Z})$ is nontrivial, and is a birational invariant (see Proposition 6.1) which is trivial for $\mathbf{P}^3_{\mathbf{C}}$.

These three papers have been extremely influential. Though they appeared around the same time, they use very different ideas; in fact, as we will see, the methods tend to apply to different types of varieties. They have been developed and extended, and applied to a number of interesting examples. Each of them has its advantages and its drawbacks; very roughly:

• The intermediate Jacobian method is quite efficient, but applies only in dimension 3 (Section 4.1).

• The computation of birational automorphisms leads to the important notion of *birational superrigidity*. However it is not easy to work out; so far, it has been applied essentially to Fano varieties whose Picard group is generated by the canonical class, which are *not* known to be unirational in dimensions > 3. We give some results in Section 4.2.

• torsion in $H^3(\bullet,\mathbf{Z})$ gives an obstruction to stable rationality (see Section 6.1). Unfortunately it applies only to very particular varieties and not to standard examples of unirational varieties, like hypersurfaces or complete intersections. We discuss in Section 7 ideas of Voisin and others that extend considerably the range of that method.

4.1. **The intermediate Jacobian.** In this section, we discuss our first irrationality criterion, which uses the intermediate Jacobian. Then we prove that smooth cubic threefolds satisfy this criterion hence give counter-examples to the Lüroth problem.

4.1.1. *The Clemens–Griffiths criterion.* We first recall the Hodge-theoretic construction of the Jacobian of a (smooth projective complex) curve C of genus g. We start from the Hodge decomposition

$$H^1(C,\mathbf{Z}) \subseteq H^1(C,\mathbf{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$$

into complex vector subspaces of the same dimension g with $H^{0,1}(C) = \overline{H^{1,0}(C)}$. The latter condition implies that the projection $H^1(C,\mathbf{R}) \to H^{0,1}(C)$ is an $\mathbf{R}$-linear isomorphism, hence that the image Γ of $H^1(C,\mathbf{Z})$ in $H^{0,1}(C)$ is a *lattice* (that is, any basis of Γ is a basis of $H^{0,1}$ over $\mathbf{R}$). The quotient

$$J(C) := H^{0,1}(C)/\Gamma$$

is a complex torus of dimension g. But there is more structure:

$$(\alpha,\beta) \longmapsto 2i\int_C \bar{\alpha}\wedge\beta$$

defines a positive definite Hermitian form H on $H^{0,1}$, and the restriction of $\mathrm{Im}(H)$ to $\Gamma \simeq H^1(C,\mathbf{Z})$ coincides with the cup-product

$$H^1(C,\mathbf{Z}) \otimes H^1(C,\mathbf{Z}) \to H^2(C,\mathbf{Z}) \simeq \mathbf{Z};$$

thus it induces on Γ a skew-symmetric, integer-valued, *unimodular* form. In other words, H is a *principal polarization* on $J(C)$. This is equivalent to the data of an ample divisor $\Theta \subseteq J(C)$ (defined up to translation) satisfying $\dim(H^0(J(C), \mathscr{O}_{J(C)}(\Theta))) = 1$. Thus $(J(C), \Theta)$ is a *principally polarized abelian variety* of dimension g called the *Jacobian* of C.

One can mimic this definition for odd dimensional varieties, starting from the middle degree cohomology; this defines the general notion of intermediate Jacobian. In general, it is only a complex torus, not an abelian variety. But for a rationally connected threefold X, we have $H^{3,0}(X) = H^0(X, \Omega_X^3) = 0$ (Proposition 2.5), hence the Hodge decomposition for H^3 becomes

$$H^3(X, \mathbf{Z})_{\mathrm{tf}} \subseteq H^3(X, \mathbf{C}) = H^{2,1}(X) \oplus H^{1,2}(X)$$

with $H^{1,2} = \overline{H^{2,1}}$ and $H^3(X, \mathbf{Z})_{\mathrm{tf}} := H^3(X, \mathbf{Z}) / \operatorname{Tors}(H^3(X, \mathbf{Z}))$. As above, $H^{1,2}(X)/H^3(X, \mathbf{Z})_{\mathrm{tf}}$ is a complex torus, with a principal polarization defined by the positive definite Hermitian form $(\alpha, \beta) \mapsto -2i \int_X \bar{\alpha} \wedge \beta$: this is the *intermediate Jacobian* $J(X)$ of X.

We will use several times the following easy result (see for instance [Vo2, Theorem 7.31]).

Lemma 4.1. *Let X be a complex manifold, let $Y \subseteq X$ a closed submanifold of codimension c, and let $\operatorname{Bl}_Y X$ the variety obtained by blowing up X along Y. For every integer p, there is a canonical isomorphism*

$$H^p(X, \mathbf{Z}) \oplus \sum_{k=1}^{c-1} H^{p-2k}(Y, \mathbf{Z}) \xrightarrow{\sim} H^p(\mathrm{Bl}_Y X, \mathbf{Z}).$$

Theorem 4.2 (Clemens–Griffiths criterion)**.** *Let X be a rational smooth projective complex threefold. The intermediate Jacobian $J(X)$ is isomorphic (as a principally polarized abelian variety) to a product of Jacobians of a curves.*

Sketch of proof. Let $\varphi\colon \mathbf{P}^3 \dashrightarrow X$ be a birational map. Hironaka's resolution of indeterminacies provides us with a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[dr, "f"] & \\ \mathbf{P}^3 \arrow[rr, dashed, "\varphi"] & & X \end{tikzcd}$$

where $\varepsilon\colon P \to \mathbf{P}^3$ is a composition of blowups, either of points or of smooth curves, and f is a birational *morphism*.

I claim that $J(P)$ is a product of Jacobians of curves. Indeed, by Lemma 4.1, blowing up a point in a threefold X does not change $H^3(X, \mathbf{Z})$, hence does not change $J(X)$ either. If we blow up a smooth curve $C \subseteq X$, Lemma 4.1 gives a canonical isomorphism $H^3(\mathrm{Bl}_C X, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(C, \mathbf{Z})$, compatible in an appropriate sense with the Hodge decompositions and the cup-products; this implies $J(\mathrm{Bl}_C X) \simeq J(X) \times J(C)$ as principally polarized abelian varieties. Thus going back to our diagram, we see that $J(P)$ is isomorphic to $J(C_1) \times \cdots \times J(C_r)$, where $C_1, \dots, C_r$ are the (smooth) curves which we have blown up in the process.

How do we relate $J(P)$ and $J(X)$? The a birational *morphism* $f\colon P \to X$ induces homomorphisms

$$f^*\colon H^3(X, \mathbf{Z}) \to H^3(P, \mathbf{Z}) \quad , \quad f_*\colon H^3(P, \mathbf{Z}) \to H^3(X, \mathbf{Z})$$

satisfying $f_* f^* = 1$, again compatible with the Hodge decompositions and the cup-products in an appropriate sense. Thus $H^3(X, \mathbf{Z})$, with its polarized Hodge structure, is a direct factor of $H^3(P, \mathbf{Z})$; this implies that JV is a direct factor of $J(P) \simeq J(C_1) \times \cdots \times J(C_p)$, in other words there exists a principally polarized abelian variety A such that $J(X)\times A\simeq J(C_1)\times\cdots\times J(C_p)$ as principally polarized abelian varieties.

How can we conclude? In most categories, the decomposition of an object as a product is not unique (think of vector spaces!). However here a miracle occurs. Let us say that a polarized abelian variety is *indecomposable* if it is nonzero and not isomorphic (as polarized abelian varieties) to a product of nontrivial polarized abelian varieties. The Jacobian of a (smooth projective connected) curve is indecomposable. One has the following general result (see [D1]; this result is actually easier to prove in our case, when the abelian varieties are principally polarized).

