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On the Hodge and Betti Numbers of Hyper-Kähler Manifolds

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Abstract. In this survey article, we review past results (obtained by Hirzebruch, Libgober–Wood, Salamon, Gritsenko, and Guan) on Hodge and Betti numbers of Kähler manifolds, and more specifically of hyper-Kähler manifolds, culminating in the bounds obtained by Guan in 2001 on the Betti numbers of hyper-Kähler fourfolds. Let X be a compact Kähler manifold of dimension m. One consequence of the Hirzebruch–Riemann–Roch theorem is that the coefficients of the χ_y-genus polynomial

$$p_X(y) := \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbb{Z}[y]$$

are (explicit) universal polynomials in the Chern numbers of X. In 1990, Libgober–Wood determined the first three terms of the Taylor expansion of this polynomial about $y = -1$ and deduced that the Chern number $\int_X c_1(X)c_{m-1}(X)$ can be expressed in terms of the coefficients of the polynomial $p_X(y)$ (Proposition 2.1). When X is a hyper-Kähler manifold of dimension $m = 2n$, this Chern number vanishes. The Hodge diamond of X also has extra symmetries which allowed Salamon to translate the resulting identity into a linear relation between the Betti numbers of X (Corollary 2.5). When X has dimension 4, Salamon's identity gives a relation between $b_2(X)$, $b_3(X)$, and $b_4(X)$. Using a result of Verbitsky's on the injectivity of the cup-product map that produces an inequality between $b_2(X)$ and $b_4(X)$, it is easy to conclude $b_2(X) \leq 23$. Guan established in 2001 more restrictions on the Betti numbers (Theorem 3.6).

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Birkhäuser

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1. Symmetries of the Hodge Diamond of a Hyper-Kähler Manifold

Let X be a compact hyper-Kähler manifold of dimension $2n$ and let σ be a holomorphic symplectic form on X. Apart from the usual symmetries

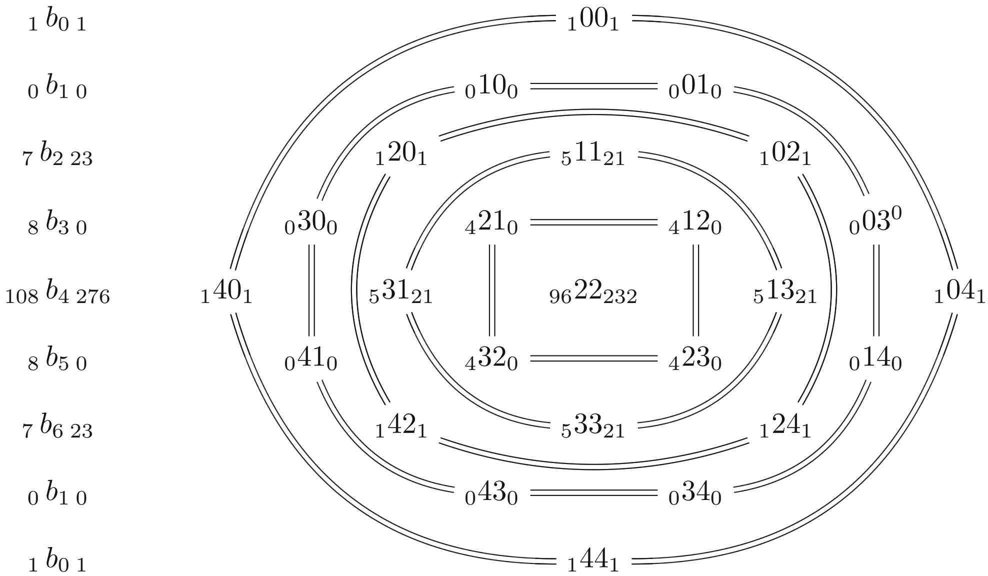
$$h^{p,q}(X) = h^{q,p}(X) = h^{2n-p,2n-q}(X)$$

coming from Kähler theory and Serre duality, there is another symmetry

$$h^{p,q}(X) = h^{2n-p,q}(X) \tag{1.1}$$

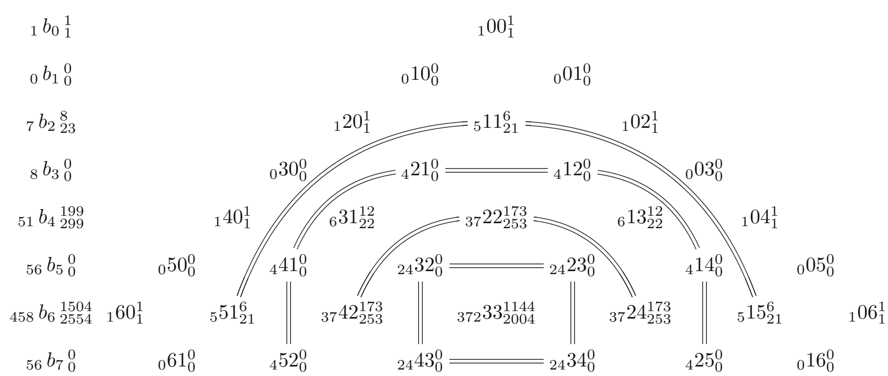
coming from the fact that the wedge product $\wedge\, \sigma^{\wedge(n-p)}$ is an isomorphism $\Omega_X^p \xrightarrow{\sim} \Omega_X^{2n-p}$. So the Hodge diamond of X has a D_8-symmetry.

Example 1.1. ($n = 2$) We represent the various symmetries of the Hodge diamond for an irreducible hyper-Kähler fourfold (note that the extra "mirror" symmetry (1.1) is only visible here on the outer edges of the diamond). In the following diagram, the Hodge numbers $h^{p,q}$ of hyper-Kähler fourfolds of Kum_2-type appear as left indices of the pq label and those for the $\mathrm{K3}^{[2]}$-type as right indices.



A priori, there are only three undetermined Hodge numbers: h^{11}, h^{21}, and h^{22}. We will see in Example 2.7 that there is a relation between them.

Example 1.2. $(n = 3)$ We represent some of the symmetries of the Hodge diamond of an irreducible hyper-Kähler sixfold. In the following diagram, the Hodge numbers $h^{p,q}$ of hyper-Kähler sixfolds of Kum_3-type appear as left indices of the pq label, those for the $\mathrm{K3}^{[3]}$-type as right indices, and those for the OG6-type as right exponents.



A priori, there are only six undetermined Hodge numbers: h^{11}, h^{21}, h^{31}, h^{22}, h^{32}, and h^{33}. We will see in Example 2.8 that there is a relation between them.

2. Salamon's Results on Betti Numbers

2.1. Hirzebruch–Riemann–Roch

Let X be a compact Kähler manifold of dimension m. Following [6], we set

$$\chi^p(X) := \sum_{q=0}^{m} (-1)^q h^{p,q}(X) = \chi(X, \Omega_X^p).$$

By Serre duality, these numbers satisfy

$$\chi^p(X) = (-1)^m \chi^{m-p}(X) \tag{2.1}$$

and we define the χ_y-genus by the formula

$$p_X(y) := \sum_{p=0}^{m} \chi^p(X) y^p = \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbb{Z}[y]. \tag{2.2}$$

For instance,

- $p_X(0) = \chi^0(X) = \chi(X, \mathcal{O}_X)$,
- $p_X(-1) = \chi_{\mathrm{top}}(X) = e(X)$,
- $p_X(1)$ is the signature of the intersection form on $H^m(X, \mathbb{R})$ (which vanishes when m is odd).

Serre duality translates into the reciprocity property $(-y)^m p_X\big(\frac{1}{y}\big) = p_X(y)$.

One consequence of the Hirzebruch–Riemann–Roch theorem is that $\chi^p(X)$ can be expressed as a universal polynomial $T_{m,p}(c_1, \ldots, c_m)$ in the Chern classes of X evaluated on X [6, Section IV.21.3, (10)], that is,

$$p_X(y) = \sum_{p=0}^{m} y^p \int_X T_{m,p}(c_1(X), \ldots, c_m(X)) = \int_X T_m(y)(c_1(X), \ldots, c_m(X)), \tag{2.3}$$

where $T_m(y) := \sum_{p=0}^{m} T_{m,p} y^p$, a polynomial with coefficients in $\mathbb{Q}[c_1, \ldots, c_m]$. One has

- $T_{m,p} = (-1)^m T_{m,m-p}$ and $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$;
- $T_{m,0} = \mathrm{td}_m(c_1, \ldots, c_m)$.

Libgober–Wood found in [11, Lemma 2.2] the first three terms of the Taylor expansion of the polynomial $T_m(y)$ about -1:

$$T_m(y-1) = c_m - \tfrac{1}{2} m c_m y + \tfrac{1}{12}\big(\tfrac{1}{2} m(3m-5) c_m + c_1 c_{m-1}\big) y^2 + \cdots \tag{2.4}$$

The following is [11, Proposition 2.3] (reproved later in [15, Theorem 4.1]).