Lemma 4.3. *Any polarized abelian variety admits a* unique *decomposition as a product of indecomposable polarized abelian varieties.*

Once we have this, we conclude as follows: since the principally polarized abelian varieties $J(C_1),\dots,J(C_r)$ are indecomposable, from the isomorphism $J(X)\times A\simeq J(C_1)\times\cdots\times J(C_r)$ and the lemma, we conclude that $J(X)$ is isomorphic to $J(C_{i_1})\times\cdots\times J(C_{i_s})$ for some subset $\{i_1,\dots,i_s\}$ of $\{1,\dots,r\}$. ☐

Remark 4.4. In the moduli space $\mathscr{A}_g$ of principally polarized abelian varieties of dimension g, the boundary $\overline{\mathscr{J}_g}\smallsetminus\mathscr{J}_g$ of the Jacobian locus $\mathscr{J}_g$ is precisely the locus of products of lower-dimensional Jacobians. So the latter can be seen as degenerations of the former.

Remark 4.5. Jacobians of curves and their products are characterized among all principally polarized abelian varieties (A,θ) by the fact that the cohomology class $\frac{\theta^{n-1}}{(n-1)!}\in H^{2n-2}(A,\mathbf{Z})$, where $n:=\dim(A)$, is represented by an (algebraic) effective 1-cycle (Matsusaka's criterion).

4.1.2. *The Schottky problem.* Thus to show that a (smooth projective) threefold X is not rational, it suffices to prove that its intermediate Jacobian $(J(X),\Theta)$ is not a product of Jacobians of curves. This is the classical Schottky problem: the characterization of Jacobians of curves among all principally polarized abelian varieties. One frequently used approach is through the singularities of the theta divisor: the codimension of $\operatorname{Sing}(\Theta)$ in J is at most 4 for a product (J,Θ) of Jacobians of curves. However, controlling $\operatorname{Sing}(\Theta)$ for an intermediate Jacobian $J(X)$ is quite difficult and requires a lot of information on the geometry of X. Let us just give a sample.

Theorem 4.6. *Let $X\subseteq\mathbf{P}^4_{\mathbf{C}}$ be a smooth cubic threefold. The theta divisor $\Theta\subseteq J(X)$ has a unique singular point p, which is a triple point. The projectified tangent cone $\mathbf{P}(TC_{\Theta,p})\subseteq\mathbf{P}(T_{J(X),p})\simeq\mathbf{P}^4_{\mathbf{C}}$ is isomorphic to $X\subseteq\mathbf{P}^4_{\mathbf{C}}$.*

This elegant result, apparently due to Mumford (see [B2] for a proof), implies the irrationality of all smooth cubic threefolds $X\subseteq\mathbf{P}^4$, because $\operatorname{codim}_{J(X)}(\operatorname{Sing}(\Theta))=5$.

There are actually very few cases where we can control so well the singular locus of the theta divisor. One of these is the case of smooth quartic double solids $X\to\mathbf{P}^3$, for which $\operatorname{Sing}(\Theta)$ has a component of codimension 5 in $J(X)$ ([Vo1]). Another case is that of *conic bundles*, that is, threefolds X with a flat morphism $p\colon X\to\mathbf{P}^2$ such that, for each closed point $s\in\mathbf{P}^2$, the fiber $p^{-1}(s)$ is isomorphic to a plane conic (possibly singular). In that case, $J(X)$ is the *Prym variety* associated with a natural double covering of the *discriminant curve* $\Delta\subseteq\mathbf{P}^2$ (the locus of $s\in\mathbf{P}^2$ such that $p^{-1}(s)$ is singular). Thanks to work of Mumford and Beauville, we have enough control on the singularities of the theta divisor of a Prym variety to show that $J(X)$ *is not a product of Jacobians of curves if* $\deg(\Delta)\geq 6$ ([B1, th. 4.9]). In particular, these conic bundles are irrational (we will see in Section 7.3 that they are not even stably rational when the discriminant curve is very general).

Unfortunately, there are many Fano threefolds that are not (or least not known to be) conic bundles. However, the Clemens–Griffiths criterion of irrationality is an *open* condition (unlike irrationality!). In fact, we have a stronger result, which follows from the properties of the *Satake compactification* of the moduli space of principally polarized abelian varieties ([B1, lemme 5.6.1]).

Lemma 4.7. *Let $\pi\colon \mathscr{X} \to B$ be a flat family of projective complex threefolds over a smooth complex curve B. Let $b_0 \in B$ and assume that*

- *the fiber $\mathscr{X}_b := \pi^{-1}(b)$ is smooth for all $b \in B \smallsetminus \{b_0\}$;*
- *the only singularities of $\mathscr{X}_{b_0}$ are ordinary double points;*
- *for some desingularization $\widetilde{\mathscr{X}_{b_0}}$ of $\mathscr{X}_{b_0}$, the intermediate Jacobian $J(\widetilde{\mathscr{X}_{b_0}})$ is not a product of Jacobians of curves.*

Then for b outside a finite subset of B, the smooth threefold $\mathscr{X}_b$ is irrational.

From this, we deduce general irrationality statements for many families of Fano threefolds: it is enough to find a degeneration as in the lemma such that $\widetilde{X}_{b_0}$ is a conic bundle with a discriminant curve of degree ≥ 6, to which the lemma applies.

Consider for example (ordinary) Gushel–Mukai threefolds: they are smooth complete intersections X of the Grassmannian $\mathsf{Gr}(2, \mathbf{C}^5) \subseteq \mathbf{P}(\bigwedge^2 \mathbf{C}^5) = \mathbf{P}^9$ in its Plücker embedding with a $\mathbf{P}^7$ and a quadric. They are Fano threefolds with canonical line bundle $\mathscr{O}_X(-1)$. When the quadric becomes singular at a point p of $\mathsf{Gr}(2, \mathbf{C}^5) \cap \mathbf{P}^7$, the threefold X acquires a node at p and projection $X \to \mathbf{P}^6$ from p is (birationally) a conic bundle with discriminant curve of degree 6. The lemma then implies that a general Gushel–Mukai threefold is irrational.

4.1.3. *Easy counterexamples.* The results of the previous section require rather involved methods. We will now discuss a more elementary approach, which however only applies to specific varieties. It is based on the so-called *Hurwitz bound* (the order of the group of automorphisms of a smooth projective curve C of genus g is $84(g-1)$ and the Torelli theorem for curves, which gives an exact sequence

$$1 \to \mathrm{Aut}(C) \to \mathrm{Aut}(J(C), \Theta) \to \mathbf{Z}/2. \tag{3}$$

Fano threefolds with very large automorphism groups will therefore not be rational.

We will give two examples. We consider first smooth complete intersections of a quadric and a cubic in $\mathbf{P}^5_{\mathbf{C}}$. They are classically known to be unirational, but a general such complete intersection is irrational (this can be proved using the degeneration result Lemma 4.7; it also follows from the proof of the theorem below and the openness of the Clemens–Griffiths criterion of irrationality). The example of the next theorem was however the first explicit irrational example ([B3]).

Theorem 4.8 (Beauville). *The Fano threefold $X \subseteq \mathbf{P}^6$ defined by the equations*

$$\sum_{i=0}^{6} x_i = \sum_{i=0}^{6} x_i^2 = \sum_{i=0}^{6} x_i^3 = 0$$

is not rational.