Proposition 2.1. (Libgober–Wood) *If X is a compact Kähler manifold of dimension m, one has the relation*

$$\int_X c_1(X) c_{m-1}(X) = \sum_{p=0}^{m} (-1)^p \big(6p^2 - \tfrac{1}{2} m(3m+1)\big) \chi^p(X). \tag{2.5}$$

Proof. The Taylor expansion of the polynomial p_X about the point -1 is

$$\begin{aligned} p_X(y-1) &= \sum_{p=0}^{m} \chi^p(X)(y-1)^p \\ &= \sum_{p=0}^{m}(-1)^p\chi^p(X) + y\sum_{p=0}^{m}(-1)^{p-1}\binom{p}{1}\chi^p(X) + y^2\sum_{p=0}^{m}(-1)^p\binom{p}{2}\chi^p(X) + \cdots \end{aligned}$$

Using the Hirzebruch–Riemann–Roch theorem (2.3) and comparing with (2.4), we get, by identifying the coefficients, the relations[1]

$$\begin{aligned} p_X(-1) &= \int_X c_m(X) = \sum_{p=0}^{m}(-1)^p\chi^p(X), \\ p'_X(-1) &= -\tfrac{1}{2}m\int_X c_m(X) = \sum_{p=0}^{m}(-1)^{p-1}p\chi^p(X), \\ p''_X(-1) &= \tfrac{1}{6}\int_X \big(\tfrac{1}{2}m(3m-5)c_m(X) + c_1(X)c_{m-1}(X)\big) = 2\sum_{p=0}^{m}(-1)^p\binom{p}{2}\chi^p(X), \end{aligned} \tag{2.6}$$

from which it is not difficult to get (2.5). □

The following consequence of Proposition 2.1 was obtained in [4, (1.14) and Proposition 2.4] using modular forms (see also [7]).[2]

Corollary 2.2. (Gritsenko) *If X is a compact Kähler manifold of dimension m that satisfies $c_1(X)_{\mathbb{R}} = 0$, one has*

$$\tfrac{1}{12}\, me(X) = \sum_{p=0}^{m}(-1)^p\big(\tfrac{1}{2}m-p\big)^2\chi^p(X) = 2\sum_{0\le p<m/2}(-1)^p\big(\tfrac{1}{2}m-p\big)^2\chi^p(X). \tag{2.7}$$

In particular, when m is even,[3] *$me(X)$ is divisible by* 24.

Proof. The first equality in (2.7) is easily obtained from the relations (2.6), and the second equality from the symmetries (2.1). □

Remark 2.3. Salamon gives in [15, p. 145] the next two terms of the expansion (2.4) (see also [10, Proposition 3.1(4)]):

[1] The first two relations are in fact formally equivalent upon using the symmetries (2.1), which give

$$p'_X(-1) = \sum_{p=0}^{m}(-1)^{p-1}(m-p)\chi^p(X) = -mp_X(-1) - p'_X(-1)$$

(see Remark 2.3).

[2] Gritsenko also gives in [4, (1.13)] relations between the $\chi^p(X)$ when $m \in \{4,6,8,10\}$, but they are all rewritings of (2.7).

[3] This assumption is missing from [4], but it is necessary: when m is odd and we write $m = 2n+1$, we have

$$\tfrac{m-3}{12}e(X) = 2\sum_{0\le p\le n}(-1)^p\Big(\big(\tfrac{1}{2}m-p\big)^2 - \tfrac{1}{4}\Big)\chi^p(X) = 2\sum_{0\le p\le n}(-1)^p\big(n(n+1) - p(2n+1) + p^2\big)\chi^p(X),$$

which is divisible by 4. So what we get is that $\frac{m-3}{2}e(X)$ is divisible by 24.

$$
\begin{aligned}
T_m(y-1) = c_m &- \tfrac{1}{2} m c_m y + \tfrac{1}{12}\big(\tfrac{1}{2}m(3m-5)c_m + c_1c_{m-1}\big)y^2 \\
&- \tfrac{1}{24}(m-2)\big(\tfrac{1}{2}m(m-3)c_m + c_1c_{m-1}\big)y^3 \\
&+ \tfrac{1}{5760}\big(m(15m^3-150m^2+485m-502)c_m + 4(15n^2-85n+108)c_1c_{m-1} \\
&+ 8(c_1^2+3c_2)c_{m-2} - 8(c_1^3-3c_1c_2+3c_3)c_{m-3}\big)y^4 + \cdots
\end{aligned}
$$

The y^3-term does not bring any new information since it is in fact a formal consequence of the reciprocity property $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$.

Using this expansion, J. Schmitt was able to find the following analogue of the Libgober–Wood formula (2.5) for a compact Kähler manifold X of dimension m:

$$
\begin{aligned}
&\int_X \big((\tfrac{1}{3}c_1^2(X)+c_2(X))c_{m-2}(X) - (\tfrac{1}{3}c_1^3(X) - c_1(X)c_2(X) + c_3(X))c_{m-3}(X)\big) \\
&\quad = \sum_{p=0}^{m} (-1)^p \big(10p^4 + (2-5m-15m^2)p^2 \\
&\quad + \frac{1}{24} m(5m+1)(15m^2+3m-2)\big)\chi^p(X). \qquad (2.8)
\end{aligned}
$$

On a hyper-Kähler manifold, where all the odd Chern classes vanish, the left side reduces to $\int_X c_2(X)c_{m-2}(X)$.

Remark 2.4. The polynomials T_m can be computed. Setting for simplicity $c_1 = 0$ (the case of interest for us), we have, for even dimensions $m \in \{2,4,6\}$ (see [11] or [2, Section 9]),

$$
\begin{aligned}
T_2(y-1) &= c_2 - c_2 y + \tfrac{1}{12}c_2 y^2, \\
T_4(y-1) &= c_4 - 2c_4 y + \tfrac{7}{6}c_4 y^2 - \tfrac{1}{6}c_4 y^3 + \tfrac{1}{720}(3c_2^2 - c_4)y^4, \\
T_6(y-1) &= c_6 - 3c_6 y + \tfrac{13}{4}c_6 y^2 - \tfrac{3}{2}c_6 y^3 + \tfrac{1}{240}(-c_3^2 + c_2c_4 + 62c_6)y^4 \\
&\quad + \tfrac{1}{720}(3c_3^2 - 3c_2c_4 - 6c_6)y^5 + \tfrac{1}{60480}(10c_2^3 - c_3^2 - 9c_2c_4 + 2c_6)y^6.
\end{aligned}
$$

Setting $\chi := \mathrm{td}_m$ (this is the constant term and leading coefficient of T_m), we get

$$
\begin{aligned}
T_2(y) &= \chi + (2\chi - c_2)y + \chi y^2, \\
T_4(y) &= \chi + (4\chi - \tfrac{1}{6}c_4)y + (6\chi + \tfrac{2}{3}c_4)y^2 + (4\chi - \tfrac{1}{6}c_4)y^3 + \chi y^4. \qquad (2.9)
\end{aligned}
$$

2.2. Application to Hyper-Kähler Manifolds

Assume now that m is even and that we have the extra "mirror" symmetry $h^{p,q}(X) = h^{m-p,q}(X)$ like we do when X is a hyper-Kähler manifold. We define polynomials

$$
h_X(s,t) := \sum_{p,q=0}^{m} h^{p,q}(X) s^p t^q \in \mathbb{Z}[s,t],
$$

$$
b_X(t) := \sum_{j=0}^{2m} b_j(X) t^j = h_X(t,t).
$$

The polynomial h_X is symmetric and $p_X(y) = h_X(-1, y)$. Now we use the evenness of m and the extra symmetry to get

$$\begin{aligned}\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) &= \sum_{p,q=0}^{m} pq(-1)^{p+q} h^{p,q}(X) \\ &= \sum_{p,q=0}^{m} (m-p)q(-1)^{m-p+q} h^{p,q}(X) \\ &= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) + m \sum_{p,q=0}^{m} q(-1)^{p+q} h^{p,q}(X) \\ &= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) - m\,\frac{\partial h_X}{\partial t}(-1,-1),\end{aligned}$$

so that

$$2\,\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) = -m\,\frac{\partial h_X}{\partial t}(-1,-1) = -m p'_X(-1). \tag{2.10}$$

In terms of the polynomial b_X, we have, by symmetry of h_X,

$$\begin{aligned} b'_X(t) &= 2\,\frac{\partial h_X}{\partial t}(t,t), \\ b''_X(t) &= 2\,\frac{\partial^2 h_X}{\partial s \partial t}(t,t) + 2\,\frac{\partial^2 h_X}{\partial t^2}(t,t), \end{aligned}$$

so that we get, using (2.10),

$$b'_X(-1) = 2p'_X(-1), \quad b''_X(-1) = -m p'_X(-1) + 2p''_X(-1). \tag{2.11}$$

Proceeding as in the proof of Proposition 2.1, we write the Taylor expansion of the polynomial b_X about the point -1:

$$\begin{aligned} b_X(t-1) &= \sum_{j=0}^{2m} b_j(X)(t-1)^j \\ &= \sum_{j=0}^{2m} b_j(X)(-1)^j + t \sum_{j=0}^{2m} b_j(-1)^{j-1}\binom{j}{1} + t^2 \sum_{j=0}^{2m} b_j(X)(-1)^j \binom{j}{2} + \cdots \end{aligned}$$

Using (2.11) and (2.6), we get

$$\sum_{j=0}^{2m} b_j(-1)^j j = -b'_X(-1) = -2p'_X(-1) = m\int_X c_m(X),$$

$$\begin{aligned}\sum_{j=0}^{2m} b_j(X)(-1)^j \binom{j}{2} &= \tfrac{1}{2} b''_X(-1) = -\tfrac{1}{2} m p'_X(-1) + p''_X(-1) \\ &= \tfrac{1}{4} m^2 \int_X c_m(X) + \tfrac{1}{6}\int_X \big(\tfrac{1}{2} m(3m-5) c_m(X) + c_1(X) c_{m-1}(X)\big).\end{aligned}$$

Putting everything together, we obtain the analogue of (2.5) [15, Theorem 4.1]:

$$2\int_X c_1(X)c_{m-1}(X) = \sum_{j=0}^{2m}(-1)^j(6j^2 - m(6m+1))b_j(X).$$

Corollary 2.5. (Salamon) *If X is a compact hyper-Kähler manifold of dimension $2n$, one has*[4]

$$\sum_{j=0}^{4n}(-1)^j(3j^2 - n(12n+1))b_j(X) = 0.$$

Using the symmetry $b_j = b_{4n-j}$, one checks that one gets the equivalent relations (in the spirit of (2.7))

$$ne(X) = 6\sum_{j=1}^{2n}(-1)^j j^2 b_{2n-j}(X), \quad nb_{2n}(X) = 2\sum_{j=1}^{2n}(-1)^j(3j^2 - n)b_{2n-j}(X).$$

Example 2.6. ($n = 1$) We obtain $b_2(X) = 22$ and $e(X) = 24$.