Proof. The group $\mathfrak{S}_7$ acts faithfully on X. One then checks that the induced action on $T_{J(X),0} = H^1(X,\Omega^2_X)$ is also faithful (it is the sum of two irreducible representations, of degrees 6 and 14), hence so is its action on the 20-dimensional principally polarized abelian variety $J(X)$. But the Hurwitz bound and the exact sequence (3) imply that the automorphism group of the Jacobian of a curve of genus 20 has order at most $2\cdot 84(20-1) = 3192 < 7! = |\mathfrak{S}_7|$. So $J(X)$ cannot be the Jacobian of a curve of genus 20.

One then needs an additional easy argument to exclude the possibility that $J(X)$ be isomorphic to a nontrivial product of Jacobians of curves. □

The same method was applied more recently in [DM] to (smooth) Gushel–Mukai threefolds $X \subseteq \mathbf{P}(\bigwedge^2\mathbf{C}^5)$ (for which, as we explained earlier, irrationality was only known for a general one). We choose coordinates $x_0,\dots,x_4$ on $\mathbf{C}^5$ and we denote by $(x_{ij})_{0\le i<j\le 4}$ the induced coordinates on $\bigwedge^2\mathbf{C}^5$.

Theorem 4.9 (Debarre–Mongardi). *The smooth Gushel–Mukai threefold $X \subseteq \mathbf{P}(\bigwedge^2\mathbf{C}^5)$ defined by the linear space*

$$x_{03}+x_{12} = x_{04}-x_{23} = 0$$

and the quadric

$$x_{01}x_{02}-x_{13}x_{14}-x_{24}x_{34} = 0$$

is irrational.

Sketch of proof. One shows that the simple group $G := \mathrm{PSL}(2,\mathbf{F}_{11})$ acts faithfully on the 10-dimensional intermediate Jacobian $J(X)$ (but not on X![4]) and that the induced action on the tangent space $T_{J(X),0}$ is an irreducible representation of dimension 10. This implies already that the principally polarized abelian variety $J(X)$ is indecomposable: by the uniqueness result Lemma 4.3, the group G permutes its m indecomposable factors and this induces a morphism $u\colon G \to \mathfrak{S}_m$ which cannot be injective since G contains elements of order 11 but not $\mathfrak{S}_m$ since $m \le 10$. The simplicity of G then implies that u is constant and that G preserves each indecomposable factor. The irreducibility of the action of G on $T_{J(X),0}$ finally implies $m = 1$.

It is known that the automorphism group of a curve of genus 10 has order at most 432 (an improvement on the Hurwitz bound). Since G is simple, any morphism $G \to \mathbf{Z}/2\mathbf{Z}$ is trivial, hence, since $|G| = 660 > 432$, the exact sequence (3) implies that G does not embed in the automorphism group of the Jacobian of a curve of genus 10. So $J(X)$ cannot be the Jacobian of a curve. Since it is indecomposable, the Clemens–Griffiths criterion implies that X is irrational. □

Corollary 4.10. *There exists a complete family, with finite moduli morphism, parametrized by a smooth projective surface, of irrational smooth Gushel–Mukai threefolds.*

This follows from a description of the moduli space of Gushel–Mukai threefolds ([DM, Corollary 5.3], [DK, Example 6.8]): through any point of the moduli space, there passes a projective surface that parametrizes mutually birationally isomorphic Gushel–Mukai threefolds. Another family of irrational Gushel–Mukai threefolds (whose intermediate Jacobian has a faithful $\mathfrak{A}_7$-action) was recently described in [BW].

A. Javanpeykar asked the following related question.[5]

[4]By [P, Theorem 1.5], G cannot act nontrivially on a smooth Gushel–Mukai threefold.

[5]It is known that families of smooth projective varieties of general type parametrized by $\mathbf{P}^1$ are isotrivial.

Question 4.11 (Javanpeykar)**.** Does there exist nonisotrivial families of smooth Fano varieties parametrized by $\mathbf{P}^1$?

4.2. **Birational rigidity.** As mentioned in the introduction, Iskovskikh–Manin proved that all smooth quartic threefolds $X \subseteq \mathbf{P}^4_{\mathbf{C}}$ are irrational by proving that any birational automorphism of X is actually biregular. But they proved much more, namely that X is *birationally superrigid* in the following sense.

Definition 4.12. Let X be a prime Fano variety with Picard number 1. We say that X is *birationally superrigid* if

(a) there is no rational dominant map $X \dashrightarrow Y$ with $0 < \dim(Y) < \dim(X)$ and with general fibers of Kodaira dimension $-\infty$;
(b) any birational isomorphism $X \overset{\sim}{\dashrightarrow} Y$ to another Fano variety Y with Picard number 1 is an isomorphism.

(The variety Y in (b) is allowed to have $\mathbf{Q}$-factorial terminal singularities.)

After the pioneering work [IM], birational superrigidity was proved for a number of Fano varieties of index 1. In particular, de Fernex extended the result of Iskovskikh–Manin and proved that any smooth hypersurface of degree n in $\mathbf{P}^n_{\mathbf{C}}$ is birationally superrigid ([dF]). We refer to the surveys [Pu] and [C] for ideas of proofs and for many more examples.

5. Rationality versus stable rationality

Now that we know that the converse of the composed implication

$$(\mathrm{R}) \Longrightarrow (\mathrm{SR}) \Longrightarrow (\mathrm{UR})$$

is false (in dimensions ≥ 3), we examine separately the two implications

$$(\mathrm{UR}) \Longrightarrow (\mathrm{SR}) \quad \text{and} \quad (\mathrm{SR}) \Longrightarrow (\mathrm{R}).$$

The implication on the right was proved to be false by Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer in [BCSS], thereby answering a question asked by Zariski in 1949 (see [Se1]).

Theorem 5.1. *Let $P(x,t) = x^3 + p(t)x + q(t)$ be an irreducible polynomial in $\mathbf{C}[x,t]$ and assume that its discriminant $\delta(t) := 4p(t)^3 + 27q(t)^2$ has degree ≥ 5. The affine hypersurface $X \subseteq \mathbf{C}^4$ defined by $y^2 - \delta(t)z^2 = P(x,t)$ is stably rational but not rational.*

The projection $X \to \mathbf{C}^2$ defined by $(x,t,y,z) \mapsto (x,t)$ makes the threefold X into an (affine) conic bundle.

The irrationality of X is proved using the intermediate Jacobian, which turns out to be the Prym variety associated with an admissible double covering of nodal curves. The stable rationality, more precisely the fact that $X \times \mathbf{P}^3_{\mathbf{C}}$ is rational, was proved in [BCSS] using some particular torsors under certain algebraic tori. A different construction of Shepherd-Barron shows that $X \times \mathbf{P}^2_{\mathbf{C}}$ is already rational ([SB]); it is not known whether $X \times \mathbf{P}^1_{\mathbf{C}}$ is rational.

6. Stable rationality versus unirationality

We prove in this section that the implication (UR) $\Rightarrow$ (SR) is also false (for smooth projective complex varieties). So we need to find unirational varieties that are not stably rational. For that, we cannot use the Clemens–Griffiths criterion since it applies only in dimension 3 hence cannot disprove stable rationality. The group of birational automorphisms is very complicated for a variety of the form $X \times \mathbf{P}^n_{\mathbf{C}}$; so the only available method is the torsion of $H^3(\bullet, \mathbf{Z})$ and its subsequent refinements, which we will examine in the next sections.

6.1. **The torsion of $H^3(\bullet, \mathbf{Z})$.** Artin and Mumford used the following property of stably rational varieties.

Proposition 6.1. *Let X be a stably rational smooth projective complex variety. The abelian group $H^3(X, \mathbf{Z})$ is torsion free.*

Proof. The Künneth formula gives an isomorphism $H^3(X \times \mathbf{P}^m, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(X, \mathbf{Z})$; since $H^1(X, \mathbf{Z})$ is always torsion free, the torsion subgroups of $H^3(X, \mathbf{Z})$ and $H^3(X \times \mathbf{P}^m, \mathbf{Z})$ are isomorphic hence, replacing X by $X \times \mathbf{P}^m$, we may assume that the variety X is rational. Let $\varphi \colon \mathbf{P}^n \overset{\sim}{\dashrightarrow} X$ be a birational map. As in the proof of the Clemens–Griffiths criterion, we have a diagram

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[dr, "f"] & \\ \mathbf{P}^n \arrow[rr, dashed, "\varphi"] & & X, \end{tikzcd}$$

where $\varepsilon \colon P \to \mathbf{P}^n$ is a composition of blowups of smooth subvarieties and f is a birational *morphism*.

By Lemma 4.1, we have $H^3(P, \mathbf{Z}) \simeq H^1(Y_1, \mathbf{Z}) \oplus \cdots \oplus H^1(Y_r, \mathbf{Z})$, where $Y_1, \ldots, Y_r$ are the subvarieties successively blown up by ε; therefore $H^3(P, \mathbf{Z})$ is torsion free. As in the proof of Theorem 4.2, $H^3(X, \mathbf{Z})$ is a direct summand of $H^3(P, \mathbf{Z})$, hence is also torsion free. □

6.2. **The Brauer group.** The torsion of $H^3(X, \mathbf{Z})$ is strongly related to the (cohomological) *Brauer group* of X. There is a huge literature on the Brauer group in algebraic geometry, starting with the three exposés by Grothendieck in [G]. We recall here the cohomological definition of this group for complex varieties.

Definition 6.2. Let X be a complex variety. We define the (cohomological) *Brauer group* $\mathrm{Br}(X)$ to be the étale cohomology group $H^2_{\text{ét}}(X, \mathscr{O}_X^\times)$.

Proposition 6.3. *Let X be a complex variety. There is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X) \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{c_1} H^2(X, \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathrm{Tors}(\mathrm{Br}(X)) \longrightarrow 0. \tag{4}$$

Proof. Let $n \in \mathbf{Z}_{>0}$. The exact sequence

$$1 \to \boldsymbol{\mu}_n \to \mathscr{O}_X^\times \xrightarrow{\bullet^n} \mathscr{O}_X^\times \to 1$$

of étale sheaves and the isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} H^1_{\text{ét}}(X, \mathscr{O}_X^\times)$ give an exact sequence

$$\mathrm{Pic}(X) \xrightarrow{\times n} \mathrm{Pic}(X) \xrightarrow{c_1} H^2_{\text{ét}}(X, \boldsymbol{\mu}_n) \longrightarrow \mathrm{Br}(X) \xrightarrow{\times n} \mathrm{Br}(X)$$

in étale cohomology (we denote all these abelian groups additively). Noting the isomorphism $H^2_{\text{ét}}(X, \boldsymbol{\mu}_n) \simeq H^2(X, \mathbf{Z}/n\mathbf{Z})$ and taking the direct limit with respect to n gives the exact sequence (4). □

It is known that when X is smooth, $\mathrm{Br}(X)$ is a torsion group (see [G, II, prop. 1.4]).

Proposition 6.4. *Let X be a smooth complex variety. There is a surjective homomorphism $\mathrm{Br}(X) \to \mathrm{Tors}(H^3(X, \mathbf{Z}))$ which is bijective when $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ is surjective.*

By Hodge theory, the condition on c_1 is satisfied in particular if X is projective and $H^2(X, \mathscr{O}_X) = 0$.

Proof. The exponential exact sequence $0 \to \mathbf{Z} \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 1$ gives an exact sequence

$$\operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathscr{O}_X) \longrightarrow \operatorname{Br}(X) \longrightarrow H^3(X, \mathbf{Z}) \longrightarrow H^3(X, \mathscr{O}_X)$$

in cohomology. Since X is smooth, $\operatorname{Br}(X)$ is a torsion group; since $H^3(X, \mathscr{O}_X)$ is torsion free (it is a complex vector space), this exact sequence reduces to

$$\operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathscr{O}_X) \longrightarrow \operatorname{Br}(X) \longrightarrow \operatorname{Tors}(H^3(X, \mathbf{Z})) \longrightarrow 0.$$

When c_1 is surjective, since $H^2(X, \mathscr{O}_X)$ has no torsion and $\operatorname{Br}(X)$ is a torsion group, we obtain $H^2(X, \mathscr{O}_X) = 0$ hence the desired result $\square$

Remark 6.5 (Birational invariance)**.** Let X be a smooth projective complex variety. The Brauer group $\operatorname{Br}(X)$ can be defined purely in terms of the function field $\mathbf{C}(X)$ as its *unramified Brauer group;* this proves that it is a birational invariant (see [B4, Section 6.5] for more details). It is even a stable birational invariant: the Brauer group of a stably rational smooth projective complex variety is trivial.

We will now describe a geometric way to construct nontrivial elements of the Brauer group.

Definition 6.6. Let X be a complex variety. A $\mathbf{P}^n$*-fibration* over X is a smooth map $P \to X$ all of whose geometric fibers are isomorphic to $\mathbf{P}^n$.

An obvious example is the projective bundle $\mathbf{P}_X(E)$ associated with a vector bundle E of rank $n+1$ on X; a vector bundle being trivial on a dense open subset of X, a projective bundle has plenty of rational sections. We will actually be interested in those $\mathbf{P}^n$-fibrations that are *not* projective bundles; for example, those that have no rational sections.

Any $\mathbf{P}^n$-fibration is locally trivial for the étale (or analytic) topology. This implies that isomorphism classes of $\mathbf{P}^n$-fibrations over X are parametrized by the étale cohomology set $H^1_{\text{ét}}(X, \mathrm{PGL}_{n+1})$ where, for an algebraic group G, we denote by G the sheaf of local maps to G. Similarly, isomorphism classes of vector bundles (analytically locally free sheaves) of rank $n+1$ over X are parametrized by the étale cohomology set $H^1_{\text{ét}}(X, \mathrm{GL}_{n+1})$.

The exact sequence

$$1 \longrightarrow \mathscr{O}_X^\times \longrightarrow \mathrm{GL}_{n+1} \longrightarrow \mathrm{PGL}_{n+1} \longrightarrow 1$$

of sheaves of groups gives rise to a sequence of pointed sets

$$H^1_{\text{ét}}(X, \mathrm{GL}_{n+1}) \xrightarrow{q} H^1_{\text{ét}}(X, \mathrm{PGL}_{n+1}) \xrightarrow{\partial} \operatorname{Br}(X)$$

which is exact in the sense that $\partial^{-1}(1) = \operatorname{Im}(q)$. Thus we associate with each $\mathbf{P}^n$-fibration $p \colon P \to X$ a class $[p] \in \operatorname{Br}(X)$ and this class is trivial if and only if p is a projective bundle. Moreover, by comparing with the exact sequence $0 \to \boldsymbol{\mu}_{n+1} \to \mathrm{SL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$, we get a commutative diagram

$$\begin{tikzcd}
H^1_{\text{ét}}(X, \mathrm{SL}_{n+1}) \arrow[r] \arrow[d] & H^1_{\text{ét}}(X, \mathrm{PGL}_{n+1}) \arrow[r] \arrow[d, equal] & H^2(X, \boldsymbol{\mu}_{n+1}) \arrow[d] \\
H^1_{\text{ét}}(X, \mathrm{GL}_{n+1}) \arrow[r] & H^1_{\text{ét}}(X, \mathrm{PGL}_{n+1}) \arrow[r, "\partial"] & \operatorname{Br}(X)
\end{tikzcd}$$

which shows that the image of ∂ is contained in the $(n+1)$-torsion subgroup of $\operatorname{Br}(X)$.