Example 2.7. ($n = 2$) Salamon's relation reads

$$b_4(X) = 46 + 10b_2(X) - b_3(X).$$

On an irreducible hyper-Kähler fourfold, because of the symmetries, there are only 3 unknown Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, and $h^{22}(X)$. One has

$$b_2(X) = 2 + h^{11}(X), \quad b_3(X) = 2h^{21}(X), \quad b_4(X) = 2 + 2h^{11}(X) + h^{22}(X).$$

Salamon's relation translates into

$$h^{22}(X) = 64 + 8h^{11}(X) - 2h^{21}(X).$$

There are two Chern numbers, $c_4 := \int_X c_4(X) = e(X)$ and $c_2^2 := \int_X c_2(X)^2$. They satisfy

$$3 = \chi(X, \mathcal{O}_X) = T_4(0) = \int_X \mathrm{td}_4(X) = \tfrac{1}{720}(3c_2^2 - c_4). \tag{2.12}$$

But we also have, using (2.9),

$$\chi^1(X) = 12 - \tfrac{1}{6}c_4, \quad \chi^2(X) = 18 + \tfrac{2}{3}c_4. \tag{2.13}$$

A priori though, the value of c_4 is not enough to determine all the Hodge numbers but, once we know c_4, one Hodge number determines all the others.

The Chern numbers for the two known deformation types of irreducible hyper-Kähler fourfolds are in the following table.

	$\chi_{\mathrm{top}} = e = c_4$	c_2^2
Kum_2	108	756
$\mathrm{K3}^{[2]}$	324	828

[4] There is a misprint in [9, 24.4.2].

Example 2.8. ($n = 3$) Salamon's relation reads

$$b_6(X) = 70 + 30b_2(X) - 16b_3(X) + 6b_4(X).$$

Because of the symmetries, there are only 6 undetermined Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, $h^{31}(X)$, $h^{22}(X)$, $h^{32}(X)$, and $h^{33}(X)$. One has

$$\begin{aligned} b_2(X) &= 2 + h^{11}(X), \\ b_3(X) &= 2h^{21}(X), \\ b_4(X) &= 2 + 2h^{31}(X) + h^{22}(X), \\ b_5(X) &= 2h^{41}(X) + 2h^{32}(X), \\ b_6(X) &= 2 + 2h^{11}(X) + 2h^{22}(X) + h^{33}(X). \end{aligned}$$

Salamon's relation translates into

$$h^{33}(X) = 140 + 28h^{11}(X) - 32h^{21}(X) + 12h^{31}(X) + 4h^{22}(X).$$

There are three Chern numbers, $c_6 := \int_X c_6(X) = e(X)$, $c_2c_4 := \int_X c_2(X)c_4(X)$, and $c_2^3 := \int_X c_2(X)^3$. They satisfy

$$4 = \chi(X, \mathcal{O}_X) = T_6(0) = \mathrm{td}_6(X) = \tfrac{1}{60480}(10c_2^3 - 9c_2c_4 + 2c_6).$$

The three known examples in dimension 6 are in the following table taken from [14, Remark 4.13] (see also [13, Appendix A]) and [12, Corollary 6.8].

	$\chi_{\mathrm{top}} = e(X) = c_6$	c_2c_4	c_2^3
Kum_3	448	6784	30208
$\mathrm{K3}^{[3]}$	3200	14720	36800
OG6	1920	7680	30720

3. Guan's Bounds for Betti Numbers of Hyper-Kähler Fourfolds

3.1. Bounds on b_2

Let X be an irreducible compact hyper-Kähler manifold of complex dimension $m = 2n$. Let σ be a symplectic form on X. One has $b_1(X) = 0$, and $b_2(X) \geq 3$ since $H^{2,0}(X) = \mathbb{C}\sigma$, $H^{0,2}(X) = \mathbb{C}\bar{\sigma}$, and $H^{1,1}(X)$ contains the class of any Kähler form.

Our aim is to prove the following upper bound for $b_2(X)$ when $m = 4$ [5, Theorem 1].

Theorem 3.1. (Guan) *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then $3 \leq b_2(X) \leq 23$. Moreover, if $b_2(X) = 23$, the Hodge numbers of X are the same as the Hodge numbers of the Hilbert square of a K3 surface.*

About the higher Betti numbers, we have the following result ([16, Theorem 1.5], [1, Theorem 1.5]).

Theorem 3.2. (Verbitsky) *Let X be an irreducible compact hyper-Kähler manifold of dimension $2n$. For all $k \leq n$, the canonical map* $\mathrm{Sym}^k H^2(X,\mathbb{R}) \to H^{2k}(X,\mathbb{R})$ *given by cup-product is injective. In particular,* $b_{2k}(X) \geq \binom{b_2(X)+k-1}{k}$.

We denote by $SH^{2k}(X) \subset H^{2k}(X,\mathbb{R})$ the image of the map above.

Proof of Theorem 3.1. Write b_j for $b_j(X)$. We have $b_3+b_4 = 46+10b_2$ (Example 2.7) and $b_4 \geq \frac{b_2(b_2+1)}{2}$ (Theorem 3.2), hence

$$\frac{b_2(b_2+1)}{2} \leq b_3 + \frac{b_2(b_2+1)}{2} \leq b_3 + b_4 = 46 + 10b_2 \tag{3.1}$$

which can be rewritten as

$$(b_2+4)(b_2-23) \leq 0,$$

so $b_2 \leq 23$. Assume now $b_2 = 23$. Substituting in the inequality above, we get $b_3 + 276 \leq 46 + 230 = 276$, so $b_3 = 0$. This implies $b_4 = 46 + 10b_2 = 276$. So the Betti numbers of X are the same as those of the Hilbert square of a K3 surface. As noted in Example 2.7, this implies that the Hodge numbers are also the same. □

3.2. Generalized Chern Numbers

For an irreducible compact hyper-Kähler manifold X of dimension $2n$, we have the Beauville–Bogomolov–Fujiki quadratic form q_X on $H^2(X,\mathbb{Q})$ ([3] or [9, Section 23]). There exists a positive rational constant c_X such that

$$\forall \beta \in H^2(X,\mathbb{Q}) \qquad \int_X \beta^{2n} = c_X q_X(\beta)^n. \tag{3.2}$$

More generally, let $\alpha \in H^{4j}(X,\mathbb{R})$ be a class that is of type $(2j,2j)$ on all small deformations of X (this is the case for example for the Chern class $c_{2j}(X)$). There is a constant $c_\alpha \in \mathbb{R}$ such that [9, Corollary 23.17]

$$\forall \beta \in H^2(X,\mathbb{R}) \qquad \int_X \alpha\beta^{2(n-j)} = c_\alpha q_X(\beta)^{n-j}. \tag{3.3}$$

For $\alpha = 1$ and $j = 0$, we recover (3.2).

We can now define the generalized Chern numbers.

Definition 3.3. Let $C \in H^{4j}(X,\mathbb{C})$ be a polynomial in the Chern classes. The number

$$N(C) := \frac{\int_X Cu^{2(n-j)}}{\left(\int_X u^{2n}\right)^{\frac{n-j}{n}}}$$

is independent of the choice of $u \in H^2(X,\mathbb{C})$ with $\int_X u^{2n} \neq 0$. We call it a *generalized Chern number* of X.

To see that $N(C)$ does not depend on the choice of u, note that $\int_X Cu^{2(n-j)} = a_C q_X(u)^{n-j}$, where a_C is the sum of the c_α as in (3.3) for all monomials α in C. Moreover, $\int_X u^{2n} = c_X q_X(u)^n$, so $N(C) = a_C c_X^{-\frac{n-j}{n}}$; it is a real number since we can always choose u in $H^{4j}(X,\mathbb{R})$.

In our case, $n = 2$, we are interested in the generalized Chern number $N(c_2(X))$. Guan rewrote [8, (1)] as follows [5, Lemma 2].

Lemma 3.4. *Let X be an irreducible compact hyper-Kähler manifold of dimension $2n$. Then*[5]

$$\frac{((2n)!)^{n-1}N(c_2(X))^n}{(24n(2n-2)!)^n} = \int_X \mathrm{td}^{\frac{1}{2}}(X). \tag{3.4}$$

Moreover $N(c_2(X)) > 0$.

Proof. For any hyper-Kähler manifold X, one has $\int_X(\sigma+\bar{\sigma})^{2n} = c_X q_X(\sigma+\bar{\sigma})^n > 0$. Hence we can write

$$N(c_2(X)) = \frac{\int_X c_2(X)(\sigma+\bar{\sigma})^{2n-2}}{(\int_X(\sigma+\bar{\sigma})^{2n})^{\frac{n-1}{n}}}.$$

The lemma therefore follows from the equality

$$\frac{\|R\|^{2n}}{(192\pi^2 n)^n \operatorname{vol}(X)^{n-1}} = \int_X \mathrm{td}^{\frac{1}{2}}(X) \tag{3.5}$$

from [8, (1)],[6] where

- $\operatorname{vol}(X) = \frac{1}{2^{2n}(2n)!}\int_X(\sigma+\bar{\sigma})^{2n}$ is the volume form on X,
- $\|R\|$ is the L^2-norm of the Riemann curvature tensor, given by

$$\|R\|^2 = \frac{8\pi^2}{2^{2n-2}(2n-2)!}\int_X c_2(X)(\sigma+\bar{\sigma})^{2n-2}.$$

Note that $\int_X c_2(X)(\sigma+\bar{\sigma})^{2n-2}$ is nonnegative, since it is a positive multiple of $\|R\|^2$. If it vanishes, X is flat, hence a torus by the Bieberbach theorem, which is absurd. □

The following proposition is [5, Lemma 3].