6.3. **The Artin–Mumford example.** The Artin–Mumford counter-example is a double cover of $\mathbf{P}^3$ branched along a *quartic symmetroid*, that is, a quartic surface defined by the vanishing of a symmetric 4×4 determinant of linear forms.

We start with a web Π of quadrics in $\mathbf{P}^3$; its elements are defined by quadratic forms $\lambda_0 q_0 + \cdots + \lambda_3 q_3$. We assume the following generality properties:

(a) the linear system Π is basepoint free;
(b) if a line in $\mathbf{P}^3$ is singular for a quadric of Π, it is not contained in another quadric of Π.

Let $\Delta \subsetneq \Pi$ be the discriminant locus, corresponding to quadrics of rank ≤ 3. It is a quartic surface (defined by $\det(\sum \lambda_i q_i) = 0$ in $\mathbf{P}(\lambda_0, \ldots, \lambda_3)$); under our hypotheses, Δ has 10 ordinary double points, corresponding to quadrics of rank 2, and no other singularities. Let $\pi \colon X' \to \Pi$ be the double covering branched along Δ. Again X' has 10 ordinary double points; blowing up these points, we obtain the Artin–Mumford (smooth projective) threefold X.

Observe that a quadric $q \in \Pi$ has two rulings by lines if $q \in \Pi \smallsetminus \Delta$, and one if $q \in \Delta \smallsetminus \mathrm{Sing}(\Delta)$. The smooth part X_0 of X' parametrizes pairs (q, λ), where $q \in \Pi$ and λ is a ruling of q.

Theorem 6.7. *The threefold X is unirational but not stably rational.*

Skectch of proof. Let $\mathbf{G}$ be the Grassmannian of lines in $\mathbf{P}^3$. A general line is contained in a unique quadric of Π, and in a unique ruling of this quadric; this defines a dominant rational map $\gamma \colon \mathbf{G} \dashrightarrow X'$, thus X is unirational. We will deduce from Proposition 6.1 that X is not stably rational by proving that $H^3(X, \mathbf{Z})$ contains an element of order 2. This is done by a direct calculation in [AM]; we use a different approach based on the Brauer group.

Consider the variety $P \subseteq \mathbf{G} \times \Pi$ consisting of pairs (ℓ, q) with $\ell \subseteq q$. The Stein factorization of the projection $P \to \Pi$ is

$$P \xrightarrow{p'} X' \xrightarrow{\pi} \Pi.$$

Set $P_0 := p'^{-1}(X_0)$. The restriction $p_0 \colon P_0 \to X_0$ of p' is a $\mathbf{P}^1$-fibration: the fiber of a point (q, λ) of X_0 is the smooth rational curve parametrizing the lines of the ruling λ. An elementary argument (see [B4, Section 6.3]) then shows that p_0 no rational sections. It is therefore not a projective bundle, hence defines a nonzero 2-torsion class in $\mathrm{Br}(X_0)$.

In the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "c_1"] \arrow[d] & H^2(X, \mathbf{Z}) \arrow[d, "r"] \\
\mathrm{Pic}^{\mathrm{an}}(X_0) \arrow[r, "c_1"] & H^2(X_0, \mathbf{Z}),
\end{tikzcd}$$

the top horizontal arrow is surjective because $H^2(X, \mathscr{O}_X) = 0$. Since $E := X \smallsetminus X_0$ is a disjoint union of quadrics (the exceptional divisors of the blowup of the 10 ordinary double points of X), the Gysin exact sequence

$$H^2(X, \mathbf{Z}) \xrightarrow{r} H^2(X_0, \mathbf{Z}) \to H^1(E, \mathbf{Z}) = 0$$

shows that r is surjective. Therefore the map $c_1 \colon \mathrm{Pic}(X_0) \to H^2(X_0, \mathbf{Z})$ is surjective and, by Proposition 6.4, we get a nonzero 2-torsion class in $H^3(X_0, \mathbf{Z})$. Using again the Gysin exact sequence

$$0 \to H^3(X, \mathbf{Z}) \to H^3(X_0, \mathbf{Z}) \to H^2(E, \mathbf{Z})$$

we find that $\mathrm{Tors}(H^3(X,\mathbf{Z}))$ is isomorphic to $\mathrm{Tors}(H^3(X_0,\mathbf{Z}))$, hence nonzero. By Proposition 6.1, the threefold X is not stably rational. □

7. The Chow group of 0-cycles

In this section, we discuss another property of stably rational varieties, namely the fact that their Chow group CH_0 parametrizing 0-cycles is *universally trivial* (Proposition 7.4). While the idea goes back to the end of the seventies (see [Bl]), its use for rationality questions is recent ([Vo4]).

This property implies that $H^3(X,\mathbf{Z})$ is torsion free (Proposition 7.9), but not conversely (see (11)). Moreover, it behaves well under deformation, even if we accept mild singularities (see Proposition 7.10).

In this section, we will need to work over nonalgebraically closed fields (of characteristic 0). We use the language of schemes.

7.1. **Chow groups.** Let X be a variety of dimension n defined over a field $\mathbf{k}$. The Chow group $\mathrm{CH}_p(X)$ is the group of dimension-p cycles on X modulo rational equivalence. More precisely, let us denote by $\Sigma_p(X)$ the set of dimension-p closed subvarieties of X. Then $\mathrm{CH}_p(X)$ is defined by the exact sequence

$$\bigoplus_{W\in\Sigma^{p+1}(X)} \mathbf{k}(W)^* \longrightarrow \mathbf{Z}^{(\Sigma^p(X))} \longrightarrow \mathrm{CH}_p(X)\to 0\,, \tag{5}$$

where the first arrow associates with $f\in\mathbf{k}(W)^*$ its divisor ([F, Section 1.3]).

Example 7.1. When $X=\mathbf{A}^1_{\mathbf{k}}=\mathrm{Spec}(\mathbf{k}[x])$, a closed point is an irreducible polynomial $P\in\mathbf{k}[x]$. The divisor of the regular function on X defined by P is P, so any point is rationally equivalent to 0 and $\mathrm{CH}_0(\mathbf{A}^1_{\mathbf{k}})=0$. More generally, one has $\mathrm{CH}_p(\mathbf{A}^n_{\mathbf{k}})=0$ for all $p\neq n$ and $\mathrm{CH}_n(\mathbf{A}^n_{\mathbf{k}})=\mathbf{Z}$.

Given a morphism $f\colon X\to Y$ between varieties, it induces pushforward homomorphisms $f_*\colon\ \mathrm{CH}_p(X)\to\mathrm{CH}_p(Y)$ when f is proper, and pullback homomorphisms $f^*\colon\ \mathrm{CH}_p(Y)\to\mathrm{CH}_{p+n}(X)$ when f is flat of relative dimension n ([F, Theorem 1.4 and Theorem 1.7]). Furthermore,

- if $Y\subseteq X$ is a closed subset, with inclusions $i\colon X\smallsetminus Y\hookrightarrow X$ and $j\colon Y\hookrightarrow X$, one has localization exact sequences ([F, Proposition 1.8])

$$\mathrm{CH}_p(Y)\xrightarrow{i_*}\mathrm{CH}_p(X)\xrightarrow{j^*}\mathrm{CH}_p(X\smallsetminus Y)\longrightarrow 0 \tag{6}$$

- for any variety X over $\mathbf{k}$, there are canonical isomorphisms ([F, Theorem 3.3(b)])

$$\mathrm{CH}_0(X)\xrightarrow{\sim}\mathrm{CH}_0(X\times\mathbf{P}^n_{\mathbf{k}}). \tag{7}$$

In particular, we have $\mathrm{CH}_p(\mathbf{P}^n_{\mathbf{k}})\simeq\mathbf{Z}$ for all $0\le p\le n$ (where the isomorphism is given by the degree of subvarieties of $\mathbf{P}^n_{\mathbf{k}}$).