Proposition 3.5. (Guan) *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then*

$$3b_2(X)N(c_2(X))^2 \leq (b_2(X)+2)\int_X c_2(X)^2. \tag{3.6}$$

Equality holds if and only if $c_2(X) \in SH^4(X)$.

Proof. The orthogonal complement $SH^4(X)^\perp$ of $SH^4(X)$ in $H^4(X,\mathbb{R})$ with respect to the intersection form consists of primitive classes. Therefore, by the second Hodge–Riemann bilinear relations, the intersection form is positive definite on $SH^4(X)^\perp$ and one has $H^4(X,\mathbb{R}) = SH^4(X) \oplus SH^4(X)^\perp$.

Let us write $c_2(X) = p + r$ with $p \in SH^4(X)$ and $r \in SH^4(X)^\perp$. As noted above, one has $\int_X r^2 \geq 0$, with equality if and only if $r = 0$.

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[5] Hitchin and Sawon, and then Guan, use the $\hat{A}^{\frac{1}{2}}$-genus instead of $\mathrm{td}^{\frac{1}{2}}$. In general, one has $\hat{A} = e^{c_1/2}\,\mathrm{td}$, so they coincide in our case since $c_1 = 0$.

[6] The authors of [5,8] use a different convention for exterior products of differential forms. The latter can be seen either as elements of the abstract exterior algebra of the space of 1-forms or as alternating multilinear forms: depending on the point of view, the two definitions of product between differential forms differ by a binomial coefficient. So, if we follow Hitchin and Sawon and we write $\operatorname{vol}(X) = \frac{1}{2^{2n}((n)!)^2}\int_X \sigma^n\bar{\sigma}^n$ and $\|R\|^2 = \frac{8\pi^2}{2^{2n-2}((n-1)!)^2}\int_X c_2(X)\sigma^{n-1}\bar{\sigma}^{n-1}$, then (3.4) becomes $\frac{((2n)!)^{n-1}N(c_2(X))^n}{(24n(2n-2)!)^n}\cdot\frac{\binom{2(n-1)}{n-1}^n}{\binom{2n}{n}^{n-1}} = \int_X \mathrm{td}^{\frac{1}{2}}(X)$.

For every $\beta \in H^2(X, \mathbb{R})$, one has, using (3.3),

$$\int_X p\beta^2 = \int_X c_2(X)\beta^2 = cq_X(\beta), \tag{3.7}$$

where $c := c_{c_2(X)}$. Write b for $b_2(X)$. Let $(e_1, \dots, e_b)$ be a basis of $H^2(X, \mathbb{C})$ which is orthonormal with respect to q_X. For all $t_1, t_2, t_3, t_4 \in \mathbb{R}$ and pairwise distinct i, j, k, l, we have

$$\int_X (t_1e_i + t_2e_j + t_3e_k + t_4e_l)^4 = c_X q_X(t_1e_i + t_2e_j + t_3e_k + t_4e_l)^2 = c_X(t_1^2 + t_2^2 + t_3^2 + t_4^2)^2,$$

which implies

$$\int_X e_i^4 = c_X, \quad \int_X e_i^2 e_j^2 = \frac{1}{3}c_X, \quad \int_X e_i^2 e_j e_k = \int_X e_i e_j e_k e_l = 0. \tag{3.8}$$

Write $p = \sum_{1 \le i \le j \le b} p_{ij} e_i \cdot e_j$. Using (3.7) and (3.8), we obtain, for $i \neq j$,

$$0 = \int_X p e_i e_j = \frac{1}{3} c_X p_{ij},$$

hence $p_{ij} = 0$. Similarly, for each i, we have

$$c = \int_X p e_i^2 = c_X p_{ii} + \frac{1}{3} c_X \sum_{j \neq i} p_{ii}.$$

Summing over $i \in \{1, \dots, b\}$, we obtain

$$bc = c_X \sum_i p_{ii} + \frac{1}{3} c_X (b-1) \sum_i p_{ii} = \frac{c_X(b+2)}{3} \sum_i p_{ii}.$$

Using these relations, we obtain

$$\int_X p^2 = \sum_i p_{ii} \int_X p e_i^2 = c \sum_i p_{ii} = \frac{3bc^2}{c_X(b+2)}.$$

Finally, Definition 3.3 gives

$$N(c_2(X)) = \frac{\int_X c_2(X) e_1^2}{\left(\int_X e_1^4\right)^{1/2}} = \frac{\int_X p e_1^2}{\left(\int_X e_1^4\right)^{1/2}} = c\, c_X^{-1/2}.$$

Putting everything together, we obtain

$$\int_X c_2(X)^2 = \int_X p^2 + \int_X r^2 \ge \int_X p^2 = \frac{3bc^2}{c_X(b+2)} = \frac{3bN(c_2(X))^2}{b+2},$$

which is the desired inequality. Equality holds if and only if $\int_X r^2 = 0$. As we saw earlier, this is equivalent to $r = 0$, that is, $c_2(X) \in SH^4(X)$. □

3.3. Bounds on b_3

Let again X be an irreducible compact hyper-Kähler manifold of dimension 4. A formal computation shows

$$\int_X \mathrm{td}^{\frac{1}{2}}(X) = \frac{1}{5760}\int_X (7c_2(X)^2 - 4c_4(X)). \tag{3.9}$$

The following result is [5, Theorem 2].

Theorem 3.6. (Guan) *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then*

$$b_3(X) \leq \frac{4(23-b_2(X))(8-b_2(X))}{b_2(X)+1}. \tag{3.10}$$

If $b_2(X) > 7$, then $(b_2(X), b_3(X)) \in \{(8,0),(23,0)\}$.

Proof. Write b_j for $b_j(X)$, c_2^2 for $\int_X c_2(X)^2$, and c_4 for $\int_X c_4(X)$. We substitute Lemma 3.4, with $n = 2$, in Proposition 3.5 to obtain

$$3b_2\,\frac{(24\cdot 4)^2}{4!}\int_X \mathrm{td}^{\frac{1}{2}}(X) \leq (b_2+2)c_2^2.$$

Substituting in (3.9) the expression for c_4 given in (2.12), we get

$$\int_X \mathrm{td}^{\frac{1}{2}}(X) = \frac{1}{5760}\big(7c_2^2 - 4(3c_2^2 - 720\cdot 3)\big) = \frac{3}{2} - \frac{c_2^2}{1152}.$$

Hence

$$(b_2+2)c_2^2 \geq 2\cdot 24^2\, b_2\int_X \mathrm{td}^{\frac{1}{2}}(X)4 = 2\cdot 24^2\, b_2\Big(\frac{3}{2} - \frac{c_2^2}{1152}\Big) = b_2(3\cdot 24^2 - c_2^2). \tag{3.11}$$

We have $h^{1,1}(X) - 2h^{2,1}(X) = \chi^1(X) = 12 - \frac{c_4}{6}$ (see (2.13)); using

$$b_2 = 2 + h^{1,1}(X), \quad b_3 = 2h^{1,2}(X),$$

we obtain $c_4 = 3(16 + 4b_2 - b_3)$. We use this in (2.12) to get $c_2^2 = 736 + 4b_2 - b_3$. Then, (3.11) becomes $(b_2+1)b_3 \leq 4(23-b_2)(8-b_2)$ as in the statement of the theorem.

If $b_2 > 7$, the right side of (3.10) is nonpositive because $b_2 \leq 23$, so it has to be zero. □

The following is [5, Corollary 1].

Corollary 3.7. (Guan) *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. If $b_2(X) \leq 7$, one of the following holds:*

- $b_2(X) = 3$ *and* $b_3(X) = 4\ell$ *with* $\ell \leq 17$;
- $b_2(X) = 4$ *and* $b_3(X) = 4\ell$ *with* $\ell \leq 15$;
- $b_2(X) = 5$ *and* $b_3(X) = 4\ell$ *with* $\ell \leq 9$;
- $b_2(X) = 6$ *and* $b_3(X) = 4\ell$ *with* $\ell \leq 4$;
- $b_2(X) = 7$ *and* $b_3(X) = 4\ell$ *with* $\ell \in \{0,2\}$.

Proof. By [3, Lemma 1.2], one has $4 \mid b_k$ for k odd. The bounds are obtained using either (3.1) or (3.10). Guan proved in [5] that the case $(b_2(X), b_3(X)) = (7,4)$ cannot occur. □

Remark 3.8. When $b_2(X) = 7$, either $b_3(X) = 0$ or the Hodge numbers of X are the same as the Hodge numbers of a generalized Kummer fourfold.

Remark 3.9. Given $(b_2(X), b_3(X))$, one can compute $N(c_2(X))$ using Lemma 3.4, since the Chern numbers of X are computed in the proof of Theorem 3.6. Then it is possible to check which pairs give an equality in (3.6). Hence, using Proposition 3.5, one can check that $c_2(X) \in SH^4(X)$ if and only if $(b_2(X), b_3(X)) \in \{(5, 36), (7, 8), (8, 0), (23, 0)\}$.

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