When X is smooth of pure dimension n, we set $\mathrm{CH}^p(X):=\mathrm{CH}_{n-p}(X)$ (the lower index denotes the dimension and the upper index the codimension) and one can define intersection products

$$\mathrm{CH}^p(X)\otimes\mathrm{CH}^q(X)\longrightarrow\mathrm{CH}^{p+q}(X)$$

satisfying various nice properties (see [F, Proposition 8.3]).

We will be particularly interested in the group $\mathrm{CH}_0(X)$ of 0-cycles. When X is proper over $\mathbf{k}$, the map

$$\sum_{n_i\in\mathbf{Z},\ p_i \text{ closed point}} n_i[p_i] \longmapsto \sum n_i[\mathbf{k}(p_i):\mathbf{k}]$$

defines a group morphism $\deg\colon \mathrm{CH}_0(X) \to \mathbf{Z}$ ([F, Example 1.6.6]). We denote its kernel by $\mathrm{CH}_0(X)_0$.

Finally, we will need the following birational invariance result. Note that, together with the isomorphism (7), it implies that if X is a stably rational smooth projective variety (over any field), one has $\mathrm{CH}_0(X)_0 = 0$.

Lemma 7.2. *Any birational isomorphism $X \overset{\sim}{\dashrightarrow} Y$ between smooth projective varieties (over any field) induces an isomorphism $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$.*

Sketch of proof. The graph $\Gamma \subseteq X \times Y$ of the birational isomorphism $X \overset{\sim}{\dashrightarrow} Y$ defines morphisms

$$\Gamma_*\colon\ \mathrm{CH}_p(X) \longrightarrow \mathrm{CH}_p(Y) \quad , \quad \alpha \longmapsto \mathsf{pr}_{2*}(\Gamma\cdot \mathsf{pr}_1^*(\alpha))$$

and

$$\Gamma^*\colon\ \mathrm{CH}_p(Y) \longrightarrow \mathrm{CH}_p(X) \quad , \quad \beta \longmapsto \mathsf{pr}_{1*}(\Gamma\cdot \mathsf{pr}_2^*(\beta))$$

where the dots represent the intersection product mentioned earlier. One shows $\Gamma^*\circ\Gamma_* = \mathrm{Id}$ on $\mathrm{CH}_0(X)$ and $\Gamma_*\circ\Gamma^* = \mathrm{Id}$ on $\mathrm{CH}_0(Y)$ (see [Vo5, Lemma 2.11] for details). □

7.2. **Universally CH_0-trivial varieties and Chow decomposition of the diagonal.** *When* $\mathbf{k}$ *is algebraically closed,* one has $\mathrm{CH}_0(X)_0 = 0$ for any (smooth projective) rationally connected variety X (any two points can be joined by a $\mathbf{P}^1_{\mathbf{k}}$ where they are rationally equivalent); we say that X is CH_0-*trivial.*[6] This does not always remain true when $\mathbf{k}$ is not algebraically closed (see (11)): being CH_0-trivial is not stable under field extensions. We make the following definition.

Definition 7.3. A smooth projective complex variety X is *universally* CH_0-*trivial* if for any field extension $\mathbf{K}/\mathbf{C}$, we have $\mathrm{CH}_0(X_{\mathbf{K}})_0 = 0$.

This property only depends on the birational isomorphism class of the variety (by Lemma 7.2) and holds for all projective spaces, hence for all (smooth projective) complex varieties. But we have even more.

Proposition 7.4. *Any stably rational smooth projective complex variety is universally* CH_0-*trivial.*

Proof. Let X be a stably rational smooth projective complex variety. For any field extension $\mathbf{K}/\mathbf{C}$, the variety $X_{\mathbf{K}}$ is again stably rational (over $\mathbf{K}$), hence $\mathrm{CH}_0(X_{\mathbf{K}})_0 = 0$ as discussed in Section 7.1. □

Proposition 7.5. *Let X be a smooth projective complex variety of dimension n and let $\Delta_X \subseteq X\times X$ be the diagonal. The following conditions are equivalent:*

(i) *the variety X is universally* CH_0-*trivial;*
(ii) *one has* $\mathrm{CH}_0(X_{\mathbf{C}(X)})_0 = 0$*;*
(iii) *there exists a point $x\in X$ such that $\delta_X - [x_{\mathbf{C}(X)}] = 0$ in $\mathrm{CH}_0(X_{\mathbf{C}(X)})$, where δ_X is the* 0*-cycle class on $X_{\mathbf{C}(X)}$ induced by the diagonal Δ_X;*
(iv) *there exist a point $x\in X$ and a dense open subset $U\subseteq X$ such that the cycle class $[\Delta_X] - [X\times\{x\}]$ restricts to* 0 *in* $\mathrm{CH}^n(U\times X)$*;*

[6] The converse is not true: a complex Enriques surface is CH_0-trivial ([BKL]) but not rationally connected.

(v) (Integral Chow decomposition of the diagonal) *there exists a point $x \in X$ such that the class*

$$[\Delta_X] - [X \times \{x\}] \tag{8}$$

in $\mathrm{CH}_n(X \times X)$ is supported on $D \times X$, for some hypersurface $D \subsetneq X$.

In (ii), (iii), (iv), and (v), the property is independent of the point $x \in X$: if it holds for one point, it holds for all points. In (iv), one says that a class $\alpha \in \mathrm{CH}_n(X \times X)$ is *supported on $D \times X$* if there exists a class $\alpha_D \in \mathrm{CH}_n(D \times X)$ such that $\alpha = i_*(\alpha_D)$, where i is the inclusion $D \times X \hookrightarrow X \times X$.

Proof. The implication (i) ⇒ (ii) is clear.

(ii) ⇒ (iii). Let η be the generic point of X. We have a diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathbf{C}(X)) \arrow[r, "\eta"] \arrow[d, "(\eta,\eta)"] & \Delta_X \arrow[d, hook] & \\
\{\eta\} \times X = X_{\mathbf{C}(X)} \arrow[r, "(\eta,\bullet)"] \arrow[d, shift right] & X \times X \arrow[r, "\mathsf{pr}_2"] \arrow[d, "\mathsf{pr}_1"', shift right] & X \arrow[d, shift right] \\
\mathrm{Spec}(\mathbf{C}(X)) \arrow[r, "\eta"] \arrow[u, bend right, "(\eta,x)"'] & X \arrow[r] \arrow[u, bend right, "(\bullet,x)"'] & \mathrm{Spec}(\mathbf{C}). \arrow[u, bend right, "x"']
\end{tikzcd}$$

The point (η, η) of $\{\eta\} \times X = X_{\mathbf{C}(X)}$ is rational (over $\mathbf{C}(X)$). Since $\mathrm{CH}_0(X_{\mathbf{C}(X)})_0 = 0$, it is rationally equivalent to any other $\mathbf{C}(X)$-point, such as $(\eta, x) = x_{\mathbf{C}(X)}$ for any closed point $x \in X$. The class $[\Delta_X] - [X \times \{x\}]$ restricts to $(\eta, \eta) - (\eta, x)$ in $\mathrm{CH}_0(X_{\mathbf{C}(X)})$, hence to 0. This shows (iii).

(iii) ⇒ (iv). An element of $\Sigma^n(\{\eta\} \times X)$ extends to an element of $\Sigma^n(X \times X)$ and two such extensions agree on $U \times X$ for some dense open subset U of X; in other words, the natural map $\varinjlim_U \Sigma^n(U \times X) \to \Sigma^n(\{\eta\} \times X)$ is an isomorphism. Thus writing down the exact sequence (5) for $U \times X$ and passing to the direct limit over U, we get a commutative diagram of exact sequences

$$\begin{tikzcd}
\varinjlim_U \bigoplus_{W \in \Sigma^{n-1}(U \times X)} \mathbf{C}(W)^* \arrow[r] \arrow[d, "\wr"] & \varinjlim_U \mathbf{Z}^{(\Sigma^n(U \times X))} \arrow[r] \arrow[d, "\wr"] & \varinjlim_U \mathrm{CH}^n(U \times X) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{W \in \Sigma^{n-1}(\{\eta\} \times X)} \mathbf{C}(X)(W)^* \arrow[r] & \mathbf{Z}^{(\Sigma^n(\{\eta\} \times X))} \arrow[r] & \mathrm{CH}^n(\{\eta\} \times X) \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical arrows are isomorphisms; therefore the third vertical arrow is also an isomorphism. We conclude that the class $[\Delta] - [X \times \{x\}]$ is zero in $\mathrm{CH}^n(U \times X)$ for some dense open subset U.

(iv) ⇒ (v). The localization exact sequence (6)

$$\mathrm{CH}_n((X \smallsetminus U) \times X) \longrightarrow \mathrm{CH}_n(X \times X) \longrightarrow \mathrm{CH}_n(U \times X) \longrightarrow 0$$

implies that the class $[\Delta] - [X \times \{x\}]$ comes from a class in $\mathrm{CH}_n((X \smallsetminus U) \times X)$. Choosing any hypersurface D in X containing $X \smallsetminus U$ and pushing forward that class to $\mathrm{CH}_n(D \times X)$ does the job.

(v) $\Rightarrow$ (i). Assume that (8) holds; then it holds in $\mathrm{CH}_n(X_{\mathbf{K}} \times X_{\mathbf{K}})$ for any extension $\mathbf{K}$ of $\mathbf{C}$, so it suffices to prove $\mathrm{CH}_0(X)_0 = 0$.

Denote by pr_1 and pr_2 the two projections from $X \times X$ to X. Any class $\delta \in \mathrm{CH}_n(X \times X)$ induces a homomorphism $\delta_*\colon\ \mathrm{CH}_0(X) \to \mathrm{CH}_0(X)$, defined by $\delta_*(z) = \mathsf{pr}_{2*}(\delta \cdot \mathsf{pr}_1^*(z))$. Let us consider the classes which appear in (8). The diagonal induces the identity of $\mathrm{CH}_0(X)$; the class of $X \times \{x\}$ maps $z \in \mathrm{CH}_0(X)$ to $\deg(z)\,[x]$, hence is 0 on $\mathrm{CH}_0(X)_0$.

Now consider the class $\alpha := [\Delta_X] - [X \times \{x\}]$ supported on $D \times X$ and write it as $(i\times 1)_*(\alpha_D)$, where $\alpha_D \in \mathrm{CH}_n(D\times X)$ and $i\colon D \hookrightarrow X$ is the inclusion. Then, for $z \in \mathrm{CH}_0(X) = \mathrm{CH}^n(X)$, one has

$$\alpha_*(z) = \mathsf{pr}_{2*}((i \times 1)_*(\alpha_D) \cdot \mathsf{pr}_1^*(z)) = \mathsf{pr}_{2*}(\alpha_D \cdot \mathsf{pr}_1^*(i^*(z))).$$

Since $\dim(D) < n$, the class $i^*(z)$ is zero, hence so is $\alpha_*(z)$. We conclude from (8) that the group $\mathrm{CH}_0(X)_0$ vanishes, since $[\Delta_X]$ induces the identity of $\mathrm{CH}_0(X)_0$ and both $[X \times \{x\}]$ and $[\Delta_X] - [X \times \{x\}]$ induce 0. □

Remark 7.6 (Rational Chow decomposition of the diagonal)**.** The original argument of Bloch–Srinivas in [BS] started from a smooth projective complex variety X such that $\mathrm{CH}_0(X)_0 = 0$ and concluded that there exists a positive integer N such that

$$N([\Delta_X] - [X \times \{x\}]) \tag{9}$$

is supported on $D \times X$ (see [S3, Section 7.2] or [Vo7, Theorem 3.5] for proofs of this result and its converse: the existence of such a decomposition implies $\mathrm{CH}_0(X)_0 = 0$). This is called a *rational Chow decomposition of the diagonal* (because it is a decomposition of the diagonal, but in $\mathrm{CH}_n(X \times X)_{\mathbf{Q}}$). The analog of Proposition 7.5(iii) is that this is equivalent to saying that the class $\delta_X - [x_{\mathbf{C}(X)}]$ in $\mathrm{CH}_0(X_{\mathbf{C}(X)})$ is N-torsion. The analog of Proposition 7.5(i) is that $\mathrm{CH}_0(X_{\mathbf{L}})_0$ is an N-torsion group (with the same positive integer N) for any field extension $\mathbf{L}/\mathbf{C}$.

Remark 7.7 (Torsion order)**.** Following [S3, Definition 7.9], one may define, for any proper variety X over a field $\mathbf{k}$, its *torsion order* $\mathrm{Tors}(X) \in \mathbf{Z}_{>0} \cup \{\infty\}$ as the smallest positive integer N such that the class δ_X can be written as $N\delta_X = z_{\mathbf{k}(X)}$ in $\mathrm{CH}_0(X_{\mathbf{k}(X)})$ for some 0-cycle z on X, and ∞ if no such integer exists. This is a stable birational invariant which is finite for rationally connected smooth projective complex varieties (see [Vo5, Corollary 4.4] for a direct proof), and Proposition 7.5 says that a smooth complex projective variety is universally CH_0-trivial if and only if its torsion order is 1. If X is unirational and there is a dominant map $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ of degree d, then $\mathrm{Tors}(X) \mid d$.

Remark 7.8 (Cubic threefolds revisited)**.** We prove in Section 4.1.2, as a consequence of the Clemens–Griffiths criterion, that no smooth cubic hypersurface $X \subseteq \mathbf{P}^4_{\mathbf{C}}$ is rational. This is because its 5-dimensional intermediate Jacobian $(J(X), \theta)$ is not isomorphic to a product of Jacobians of curves. As explained in Remark 4.5, this is equivalent to saying that the minimal cohomology class $\theta^4/4!$ is not the class of an *effective* 1-cycle.

In [Vo6, Theorem 1.7], Voisin proves the remarkable result that X *is universally* CH_0*-trivial if and only if the class* $\theta^4/4!$ *is the class of a 1-cycle.* Whether this holds for all smooth cubic threefolds is an open problem, but she constructs large families of cubic threefolds for which this holds. She also constructs large families of smooth cubic fourfolds that are universally CH_0-trivial.

In general, by Example 2.1 and Remark 7.7, the torsion order of any smooth cubic hypersurface of dimension ≥ 2 is either 1 (if it is is universally CH_0-trivial) or 2 (if it is not).

7.3. **Applications.** Despite its technical aspect, Proposition 7.5 has remarkable consequences, which were worked out by Bloch–Srinivas in [BS].

Proposition 7.9. *Let X be a smooth projective complex variety. Suppose X is universally* CH_0*-trivial.*

(a) *We have $H^0(X,\Omega^r_X)=0$ for all $r>0$.*

(b) *The group $H^3(X,\mathbf{Z})$ is torsion free.*

Proof. The proof is very similar to that of the implication (v) ⇒ (i) in Proposition 7.5; we use the same notation. Again a class δ in $\mathrm{CH}^n(X\times X)$ induces a homomorphism $\delta^*\colon H^r(X,\mathbf{Z})\to H^r(X,\mathbf{Z})$, defined by $\delta^*(z):=\mathsf{pr}_{1*}(\delta\cdot\mathsf{pr}_2^*(z))$. The diagonal induces the identity, the class $[X\times\{x\}]$ gives 0 for $r>0$, and the class $(i\times 1)_*\alpha_D$ gives the homomorphism $z\mapsto i_*(\mathsf{pr}_{1*}(\alpha_D\cdot\mathsf{pr}_2^*(z)))$. Thus formula (8) gives for $r>0$ a commutative diagram[7]

(10)
$$\begin{tikzcd} & H^\bullet(D,\mathbf{Z}) \arrow[dr, "i_*"] & \\ H^r(X,\mathbf{Z}) \arrow[ur, "\mathsf{pr}_{1*}(\alpha_D\cdot\mathsf{pr}_2^*(-))"] \arrow[rr, "\mathrm{Id}"] & & H^\bullet(X,\mathbf{Z}). \end{tikzcd}$$

Since $D\subseteq X$ is a hypersurface, the homomorphism $i_*\colon H^\bullet(D,\mathbf{C})\to H^\bullet(X,\mathbf{C})$ is a morphism of Hodge structures of bidegree $(1,1)$. Therefore its image intersects trivially the subspace $H^{r,0}(X)$ of $H^r(X,\mathbf{C})$. Since i_* is surjective by (10), its image contains $H^r(X,\mathbf{C})$, hence $H^{r,0}(X)=0$.

Now we take $r=3$ in (10). The only possible part of $H^\bullet(D,\mathbf{Z})$ with a nontrivial contribution in (10) is $H^1(D,\mathbf{Z})$, which is torsion free. Any torsion element in $H^3(X,\mathbf{Z})$ goes to 0 in $H^1(D,\mathbf{Z})$, hence is zero. □

Observe that in the proof, we use only formula (8) in cohomology and not in the Chow group. The relation between these two properties is discussed in Voisin's papers [Vo3, Vo4, Vo6].

It is a fundamental conjecture of Bloch that the vanishing (a) in the proposition should imply that X is CH_0-trivial.

We summarize in the following diagram the implications that we proved between the various properties of a smooth projective complex variety that we defined.

$$\begin{tikzcd}[arrows=Rightarrow] \text{(R)} \arrow[d] & & & \\ \text{(SR)} \arrow[r] \arrow[d] & \text{univ. } \mathrm{CH}_0\text{-trivial} \arrow[r, Leftrightarrow] & \mathrm{Tors}=1 \arrow[r] \arrow[d] & H^3(\bullet,\mathbf{Z}) \text{ torsion free} \\ \text{(UR)} \arrow[rr] \arrow[d] & & \mathrm{Tors}\mid\deg \arrow[d] & \\ \text{(RC)} \arrow[r] & \mathrm{CH}_0\text{-trivial} \arrow[r] & \mathrm{Tors}<\infty. & \end{tikzcd}$$

[7]To be entirely correct, one should work on a desingularization of D.

The reason why universal CH_0-triviality has been so successful at proving new non stable rationality results is that, as the Clemens–Griffiths criterion, it behaves well under deformation (compare with Lemma 4.7).

Proposition 7.10 (Voisin). *Let $\pi\colon \mathscr{X} \to B$ be a proper flat family over a smooth variety B, with $\dim(\mathscr{X}) \geq 3$. Let $b_0 \in B$ and assume that*

- *the general fiber $\mathscr{X}_b$ is smooth;*
- *the only singuarities of $\mathscr{X}_{b_0}$ are ordinary double points;*
- *some desingularization $\widetilde{\mathscr{X}_{b_0}}$ of $\mathscr{X}_{b_0}$ is not universally CH_0-trivial.*

Then $\mathscr{X}_b$ is not universally CH_0-trivial for a very general point b of B.

We refer to [Vo4] for the proof. The idea is that there cannot exist a decomposition (8) as in Proposition 7.5 for b general in B, because it would extend to an analogous decomposition over $\mathscr{X}$, then specialize to $\mathscr{X}_{b_0}$, and finally extend to $\widetilde{\mathscr{X}_{b_0}}$. One concludes by observing that the locus of points $b \in B$ such that $\mathscr{X}_b$ is smooth and CH_0-trivial is a countable union of closed subsets.

Corollary 7.11. *The double cover of $\mathbf{P}^3$ branched along a very general quartic surface is not stably rational.*

Proof. Consider the pencil of quartic surfaces in $\mathbf{P}^3$ spanned by a smooth quartic and a quartic symmetroid, and the family of double covers of $\mathbf{P}^3$ branched along the members of this pencil. By Proposition 7.9(b), the Artin–Mumford threefold is not universally CH_0-trivial. Applying the proposition, we conclude that a very general quartic double solid is not universally CH_0-trivial, hence not stably rational. □

Any smooth quartic double solid X is a Fano variety, hence rationally connected; it is in fact even unirational (see [IP, Example 10.1.3(iii)]). Since the group $H^3(X, \mathbf{Z})$ is torsion free, Proposition 7.10 implies that both implications

$$\begin{array}{lcl} H^3(\bullet, \mathbf{Z}) \text{ torsion free} & \Longrightarrow & \text{univ. } \mathrm{CH}_0\text{-trivial} \\ & & \\ \text{(UR)} & \Longrightarrow & \text{univ. } \mathrm{CH}_0\text{-trivial} \end{array} \qquad \text{are false} \tag{11}$$

for very general quartic double solids.

More generally, Voisin shows that the desingularization of a very general quartic double solid with at most seven nodes is not stably rational. We do not know whether there exist smooth quartic double solids that are universally CH_0-trivial.

Voisin's technique has given rise to a number of other results. Colliot-Thélène and Pirutka have extended Proposition 7.10 to the case where the singular fiber $\mathscr{X}_{b_0}$ has (still sufficiently nice) nonisolated singularities and applied this to prove that a very general quartic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$ is not stably rational ([CP]). Hassett, Kresch, and Tschinkel have shown that a conic bundle with discriminant a very general plane curve of degree ≥ 6 is not stably rational ([HKT]; compare with Section 4.1.2). This allowed them to produce the first examples of smooth irrational varieties that deform to rational ones.

In [S1], Schreieder introduced a variant of the method of Voisin and Colliot-Thélène–Pirutka, which allows one to prove non stable rationality via a degeneration argument where a non universally CH_0-trivial resolution of the special fibre is not needed. He used this technique to simplify the arguments in [HPT1, HPT2, HPT3] and to apply them to large classes of quadric surface bundles. He also obtained in [S2] a dramatic improvement of the range of degrees for which very general hypersurfaces are known to be not stably rational (see Example 2.3). I recommend the excellent survey [S3] to the interested reader.

The literature on rationality questions is extremely vast and I have barely touched its surface. Also, I completely left aside many techniques such as the derived category approach to rationality problems (see [Ku] for an account).

It is therefore extremely frustating that, despite all these efforts, one cannot still answer the simple question: *are there irrational smooth cubic hypersurfaces in* $\mathbf{P}^5_{\mathbf{C}}$*?*

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